# STABILITY IN GAGLIARDO-NIRENBERG INEQUALITIES

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**Abstract** *The purpose of this paper is to establish a quantitative and constructive stability result for a class of subcritical Gagliardo-Nirenberg inequalities. We develop a new strategy, in which the flow of the fast diffusion equation is used as a tool: a stability result in the inequality is equivalent to an improved rate of convergence to equilibrium for the flow. In both cases, the tail behaviour plays a key role. The regularity properties of the parabolic flow allow us to connect an improved entropy - entropy production inequality during the initial time layer to spectral properties of a suitable linearized problem which is relevant for the asymptotic time layer. Altogether, the stability in the inequalities is measured by a deficit which controls in strong norms the distance to the manifold of optimal functions.*

## 1 Introduction and main results

In the study of functional inequalities, the existence of an optimal function and its characterization is a standard problem of the calculus of variations. Let us consider the following Gagliardo-Nirenberg inequality, also known in the literature as the Gagliardo-Nirenberg-Sobolev inequality, that can be written as

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\ge\mathcal{C}_{\rm GN}\,\|f\|_{2\,p}\quad\forall\,f\in\mathcal{D}(\mathbb{R}^d)\,,\tag{1}$$

where $\mathcal{D}(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ with compact support. The exponent $p$ is in the range $(1,+\infty)$ if $d=1$ or 2, and $p\in\big(1,d/(d-2)\big]$ if $d\ge3$. The exponent $\theta=\frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$ is determined by the scaling invariance. According to [27],

$$\mathsf{g}(x)=\Big(1+|x|^2\Big)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d\tag{2}$$

is an optimal function of (1), and the set of all optimal functions is the manifold of the *Aubin-Talenti type* functions $g_{\lambda,\mu,y}(x):=\mu\,\mathsf{g}\,((x-y)/\lambda)$ parametrized by $(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d$.

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Inequality (1) can also be written in non-scale invariant form as

$$\delta[f] := (p-1)^2 \, \|\nabla f\|_2^2 + 4\,\frac{d-p\,(d-2)}{p+1}\, \|f\|_{p+1}^{p+1} - 2\,\mathcal{K}_{\mathrm{GN}}\, \|f\|_{2p}^{2p\gamma} \geq 0 \tag{3}$$

where

$$\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}\,,$$

and equality is again achieved by $\mathsf{g}$. See Section 4.3 for the equivalence of (1) and (3). We will call $\delta$ the *deficit functional*.

In this work we study the *stability* properties of the Gagliardo-Nirenberg inequality (1). The main question that we want to address here is

$$\textit{If } \delta[f] \textit{ is small, in what sense, if any, is } f \textit{ close to } \mathsf{g}? \tag{Q}$$

In the critical case $p = p^*$, with $p^* := d/(d-2) = 2^*/2$, $d \geq 3$, G. Bianchi and H. Egnell proved in [4] the existence of a positive constant $\mathcal{C}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d \, \|f\|_{2^*}^2 \geq \mathcal{C} \, \inf \|\nabla f - \nabla g\|_2^2 \,, \tag{4}$$

where $\mathsf{S}_d$ is the optimal constant in Sobolev's inequality and the infimum is taken over the $(d+2)$-dimensional manifold of the Aubin-Talenti functions $g_{\lambda,\mu,y}$. This result was immediately recognized as a major breakthrough, with the irritating drawback that the constant $\mathcal{C}$ is still unknown, because the existence of $\mathcal{C}$ is obtained by contradiction and no constructive estimate has been obtained so far.

Our goal is to establish a *quantitative and constructive* analogue of the estimate of G. Bianchi and H. Egnell in the subcritical range $p \in (1, p^*)$, where we adopt the convention that $p^* = +\infty$ if $d = 1$ or $d = 2$. More specifically, we aim at proving that $\delta[f]$ controls a distance to the function $\mathsf{g}$ under some suitable assumptions, up to an explicit multiplicative constant. Here we devise an entirely new strategy based on a nonlinear flow and its fine regularity properties. We relate in a quantitative way an initial time layer with a properly linearized problem in the asymptotic regime. Our method applies to the entire family of Gagliardo-Nirenberg inequalities (1) with subcritical exponent $p$ and it breaks in the critical case, that is, in the case of Sobolev's inequality. In any case, we feel that our result sheds a new light on a quantitative stability theory in functional interpolation inequalities.

Before stating our main result, we need to introduce the *free energy* or *relative entropy functional*

$$\mathcal{E}[f] := \frac{2\,p}{1-p} \int_{\mathbb{R}^d} \left( |f|^{p+1} - \mathsf{g}^{p+1} - \tfrac{1+p}{2\,p}\, \mathsf{g}^{1-p} \left( |f|^{2\,p} - \mathsf{g}^{2\,p} \right) \right) dx \,,$$

which is a nonnegative functional. The free energy $\mathcal{E}$ is naturally associated to the fast diffusion flow and in Section 2 the relation between $\delta$ and $\mathcal{E}$ will be clarified. It is interesting to notice that if $\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|\mathsf{g}\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$, then, by the Csiszár-Kullback inequality, the entropy controls the $\mathrm{L}^1$ distance between $|f|^{2p}$ and $\mathsf{g}^{2p}$, namely there exists a constant $C_p > 0$ such that

$$\left\||f|^{2p} - \mathsf{g}^{2p}\right\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p \sqrt{\mathcal{E}[f]}\,.$$

For details, see Appendix D.

Let us denote by $\mathcal{W}$ the Sobolev space of measurable functions $f$ on $\mathbb{R}^d$ such that $|\nabla f|$ is in $\mathrm{L}^2(\mathbb{R}^d)$ and $x \mapsto |x|^2\,|f|^{2p}$ is integrable. This moment condition is subtle and will be discussed in Section 4.5. We are now in the position of stating our main result.

**Theorem 1.** *Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is a positive constant $\mathcal{C}$ such that*

$$\delta[f] \ge \mathcal{C}\,\mathcal{E}[f]$$

*for any $f \in \mathcal{W}$ such that*

$$\int_{\mathbb{R}^d} |f|^{2p}\,dx = \int_{\mathbb{R}^d} |\mathsf{g}|^{2p}\,dx \quad \textit{and} \quad \int_{\mathbb{R}^d} x\,|f|^{2p}\,dx = 0\,, \tag{5}$$

$$\sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f|^{2p}\,dx \le A \quad \textit{and} \quad \mathcal{E}[f] \le G\,. \tag{6}$$

A non-optimal yet constructive estimate of the constant $\mathcal{C}$ is given by

$$\mathcal{C} = \mathcal{C}_\star \left(1 + A^{\frac{2\,(p-1)}{d-p\,(d-4)}} + G\right)^{-1}$$

for some constant $\mathcal{C}_\star$ which depends only on $p$ and $d$. Moreover $\mathcal{C}_\star$ is positive and finite for any $p \in (1, p^*)$. Limit cases are discussed at the end of Section 4.2. Condition (5) is intended for selecting specific functions among all *Aubin-Talenti type* functions $g_{\lambda,\mu,y}$ and a more general statement can be written by playing with normalization, translation and scaling: see Theorem 15, which is our most general and deepest result on stability for Gagliardo-Nirenberg inequalities. The statement requires preliminary results and additional notation: for these reasons, it is stated in Section 4. Also see important remarks on the functional spaces in Section 4.5. The restriction (6) is more severe than (5) as it is needed for a regularity result which is at the core of our method: see Section 3. Our proof is *constructive* in the sense that the constant $\mathcal{C}_\star$ can be computed: we do not give a fully explicit expression here, as it is rather complicated and has no interest by itself, but refer to [10] for an expression and details on all intermediate steps in the proofs.

It seems that (4) involves a stronger norm in the right-hand side, but one can play the same game as in [4] and prove that any nonnegative function $f \in \mathcal{W}$ satisfies

$$\delta[f] \geq \frac{(p-1)^2\,(p+1)\,\mathcal{C}}{(p+1)\,\mathcal{C}+4\,(p-1)}\int_{\mathbb{R}^d}\left|\nabla f+\tfrac{1}{p-1}\,f^p\,\nabla \mathsf{g}^{1-p}\right|^2 dx \tag{7}$$

under the assumptions of Theorem 1: see Section 4.2.

The central idea of our method is to use *entropy methods* and nonlinear flows, take advantage of improved entropy – entropy production inequalities during an initial time layer (due to the nonlinearity of the flow) and during an asymptotic time layer (due to an improved spectral gap in an associated spectral problem). Because of the regularization properties, we obtain an estimate of a threshold time that connects the two regimes and allows us to obtain explicit estimates. It is known indeed from [27] that Gagliardo-Nirenberg inequalities (1) and the *fast diffusion equation*

$$\frac{\partial u}{\partial t} = \Delta u^m\,,\quad u(t=0,\cdot) = u_0 \tag{8}$$

are deeply connected through entropy - entropy production inequalities which measure the decay rate of the relative entropy along the fast diffusion flow. The set of nonnegative functions $u \in \mathrm{L}^1(\mathbb{R}^d)$ such that

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}}\int_{|x|>r} u\,dx \leq A < \infty \tag{$\mathrm{H}_A$}$$

where $m_c = (d-2)/d$ and $A$ is a positive parameter, is stable under the action of the flow, see [13, Proposition 5.3] and [81, Chapter 4]. Our strategy is based on this stability property and on the uniform convergence in relative error which arises from the Hölder regularity of the parabolic flow (8) and a *global Harnack Principle*, see Section 3.

As we borrow tools from various fields of parabolic regularity theory, calculus of variations, and entropy methods, we cannot pretend to any exhaustivity in our list of references. The flow associated with (8) is used as a tool for proving a stability result. Nonlinear flows have already been exploited to identify optimality cases in functional inequalities. Much less has been achieved so far for proving a *quantitative stability result* in a fundamental inequality such as (1). We list a few results in this direction below but claim that our method is the first one to relate the estimates based on entropy methods and the regularity properties of the flow. We have tried to keep notations consistent with the literature as much as possible, but the reader is invited to pay attention to normalizations or definitions, which may slightly differ from some earlier papers, for instance by numerical constants.

Now let us give a synthetic overview of the literature. The characterization of optimal functions in functional inequalities is a standard problem in nonlinear analysis and in the calculus of variations: see for instance [64, 66, 67]. The issue of the *stability* of the optimal functions started with the study of solitary waves obtained by minimization methods as in [22, 58, 87]. In recent years, the problem of finding stability estimates for the various sharp inequalities in analysis and geometry, such as the isoperimetric inequality, the Brunn–Minkowski inequality, the Sobolev inequality, the logarithmic Sobolev inequality, *etc.*, has been intensively studied. See for instance [51, 52, 49, 50]. In the case of Sobolev and Gagliardo-Nirenberg inequalities, some pioneering results were obtained in bounded domains in [16, 45], but the breakthrough came with the result [4] of G. Bianchi and H. Egnell. In the same spirit, several other results have been obtained. The stability of the Sobolev inequalities in the case of an $\mathrm{L}^q$ norm of the gradient with $q \neq 2$ was proven by A. Cianchi, N. Fusco, F. Maggi and A. Pratelli in [23] and more recently by A. Figalli, R. Neumayer and Y. R.-L. Zhang in [53, 70, 54] by a different method. Also see [48] for a related result for $q = 2$. For the Gagliardo-Nirenberg inequality (1), some stability results have been obtained by A. Figalli and E. Carlen in [17], F. Seuffert in [80] and V.H. Nguyen in [72], with non-constructive constants. Also see [47] for an introduction to stability issues and some consequences of the known results, and [18, 77] for stability results for inequalities other than (1). All these results are *quantitative*, in the sense that a precise notion of distance is controlled by the deficit of the inequality, but the proportionality constant is achieved by compactness, through a contradiction argument, and no estimate on the constant is given. In this sense, these methods are *not constructive*. By duality and flows, quantitative and constructive results were obtained in [33, 40], however by controlling a weaker norm than in the Bianchi-Egnell approach.

In the subcritical regime, the situation is slightly better than for the critical case of Sobolev inequalities. *Constructive* results have been obtained in [7] and can also be deduced from [6] in a very restricted neighborhood of the manifold of the Aubin-Talenti functions (as it is measured in the uniform norm associated with the relative error). The global result of [42] is clearly sub-optimal as the remainder terms is of the order of the square of the entropy while one would expect a linear term in the entropy. This result has been rephrased in [44] and relies on scaling considerations, which are actually not so deep. Progresses have been achieved in subcritical interpolation inequalities on the sphere in [38, 34], with a constant which is obtained through an explicit, standard variational problem whose value is however not known, except in the limit case of the logarithmic Sobolev inequality or if additional symmetry assumptions are imposed.

Various proofs of the optimality of the Aubin-Talenti type function defined by (2) are known: by direct variational methods in [27]; using the *carré du champ method* first at formal level in [19, 20] and then with a complete proof in [21] (also see the simpler presentation of [60]); by mass transport in [24]; by a continuous dimension argument in [3, 80]. As far as the evolution problem is concerned, the global existence of a nonnegative solution of (8) is established in [59] for any $0 < m < 1$. The role of self-similar solutions in large time asymptotics is, for instance, studied in [55, 27]. Much more is known on (8) and we refer to [85, 86] for a global overview. Asymptotic rates of convergence have been studied in various papers among which we can quote [6, 7, 41, 28, 61, 21].

Entropy - entropy production inequalities as in [27] and the concavity of the generalized Rényi entropy powers along the fast diffusion flow as in [43, 78] are equivalent as shown in [36]. To our knowledge, the use of the quotient associated with the entropy - entropy production inequality has not been invoked yet. Although elementary, this is one of the key points for controlling the decay rate of the entropy during the *initial time layer* (see Section 2.2). The analysis of the decay rate of the entropy in the *asymptotic time layer* is more classical and essentially known from [6]. It relies on a spectral gap property that goes back to [79, 29, 30, 5]. The key issue of this paper is to control the *threshold time* between these two regimes and relies on a quantitative regularity theory. As a side remark, let us recall that the fast diffusion equation can be formally interpreted as the gradient flow of the entropy with respect to the Wasserstein distance, according to [74]. Such an interpretation is possible only if the second moment is controlled, which corresponds to the functional framework discussed in Section 4.5.

A very interesting extension of the Gagliardo-Nirenberg inequalities (1) is the case of Caffarelli-Kohn-Nirenberg-Sobolev inequalities, with an important issue concerning symmetry breaking studied in [37, 39], and a deep connection with a generalization of the flow of (8): see [8, 9] and [12, 13] for regularity and asymptotics issues. These last two papers, and [14, 15], are at the basis of our quantitative estimates of Section 3 dealing with the regularity of the solutions of (8), see also [10]. This is the crux of the paper and definitely its most technical part. Some of the results presented there were already contained in the PhD thesis of N. Simonov, [81, Chapters 4 and 6]. According to [13, 81], ($\mathrm{H}_A$) is not only sufficient but also necessary to establish a result valid for all initial data, at least by our method. Concerning the *global Harnack Principle*, *i.e.*, the comparison with Barenblatt functions on the whole Euclidean space, we shall primarily refer to [85, Theorem 4.8] and [14, 13]. Also see [84] for an early application. For more introductory references on Harnack's inequalities in the framework of nonlinear diffusions, which usually refer to local results, we shall quote [26, Chapter 1] and [32, Chapter 6]. *Convergence in relative error* has already been addressed in [21, 84, 85], however without any construction of the constants. This convergence has also been exploited in the framework of entropy methods in [6], but in a much more restricted functional framework. The precise statement of the *global Harnack Principle* and its application to the uniform convergence in relative error is the main topic addressed in [14] and has recently been characterized by two of the authors in [13]. This has been exploited in [5, 6, 11, 7, 9, 8, 13] to study sharp decay rates for fast diffusion flows, including in the more general case of Caffarelli-Kohn-Nirenberg weights, however with no explicit and constructive estimates. Building such explicit estimates is the main technical task of this paper.

For a number of classical or elementary statements concerning constants for which we are not aware of published work or detailed enough expressions, we provide supplementary material in [10]. In most of the cases, these estimates follow from classical techniques but their proofs involve lengthy and tedious computations, and are not available from the existing literature. Some simple constants and useful identities are also collected in Appendix E. We shall speak of *numerical constants* for explicit expressions depending only on $d$ and $m$ or $p$.

This paper is organized as follows. In Section 2, we prove some estimates on entropy – entropy production inequalities and explain the role of the initial and asymptotic time layers in our method. Section 3 is devoted to a proof of the *uniform convergence in relative error* of the solution (8) to a Barenblatt profile, based a quantitative Harnack inequality, and an explicit estimate of the threshold time between the initial and asymptotic time layers. The proof of Theorem 1 and some additional results are exposed in Section 4. Various interpolation inequalities of independent interest and useful identities are collected in a series of appendices.

# 2 Relative entropy and fast diffusion flow

## 2.1 Definitions and basic results

For any $m \in (m_1, 1)$ with $m_1 := (d-1)/d$, let us consider the *fast diffusion equation* in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \cdot \left( v \, \nabla v^{m-1} \right) = 2 \, \nabla \cdot (x \, v) \,, \quad v(t=0,\cdot) = v_0 \tag{9}$$

with a nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$. The relation between the flow in (8) and the one in (9) are explained in Section 2.4. Equation (9) admits the Barenblatt profile

$$\mathcal{B}(x) = \left( 1 + |x|^2 \right)^{\frac{1}{m-1}} \quad \forall \, x \in \mathbb{R}^d$$

as a stationary solution of mass

$$\mathcal{M} := \int_{\mathbb{R}^d} \mathcal{B} \, dx = \pi^{\frac{d}{2}} \, \frac{\Gamma\left( \frac{1}{1-m} - \frac{d}{2} \right)}{\Gamma\left( \frac{1}{1-m} \right)} \,. \tag{10}$$

Further properties of $\mathcal{B}$ are listed in Appendix E. We shall assume that the mass of $v_0$ is taken equal to $\mathcal{M}$. It is then well known that the mass is conserved, *i.e.*,

$$\int_{\mathbb{R}^d} v(t,x) \, dx = \mathcal{M} \quad \forall \, t \geq 0 \,,$$

as well as the center of mass

$$\int_{\mathbb{R}^d} x \, v(t,x) \, dx = \int_{\mathbb{R}^d} x \, v_0(x) \, dx \quad \forall \, t \geq 0 \,.$$

The *free energy* (or *relative entropy*) and the *Fisher information* (or *relative entropy production*) are defined respectively by

$$\mathcal{F}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \Big( v^m - \mathcal{B}^m - m\, \mathcal{B}^{m-1} \, (v - \mathcal{B}) \Big) \, dx$$

and

$$\mathcal{I}[v] := \frac{m}{1-m} \int_{\mathbb{R}^d} v \, \left| \nabla v^{m-1} - \nabla \mathcal{B}^{m-1} \right|^2 dx\,.$$

Notice that $(1+|x|^2)\, v = \mathcal{B}^{m-1}\, v$ is integrable under assumption $(\mathrm{H}_A)$ for any $m > m_1$ (and $m > 1/2$ if $d = 1$). It is known from [27] that inequality (3) is equivalent to the *entropy - entropy production* inequality

$$\mathcal{I}[v] \geq 4\, \mathcal{F}[v]\,. \tag{11}$$

The quotient

$$\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$$

is well defined if $v \neq \mathcal{B}$ and, as a consequence of (11), we have the bound

$$\mathcal{Q}[v] \geq 4\,. \tag{12}$$

We also learn from [7] that $\lim_{\varepsilon \to 0} \mathcal{Q}[\mathcal{B} + \varepsilon\, b_1] = 4$ where $b_1 := \partial \mathcal{B}/\partial x_1$. In [71, 75], it was proved that for a solution $v$ of (9), $\frac{d}{dt}\mathcal{F}[v(t,\cdot)] = -\,\mathcal{I}[v(t,\cdot)]$ and we read from [20, 21, 19] that $\frac{d}{dt}\mathcal{I}[v(t,\cdot)] \leq -\,4\,\mathcal{I}[v(t,\cdot)]$, which can be rewritten as

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}\,(\mathcal{Q} - 4)\,. \tag{13}$$

Here, with a slight abuse of notations, $\mathcal{Q}(t)$ stands for $Q[v(t,\cdot)]$. Our goal in this section is to prove that the bound (12) can be improved under additional conditions. We distinguish an *initial time layer* $(0,T)$ and an *asymptotic time layer* $(T, +\infty)$. In the first case (see Section 2.2) we exploit (13) while the improvement for large values of $t$ is based on spectral considerations (see Section 2.3) as in [6].

### 2.2 The initial time layer improvement

On the interval $(0,T)$, we prove a uniform positive lower bound on $Q[v(t,\cdot)] - 4$ if we know that $Q[v(T,\cdot)] - 4 > 0$. The precise result goes as follows.

**Lemma 2.** *Assume that $v$ is a solution to (9) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] < +\infty$ and $\int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M}$. If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T,\cdot)] \geq 4 + \eta$, then we also have*

$$\mathcal{Q}[v(t,\cdot)] \geq \; 4 + \frac{4\,\eta\, e^{-4\,(T-t)}}{4 + \eta - \eta\, e^{-4\,(T-t)}} \quad \forall\, t \in [0,T]\,. \tag{14}$$

Notice that the right-hand side in (14) is monotone increasing in $t$, so that we also have the lower bound

$$\mathcal{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}} \quad \forall\,t\in[0,T]\,.$$

*Proof.* The estimate (14) follows by integrating the Bernouilli differential inequality (13) on the interval $[t,T]$. □

### 2.3 The asymptotic time layer improvement

Let us define the *linearized free energy* and the *linearized Fisher information* by

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} |g|^2\,\mathcal{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathcal{B}\,dx\,,$$

in such a way that

$$\mathsf{F}[g] = \lim_{\varepsilon\to 0}\varepsilon^{-2}\,\mathcal{F}[\mathcal{B}+\varepsilon\,\mathcal{B}^{2-m}\,g] \quad \text{and} \quad \mathsf{I}[g] = \lim_{\varepsilon\to 0}\varepsilon^{-2}\,\mathcal{I}[\mathcal{B}+\varepsilon\,\mathcal{B}^{2-m}\,g]\,.$$

By the *Hardy-Poincaré inequality* of [6], which can also be obtained as a consequence of (11), if $d\geq 1$ and $m\in(m_1,1)$, then for any function $g\in \mathrm{L}^2(\mathbb{R}^d,\mathcal{B}^{2-m}\,dx)$ such that $\nabla g\in \mathrm{L}^2(\mathbb{R}^d,\mathcal{B}\,dx)$ and $\int_{\mathbb{R}^d} g\,\mathcal{B}^{2-m}\,dx = 0$, we have

$$\mathsf{I}[g] \geq 4\,\mathsf{F}[g]\,.$$

According to the *improved Hardy-Poincaré inequality* of [7, Lemma 1] (also see [41, Proposition 1] and [79, 29, 30] for related spectral results), if additionally we assume that $\int_{\mathbb{R}^d} x\,g\,\mathcal{B}^{2-m}\,dx = 0$, then we have

$$\mathsf{I}[g] \geq 4\,\alpha\,\mathsf{F}[g]\,. \tag{15}$$

where $\alpha = 2-d\,(1-m)$. Our purpose is to use (15) in order to establish an improved lower bound for $\mathcal{Q}[v(t,\cdot)]$ in the asymptotic time layer as $t\to+\infty$. We have the following result on a time-interval $(T,+\infty)$.

**Proposition 3.** *Let $m\in(m_1,1)$ if $d\geq 2$, $m\in(1/3,1)$ if $d=1$, $\eta = 2\,d\,(m-m_1)$ and $\chi = m/(266+56\,m)$. If $v$ is a nonnegative solution to (9) of mass $\mathcal{M}$, with*

$$(1-\varepsilon)\,\mathcal{B} \leq v(t,\cdot) \leq (1+\varepsilon)\,\mathcal{B} \quad \forall\,t\geq T \tag{$H_{\varepsilon,T}$}$$

*for some $\varepsilon\in(0,\chi\,\eta)$ and $T>0$, and such that $\int_{\mathbb{R}^d} x\,v(t,x)\,dx = 0$, then we have*

$$\mathcal{Q}[v(t,\cdot)] \geq 4+\eta \quad \forall\,t\geq T\,. \tag{16}$$

*Proof.* We estimate the free energy $\mathcal{F}$ and the Fisher information $\mathcal{I}$ in terms of their linearized counterparts $\mathsf{F}$ and $\mathsf{I}$ as in [6]. Notice that Assumption ($H_{\varepsilon,T}$) is not exactly the same as in [6], which motivates a restriction on $m$ if $d=1$. Let

$$g := v\,\mathcal{B}^{m-2} - \mathcal{B}^{m-1}\,.$$

Under Assumption ($H_{\varepsilon,T}$), we learn from [6, Lemma 3] that

$$(1+\varepsilon)^{-a}\,\mathsf{F}[g(t,\cdot)] \le \mathcal{F}[v(t,\cdot)] \le (1-\varepsilon)^{-a}\,\mathsf{F}[g(t,\cdot)] \quad \forall\,t\ge T\,, \tag{17}$$

where $a=2-m$, and

$$\mathsf{I}[g] \le s_1(\varepsilon)\,\mathcal{I}[v] + s_2(\varepsilon)\,\mathsf{F}[g] \tag{18}$$

from [6, Lemma 7], where

$$s_1(\varepsilon) := \frac{(1+\varepsilon)^{2\,a}}{1-\varepsilon} \quad \text{and} \quad s_2(\varepsilon) := \frac{2\,d}{m}\,(1-m)^2\left(\frac{(1+\varepsilon)^{2\,a}}{(1-\varepsilon)^{2\,a}}-1\right).$$

The estimate (17) follows from a simple Taylor expansion while (18) is a consequence of some slightly more complicated but still elementary computations. Collecting (15), (17) and (18), elementary computations show that (16) holds if $\varepsilon\in(0,\chi\,\eta)$. Details can be found in [10, Section 2]. $\square$

## 2.4 Entropy, flow, self-similarity and inequalities

Let us conclude this section by collecting some miscellaneous properties which will be useful in the sequel.

A very standard consequence of (11) is that a solution of (9) with nonnegative initial datum $v_0\in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0]<+\infty$ satisfies the estimate

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[v_0]\,e^{-4\,t} \quad \forall\,t\ge 0\,.$$

Under the assumptions of Lemma 2 and Proposition 3, we have

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\,t\in[0,T]\,, \quad \text{with} \quad \zeta = \frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}$$

and

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[v(T,\cdot)]\,e^{-(4+\eta)\,(t-T)} \quad \forall\,t\in[T,+\infty)\,.$$

Solutions of (8) are transformed into solutions of (9) with same initial data $u_0=v_0$ by the *self-similar change of variables*

$$u(t,x) = \frac{\mu^d}{R(t)^d}\,v\left(\frac{1}{2}\log R(t), \frac{\mu\,x}{R(t)}\right) \tag{19}$$

where $\mu$ and $R$ are given respectively

$$\mu := \left(\tfrac{1-m}{2\,m}\right)^{\frac{1}{\alpha}} \tag{20}$$

and

$$R(t) = (1+\alpha\,t)^{1/\alpha}\,, \quad \alpha = d\,(m-m_c)\,, \quad m_c = (d-2)/d\,. \tag{21}$$

Reciprocally, the function

$$B(t,x) = \frac{\mu^d}{R(t)^d}\,\mathcal{B}\left(\frac{\mu\,x}{R(t)}\right) \tag{22}$$

is a self-similar solution of (8) which describes the so-called *intermediate asymptotics* of the solution $u$ of (8), that is, the large time behaviour of $u$ under the condition that $\int_{\mathbb{R}^d} u_0\,dx = \mathcal{M}$. If we relax this condition, any nonnegative solution $u$ of (8) such that $\int_{\mathbb{R}^d} u_0\,dx = M$ is attracted by the Barenblatt self-similar solution of (8) of mass $M$ defined by

$$B\Big(t\,,\,x\,;\,M\Big) := \left(\tfrac{M}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\,\frac{\mu^d}{R(t)^d}\,\mathcal{B}\left(\left(\tfrac{M}{\mathcal{M}}\right)^{\frac{1-m}{\alpha}}\,\frac{\mu\,x}{R(t)}\right)\,. \tag{23}$$

The above profile $B(t,x;M)$ translated in time by a parameter $\tau$ is still a solution to (8). The choice $\tau = -\frac{1}{\alpha}$ is a remarkable one since

$$B\Big(t-\tfrac{1}{\alpha}\,,\,x\,;\,M\Big) \rightharpoonup M\,\delta_{x=0} \quad \text{as} \quad t\to 0_+\,,$$

in the sense of distributions. Such a profile can be written as

$$B\left(t-\tfrac{1}{\alpha}\,,\,x\,;\,M\right) = \left(\tfrac{M}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\,\frac{\mathsf{b}^d}{t^{\frac{d}{\alpha}}}\,\mathcal{B}\left(\left(\tfrac{M}{\mathcal{M}}\right)^{\frac{1-m}{\alpha}}\,\frac{\mathsf{b}}{t^{\frac{1}{\alpha}}}\,x\right) \quad \text{with} \quad \mathsf{b} = \left(\tfrac{1-m}{2\,m\,\alpha}\right)^{\frac{1}{\alpha}}\,. \tag{24}$$

The *entropy - entropy production* inequality (11) is equivalent to (1). With

$$p = \frac{1}{2\,m-1}\,, \quad v = |f|^{2\,p}\,, \quad \gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)} \quad \text{and} \quad \|f\|_{2\,p}^{2\,p} = \mathcal{M}\,, \tag{25}$$

we have indeed the identity

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] = \tfrac{p+1}{p-1}\left((p-1)^2\,\|\nabla f\|_2^2 + 4\,\tfrac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - 2\,\mathcal{K}_{\rm GN}\,\|f\|_{2\,p}^{2\,p\,\gamma}\right) \tag{26}$$

where both sides vanish if $\|f\|_{2p}^{2p} = \mathcal{M}$ and $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$ and $f = \mathsf{g}$, with $\mathsf{g}$ given by (2). Notice that the restriction $m > 1/2$ in dimension $d = 1$ guarantees that $p$ is well defined and positive. In higher dimensions, we always have $m_1 \geq 1/2$. The constant $\mathcal{K}_{\rm GN}$ is related to $\mathcal{C}_{\rm GN}$ by identity (97), see further details in Section 4.3. Notice that the range $p \in (1,p^*)$ corresponds to $m \in (m_1,1)$. We look for an improvement of (1), that we shall actually prove on $\mathcal{I}[u] - 4\,\mathcal{F}[u]$ using the initial and the asymptotic time layers as in Sections 2.2 and 2.3. The difficulty is of course to prove that ($H_{\varepsilon,T}$) is satisfied for some threshold time $T > 0$, which is the purpose of the next section.

## 3 Uniform convergence in relative error

### 3.1 Statement and strategy of the proof

The main result of the section is the following estimate on the *uniform convergence in relative error.*

**Theorem 4.** *Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit time $t_\star \geq 0$ such that, if $u$ is a solution of (8) with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying ($\mathrm{H}_A$), $\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,dx$ and $\mathcal{F}[u_0] \leq G$, then*

$$\sup_{x\in\mathbb{R}^d} \left|\frac{u(t,x)}{B(t,x)} - 1\right| \leq \varepsilon \quad \forall\, t \geq t_\star\,.$$

The main point of this result is that $t_\star$ is uniform with respect to the initial datum $u_0$ and depends only on $d$, $m$, $\varepsilon$, $A > 0$, and $G$. The restriction $m > 1/3$ when $d = 1$ is needed to ensure that $x \mapsto |x|^2\, u(t,x)$ is integrable for $t \geq 0$ a.e. The range needed for the proof of Theorem 1 is anyway limited to $(1/2, 1)$ because the exponent $p$ given by (25) is taken positive. The parameter $\varepsilon$ is allowed to take any value in an interval $(0, \varepsilon_{m,d})$ for some $\varepsilon_{m,d} \in (0, 1/2)$ which will be explicitly given later.

The proof relies first on local estimates, from above and from below, for which we provide explicit constants in Section 3.2. In Section 3.3, we state a *global Harnack Principle* which allows us to compare the solution $u$ with Barenblatt functions, however with different masses. The novelty is that we make the dependence on $u_0$ explicit. To prove the uniform convergence in relative error, we have to compare $u$ with the Barenblatt function $\mathcal{B}$ with same mass as $u$. This is done in Section 3.4 outside of a large ball in $x$, or for large values of $t$ and up to a multiplicative constant. Explicit Hölder continuity estimates are then needed to obtain uniform estimates in relative error in a ball in $x$: see Section 3.5. Collecting all estimates in Section 3.6, we prove Theorem 4 and establish an explicit estimate of $t_\star$, which is stated in Proposition 12.

### 3.2 Local estimates

Here we state local upper and lower bounds and provide explicit constants. Additional details can be found in [10]. Let us start by an $\mathrm{L}^1$-$\mathrm{L}^\infty$ estimate.

**Lemma 5.** *Under the assumptions of Theorem 4, there exists a numerical positive constant $\overline{\kappa}$ such that any solution $u$ of (8), with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, satisfies for all $(t, R) \in (0, +\infty)^2$ the estimate*

$$\sup_{y\in B_{R/2}(x)} u(t,y) \leq \overline{\kappa}\left(\frac{1}{t^{d/\alpha}}\left(\int_{B_R(x)} u_0(y)\,dy\right)^{2/\alpha} + \left(\frac{t}{R^2}\right)^{\frac{1}{1-m}}\right). \tag{27}$$

Lemma 5 is is well known, *cf.* [31, 32, 26, 15], except for the value of the numerical constant $\overline{\kappa}$. A constructive proof, however with no detailed expression of $\overline{\kappa}$, can be found in [12, 15]. The novelty is that (27) holds with

$$\overline{\kappa} = \mathsf{k}\,\mathcal{K}^{\frac{2\,q}{\beta}} \tag{28}$$

where $\mathcal{K}$ is the constant in the interpolation inequality

$$\|f\|^2_{\mathrm{L}^{p_m}(B)} \le \mathcal{K}\left(\|\nabla f\|^2_{\mathrm{L}^2(B)} + \|f\|^2_{\mathrm{L}^2(B)}\right) \tag{29}$$

on the unit ball $B \subset \mathbb{R}^d$. Here $\mathsf{k} = \mathsf{k}(m, d, \beta, q)$ is a numerical constant given by

$$\mathsf{k}^\beta = \left(\tfrac{4\,\beta}{\beta+2}\right)^\beta \left(\tfrac{4}{\beta+2}\right)^2 \pi^{\,8\,(q+1)}\, e^{8\sum_{j=0}^\infty \log(j+1)\left(\frac{q}{q+1}\right)^j}\, 2^{\frac{2\,m}{1-m}}\,(1+\mathfrak{a}\,\omega_d)^2\,\mathfrak{b}$$

where

$$\omega_d := |\mathbb{S}^{d-1}| = \tfrac{2\,\pi^{d/2}}{\Gamma(d/2)}\,,$$

$$\mathfrak{a} = \tfrac{3\,(16\,(d+1)\,(3+m))^{\frac{1}{1-m}}}{(2-m)\,(1-m)^{\frac{m}{1-m}}} + \tfrac{2^{\frac{d-m\,(d+1)}{1-m}}}{3^d\,d} \quad \text{and} \quad \mathfrak{b} = \tfrac{38^{\,2\,(q+1)}}{\left(1-(2/3)^{\frac{\beta}{4\,(q+1)}}\right)^{4\,(q+1)}}\,.$$

The parameters $\beta$ and $q$ depend on the dimension $d$ and are given in Table 1 where we recall that $\alpha = d\,(m - m_c)$. The value of $\mathcal{K}$ is deduced from Lemma 17 (in Appendix B) for $d \ge 3$ and otherwise from the results of Appendix C. It is, to our knowledge, new. The details of the computation of $\overline{\kappa}$ can be found in [10].

| | $p_m$ | $\mathcal{K}$ | $q$ | $\beta$ |
|---|---|---|---|---|
| $d \ge 3$ | $\frac{2\,d}{d-2}$ | $\frac{2}{\pi}\,\Gamma(\frac{d}{2}+1)^{2/d}$ | $\frac{d}{2}$ | $\alpha$ |
| $d = 2$ | 4 | $\frac{2}{\sqrt{\pi}}$ | 2 | $2\,(\alpha-1)$ |
| $d = 1$ | $\frac{4}{m}$ | $2^{1+\frac{m}{2}}\ \max\left(\frac{2\,(2-m)}{m\,\pi^2}, \frac{1}{4}\right)$ | $\frac{2}{2-m}$ | $\frac{2\,m}{2-m}$ |

Table 1: Table of the parameters and the constant in inequality(29) in dimensions $d = 1$, $d = 2$ and $d \ge 3$. The latter case corresponds to the critical Sobolev exponent while the inequality for $d \le 2$ is subcritical.

The second estimate is a lower bound in which, again, the novelty is the explicit form of the numerical constants.

**Lemma 6.** *Under the assumptions of Theorem 4, there exists two positive numerical constants $\kappa_\star$ and $\kappa$ such that any solution $u$ of (8), with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, satisfies for all $R > 0$ the estimate*

$$\inf_{|x-x_0|\le R} u(t,x) \ge \kappa\left(R^{-2}\,t\right)^{\frac{1}{1-m}} \quad \forall\, t \in [0, 2\,\underline{t}] \ \textit{with}\ \underline{t} := \tfrac{\kappa_\star}{2}\,\|u_0\|^{1-m}_{\mathrm{L}^1(B_R(x_0))}\,R^\alpha\,. \tag{30}$$

With $\overline{\kappa}$ as in Lemma 5, the precise form of the constants is

$$\kappa_\star = 2^{3\,\alpha+2}\,d^{\alpha} \quad \text{and} \quad \kappa = \alpha\,\omega_d\left(\frac{(1-m)^4}{2^{38}\,d^{4}\,\pi^{16\,(1-m)\,\alpha}\,\overline{\kappa}^{\,\alpha^2\,(1-m)}}\right)^{\frac{2}{(1-m)^2\,\alpha\,d}}. \tag{31}$$

We refer to [14, 85, 12] for a constructive proof of Lemma 6. The details of the computation of $\kappa_\star$ and $\kappa$ can be found in [10].

### 3.3 Global Harnack Principle

We prove that the solution of (8) is bounded from above (Proposition 7) and from below (Proposition 8) by two Barenblatt functions as defined in (23). Compared to the existing literature [14, 12, 85], we provide a simpler proof and explicit constants.

**Proposition 7.** *Under the assumptions of Theorem 4, there exist positive constants $\overline{t}$ and $\overline{M}$ such that any solution $u$ satisfies*

$$u(t,x) \le\; B\Big(t+\overline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\overline{M}\Big) \quad \forall\,(t,x)\in[\,\overline{t},+\infty)\times\mathbb{R}^d\,. \tag{32}$$

The expressions of $\overline{t}$ and $\overline{M}$ are given in (36) and in (37) respectively. Here $\overline{M}$ is a numerical constant. The reader may notice that the factor $1/\alpha$ causes no harm in the definition of the Barenblatt profile, see (24).

*Proof of Proposition 7.* The proof is divided in several steps, and follows the standard strategy, but here we keep track of all the constants.

*Step 1. A priori estimates on the solution.* By taking $R\to\infty$ in (27), we deduce that

$$u(t,x) \le \overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,t^{-\frac{d}{\alpha}} \quad \forall\,(t,x)\in(0,+\infty)\times\mathbb{R}^d\,, \tag{33}$$

where $\overline{\kappa}$ is as in Lemma 5. Let us choose

$$t_0 := \,A^{1-m}\,, \tag{34}$$

$x_0\neq 0$ and $R=|x_0|/4$, so that $B_R(x_0)\subset B_R^c(0)$. Using ($\mathrm{H}_A$) and (34), we deduce from (27) that

$$u(t_0,x_0) \le \overline{\kappa}\left(\frac{4^{\frac{2}{1-m}}}{t_0^{\frac{d}{\alpha}}}\,\frac{A^{\frac{2}{\alpha}}}{|x_0|^{\frac{2}{1-m}}} + 2^{\frac{4}{1-m}}\left(\frac{t_0}{|x_0|^2}\right)^{\frac{1}{1-m}}\right) \le 2^{1+\frac{4}{1-m}}\,\frac{t_0^{\frac{1}{1-m}}}{|x_0|^{\frac{2}{1-m}}}\,\overline{\kappa}\,. \tag{35}$$

*Step 2. Proof of (32) at time $t_0$.* Let us define

$$c := \max\Big\{1,\,2^{5-m}\,\overline{\kappa}^{1-m}\,\mathsf{b}^{\alpha}\Big\}\,,\quad \overline{t} := c\,t_0\,, \tag{36}$$

and

$$\overline{M} := 2^{\frac{\alpha}{2\,(1-m)}}\,\overline{\kappa}^{\frac{\alpha}{2}}\,(1+c)^{\frac{d}{2}}\,\mathtt{b}^{-\frac{d\,\alpha}{2}}\,\mathcal{M}^2\,, \tag{37}$$

where $\mathtt{b}$ is as in (24). Let us also define the auxiliary function

$$\lambda(t) := \left(\tfrac{\overline{M}}{\mathcal{M}}\right)^{\frac{1-m}{\alpha}}\mathtt{b}\,t^{-\frac{1}{\alpha}}\,(1+c)^{-\frac{1}{\alpha}}\quad\text{so that}\quad\lambda(t_0) = \left(\tfrac{\overline{M}}{\mathcal{M}}\right)^{\frac{1-m}{\alpha}}\mathtt{b}\,(t_0+\overline{t})^{-\frac{1}{\alpha}}\,.$$

If $\lambda(t_0)\,|x|\le 1$, we deduce from (33), (37) and (24) that

$$u(t_0,x)\le\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,t_0^{-\frac{d}{\alpha}} = \left(\tfrac{\overline{M}}{\mathcal{M}}\right)^{\frac{2}{\alpha}}\frac{\mathtt{b}^d}{2^{\frac{1}{1-m}}}\,t_0^{-\frac{d}{\alpha}}\,(1+c)^{-\frac{d}{\alpha}}\le B\Big(t_0+\overline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\overline{M}\Big)\,.$$

If $\lambda(t_0)\,|x|\ge 1$, we deduce from (35), (36) and (24) that

$$u(t_0,x)\le 2^{1+\frac{4}{1-m}}\,\frac{t_0^{\frac{1}{1-m}}}{|x_0|^{\frac{2}{1-m}}}\,\overline{\kappa}\le\left(\frac{1+c}{2\,\mathtt{b}^\alpha}\right)^{\frac{1}{1-m}}\frac{t_0^{\frac{1}{1-m}}}{|x_0|^{\frac{2}{1-m}}}\le B\Big(t_0+\overline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\overline{M}\Big)\,.$$

*Step 3. Comparison.* Once we have obtained (32) at time $t=t_0$, by comparison it also holds for any $t\ge t_0$. In particular (32) holds for any $t\ge\overline{t}\ge t_0$, which completes the proof. □

We are now in the position to prove a global lower bound.

**Proposition 8.** *Under the assumptions of Theorem 4, there exist positive constants $\underline{t}$ and $\underline{M}$ such that any solution $u$ satisfies*

$$u(t,x)\ge B\Big(t-\underline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\underline{M}\Big)\quad\forall\,(t,x)\in[2\,\underline{t},+\infty)\times\mathbb{R}^d\,. \tag{38}$$

Compared to [14, 85, 12], the novelty here is that, again, we provide constructive estimates of the constants. The value of the constant $\underline{M}$ is given below by (47): it is a numerical constant, which is independent of $u$. An upper bound on $\underline{t}$ is given by (43). This bound depends only on $A$ and various numerical constants. As an intermediate quantity, we define $R_\star>0$ such that

$$\int_{|x|\le R_\star}u_0\,dx = \frac{1}{2}\,\mathcal{M}\,. \tag{39}$$

*Proof.* Our task is essentially to keep track of the constants and we claim no originality in the strategy of the proof.

*Step 1.* Let $R_\star>0$ be as in (39). We read from ($\mathrm{H}_A$) that

$$R_\star^\alpha\le\left(\frac{2\,A}{\mathcal{M}}\right)^{1-m}. \tag{40}$$

From Lemma 6 we have that

$$\inf_{|x|\le R_\star} u(t,x) \ge \kappa \left(R_\star^{-2}\,t\right)^{\frac{1}{1-m}} \quad \forall\, t\in[0,2\,\underline{t}] \tag{41}$$

for $\kappa$ and $\kappa_\star$ given in (31) and $\underline{t}$ given by

$$\underline{t} = \frac{1}{2}\,\kappa_\star \left(\frac{\mathcal{M}}{2}\right)^{1-m} R_\star^\alpha\,. \tag{42}$$

After taking into account (40), we obtain

$$\underline{t} \le \frac{1}{2}\,\kappa_\star \left(\frac{\mathcal{M}}{2}\right)^{1-m} \left(\frac{2\,A}{\mathcal{M}}\right)^{1-m} = \frac{\kappa_\star}{2}\,A^{1-m}. \tag{43}$$

*Step 2.* If $|x|\le R_\star$, we deduce from (24), (40) and (41) that

$$u(2\,\underline{t},x) \ge \kappa \left(\frac{2\,\underline{t}}{R_\star^2}\right)^{\frac{1}{1-m}} \ge B\Big(t-\tfrac{1}{\alpha}\,,0\,;\,M\Big) = \left(\frac{M}{\mathcal{M}}\right)^{\frac{2}{\alpha}} \mathsf{b}^d\,t^{-\frac{d}{\alpha}} \ge B\Big(t-\tfrac{1}{\alpha}\,,x\,;\,M\Big)$$

for any $t>0$ and $M>0$ such that

$$M^{\frac{2}{\alpha}}\,t^{-\frac{d}{\alpha}} \le \frac{\kappa}{\mathsf{b}^d}\left(\frac{2\,\underline{t}}{R_\star^2}\right)^{\frac{1}{1-m}} \mathcal{M}^{\frac{2}{\alpha}}\,. \tag{44}$$

Let us notice that (44) at $t=\underline{t}$ with $\underline{t}$ given by (42) amounts to

$$M \le \frac{\kappa_\star^{\frac{1}{1-m}}}{2^{\frac{d}{2}}}\left(\frac{\kappa}{\mathsf{b}^d}\right)^{\frac{\alpha}{2}} \mathcal{M}^2\,. \tag{45}$$

This condition is independent of $R_\star$.

*Step 3.* If $|x|=R_\star$, we enforce the condition that

$$u(t+\underline{t},x) \ge \kappa \left(\frac{t+\underline{t}}{R_\star^2}\right)^{\frac{1}{1-m}} \ge B\Big(t-\tfrac{1}{\alpha}\,,x\,;\,M\Big)$$

for any $t\in[0,\underline{t}]$ by requesting that

$$\kappa \left(\frac{\underline{t}}{R_\star^2}\right)^{\frac{1}{1-m}} \ge \frac{M}{\mathcal{M}}\,R_\star^{-d} \sup_{\lambda>0} \lambda^d \left(1+\lambda^2\right)^{\frac{1}{m-1}} \ge \frac{M}{\mathcal{M}}\,\frac{\lambda(t)^d}{R_\star^d}\left(1+\lambda(t)^2\right)^{\frac{1}{m-1}}$$

where the left-hand side is the estimate of $u(\underline{t},x)$ deduced from (41), while the right-hand side is the value of $B(t-\frac{1}{\alpha},x;M)$ for $|x|=R_\star$ and

$$\lambda(t) := \left(\tfrac{M}{\mathcal{M}}\right)^{\frac{1-m}{\alpha}} \mathsf{b}\,t^{-\frac{1}{\alpha}}\,R_\star\,.$$

After taking into account (42), we obtain the condition

$$M \leq \frac{\kappa\,\kappa_\star^{\frac{1}{1-m}}}{\left(d\,(1-m)\right)^{d/2}\;\alpha^{\frac{\alpha}{2\,(1-m)}}}\,\mathcal{M}^2\,, \tag{46}$$

which is also independent of $R_\star$.

*Step 4.* We adapt [59, Lemma 3.4] as follows. We choose

$$\underline{M} := \min\left\{2^{-\,d/2}\left(\frac{\kappa}{\mathsf{b}^d}\right)^{\alpha/2},\frac{\kappa}{\left(d\,(1-m)\right)^{d/2}\;\alpha^{\frac{\alpha}{2\,(1-m)}}}\right\}\,\kappa_\star^{\frac{1}{1-m}}\,\mathcal{M}^2 \tag{47}$$

so that (45) and (46) are simultaneously true. Notice that $\underline{M}$ is independent of $R_\star$.

The function $\underline{u}(t,x) := B(t-\underline{t}-\frac{1}{\alpha},x;\underline{M})$ is such that

$$\underline{u}(2\,\underline{t},x) \leq u(2\,\underline{t},x) \quad \text{if} \quad |x| \leq R_\star \tag{48}$$

by Step 2,

$$\underline{u}(t,x) \leq u(t,x) \quad \text{if} \quad (t,x) \in (\underline{t},\,2\,\underline{t}) \times \mathbb{R}^d\,, \quad |x| = R_\star$$

by Step 3, and, in the sense of distributions,

$$\underline{u}(t,\cdot) \rightharpoonup \underline{M}\,\delta_{x=0} \quad \text{as} \quad t \to \underline{t}_+\,.$$

As a consequence, we also have that

$$\lim_{t\to\underline{t}_+}\underline{u}(t,\cdot) \leq u\,(\underline{t}\,,x) \quad \text{for any} \quad x \in \mathbb{R}^d \quad \text{such that} \quad |x| \geq R_\star\,.$$

The functions $\underline{u}$ and $u$ solve (8). By arguing as in [59, Lemma 3.4], we find that

$$\underline{u}(t,x) \leq u(t,x) \quad \forall\,(t,x) \in [\underline{t},\,2\,\underline{t}] \times \mathbb{R}^d \quad \text{such that} \quad |x| \geq R_\star\,.$$

This inequality holds in particular for $t = 2\,\underline{t}$, which can be combined with (48) to prove that

$$\underline{u}(2\,\underline{t},x) \leq u(2\,\underline{t},x) \quad \forall\,x \in \mathbb{R}^d\,.$$

Notice that [59, Lemma 3.4] holds only for smooth functions, so that an approximation scheme is needed, which is standard and will be omitted here.

*Step 5.* By standard comparison methods, if (38) is true at $t = 2\,\underline{t}$, it is also true at any $t \geq 2\,\underline{t}$. This completes the proof of (38). □

So far, the upper and lower estimates of Propositions 7 and 8 correspond to Barenblatt functions which do not have the same mass $\mathcal{M}$ as $u$. The next two subsections are devoted to the comparison of the solution $u$ of (8) with the Barenblatt function $\mathcal{B}$ of mass $\mathcal{M}$.

### 3.4 The outer estimates

As a byproduct of Proposition 8, by integrating over $\mathbb{R}^d$, we deduce from (38) that $\underline{M}\,/\mathcal{M} < 1$, which proves that

$$\underline{\varepsilon} := 1 - (\underline{M}\,/\mathcal{M})^{\frac{2}{\alpha}} > 0\,.$$

With this definition, notice that $\underline{\varepsilon}$ is a numerical constant. Let us compare the solution $u$ of (8) with the Barenblatt function $B$ with same mass as $u$ outside of a large ball in $x$, or for large values of $t$ but up to a multiplicative factor. With the notation of (22) and (23), we recall that $B(t,x) = B(t\,,\,x\,;\,\mathcal{M})$.

**Corollary 9.** *Under the assumptions of Theorem 4 and for any $\varepsilon \in (0,\underline{\varepsilon})$, there are some $\underline{T}(\varepsilon)$ and $\underline{\rho}(\varepsilon)$ for which any solution $u$ of (8) satisfies*

$$u(t,x) \geq (1-\varepsilon)\,B(t\,,\,x\,) \quad \textit{if} \quad |x| \geq R(t)\,\underline{\rho}(\varepsilon) \quad \textit{and} \quad t \geq \underline{T}(\varepsilon)\,. \tag{49}$$

*Furthermore, there exists $\underline{C} > 0$ such that, for all $x \in \mathbb{R}^d$,*

$$u(t,x) \geq \underline{C}\,B\Big(t - \tfrac{1}{\alpha}\,,\,x\Big) \quad \textit{if} \quad t \geq \underline{T}(\varepsilon)\,. \tag{50}$$

The constants $\underline{T}(\varepsilon)$, $\underline{\rho}(\varepsilon)$ and $\underline{C}$ have an explicit expression which will be given below in (51), (52) and (53) respectively.

*Proof.* The Barenblatt solution of mass $M$ as defined in (23) can be rewritten as

$$B(t\,,\,x\,;\,M) = \lambda(t)^d \left((\mathcal{M}/M)^{2\,\frac{1-m}{\alpha}} + \lambda(t)^2\,|x|^2\right)^{\frac{1}{m-1}}$$

where $\lambda(t) := \mu\,R(t)^{-1}$, $\mu$ is a constant given by (20), and $R(t) = (1+\alpha\,t)^{1/\alpha}$, so that

$$\frac{B\Big(t - \underline{t} - \frac{1}{\alpha}\,,\,x\,;\,\underline{M}\Big)}{B(t\,,\,x\,)} = \frac{\lambda\Big(t-\underline{t}-\frac{1}{\alpha}\Big)^d}{\lambda(t)^d}\left(\frac{1+\lambda(t)^2\,|x|^2}{(1-\underline{\varepsilon})^{m-1} + \lambda\Big(t-\underline{t}-\frac{1}{\alpha}\Big)^2\,|x|^2}\right)^{\frac{1}{1-m}}.$$

With

$$\eta(t) := \left(\frac{t+\frac{1}{\alpha}}{t-\underline{t}}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad s(t,x) := \lambda(t)^2\,|x|^2\,,$$

Inequality (49) amounts to

$$\eta^d \left(\frac{1+s}{(1-\underline{\varepsilon})^{m-1} + \eta^2\,s}\right)^{\frac{1}{1-m}} \geq 1-\varepsilon\,.$$

It is sufficient to have

$$\eta(t)^\alpha\,(1-\varepsilon)^{1-m} < 1 \quad \text{and} \quad s(t,x) = \left(\frac{\mu\,|x|}{R(t)}\right)^2 \geq \frac{\eta(t)^{-d\,(1-m)}\left(\frac{1-\varepsilon}{1-\underline{\varepsilon}}\right)^{1-m} - 1}{1-\eta(t)^\alpha\,(1-\varepsilon)^{1-m}}\,.$$

Using (43), the first condition is satisfied if $t \geq \underline{T}(\varepsilon)$ with

$$\tau(\varepsilon) := \frac{2\,\underline{t} + \frac{1}{\alpha}\Big(1 + (1-\varepsilon)^{1-m}\Big)}{1-(1-\varepsilon)^{1-m}} \leq \frac{\kappa_\star\,(2\,A)^{1-m} + \frac{2}{\alpha}}{1-(1-\varepsilon)^{1-m}} =: \underline{T}(\varepsilon)\,. \tag{51}$$

Notice that $\underline{T}(\varepsilon) = O(1/\varepsilon)$ as $\varepsilon \to 0$ and also that Condition (51) guarantees that $\underline{T}(\varepsilon) \geq 2\,\underline{t}$.

Next, using

$$1 \leq \eta(t) \leq \eta\Big(\underline{T}(\varepsilon)\Big) \leq \frac{2^{\frac{1}{\alpha}}}{\Big(1 + (1-\varepsilon)^{1-m}\Big)^{\frac{1}{\alpha}}} = \eta\Big(\tau(\varepsilon)\Big)$$

for any $t \geq \underline{T}(\varepsilon)$, the second condition follows from $s(t,x) \geq \mu^2\,\underline{\rho}^2(\varepsilon)$ with

$$\underline{\rho}(\varepsilon) := \frac{1}{\mu}\left(\Big(1 + (1+\varepsilon)^{1-m}\Big)\,\frac{\Big(\frac{1-\varepsilon}{1-\underline{\varepsilon}}\Big)^{1-m} - 1}{1-(1-\varepsilon)^{1-m}}\right)^{1/2}\,. \tag{52}$$

It follows from a Taylor expansion that $\underline{\rho}(\varepsilon) = O(1/\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

With the above notation, we remark that

$$\frac{B\Big(t - \underline{t} - \frac{1}{\alpha}\,,\,x\,;\,\underline{M}\Big)}{B\Big(t - \frac{1}{\alpha}\,,\,x\Big)} = \gamma(t)^d\,\left(\frac{1+\sigma}{(1-\underline{\varepsilon})^{m-1} + \gamma^2\,\sigma}\right)^{\frac{1}{1-m}}$$

where

$$\gamma(t) = \left(\frac{t}{t-\underline{t}}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \sigma = \lambda\left(t - \tfrac{1}{\alpha}\right)^2\,|x|^2\,.$$

Since $\gamma > 1$, inequality (50) amounts to find

$$\inf_{\sigma \geq 0}\left(\frac{1+\sigma}{(1-\underline{\varepsilon})^{m-1} + \gamma^2\,\sigma}\right)^{\frac{1}{1-m}}\,.$$

A straightforward computation shows that such an infimum is achieved either at 0 or at infinity. Since for any $t \geq \underline{T}(\varepsilon) \geq 2\,\underline{t}$ we have that $\gamma(t) \leq \gamma(2\,\underline{t}) = 2^{\frac{1}{\alpha}}$, we obtain

$$\inf_{\sigma \geq 0}\left(\frac{1+\sigma}{(1-\underline{\varepsilon})^{m-1} + \gamma^2\,\sigma}\right)^{\frac{1}{1-m}} \geq \min\Big\{(1-\underline{\varepsilon})\,,\,\gamma^{-\frac{2}{1-m}}\Big\} \geq \frac{1-\underline{\varepsilon}}{2^{\frac{2}{(1-m)\,\alpha}}} =: \underline{C}\,, \tag{53}$$

where we have used that $\underline{\varepsilon} < 1$ and $2^{\frac{2}{(1-m)\,\alpha}} > 1$. The proof is completed. □

Next we prove lower bounds. As a byproduct of Proposition 7, by integrating over $\mathbb{R}^d$, we deduce from (32) that $\overline{M}\,/\mathcal{M} > 1$, which proves that

$$\overline{\varepsilon} := \Big(\overline{M}/\mathcal{M}\Big)^{\frac{2}{\alpha}} - 1 > 0\,.$$

With this definition, notice that $\overline{\varepsilon}$ is a numerical constant.

**Corollary 10.** *Under the assumptions of Theorem 4 and for any $\varepsilon \in (0, \overline{\varepsilon})$, there are some $\overline{T}(\varepsilon)$ and $\overline{\rho}(\varepsilon)$ for which any solution $u$ of (8) satisfies*

$$u(t,x) \le (1+\varepsilon)\, B(t\,,\,x\,) \quad \textit{if} \quad |x| \ge\, R(t)\, \overline{\rho}(\varepsilon) \quad \textit{and} \quad t \ge \overline{T}(\varepsilon)\,. \tag{54}$$

*Furthermore, there exists $\overline{C} > 0$ such that, for all $x \in \mathbb{R}^d$,*

$$u(t,x) \le \overline{C}\, B\Big(t - \tfrac{1}{\alpha}\,,\,x\Big) \quad \textit{if} \quad t \ge \overline{T}(\varepsilon)\,.$$

The constants $\overline{T}(\varepsilon)$, $\overline{\rho}(\varepsilon)$ and $\overline{C}$ have an explicit expression given below in (56), (57) and (58) respectively.

*Proof.* The beginning of the proof is the same as for Corollary 9. With the same notation except for $\eta$ which is now defined by

$$\eta(t) := \left(\frac{\frac{1}{\alpha} + t}{t + \overline{t}}\right)^{\frac{1}{\alpha}}\,,$$

where $\overline{t}$ is as in (36), Inequality (54) amounts to

$$\eta^d \left(\frac{1+s}{(1+\overline{\varepsilon})^{m-1} + \eta^2\, s}\right)^{\frac{1}{1-m}} \le 1+\varepsilon\,.$$

To prove the above inequality it is sufficient to have

$$\eta(t)^\alpha\,(1+\varepsilon)^{1-m} > 1 \quad \text{and} \quad s(t,x) = \left(\frac{\mu\,|x|}{R(t)}\right)^2 \ge \frac{1 - \eta(t)^{-d\,(1-m)}\left(\frac{1+\varepsilon}{1+\overline{\varepsilon}}\right)^{1-m}}{\eta(t)^\alpha\,(1+\varepsilon)^{1-m} - 1}\,. \tag{55}$$

Let us define

$$\overline{T}(\varepsilon) := \frac{2\,\overline{t}}{(1+\varepsilon)^{1-m} - 1} \tag{56}$$

where $\overline{t}$ is as in (36) and $\overline{T}(\varepsilon) = O(1/\varepsilon)$ as $\varepsilon \to 0$ follows from a Taylor expansion. If $\overline{t} < 1/\alpha$ then the first condition in (55) is always satisfied, while in the case $\overline{t} \ge 1/\alpha$, we need to ask that $t \ge \overline{T}(\varepsilon)$. In both cases we have that

$$\eta(t) \ge \left(\frac{2}{(1+\varepsilon)^{1-m} + 1}\right)^{\frac{1}{\alpha}}$$

for any $t \ge \overline{T}(\varepsilon)$. As a consequence, a sufficient condition the second inequality in (55) is $s(t,x) \ge \mu^2\, \overline{\rho}^2(\varepsilon)$ with

$$\overline{\rho}(\varepsilon) := \frac{1}{\mu}\left(\frac{(1+\varepsilon)^{1-m} + 1}{(1+\varepsilon)^{1-m} - 1}\right)^{\frac{1}{2}}\,. \tag{57}$$

It follows from a second order Taylor expansion that $\overline{\rho}(\varepsilon) = O(1/\sqrt{\varepsilon})$ as $\varepsilon \to 0$.

As in Corollary 9, we remark that

$$\frac{B\Big(t+\overline{t}-\frac{1}{\alpha}\,,\,x\,;\,\underline{M}\Big)}{B\Big(t-\frac{1}{\alpha}\,,\,x\Big)}=\gamma(t)^{d}\,\left(\frac{1+\sigma}{(1+\overline{\varepsilon})^{m-1}+\gamma^{2}\,\sigma}\right)^{\frac{1}{1-m}}$$

where

$$\gamma(t)=\left(\frac{t}{t+\overline{t}}\right)^{\frac{1}{\alpha}}\quad\text{and}\quad\sigma=\lambda\left(t-\tfrac{1}{\alpha}\right)^{2}\,|x|^{2}\,.$$

Since $\gamma(t)\le 1$, inequality (50) amounts to find

$$\sup_{\sigma\ge 0}\left(\frac{1+\sigma}{(1+\overline{\varepsilon})^{m-1}+\gamma^{2}\,\sigma}\right)^{\frac{1}{1-m}}\,.$$

A straightforward computation shows that such infimum is achieved either at 0 or at infinity. Since $\gamma(t)\ge(2/3)^{1/\alpha}$ for any $t\ge\overline{T}(\varepsilon)\ge 2\,\overline{t}$, we can argue that

$$\sup_{\sigma\ge 0}\left(\frac{1+\sigma}{(1+\overline{\varepsilon})^{m-1}+\gamma^{2}\,\sigma}\right)^{\frac{1}{1-m}}\le\max\left\{(1+\overline{\varepsilon})\,,\gamma^{-\frac{2}{1-m}}\right\}\le(1+\overline{\varepsilon})\,\left(\tfrac{3}{2}\right)^{\frac{2}{(1-m)\,\alpha}}=:\overline{C}\,. \tag{58}$$

The proof is completed. □

### 3.5 The inner estimate

Here we prove the uniform convergence in relative error inside a finite ball. Let

$$\varepsilon_{m,d}:=\min\left\{\overline{\varepsilon},\ \underline{\varepsilon},\ \tfrac{1}{2}\right\} \tag{59}$$

where, as in Section 3.4, $\overline{\varepsilon}=\left(\overline{M}/\mathcal{M}\right)^{\frac{2}{\alpha}}-1$ and $\underline{\varepsilon}=1-(\underline{M}/\mathcal{M})^{\frac{2}{\alpha}}$ are given in terms of $\overline{M}$ and $\underline{M}$ defined respectively by (37) and (47). For any $\varepsilon\in(0,\varepsilon_{m,d})$, let us define

$$\rho(\varepsilon):=\max\left\{\overline{\rho}(\varepsilon),\ \underline{\rho}(\varepsilon)\right\}\quad\text{and}\quad T(\varepsilon):=\max\left\{\overline{T}(\varepsilon),\ \underline{T}(\varepsilon)\right\} \tag{60}$$

where $\overline{\rho}(\varepsilon)$, $\underline{\rho}(\varepsilon)$, $\overline{T}(\varepsilon)$, and $\underline{T}(\varepsilon)$ are defined by (51), (52), (56), and (57). We know that $\rho(\varepsilon)=O(1/\sqrt{\varepsilon})$ and $T(\varepsilon)=O(1/\varepsilon)$ as $\varepsilon\to 0$. The main result of this section is the following.

**Proposition 11.** *Under the assumptions of Theorem 4, there exist a numerical constant* $\mathsf{K}>0$ *and an exponent* $\vartheta\in(0,1)$ *such that, for any* $\varepsilon\in(0,\varepsilon_{m,d}$ *and for any* $t\ge 4\,T(\varepsilon)$*, any solution* $u$ *of* (8) *satisfies*

$$\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}}\left(\frac{1}{t}+\frac{\sqrt{G}}{R(t)}\right)^{\vartheta}\quad\text{if}\quad|x|\le 2\,\rho(\varepsilon)\,R(t)\,. \tag{61}$$

The exponent is $\vartheta = \nu/(d+\nu)$ and the numerical constants $\nu = \nu(m,d)$ and $\mathsf{K}(m,d)$ are explicit and given below in (67) and (84) respectively.

*Proof.* By the triangle inequality, the left-hand-side of (61) can be estimated by

$$\left|\frac{u(t,x)}{B(t,x)}-1\right| \le \left|\frac{B\left(t-\frac{1}{\alpha},x\right)}{B(t,x)}\right| \left(\left|\frac{u(t,x)}{B(t-\frac{1}{\alpha}\,,\,x\,)}-1\right| + \left|\frac{B(t,x)}{B(t-\frac{1}{\alpha}\,,\,x\,)}-1\right|\right). \tag{62}$$

The supremum of the quotient $B(t-1/\alpha,x)/B(t,x)$ is achieved at $x=0$ for any $t\ge 0$. Using (22) and (24), we have that

$$\left\|\frac{B\left(t-\frac{1}{\alpha},x\right)}{B(t,x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{R(t)^d}{\alpha^{\frac{d}{\alpha}}\,t^{\frac{d}{\alpha}}} =: \overline{c_1}(t)\,. \tag{63}$$

The supremum of the quotient $B(t,x)/B(t-1/\alpha,x)$ is achieved at infinity, therefore a simple computation shows that

$$\left\|\frac{B(t,x)}{B(t-\frac{1}{\alpha}\,,\,x\,)}-1\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \left(1+\frac{1}{\alpha\,t}\right)^{\frac{1}{1-m}}-1 \le \frac{\overline{c}_3}{t}+\frac{\overline{c}_2}{t^2}\,.$$

A Taylor expansion shows that the values of $\overline{c}_2$ and $\overline{c}_3$ are

$$\overline{c}_3 = \frac{1}{1-m} \quad\text{and}\quad \overline{c}_2 = \frac{m}{2\,(1-m)^2\,\alpha^2}\,.$$

Our task is to estimate the missing term $|u(t,x)/B(t-1/\alpha\,,\,x\,)-1|$. This is done by interpolating the above quantity between its $\mathrm{L}^p$ and $C^\nu$ norms, by means of inequality (102), in Appendix A. In order to do so, we use parabolic regularity theory to estimate the $C^\nu$ norm of the quotient $u(t,x)/B(t-1/\alpha\,,\,x\,)$.

*Step 1.* We recall some elements of linear parabolic regularity theory. More details are given in [10]. Let $\Omega$ be an open domain and let us consider positive *weak solution* to

$$\frac{\partial v}{\partial t} = \nabla\cdot\Big(A(t,x)\,\nabla v\Big) \tag{64}$$

on $\Omega_T := (0,T)\times\Omega$, where $A(t,x)$ is a real symmetric matrix with bounded measurable coefficients satisfying a uniform ellipticity condition, *i.e.*,

$$0<\lambda_0\,|\xi|^2 \le \sum_{i,j=1}^d A_{i,j}(t,x)\,\xi_i\,\xi_j \le \lambda_1\,|\xi|^2$$

for some positive constants $\lambda_0$ and $\lambda_1$. A definition of weak solution is given in [69, p. 728], see also [63, Chapter 3] or [1]. Since the celebrated works of

J. Moser [68, 69], it is known that, whenever $(t_0-R^2,t_0+R^2)\times B_{2R}(x_0)\subset\Omega_T$, then the following Harnack inequality holds for positive weak solutions, to (64)

$$\sup_{D_R^-(t_0,x_0)} v \le \mathsf{h}^{\lambda_1+\lambda_0^{-1}} \inf_{D_R^+(t_0,x_0)} v\,,$$

where

$$D_R^+(t_0,x_0) := (t_0+\tfrac{3}{4}R^2, t_0+R^2)\times B_{R/2}(x_0)\,,$$
$$D_R^-(t_0,x_0) := \left(t_0-\tfrac{3}{4}R^2, t_0-\tfrac{1}{4}R^2\right)\times B_{R/2}(x_0)\,.$$

The value of the constant $\mathsf{h}$ is computed in [10]:

$$\mathsf{h} := \exp\left[2^{d+4}\,3^d\,d + c_0^3\,2^{2\,d+7}\left(1+\frac{2^{d+2}}{(\sqrt{2}-1)^{2\,d+4}}\right)\sigma\right] \tag{65}$$

where

$$c_0 = 3^{\frac{2}{d}}\,2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}}\left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}}\right)^{(d+1)(d+2)}\mathcal{K}^{\frac{2\,d+4}{d}}\,,$$

$$\sigma = \sum_{j=0}^{\infty}\left(\tfrac{3}{4}\right)^j\left((2+j)\,(1+j)\right)^{2\,d+4}$$

and $\mathcal{K}$ is as in Table 1. Let us define

$$Q_1 := (1/2,3/2)\times B_1(0)\,,\quad Q_2 := (1/4,2)\times B_8(0)\,,$$
$$Q_3 := (1/2,3/2)\times B_1(0)\setminus B_{1/2}(0)\quad\text{and}\quad Q_4 := (1/4,2)\times B_8(0)\setminus B_{1/4}(0)\,.$$

Arguing as in [68, p. 108-109], see also [13] for more details, we can show that a nonnegative weak solution to (64) defined on $Q_2$ satisfies the following inequality

$$\sup_{(t,x),(s,y)\in Q_i}\frac{|v(t,x)-v(s,y)|}{\left(|x-y|+|t-s|^{1/2}\right)^{\nu}} \le c_1\,\|v\|_{\mathrm{L}^\infty(Q_{i+1})}\quad \forall\, i\in\{1,2\} \tag{66}$$

where

$$c_1 := 2^{10}\,,\quad \nu := \log_4\left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}}-1}\right)\in(0,1)\quad\text{and}\quad \overline{\mathsf{h}} := \mathsf{h}^{\lambda_1+1/\lambda_0}\,. \tag{67}$$

We refer to [10] on how the constant $c_1$ is computed. Notice that $\nu\ge 1/\overline{\mathsf{h}}$

$$\nu \ge \frac{1}{\mathsf{h}^{\lambda_1+\lambda_0^{-1}}}$$

is a positive number, which only depends only on $d$, through $\mathsf{h}$ as defined in (65), and on $\lambda_0$, $\lambda_1$. In what follows, we apply this linear theory to nonlinear equations by choosing $\lambda_0$ and $\lambda_1$ appropriately, depending only on $m$ and $d$: see below (73). Let us define the $C^\nu(\Omega)$ semi-norm as

$$\lfloor u\rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|u(x)-u(y)|}{|x-y|^\nu}\,. \tag{68}$$

From inequality (66) we deduce that a nonnegative weak solution to (64) defined on $Q_2$ satisfies, for any $s \in (1/2, 3/2)$

$$\max\{\lfloor v(s,\cdot)\rfloor_{C^\nu(B_1(0))}, \lfloor v(s,\cdot)\rfloor_{C^\nu\left(B_1(0)\setminus B_{1/2}(0)\right)}\} \le c_1 \, \|v\|_{\mathrm{L}^\infty(Q_2)} \,. \tag{69}$$

*Step 2.* We estimate the $C^\nu$ norm of $u(t,x)$. For any $k>0$ and $\tau>0$, let us define the re-scaled function

$$\hat{u}_{\tau,k}(t,x) := k^{\frac{2}{1-m}} \, \tau^{\frac{d}{\alpha}} \, u\left(\tau \, t, k \, \tau^{\frac{1}{\alpha}} \, x\right) \,. \tag{70}$$

The function $\hat{u}_{\tau,k}$ solves (8). Similarly, the Barenblatt profile $B$ as defined in (24) is rescaled according to

$$\hat{B}_{\tau,k}\left(t-\tfrac{1}{\alpha}, x\right) = B\left(t-\tfrac{1}{\alpha}, x; k^{\frac{\alpha}{1-m}}\,\mathcal{M}\right) \,.$$

In this step we obtain estimates for the $C^\nu$-norm of $\hat{u}_{\tau,1}(1,\cdot)$ and $B(1-\frac{1}{\alpha},\cdot)$. Let us begin with the latter: for any $\gamma \in (0,1)$, we have

$$\begin{aligned}\lfloor B(1-\tfrac{1}{\alpha},x)\rfloor_{C^\gamma\left(\mathbb{R}^d\right)} &\le 2\max\left\{\|B(1-\tfrac{1}{\alpha},\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}, \|\nabla B(1-\tfrac{1}{\alpha},\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\right\}\\ &= 2\,\mathtt{b}\,\max\left\{1\,, 2^{\frac{3-2\,m}{1-m}} \frac{\mathtt{b}^d\,(2-m)^{2-m}}{\sqrt{1-m}\,(3-m)^{\frac{5-3\,m}{2\,(1-m)}}}\right\} =: c_2\,,\end{aligned} \tag{71}$$

where $\mathtt{b}$ is as in (24). By the results of Corollaries 9 and 10, there exists positive constants $\underline{C}$ and $\overline{C}$ such that, for all $x \in \mathbb{R}^d$, all $t \ge T(\varepsilon)/\tau$ and all $k \ge 1$,

$$0 < \underline{C} \le \frac{\hat{u}_{\tau,k}(t,x)}{B(t-\frac{1}{\alpha},\, x;\, k^{\frac{\alpha}{1-m}}\,\mathcal{M})} \le \overline{C} < \infty\,, \tag{72}$$

where the expressions of $\underline{C}$ and $\overline{C}$ are given in (53) and in (58) respectively, and depend only on $m$ and $d$. Let us define

$$\begin{aligned}\lambda_0^{\frac{1}{m-1}} &:= m^{\frac{1}{m-1}}\,\overline{C}\,\max\left\{\sup_{Q_2} B\left(t-\tfrac{1}{\alpha},x\right),\, \sup_{k\ge1}\sup_{Q_4} B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M})\right\},\\ \lambda_1^{\frac{1}{m-1}} &:= m^{\frac{1}{m-1}}\,\underline{C}\,\min\left\{\inf_{Q_2} B\left(t-\tfrac{1}{\alpha},x\right),\, \inf_{k\ge1}\inf_{Q_4} B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M})\right\}.\end{aligned} \tag{73}$$

We remark that

$$\sup_{k\ge1}\sup_{Q_4} B\left(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M}\right) \quad \text{and} \quad \inf_{k\ge1}\inf_{Q_4} B\left(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M}\right)$$

are bounded and bounded away from zero, see details in [10]. As a consequence of (72) we obtain that, for any $\tau \ge 4\,T(\varepsilon)$ and for any $k \ge 1$, we have

$$\left(\tfrac{\lambda_1}{m}\right)^{\frac{1}{m-1}} \le \hat{u}_{\tau,k}(t,x) \le \left(\tfrac{\lambda_0}{m}\right)^{\frac{1}{m-1}} \quad \forall\,(t,x)\in\, Q_2\,,\, Q_4\,.$$

The function $\hat{u}_{\tau,k}$ is a nonnegative weak solution to (64) and, as a consequence of inequality (69), we get that for any $\tau \geq 4\,T(\varepsilon)$,

$$\begin{aligned} &\lfloor \hat{u}_{\tau,1}(1,\cdot)\rfloor_{C^\nu(B_1(0))} \leq c_1\,\|\hat{u}_{\tau,1}\|_{\mathrm{L}^\infty(Q_2)}\,,\\ &\lfloor \hat{u}_{\tau,k}(1,\cdot)\rfloor_{C^\nu\left(B_1(0)\setminus B_{1/2}(0)\right)} \leq c_1\,\|\hat{u}_{\tau,k}\|_{\mathrm{L}^\infty(Q_4)} \quad \forall\, k\geq 2\,, \end{aligned} \tag{74}$$

where $\nu$ is as in (67) and $\lambda_0$, $\lambda_1$ are as in (73). We observe that

$$\begin{aligned} &\lfloor \hat{u}_{\tau,k}(1,\cdot)\rfloor_{C^\nu\left(B_1(0)\setminus B_{1/2}(0)\right)} = k^{\frac{2}{1-m}+\nu}\,\lfloor \hat{u}_{\tau,1}(1,\cdot)\rfloor_{C^\nu\left(B_k(0)\setminus B_{k/2}(0)\right)}\,,\\ &\|\hat{u}_{\tau,k}\|_{\mathrm{L}^\infty(Q_4)} = k^{\frac{2}{1-m}}\,\|\hat{u}_{\tau\,,1}\|_{\mathrm{L}^\infty\left([1/4\,,2]\times B_k(0)\setminus B_{k/2}(0)\right)} \leq k^{\frac{2}{1-m}}\,\|\hat{u}_{\tau,1}\|_{\mathrm{L}^\infty\left([1/4\,,2]\times\mathbb{R}^d\right)}\,. \end{aligned} \tag{75}$$

We finally estimate the $C^\nu$-norm of $\hat{u}_{\tau,1}(1,\cdot)$, combining (74) with (75) we get

$$\begin{aligned} \lfloor \hat{u}_{\tau\,,1}(1,\cdot)\rfloor_{C^\nu\left(\mathbb{R}^d\right)} &\leq \lfloor \hat{u}_{\tau\,,1}(1,\cdot)\rfloor_{C^\nu(B_1(0))} + \sum_{j=0}^{\infty}\lfloor \hat{u}_{\tau\,,1}(1,\cdot)\rfloor_{C^\nu\left(B_{2^{j+1}}(0)\setminus B_{2^j}(0)\right)}\\ &\leq c_1\,\|\hat{u}_{\tau\,,1}\|_{\mathrm{L}^\infty\left([1/4\,,2]\times\mathbb{R}^d\right)}\,\frac{2^\nu}{2^\nu-1}\,. \end{aligned}$$

Lastly, we notice that, as a consequence of (70) and inequality (27) where we take the limit $R\to\infty$, we have

$$\|\hat{u}_{\tau\,,1}\|_{\mathrm{L}^\infty\left([1/4\,,2]\times\mathbb{R}^d\right)} \leq \tau^{\frac{d}{\alpha}}\,\|u\|_{\mathrm{L}^\infty([1/4\,,2]\times\mathbb{R}^d)} \leq \tau^{\frac{d}{\alpha}}\,\overline{\kappa}\,\frac{4^{\frac{d}{\alpha}}\,\mathcal{M}^{\frac{2}{\alpha}}}{\tau^{\frac{d}{\alpha}}} = 4^{\frac{d}{\alpha}}\,\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,. \tag{76}$$

*Step 3.* In this step we shall show that for any $t\geq 4\,T(\varepsilon)$, the following inequality

$$\left|\frac{u(t,x)}{B(t-\frac{1}{\alpha}\,,\,x\,)}-1\right| \leq C\,\|u(t,x)-B(t-\tfrac{1}{\alpha}\,,\,x\,)\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\vartheta} \quad \text{if} \quad |x|\leq 2\,Z\,\rho(\varepsilon)\,t^{\frac{1}{\alpha}} \tag{77}$$

holds for any $Z\geq 1$, with $C$ as in (80) and

$$\vartheta = \frac{\nu}{d+\nu}\,. \tag{78}$$

Let us define

$$\mathsf{C} := C_{d,\nu,1}\left(\left(c_1\,4^{\frac{d}{\alpha}}\,\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^\nu}{2^\nu-1}+c_2\right)^{\frac{d}{d+\nu}} + \frac{1}{(2\,Z\,\rho(\varepsilon))^d}\,(2\,\mathcal{M})^{\frac{d}{d+\nu}}\right)\,.$$

By inequalities (71), (74) - (76) and (102) we deduce that for any $\tau\geq 4\,T(\varepsilon)$

$$\|\hat{u}_{\tau\,,1}(1,x)-\hat{B}_{\tau\,,1}(1-\tfrac{1}{\alpha},x)\|_{\mathrm{L}^\infty(B_{2\,Z\,\rho(\varepsilon)})} \leq \mathsf{C}\,\|\hat{u}_{\tau\,,1}(1,x)-\hat{B}_{\tau\,,1}(1-\tfrac{1}{\alpha},x)\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\vartheta}\,, \tag{79}$$

where $C_{d,\nu,1}$ is as in (102) and $\vartheta$ as in (78). Let us define

$$C := \mathsf{b}^d \left(1 + 4\,\mathsf{b}^2\,Z^2\,\rho(\varepsilon)^2\right)^{\frac{1}{1-m}}\,\mathsf{C} = \left\|\frac{1}{\hat{B}_\tau(1-\frac{1}{\alpha},\cdot)}\right\|_{\mathrm{L}^\infty(B_{2\,Z\,\rho(\varepsilon)})}\mathsf{C}\,. \tag{80}$$

From inequality (79), we deduce that, for any $x \in \mathbb{R}^d$ such that $|x| \le 2\,Z\,\rho(\varepsilon)$,

$$\left|\frac{\hat{u}_{\tau,1}(1,x) - \hat{B}_{\tau,1}(1-\frac{1}{\alpha},x)}{\hat{B}_{\tau,1}(1-\frac{1}{\alpha},x)}\right| \le C\,\left\|\hat{u}_{\tau,1}(1,\cdot) - \hat{B}_{\tau,1}(1-\tfrac{1}{\alpha},\cdot)\right\|_{\mathrm{L}^1(\mathbb{R}^d)}^\vartheta\,. \tag{81}$$

Let us define $y = \tau^{\frac{1}{\alpha}}\,x$. By using (70), we can see that the left-hand-side of (81) is as the left-hand-side of (77), indeed we can write

$$\left|\frac{u(\tau,y)}{B(\tau-\frac{1}{\alpha}\,,\,y\,)} - 1\right| = \left|\frac{\hat{u}_{\tau,1}(1,x) - \hat{B}_{\tau,1}(1-\frac{1}{\alpha},x)}{\hat{B}_{\tau,1}(1-\frac{1}{\alpha},x)}\right|.$$

The same holds for the right-hand-sides of those inequalities, indeed from (70) we deduce that

$$\left\|\hat{u}_{\tau,1}(1,\cdot) - \hat{B}_{\tau,1}(1-\tfrac{1}{\alpha},\cdot)\right\|_{\mathrm{L}^1(\mathbb{R}^d)} = \left\|u(\tau,\cdot) - B(\tau-\tfrac{1}{\alpha},\cdot)\right\|_{\mathrm{L}^1(\mathbb{R}^d)}\,.$$

Combining the above observation we deduce that inequality (77) holds.

*Step 4.* In order to get an estimate of the relative error as in Theorem 4, we need some additional information coming from Section 2, namely that $\|u(t,\cdot) - B(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d)}$ can be estimated from the free energy of the initial datum $u_0$.

For $t \ge T(\varepsilon) \ge 2/\alpha$ we have that $R(t) \le (2\,\alpha)^{\frac{1}{\alpha}}\,t^{1/\alpha}$. By combining (77) (where we set $Z = (2\,\alpha)^{1/\alpha}$) and (62) (where we have estimated $\overline{c}_1(t) \le 2^{d/\alpha}$) we find that for any $t \ge T(\varepsilon) \ge 2/\alpha$, we have that

$$\left|\frac{u(t,x)}{B(t,x)} - 1\right| \le 2^{\frac{d}{\alpha}}\,\left(C\,\left\|u(t,\cdot) - B(t-\tfrac{1}{\alpha},\cdot)\right\|_1^\vartheta + \left(\overline{c}_3 + \tfrac{2}{\alpha}\,\overline{c}_2\right)\frac{1}{t}\right) \tag{82}$$

if $|x| \le 2\,R(t)\,\rho(\varepsilon)$. By the triangle inequality and the above estimate on the quotients of two delayed Barenblatt solutions we obtain that for any $t \ge T(\varepsilon)$

$$\left\|u(t,x) - B\Big(t-\tfrac{1}{\alpha},x\Big)\right\|_1 \le \|u(t,x) - B(t,x)\|_1 + \left(\overline{c}_3 + \tfrac{2}{\alpha}\,\overline{c}_2\right)\frac{\mathcal{M}}{t}\,.$$

We take advantage of the Csiszár-Kullback inequality (109) for the flow in self-similar variables (9), namely

$$\|u(t,x) - B(t,x)\|_1 = \|v(\tau) - \mathcal{B}\|_1 \le \sqrt{\tfrac{4\,\alpha\,\mathcal{M}}{m}}\,\frac{\sqrt{G}}{R(t)}\,,$$

where $R(t)$ is as in (21). This yields (61) with a constant in the right-hand-side given by

$$\overline{C} \leq 2^{\frac{d}{\alpha}} \left(C + \overline{c}_3 + \tfrac{2}{\alpha}\,\overline{c}_2\right)\left(\sqrt{\tfrac{4\,\alpha\,\mathcal{M}}{m}} + \left(\overline{c}_3 + \tfrac{2}{\alpha}\,\overline{c}_2\right)\mathcal{M}\right)^{\vartheta} \leq \frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}} \tag{83}$$

and

$$\begin{aligned}\mathsf{K} := {} & 2^{\frac{3\,d}{\alpha}+\frac{3+6\,\alpha}{\alpha\,(1-m)}+\vartheta+10}\,\frac{(\alpha+\mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2\,(1+\vartheta)+\frac{2}{1-m}}}\\ & \times\left[1+\mathsf{b}^{d}\,C_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c_2\right)^{\frac{d}{d+\nu}}+\frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathcal{M}^{\frac{d}{d+\nu}}\right)\right].\end{aligned} \tag{84}$$

We recall that $c_2$, $\overline{\kappa}$, $\mu$ and $\mathsf{b}$ are all numerical constants, which have been introduced earlier in (71), (28), (20) and (24). See [10] for further details and estimates. □

### 3.6 Computation of $t_\star$

In order to prove the uniform convergence in relative error (Theorem 4), our task is to combine the outer estimates (49) and (54), which provides us with a control of the tail, with inequality (61), which gives us an explicit inner estimate. We have the following estimate, which is slightly more precise than Theorem 4, as it shows the dependence of $t_\star$ in $\varepsilon$, $A$ and $G$ as well as the behavior of the numerical constant as $m \to 1_-$. We recall that $\varepsilon_{m,d}$ is defined by (59). We refer to [10, Appendix A.1] for a "user guide" which collects the formulae needed in the computation of $t_\star$.

**Proposition 12.** *Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$. With the notation of Theorem 4, we have*

$$t_\star = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}\,, \tag{85}$$

*where $\mathsf{c}_\star$ is a positive and finite numerical constant, $\mathsf{a}$ and $\mathsf{b}$ are given by*

$$\mathsf{a} = \frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}\,,$$

*where $\vartheta$ is as in Proposition 11.*

An explicit expression of $t_\star$ is given by (88) and $\mathsf{c}_\star$ is defined by (89). See [10] for a detailed estimate.

*Proof.* By definitions (51), (56) and (60), we have

$$T(\varepsilon) = \max\left\{\frac{2\,c\,A^{1-m}}{(1+\varepsilon)^{1-m}-1}\,, \frac{\kappa_\star\,(2\,A)^{1-m}+\frac{2}{\alpha}}{1-(1-\varepsilon)^{1-m}}\right\} \leq \frac{1}{4}\Big(\kappa_1(\varepsilon,m)\,A^{1-m} + \kappa_3(\varepsilon,m)\Big)$$

where $c$ and $\kappa_\star$ are as in (36) and (31), and

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1}\,,\,\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\},\ \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}\,.$$

From Corollaries 9 and 10, we obtain that the inequality

$$\left|\frac{u(t,x)}{B(t,x)}-1\right| \le \varepsilon \tag{86}$$

holds if $t \ge \kappa_1(\varepsilon,m)\,A^{1-m} + \kappa_3(\varepsilon,m)$ and $|x| \ge \rho(\varepsilon)\,R(t)$. We also know from inequality (61) that (86) also holds for any $|x| \le 2\,\rho(\varepsilon)\,R(t)$ if $t \ge 4\,T(\varepsilon)$ and $t$ is such that

$$\frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}}\left(\frac{1}{t}+\frac{\sqrt{G}}{R(t)}\right)^{\vartheta} \le \varepsilon\,. \tag{87}$$

Since $R(t) \le 2\,\alpha\,t$ for any $t \ge 2/\alpha$ and $2^{\alpha-1}\,(1+G^{\alpha/2}\,) \ge (1+\sqrt{G}\,)^\alpha$, (87) holds if

$$t \ge \max\left\{\kappa_2(\varepsilon,m)\,\left(1+G^{\frac{\alpha}{2}}\right)\,,\tfrac{2}{\alpha}\right\} \quad\text{with}\quad \kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\,.$$

Since $\kappa_2(\varepsilon,m)+\kappa_3(\varepsilon,m) \ge 2/\alpha$, then (86) holds for any $x \in \mathbb{R}^d$ if

$$\begin{aligned} t \ge t_\star = \mathsf{c}_\star\,&\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}\,,\\ &\ge \kappa_1(\varepsilon,m)\,A^{1-m} + \kappa_2(\varepsilon,m)\,G^{\frac{\alpha}{2}} + \kappa_2(\varepsilon,m) + \kappa_3(\varepsilon,m)\,. \end{aligned} \tag{88}$$

With $(m,\varepsilon) \in (0,1)\times(0,\varepsilon_{m,d})$, we deduce from the elementary estimates

$$\tfrac{1}{(1-m)\,\varepsilon} \le \tfrac{1}{(1+\varepsilon)^{1-m}-1} \le \tfrac{4}{(1-m)\,\varepsilon} \quad\text{and}\quad \tfrac{1}{2}\,\tfrac{1}{(1-m)\,\varepsilon} \le \tfrac{1}{1-(1-\varepsilon)^{1-m}} \le \tfrac{1}{(1-m)\,\varepsilon}$$

that $\kappa_2$ dominates $\kappa_1$ and $\kappa_3$ as either $\varepsilon \to 0_+$. Up to elementary computations (see [10, Section 3.5] for details), this proves that

$$\mathsf{c}_\star(m,d) = \sup_{\varepsilon\in(0,\varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\ \varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\ \varepsilon\,\kappa_3(\varepsilon,m)\right\} \tag{89}$$

is finite, which completes the proof. □

From the expression of $\mathsf{c}_\star(m,d)$ we obtain that

$$\mathsf{c}_\star(m,d) \ge \varepsilon\,\kappa_3(\varepsilon,m) \ge \frac{8}{\alpha\,(1-m)} \to \infty \quad\text{as}\quad m \to 1^-\,. \tag{90}$$

# 4 Stability

## 4.1 Improved entropy-entropy production inequality

As a consequence of Sections 2 and 3, we have a first result which goes as follows.

**Theorem 13.** *Let* $m \in (m_1, 1)$ *if* $d \geq 2$*,* $m \in (1/2, 1)$ *if* $d = 1$*,* $A > 0$ *and* $G > 0$*. Then there is a positive number* $\zeta$ *such that*

$$\mathcal{I}[v] \geq (4 + \zeta)\,\mathcal{F}[v] \tag{91}$$

*for any nonnegative function* $v \in \mathrm{L}^1(\mathbb{R}^d)$ *such that* $\mathcal{F}[v] = G$*,* $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$*,* $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and* $v$ *satisfies* ($\mathrm{H}_A$)*.*

An expression of $\zeta$ is given below in (95) in terms of $A$ and $G$. Inequality (91) is an improvement of the *entropy - entropy production* inequality (11). We prove that the inequality holds at any time $t \geq 0$ for any solution of the evolution equation (9) and, as a special case, for its initial datum.

*Proof.* Let us consider a solution $v$ of (9) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and $v_0$ satisfies ($\mathrm{H}_A$). Let us choose some $\varepsilon \in (0, \chi\,\eta)$, with $\eta = 2\,d\,(m - m_1)$ and $\chi$ as in Proposition 3. By the change of variables (19), the function $u$ solves (8) and we learn from Theorem 4 that

$$(1 - \varepsilon)\,\mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\,\mathcal{B} \quad \forall\, t \geq T$$

where $T = \frac{1}{2}\log R(t_\star)$ follows from the definition (21) of $R$, and $t_\star$ is computed from $\varepsilon \in (0, \varepsilon_{m,d})$ as in Theorem 4, *i.e.*, given by (88). This determines an asymptotic time layer improvement: according to Proposition 3, (16) holds with $\eta = 2\,d\,(m - m_1)$ for $\varepsilon \in (0, \chi\,\eta)$, that is,

$$\mathcal{I}[v(t, .)] \geq (4 + \eta)\,\mathcal{F}[v(t, .)] \quad \forall\, t \geq T\,.$$

With the initial time layer improvement of Lemma 2, we obtain that

$$\mathcal{I}[v(t, .)] \geq (4 + \zeta)\,\mathcal{F}[v(t, .)] \quad \forall\, t \in [0, T]\,, \quad \text{where} \quad \zeta = \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}}\,. \tag{92}$$

As a consequence, (91) holds for $v(t, .)$, for any $t \geq 0$, because $\zeta \leq \eta$, under the condition

$$\varepsilon \in (0,\, 2\,\varepsilon_\star) \quad \text{with} \quad \varepsilon_\star := \frac{1}{2}\,\min\left\{\varepsilon_{m,d},\, \chi\,\eta\right\}.$$

As a special case, it is true at $t = 0$ with $\varepsilon = \varepsilon_\star$ and for an arbitrary initial datum satisfying the assumptions of Theorem 13. This completes the proof. □

Of course the fact that the inequality holds true at any $t \geq 0$ for a solution of (9) is a stability property under the action of the nonlinear fast diffusion flow. The improvement in inequality (91) has an interesting counterpart in terms of rates, which goes as follows.

**Corollary 14.** *Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$ and let $\zeta$ be as in Theorem 13. If $v$ is a solution of (9) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and $v_0$ satisfies ($\mathrm{H}_A$), then*

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0]\,e^{-\,(4+\zeta)\,t} \quad \forall\,t \geq 0\,. \tag{93}$$

Let us give the sketch of a proof and some comments. We know from Theorem 13 that

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)] = -\,\mathcal{I}[v(t,\cdot)] \leq -\,(4+\zeta)\,\mathcal{F}[v(t,\cdot)]$$

and obtain (93) by a Grönwall estimate. Inequality (91) can be recovered as a consequence of Corollary 14. It is indeed enough to notice that (93) is an equality at $t = 0$ and differentiate it at $t = 0_+$. Notice that the optimal decay rate in (93) is the optimal constant in (91), as in [27] in the non-improved version of the inequality.

### 4.2 Stability of Gagliardo-Nirenberg inequalities: proof of Theorem 1

This section is devoted to the proof of the main result of Section 1.

*Proof of Theorem 1.* Using (25) where, $p = 1/(2\,m-1)$ and $v = |f|^{2\,p}$, we learn from (26) and from Theorem 13 that

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] = \tfrac{p+1}{p-1}\,\delta[f] \geq \zeta\,\mathcal{F}[v] = \zeta\,\mathcal{E}[f]$$

under Condition (5). As a consequence, Theorem 1 holds with $\mathcal{C} = \frac{p-1}{p+1}\,\zeta$.

We can also notice that (91) can be rewritten as

$$\mathcal{I}[v] -\; 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]\,. \tag{94}$$

After taking into account

$$\mathcal{I}[v] = \left(p^2-1\right)\int_{\mathbb{R}^d}\left|\nabla f + \tfrac{1}{p-1}\,f^p\,\nabla \mathsf{g}^{1-p}\right|^2 dx\,,$$

this proves (7) in the case of non-negative functions.

The expression of $\zeta$ given in (92) can be rewritten using $T = \frac{1}{2}\,\log R(t_\star)$ as

$$\zeta = \frac{4\,\eta}{(4+\eta)\,(1+\alpha\,t_\star)^{2/\alpha} - \eta}\,,$$

where $t_\star$ is given by (85). Since $t_\star \geq 2/\alpha$, using $R(t_\star) = (1+\alpha\,t_\star)^{\frac{1}{\alpha}} \leq (2\,\alpha\,t_\star)^{1/\alpha}$ we obtain

$$\zeta \geq \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}}\left(1+A^{1-m}+G^{\frac{\alpha}{2}}\right)^{-\frac{2}{\alpha}}$$

with the notation of Proposition 12. Let

$$c_\alpha := \inf_{x,\,y>0}\frac{1+x^{2/\alpha}+y}{\left(1+x+y^{\alpha/2}\right)^{2/\alpha}}\,.$$

Then we have

$$\zeta \geq \mathsf{Z}\Big(A,\mathcal{F}[u_0]\Big)$$

with

$$\mathsf{Z}(A,G) := \frac{\zeta_\star}{1+A^{(1-m)\,\frac{2}{\alpha}}+G}\,. \tag{95}$$

Here we make the choice $\varepsilon = \varepsilon_\star$ as in Section 4.1, so that the numerical constant $\zeta_\star$ is defined as

$$\zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha\,. \tag{96}$$

This is the explicit expression of the constant of our main result, Theorem 1. □

The constant $\zeta_\star$ deserves some comments. First of all, we know that $\varepsilon_\star \leq \frac{\chi}{2}\,\eta$ so that $\zeta_\star = \zeta_\star(m)$ is at most of the order of $(m-m_1)^{1+2\,\mathsf{a}/\alpha}$ where $\frac{\mathsf{a}}{\alpha} = \frac{2-m}{\vartheta\,(1-m)}$. As a consequence, we know that $\lim_{m\to m_1}\zeta_\star(m) = 0$. This also means that the estimate of the constant $\mathcal{C}$ in Theorem 1 decays to 0 if $p \to p^*$ if $d \geq 2$. On the other hand, it appears from (96) that $\lim_{m\to 1_-}\zeta_\star(m) = 0$. Our method is therefore limited to the strictly subcritical range $\max\{1/2, m_1\} < m < 1$, or $1 < p < p^*$.

### 4.3 Scale invariance and Gagliardo-Nirenberg inequalities

How $\mathcal{C}_{\rm GN}$ and $\mathcal{K}_{\rm GN}$ are related has already been dealt with in [44, Section 4.1], but further details are needed in preparation for Section 4.4. The Gagliardo-Nirenberg inequality (1) is equivalent to Inequality (3), both with optimal constants, are equivalent. Indeed, with

$$a = \frac{d}{p}+2-d \quad \text{and} \quad b = d\,\frac{p-1}{2\,p}\,,$$

the optimization with respect to $\lambda > 0$ of $h(\lambda) := \lambda^a\,X + \lambda^{-b}\,Y$ indeed shows that

$$h(\lambda) \geq c(p,d)\,X^{\frac{b}{a+b}}\,Y^{\frac{a}{a+b}} \quad \text{where} \quad c(p,d) := \left(\tfrac{b}{a}\right)^{\frac{a}{a+b}} + \left(\tfrac{a}{b}\right)^{\frac{b}{a+b}}\,,$$

with equality if and only if $\lambda = \left(\frac{b\,Y}{a\,X}\right)^{1/(a+b)}$, and we can check that

$$\tfrac{2\,b}{a+b} + \tfrac{(p+1)\,a}{a+b} = 2\,p\,\gamma\,.$$

We apply this optimization to $h(\lambda)=\delta[f_\lambda]$, where $f_\lambda$ is given by the scaling

$$f_\lambda(x)=\lambda^{\frac{d}{2\,p}}\,f(\lambda\,x)\quad\forall\,x\in\mathbb{R}^d\,.$$

We have $\|f_\lambda\|_{2\,p}=\|f\|_{2\,p}$ and, for the optimal choice of $\lambda$, that is,

$$\lambda=\left(\frac{b\,Y}{a\,X}\right)^{\frac{1}{a+b}}\quad\text{where}\quad X=(p-1)^2\,\|\nabla f\|_2^2\quad\text{and}\quad Y=4\,\tfrac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1}\,,$$

we have the equality

$$\begin{aligned}\delta[f_\lambda]=(p-1)^2\,\|\nabla f_\lambda\|_2^2+4\,\tfrac{d-p\,(d-2)}{p+1}\,\|f_\lambda\|_{p+1}^{p+1}-2\,\mathcal{K}_{\rm GN}\,\|f_\lambda\|_{2\,p}^{2\,p\,\gamma}\\ =C(p,d)\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2\,p\,\gamma}-2\,\mathcal{K}_{\rm GN}\,\|f\|_{2\,p}^{2\,p\,\gamma}\end{aligned}$$

with

$$C(p,d)=c(p,d)\left((p-1)^\theta\left(4\,\tfrac{d-p\,(d-2)}{p+1}\right)^{\frac{1-\theta}{p+1}}\right)^{2\,p\,\gamma}\,.$$

Concerning the optimal constants in (1) and (3), we infer the relation

$$C(p,d)\,\mathcal{C}_{\rm GN}^{2\,p\,\gamma}=2\,\mathcal{K}_{\rm GN}\,. \tag{97}$$

### 4.4 Stability of scale invariant Gagliardo-Nirenberg inequalities

We conclude this paper by a consequence of Theorem 1 which is a deep but technical result that requires further notation. For any $f\in\mathcal{D}(\mathbb{R}^d)$, let us consider the *best matching Aubin-Talenti profile* $\mathsf{g}_f$ in the sense that

$$\int_{\mathbb{R}^d}\left(|f|^{p+1}-\mathsf{g}_f^{p+1}\right)dx=\min_{\mathsf{g}\in\mathfrak{M}}\int_{\mathbb{R}^d}\left(\mathsf{g}^{p+1}-|f|^{p+1}+\tfrac{1+p}{2\,p}\,\mathsf{g}^{1-p}\left(|f|^{2\,p}-\mathsf{g}^{2\,p}\right)\right)dx\,,$$

where the minimum is taken over the manifold $\mathfrak{M}$ of all Aubin-Talenti profiles. See [42, 43] for details. If $x_f$ denotes the center of mass of $|f|^{2\,p}$, we recall that

$$\int_{\mathbb{R}^d}|f|^{2\,p}\,dx=\int_{\mathbb{R}^d}\mathsf{g}_f^{2\,p}\,dx\,,\quad x_f\int_{\mathbb{R}^d}|f|^{2\,p}\,dx=\int_{\mathbb{R}^d}x\,|f|^{2\,p}\,dx=\int_{\mathbb{R}^d}x\,\mathsf{g}_f^{2\,p}\,dx\,,$$
$$\text{and}\quad\int_{\mathbb{R}^d}|x-x_f|^2\,|f|^{2\,p}\,dx=\int_{\mathbb{R}^d}|x-x_f|^2\,\mathsf{g}_f^{2\,p}\,dx\,.$$

With

$$\kappa[f]:=\frac{\mathcal{M}^{\frac{1}{2\,p}}}{\|f\|_{2\,p}}\quad\text{and}\quad\sigma[f]:=\frac{\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx}{\kappa[f]^{2\,p}\,\int_{\mathbb{R}^d}|x-x_f|^2\,|f|^{2\,p}\,dx}\,,$$

we notice that

$$\mathsf{g}_f(x)=\frac{1}{\kappa[f]\,\sigma[f]^{\frac{d}{4\,p}}}\,\mathsf{g}\left(\frac{x-x_f}{\sqrt{\sigma[f]}}\right)\quad\forall\,x\in\mathbb{R}^d\,.$$

We also define

$$\lambda[f] := \left( \frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2\,p}{d-p\,(d-4)}}\,,$$

$$\mathsf{A}[f] := \frac{\mathcal{M}}{\lambda[f]^{\frac{d-p\,(d-4)}{p-1}}\,\|f\|_{2\,p}^{2\,p}}\,\sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2\,p}\,dx$$

and

$$\mathsf{E}[f] := \frac{2\,p}{1-p} \int_{\mathbb{R}^d} \left( \frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2\,p}}}\,|f|^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2\,p}\,\mathsf{g}^{1-p} \left( \frac{\kappa[f]^{2\,p}}{\lambda[f]^2}\,|f|^{2\,p} - \mathsf{g}^{2\,p} \right) \right) dx\,.$$

Our final result and deepest result is an inequality that can be interpreted as a general stability result in $\mathcal{W}$ of the Gagliardo-Nirenberg inequality (1).

**Theorem 15.** *Let $d \ge 1$ and $p \in (1, p^*)$. For any $f \in \mathcal{W}$, we have*

$$\left( \|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta} \right)^{2\,p\,\gamma} - \left( \mathcal{C}_{\mathrm{GN}}\,\|f\|_{2\,p} \right)^{2\,p\,\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2\,p}^{2\,p\,\gamma}\,\mathsf{E}[f] \tag{98}$$

*where*

$$\mathfrak{S}[f] = \frac{\mathcal{M}^{\frac{p-1}{2\,p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}\left(\mathsf{A}[f],\,\mathsf{E}[f]\right)\,.$$

*Here $\mathsf{Z}$ is defined as in (95).*

*Proof.* We rewrite the result of Theorem 13 applied to

$$f_\lambda(x) = \lambda^{\frac{d}{2\,p}}\,\kappa[f]\,f(\lambda\,x + x_f) \tag{99}$$

with $\lambda = \lambda[f]$. For simplicity, we use the notation $f_* = f_{\lambda[f]}$ and rely on the computations of Section 4.3. We notice that $f_*$ is such that $\|f_*\|_{2\,p}^{2\,p} = \mathcal{M}$ and $\int_{\mathbb{R}^d} x\,|f_*|^{2\,p}\,dx = 0$, so that Theorem 13 applies and, as a consequence,

$$\delta[f_*] \ge \mathcal{C}\,\mathcal{E}[f_*]\,.$$

We learn from Section 4.2 that

$$\mathcal{C} = \tfrac{p-1}{p+1}\,\mathsf{Z}\Big(A, \mathcal{E}[f_*]\Big) \quad \text{where} \quad A = \sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} |f_*|^{2\,p}\,dx\,.$$

By undoing the change of variables (99), we find

$$A = \mathsf{A}[f_*] = \mathsf{A}[f] \quad \text{and} \quad \mathcal{E}[f_*] = \mathsf{E}[f]\,,$$

which completes the proof. □

We also have a scale invariant form of (7). By writing (94) for $f_*$, we find that

$$\left( \|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta} \right)^{2\,p\,\gamma} - \left( \mathcal{C}_{\mathrm{GN}}\,\|f\|_{2\,p} \right)^{2\,p\,\gamma} \\ \ge \|f\|_{2\,p}^{2\,p\,\gamma}\,\tfrac{(p^2-1)}{C(p,d)}\,\tfrac{\mathsf{Z}(\mathsf{A}[f],\,\mathsf{E}[f])}{4+\mathsf{Z}(\mathsf{A}[f],\,\mathsf{E}[f])} \int_{\mathbb{R}^d} \left| \kappa[f]\,\lambda[f]^{\frac{d-p\,(d-2)}{2\,p}}\,\nabla f + \tfrac{\kappa[f]^p}{\lambda[f]}\,x\,|f|^p \right|^2 dx \tag{100}$$

for any function $f \in \mathcal{W}$.

### 4.5 Concluding remarks

Theorem 15 is not straightforward to read and deserves some comments.

(i) The constant $\mathfrak{S}[f]$ in the right-hand side of (98) measures the stability. Although it has a complicated expression, we have shown that it can be written in terms of well-defined quantities depending on $f$ and purely numerical constants. As a special case, it is straightforward to check that

$$\mathfrak{S}[\mathsf{g}] > 0$$

where $\mathsf{g}$ is the Aubin-Talenti function (2). Stability results known so far from [44, 42] involve in the right-hand side an $\mathcal{E}[f]^2$ term, while here we achieve a linear lower estimate in terms of $\mathcal{E}[f]$.

(ii) An easy consequence of (98) is an estimate of the deficit in the Gagliardo-Nirenberg inequality (1), namely

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} - \mathcal{C}_{\mathrm{GN}} \, \|f\|_{2p} \ge \frac{(2\,p\,\gamma)^{-1} \, \mathfrak{S}[f] \, \|f\|_{2p}^{2\,p\,\gamma}}{\left(\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta}\right)^{2\,p\,\gamma-1}} \, \mathsf{E}[f] \quad \forall\, f \in \mathcal{W}\,.$$

(iii) While the restriction of (5) has been lifted in Theorem 15, Condition (6) is deeply rooted in our method. It is an open question to decide if this assumption can be removed. Note that it is present in Theorem 15 because $\mathsf{A}[f] = +\infty$ or $\mathsf{E}[f] = +\infty$ means $\mathfrak{S}[f] = 0$. The same remark applies to (100).

(iv) More subtle is the fact the natural space is not the space of functions $f$ in $\mathrm{L}^{2p}(\mathbb{R}^d)$ with gradient in $\mathrm{L}^2(\mathbb{R}^d)$, but we also need that $\int_{\mathbb{R}^d} |x|^2 \, |f|^{2p} \, dx$ is finite, for instance to define the free energy. Up to the condition that $\mathsf{A}[f] < +\infty$, we are therefore working in the space

$$\mathcal{W} := \left\{ f \in \mathrm{L}^{2p}\left(\mathbb{R}^d, (1+|x|^2)\,dx\right) \,:\, \nabla f \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$$

obtained as the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to the natural norm. If $p = p^*$, this is not the space of the stability result in the critical case by G. Bianchi and H. Egnell. It is however consistant with the use of the Fisher information. We can for instance notice that the Fisher information being nonnegative, that is,

$$0 \le \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + |f|^{p-1} f \, \nabla \mathsf{g}^{1-p}\right|^2 dx = \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + 2\,|f|^{p-1} f \, x\right|^2 dx\,,$$

after expanding the square, we obtain the inequality

$$\int_{\mathbb{R}^d} |f|^{p+1} \, dx \le \frac{p-1}{4} \int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \frac{1}{p-1} \int_{\mathbb{R}^d} |x|^2 \, |f|^{2p} \, dx\,.$$

An optimization under scaling proves that

$$\left(\int_{\mathbb{R}^d} |f|^{p+1} \, dx\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2 \, dx \int_{\mathbb{R}^d} |x|^2 \, |f|^{2p} \, dx\,. \tag{101}$$

This is for $p>1$ a nonlinear extension of the *Heisenberg uncertainty principle* whose standard form corresponds to $p=1$. Such an extension is known, including in the presence of weights, see for instance [88]. Hence $\mathcal{W}$ is a natural space for stability properties in Gagliardo-Nirenberg inequalities.

(v) The two quantities $\lambda[f]$ and $\sqrt{\sigma[f]}$ define length scales. They are equal only in the equality case of (101) which is achieved if and only if $f$ is an Aubin-Talenti function. This can also be read on (98) and (100).

(vi) The right-hand side term $\mathsf{E}[f]$ in (98) is the relative entropy of $|f_*|^{2p}$ with respect to $|\mathsf{g}|^{2p}$ and we can write that

$$\mathsf{E}[f] \geq \frac{2\,p}{p-1}\,\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2\,p}}}\,\mathsf{D}[f] \quad \text{where} \quad \mathsf{D}[f] := \int_{\mathbb{R}^d}\left(\mathsf{g}_f^{p+1}-|f|^{p+1}\right)dx\,.$$

We recall that $\mathsf{g}_f$ denotes the *best matching Aubin-Talenti profile* and, up to a multiplicative constant, $\mathsf{D}$ is the relative entropy of $|f|^{2p}$ to the best matching Barenblatt profile $|\mathsf{g}_f|^{2p}$, which minimizes the relative entropy with respect to all Barenblatt profiles. We emphasize the fact that, according to our conventions, Barenblatt profiles are the same as Aubin-Talenti profiles raised to the power $2\,p$. As a consequence of the Csiszár-Kullback inequality, we have

$$\mathsf{D}[f] \geq c_p\,\mathcal{M}\,\frac{\left\||f|^{2\,p}-|\mathsf{g}_f|^{2\,p}\right\|_1^2}{\|f\|_{2\,p}^{4\,p}}\,,$$

with $c_p = 2\,\frac{p-1}{p}\,\frac{d-p\,(d-4)}{p+1}$. For more details, see Appendix D. This estimate means that $\mathsf{D}$ provides us with an estimate of a distance to the manifold $\mathfrak{M}$.

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# Appendices

## A An interpolation between $\mathrm{L}^p$ and $C^\nu$ norms

The purpose of this appendix is to establish a simple interpolation lemma which is at the core of our result of uniform convergence in relative error (Theorem 4). We give an explicit constant as well as an elementary proof. Such a result goes back to [56] and to [73, p. 126]. We recall that $\lfloor\cdot\rfloor_{C^\nu(\mathbb{R}^d)}$ is defined in (68). We claim no originality except for the computation of the constant.

**Lemma 16.** *Let $p \geq 1$ and $\nu \in (0,1)$. Then there exists a positive constant $C_{d,\nu,p}$ such that, for any $u \in \mathrm{L}^p(B_{2R}(x)) \cap C^\nu(B_{2R}(x))$, $R > 0$ and $x \in \mathbb{R}^d$*

$$\|u\|_{\mathrm{L}^\infty(B_R(x))} \leq C_{d,\nu,p} \left( \lfloor u \rfloor_{C^\nu(B_{2R}(x))}^{\frac{d}{d+p\,\nu}} \|u\|_{\mathrm{L}^p(B_{2R}(x))}^{\frac{p\,\nu}{d+p\,\nu}} + R^{-\frac{d}{p}} \|u\|_{\mathrm{L}^p(B_{2R}(x))} \right). \tag{102}$$

*Analogously, we have*

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq C_{d,\nu,p} \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\,\nu}} \|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p\,\nu}{d+p\,\nu}} \quad \forall\, u \in \mathrm{L}^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d). \tag{103}$$

*In both cases, the inequalities hold with the constant*

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\,\nu)+dp}{p(d+p\,\nu)}} \left(1 + \tfrac{d}{\omega_d}\right)^{\frac{1}{p}} \left(1 + \left(\tfrac{d}{p\,\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\,\nu}} \left( \left(\tfrac{d}{p\,\nu}\right)^{\frac{p\,\nu}{d+p\,\nu}} + \left(\tfrac{p\,\nu}{d}\right)^{\frac{d}{d+p\,\nu}} \right)^{1/p}.$$

*Proof.* For any $z$, $y \in B_R(x)$, by the triangle inequality and by definition of $\lfloor \cdot \rfloor_{C^\nu(B_{2R})}$ given in (68), we have that

$$\begin{aligned} |u(z)|^p &\leq \big(|u(z) - u(y)| + |u(y)|\big)^p \\ &\leq 2^{p-1}\big(|u(z) - u(y)|^p + |u(y)|^p\big) \\ &\leq 2^{p-1}\left[\left(C + \lfloor u \rfloor_{C^\nu(B_{2R}(x))}\right)^p |z-y|^{p\,\nu} + |u(y)|^p\right] \end{aligned}$$

for some $C > 0$ to be chosen later. Let $0 \leq \rho < R$. By averaging on a ball $B_\rho(z)$, we have

$$\begin{aligned} |u(z)|^p &\leq \frac{2^{p-1} d}{\omega_d\, \rho^d} \left[\left(C + \lfloor u \rfloor_{C^\nu(B_{2R}(x))}\right)^p \int_{B_\rho(z)} |z-y|^{p\,\nu}\, dy + \int_{B_\rho(z)} |u(y)|^p\, dy\right] \\ &\leq 2^{p-1}\left(1 + \frac{d}{\omega_d}\right)\left[\rho^{p\,\nu}\left(C + \lfloor u \rfloor_{C^\nu(B_{2R}(x))}\right)^p + \rho^{-d} \|u\|_{\mathrm{L}^p(B_{2R}(x))}^p\right]. \end{aligned} \tag{104}$$

The right-hand side of the above inequality achieves its minimum w.r.t. $\rho > 0$ at

$$\rho_\star := \left( \frac{d\, \|u\|_{\mathrm{L}^p(B_{2R}(x))}^p}{p\,\nu \left(C + \lfloor u \rfloor_{C^\nu(B_{2R}(x))}\right)^p} \right)^{\frac{1}{d+p\,\nu}}.$$

With $C > 0$, the denominator in the right-hand side is never zero. With the choice

$$C := \left(\tfrac{d}{p\,\nu}\right)^{\frac{1}{p}} \frac{\|u\|_{\mathrm{L}^p(B_{2R}(x))}}{R^{\frac{d+p\,\nu}{p}}},$$

we are sure that $\rho_\star < R$. Hence, by evaluating (104) at $\rho_\star$ we obtain

$$\|u\|_{\mathrm{L}^\infty(B_R(x))} \leq 2^{1-\frac{1}{p}} \left(1 + \frac{d}{\omega_d}\right)^{\frac{1}{p}} \left( \left(\tfrac{d}{p\,\nu}\right)^{\frac{p\,\nu}{d+p\,\nu}} + \left(\tfrac{p\,\nu}{d}\right)^{\frac{d}{d+p\,\nu}} \right)^{1/p} \|u\|_{\mathrm{L}^p(B_{2R}(x))}^{\frac{p\,\nu}{d+p\,\nu}} \left(C + \lfloor u \rfloor_{C^\nu(B_{2R}(x))}\right)^{\frac{d}{d+p\,\nu}}.$$

Inequality (102) is deduced from the above one. Inequality (103) can be deduced from (102) by taking $R \to \infty$. The proof is completed. □

## B Optimal constant in a Sobolev inequality

Let $B$ be the unit ball in $\mathbb{R}^d$ with $d \geq 3$, $2^* = 2\,d/(d-2)$ and consider the Sobolev inequality associated with the embedding $\mathrm{H}^1(B) \hookrightarrow \mathrm{L}^{2^*}(B)$ with optimal constant $\mathcal{S}_\lambda$, that is,

$$\|f\|_{\mathrm{L}^{2^*}(B)} \leq \mathcal{S}_\lambda \left(\|\nabla f\|_{\mathrm{L}^2(B)} + \lambda\,\|f\|_{\mathrm{L}^2(B)}\right) \quad \forall\, f \in \mathrm{H}^1(B) \tag{105}$$

where $\lambda > 0$ is an arbitrary parameter. This inequality is standard and can be proved using an extension of $\mathrm{H}^1(B)$ to $\mathrm{H}^1(\mathbb{R}^d)$ as in [46, Section 5.6]. For our purpose, we need an explicit estimate of $\mathcal{S}_\lambda$. Let

$$\lambda_{\mathrm{S}} := \frac{\sqrt{d\,(d-2)}}{2}\left(2\sqrt{\pi}\,\frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{1/d}.$$

**Lemma 17.** *For any $d \geq 3$, the optimal constant $\mathcal{S}_\lambda$ in (105) is given by*

$$\mathcal{S}_\lambda = \begin{cases} \frac{2}{\sqrt{\pi\,d\,(d-2)}}\left(\frac{1}{2\sqrt{\pi}}\,\Gamma\left(\frac{d+1}{2}\right)\right)^{\frac{1}{d}} = \mathsf{S}_d & \text{if} \quad \lambda \geq \lambda_{\mathrm{S}}\,, \\ \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{d}{2}+1\right)^{\frac{1}{d}}\,\frac{1}{\lambda} = \frac{\mathsf{S}_d\,\lambda_{\mathrm{S}}}{\lambda} & \text{if} \quad \lambda \leq \lambda_{\mathrm{S}}\,. \end{cases}$$

We may notice that $\mathcal{S}_1 = \frac{1}{\sqrt{\pi}}\,\Gamma\left(\frac{d}{2}+1\right)^{1/d} > \mathcal{S}_{\lambda_{\mathrm{S}}}$ because $\lambda_{\mathrm{S}} > 1$ and $\mathcal{S}_{\lambda_{\mathrm{S}}} = \mathsf{S}_d$ is the usual optimal Sobolev constant in $\mathbb{R}^d$ such that

$$\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, f \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,, \quad \mathsf{S}_d = \frac{2\,(\omega_{d+1})^{-1/d}}{\sqrt{d\,(d-2)}}\,, \tag{106}$$

according to [2, 76, 82]. It is worth mentioning that $\lambda\,\mathcal{S}_\lambda = |B|^{\frac{1}{2^*}-\frac{1}{2}}$ for any $\lambda \leq \lambda_{\mathrm{S}}$ where $|B|$ is the Lebesgue measure of the unit ball.

*Proof.* We will first prove the inequality for $\lambda = \lambda_{\mathrm{S}}$ then we generalize it. Up to replacing $f$ by $|f|$, we can reduce the proof of the inequality to the case of nonnegative functions. By standard non-increasing rearrangements (see, *e.g.*, [65]), we can further reduce it to the case of a radial, non-increasing function $f$ with boundary value $f_0$. On the other hand, from the Minkowski inequality, we have

$$\|f\|_{\mathrm{L}^{2^*}(B)} = \|(f-g)+g\|_{\mathrm{L}^{2^*}(B)} \leq \|f-g\|_{\mathrm{L}^{2^*}(B)} + \|g\|_{\mathrm{L}^{2^*}(B)}\,.$$

Applied to the constant function $g = f_0 \leq f$, this means that

$$\|f\|_{\mathrm{L}^{2^*}(B)} \leq \|f-f_0\|_{\mathrm{L}^{2^*}(B)} + |B|^{\frac{1}{2^*}-\frac{1}{2}}\,\|f\|_{\mathrm{L}^2(B)}$$

because $\|f_0\|_{\mathrm{L}^2(B)} = \sqrt{|B|}\, f_0 \le \|f\|^2_{\mathrm{L}^2(B)}$ and $\|f_0\|_{\mathrm{L}^{2^*}(B)} = |B|^{1/2^*} f_0$. On the other hand, $f - f_0$ is a nonnegative function on $B$ with boundary value 0, that we can extend by 0 on $\mathbb{R}^d \setminus B$. By Sobolev's inequality (106), we have

$$\|f - f_0\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \le \mathsf{S}_d \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)} \,.$$

We recall that $|B| = \omega_d/d$ and conclude that

$$\|f\|_{\mathrm{L}^{2^*}(B)} \le \mathsf{S}_d \, \|\nabla f\|_{\mathrm{L}^2(B)} + \mathsf{S}_d \, \lambda_{\mathrm{S}} \, \|f\|_{\mathrm{L}^2(B)} \,, \tag{107}$$

which proves (105) in the case $\lambda = \lambda_{\mathrm{S}}$.

• if $\lambda \ge \lambda_{\mathrm{S}}$, the following inequality follows

$$\|f\|_{\mathrm{L}^{2^*}(B)} \le \mathsf{S}_d \, \|\nabla f\|_{\mathrm{L}^2(B)} + \mathsf{S}_d \, \lambda_{\mathrm{S}} \, \|f\|_{\mathrm{L}^2(B)} \le \mathcal{S}_\lambda \, \|\nabla f\|_{\mathrm{L}^2(B)} + \mathcal{S}_\lambda \, \lambda \, \|f\|_{\mathrm{L}^2(B)} \,,$$

where we only have used the fact that $\mathcal{S}_\lambda = \mathsf{S}_d$ and that $\lambda \ge \lambda_{\mathrm{S}}$. In order to establish optimality we define a smooth truncation function $\varphi$ such that $\varphi(r) = 1$ if $r \le 1/2$, $\varphi(r) = 0$ if $r \ge 1$, and $0 \le \varphi \le 1$. We consider the test functions

$$f_\mu(x) = \varphi(|x|) \left(\mu^{-1} + \mu \, |x|^2\right)^{-\frac{d-2}{2}} \quad \forall \, x \in B$$

and let $\mu \to +\infty$. The result follows from

$$\lim_{\mu \to +\infty} \frac{\|f_\mu\|_{\mathrm{L}^{2^*}(B)}}{\|\nabla f_\mu\|_{\mathrm{L}^2(B)}} = \mathsf{S}_d \quad \text{and} \quad \lim_{\mu \to +\infty} \frac{\|f_\mu\|_{\mathrm{L}^2(B)}}{\|\nabla f_\mu\|_{\mathrm{L}^2(B)}} = 0 \,.$$

• if $\lambda \le \lambda_{\mathrm{S}}$, from(107) we deduce

$$\|f\|_{\mathrm{L}^{2^*}(B)} \le \frac{\mathsf{S}_d \, \lambda_{\mathrm{S}}}{\lambda} \, \frac{\lambda}{\lambda_{\mathrm{S}}} \, \|\nabla f\|_{\mathrm{L}^2(B)} + \frac{\mathsf{S}_d \, \lambda_{\mathrm{S}}}{\lambda} \, \lambda \, \|f\|_{\mathrm{L}^2(B)} \le \mathcal{S}_\lambda \, \|\nabla f\|_{\mathrm{L}^2(B)} + \mathcal{S}_\lambda \, \lambda \, \|f\|_{\mathrm{L}^2(B)} \,,$$

where we have used that $\mathcal{S}_\lambda = \mathsf{S}_d \, \lambda_{\mathrm{S}}/\lambda$ and that $\lambda/\lambda_{\mathrm{S}} \le 1$. To establish optimality we observe that $\lambda \, \mathcal{S}_\lambda = |B|^{\frac{1}{2^*} - \frac{1}{2}}$ so equality is achieved by $f \equiv f_0 = 1$. □

# C Interpolation inequalities in low dimensions

In dimension $d = 1$ and $d = 2$, we cannot rely on the Sobolev inequality of Appendix B. This is why direct proofs for subcritical cases have to be established.

## C.1 One-dimensional interpolation inequalities

For an introduction to Gagliardo-Nirenberg inequality in dimension $d = 1$, we refer to [57, 35]. Here we prove the following elementary result on an interval.

**Lemma 18.** *Let $p \in (2,\infty)$. Then for all $u \in W^{1,2}(I_R)$, where $I_R=(-R,R)$ we have*

$$\|u\|^2_{\mathrm{L}^p(I_R)} \le (2\,R)^{1+\frac{2}{p}} \left(\tfrac{p-2}{\pi^2}\,\|u'\|^2_{\mathrm{L}^2(I_R)} + \tfrac{1}{4\,R^2}\,\|u\|^2_{\mathrm{L}^2(I_R)}\right)$$

*and this inequality is sharp.*

By sharp, we mean that the infimum of the quotient

$$\mathcal{Q}_R[u] := \frac{4\,R^2\,\|u'\|^2_{\mathrm{L}^2(I_R)}}{(2\,R)^{1-\frac{2}{p}}\,\|u\|^2_{\mathrm{L}^p(I_R)} - \|u\|^2_{\mathrm{L}^2(I_R)}}$$

is achieved by $\lim_{n\to+\infty}\mathcal{Q}_R[u_n] = \frac{\pi^2}{p-2}$ with $u_n(x) = 1+\frac{1}{n}\,\sin\left(\frac{\pi\,x}{2\,R}\right)$.

*Proof.* Let us denote by $\mathcal{C}_R$ the infimum of $\mathcal{Q}_R$ on the set $\mathcal{W}_R$ of the non-constant functions in $W^{1,2}(I_R)$. To a function $u \in \mathcal{W}_R$, we associate a function $v$ on $I_{2R}$ by considering $v(x-R) = u(x)$ in $I_R$ and $v(x-R) = u(2\,R-x)$ in $(R,3\,R)$. Since $v(2\,R) = v(-\,2\,R)$, $v$ the function can be repeated periodically and considered as a $4\,R$-periodic function on $\mathbb{R}$, or simply a function on $I_{2\,R}$ with periodic boundary conditions. We can easily check that

$$\mathcal{Q}_R[u] = \frac{1}{4}\,\mathcal{Q}_{2R}[v]\,,$$

and deduce that $\mathcal{C}_R = \inf \mathcal{Q}_{2R}[v]$ where the infimum is taken on the set of the even functions in $\mathcal{W}_{2R}$. Hence

$$\mathcal{C}_R \ge \frac{1}{4} \inf_{\substack{v\in\mathcal{W}_{2R},\\ v \text{ is periodic}}} \mathcal{Q}_{2R}[v]\,, \tag{108}$$

where the inequality arises because we relax the symmetry condition $v(x) = v(-x)$. With the scaling $v(x) = w\left(\frac{\pi\,x}{2\,R}\right)$, we reduce the problem on the periodic functions in $\mathcal{W}_{2R}$ to the interpolation on the circle $\mathbb{S}^1$ with the uniform probability measure. The optimal inequality on $\mathbb{S}^1$ is

$$\|w\|^2_{\mathrm{L}^p(\mathbb{S}^1)} - \|w\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \le (p-2)\,\|w'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

for any $p>2$, where $\mathbb{S}^1 \approx I_\pi$ (with periodic boundary conditions), the measure is $d\mu = \frac{dx}{2\,\pi}$ and

$$\|w\|^2_{\mathrm{L}^p(\mathbb{S}^1)} = \left(\int_{-\pi}^{+\pi} |w|^p\,d\mu\right)^{2/p}.$$

Moreover, the inequality in (108) is actually an equality, because the infimum is obtained on $\mathbb{S}^1$ among functions which satisfy the symmetry condition $v(x) = v(-x)$: a minimizing sequence is for instance given by $w_n(x) = 1+\frac{1}{n}\,\cos x$.

With $v(x)=w\left(\frac{\pi\,x}{2\,R}\right)$, we find that

$$\left(\int_{-2R}^{+2R}|v|^p\,dx\right)^{2/p}\le(4\,R)^{\frac{2}{p}-1}\left((p-2)\,\frac{4\,R^2}{\pi^2}\int_{-2R}^{+2R}|v'|^2\,dx+\int_{-2R}^{+2R}|v|^2\,dx\right).$$

With no restriction, as far as optimal constants are concerned, we can assume that $v(x)=v(-x)$, so that each of the integral in $v$ is twice as big as the integral computed with the restriction $u$ of $v$ to $I_R$:

$$\left(2\int_{-R}^{+R}|u|^p\,dx\right)^{2/p}\le2\,(4\,R)^{\frac{2}{p}-1}\left((p-2)\,\frac{4\,R^2}{\pi^2}\int_{-R}^{+R}|u'|^2\,dx+\int_{-R}^{+R}|u|^2\,dx\right).$$

This proves that $\mathcal{C}_R=\frac{p-2}{\pi^2}$. □

As an easy consequence of Lemma 18 and to fit better the purpose of Section 3.2, we can observe that the following (non optimal) inequality holds

$$\|u\|^2_{\mathrm{L}^p(I_R)}\le(2\,R)^{1+\frac{2}{p}}\max\left(\tfrac{p-2}{\pi^2},\tfrac14\right)\left(\|u'\|^2_{\mathrm{L}^2(I_R)}+\tfrac{1}{R^2}\,\|u\|^2_{\mathrm{L}^2(I_R)}\right).$$

### C.2 A two-dimensional interpolation inequality

**Lemma 19.** *Let $d=2$. For any $R>0$, we have*

$$\|u\|^2_{\mathrm{L}^4(B_R)}\le\frac{2\,R}{\sqrt{\pi}}\left(\|\nabla u\|^2_{\mathrm{L}^2(B_R)}+\frac{1}{R^2}\,\|u\|^2_{\mathrm{L}^2(B_R)}\right)\quad\forall\,u\in\mathrm{H}^1(B_R)\,.$$

The constant $2/\sqrt{\pi}$ is not optimal. Numerically, we find in [10] that the optimal constant is approximatively $0.0564922...<2/\sqrt{\pi}\approx1.12838$.

*Proof.* Let $\Omega=B_R$ (the proof applies to more general domains, but we do not need such a result). We use the method of Gagliardo and Nirenberg in [56, 73]. As a first step, we prove the inequality corresponding to the embedding $\mathrm{W}^{1,1}(\Omega)\hookrightarrow\mathrm{L}^2(\Omega)$.

Using Lebesgue's version of the fundamental theorem of calculus, we get

$$u(x,y)=u(x_0,y)+\int_{x_0}^x u_x(\xi,y)\,d\xi\quad\text{and}\quad u(x,y)=u(x,y_0)+\int_{y_0}^y u_y(x,\eta)\,d\eta\,,$$

which implies (letting $\Omega_x$ and $\Omega_y$ be $x$ and $y$ sections of $\Omega$ respectively)

$$|u(x,y)|\le|u(x_0,y)|+\int_{\Omega_y}f(\xi,y)\,d\xi\quad\text{and}\quad|u(x,y)|\le|u(x,y_0)|+\int_{\Omega_x}g(x,\eta)\,d\eta$$

where $f(\xi,y)=|u_x(\xi,y)|$ and $g(x,\eta):=|u_y(x,\eta)|$. Multiplying the two above expressions, we get

$$|u(x,y)|^2\le A(x_0,y)\,B(x,y_0)$$

where

$$A(x_0,y) := |u(x_0,y)| + \int_{\Omega_y} f(\xi,y)\,d\xi \quad \text{and} \quad B(x,y_0) := |u(x,y_0)| + \int_{\Omega_x} g(x,\eta)\,d\eta\,.$$

Integrating over $\Omega$ in $dx\,dy$ and then again in $\Omega$ in $dx_0\,dy_0$ we obtain

$$\begin{aligned} |\Omega|\;\|u\|^2_{\mathrm{L}^2(\Omega)} &= \iint_\Omega \iint_\Omega |u(x,y)|^2\,dx\,dy\;dx_0\,dy_0 \\ &\le \iint_\Omega A(x_0,y)\,dx_0\,dy \iint_\Omega B(x,y_0)\,dx\,dy_0\,. \end{aligned}$$

Finally, notice that

$$\begin{aligned} \iint_\Omega A(x_0,y)\,dx_0\,dy &= \iint_\Omega \left( |u(x_0,y)| + \int_{\Omega_y} f(\xi,y)\,d\xi \right) dx_0\,dy \\ &\le \|u\|_{\mathrm{L}^1(\Omega)} + \mathrm{diam}(\Omega)\;\|f\|_{\mathrm{L}^1(\Omega)} \end{aligned}$$

and

$$\begin{aligned} \iint_\Omega B(x,y_0)\,dx\,dy_0 &= \iint_\Omega \left( |u(x,y_0)| + \int_{\Omega_x} g(x,\eta)\,d\eta \right) dx\,dy_0 \\ &\le \|u\|_{\mathrm{L}^1(\Omega)} + \mathrm{diam}(\Omega)\;\|g\|_{\mathrm{L}^1(\Omega)}\;. \end{aligned}$$

Summing up, we obtain

$$\|u\|^2_{\mathrm{L}^2(\Omega)} \le \frac{1}{|\Omega|} \left( \|u\|_{\mathrm{L}^1(\Omega)} + \mathrm{diam}(\Omega)\;\|f\|_{\mathrm{L}^1(\Omega)} \right) \left( \|u\|_{\mathrm{L}^1(\Omega)} + \mathrm{diam}(\Omega)\;\|g\|_{\mathrm{L}^1(\Omega)} \right)\;.$$

We recall that $f = |u_x|$, $g = |u_y|$ and $|\nabla u| = \sqrt{f^2+g^2}$. It is straightforward to check that

$$f + g \le \sqrt{2\,(f^2+g^2)} = \sqrt{2}\,|\nabla u|\,.$$

Since $h \mapsto \sqrt{1+h^2}$ is a convex function, we may apply Jensen's inequality to $h = f/g$, $d\mu = g\,dx\,dy/\iint_\Omega g\,dx\,dy$ and obtain

$$\begin{aligned} \|\nabla u\|^2_{\mathrm{L}^1(\Omega)} &= \left( \iint_\Omega \sqrt{1+h^2}\,d\mu\;\iint_\Omega g\,dx\,dy \right)^2 \\ &\ge \left( 1 + \left( \iint_\Omega h\,d\mu \right)^2 \right) \left( \iint_\Omega g\,dx\,dy \right)^2 = \|f\|^2_{\mathrm{L}^1(\Omega)}\,\|g\|^2_{\mathrm{L}^1(\Omega)}\;. \end{aligned}$$

Hence

$$\|u\|^2_{\mathrm{L}^2(\Omega)} \le \frac{1}{|\Omega|} \left( \|u\|_{\mathrm{L}^1(\Omega)} + \frac{1}{\sqrt{2}}\,\mathrm{diam}(\Omega)\;\|\nabla u\|_{\mathrm{L}^1(\Omega)} \right)^2\;.$$

We apply this estimate to $u^2$ to get

$$\begin{aligned} \|u\|^2_{\mathrm{L}^4(\Omega)} &\le \frac{1}{\sqrt{|\Omega|}} \left( \|u\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{\sqrt{2}}\,\mathrm{diam}(\Omega)\,\|\nabla u^2\|_{\mathrm{L}^1(\Omega)} \right) \\ &\le \frac{1}{\sqrt{|\Omega|}} \left( \|u\|^2_{\mathrm{L}^2(\Omega)} + \sqrt{2}\,\mathrm{diam}(\Omega)\;\|\nabla u\|_{\mathrm{L}^2(\Omega)}\;\|u\|_{\mathrm{L}^2(\Omega)} \right)\;. \end{aligned}$$

We use the elementary estimate

$$\sqrt{2}\,\mathrm{diam}(\Omega)\;\|\nabla u\|_{\mathrm{L}^2(\Omega)}\;\|u\|_{\mathrm{L}^2(\Omega)}\le\|u\|^2_{\mathrm{L}^2(\Omega)}+\frac{1}{2}\,\mathrm{diam}(\Omega)^2\;\|\nabla u\|^2_{\mathrm{L}^2(\Omega)}$$

and finally obtain

$$\|u\|^2_{\mathrm{L}^4(\Omega)}\le\frac{\mathrm{diam}(\Omega)^2}{2\,\sqrt{|\Omega|}}\left(\|\nabla u\|^2_{\mathrm{L}^2(B_R)}+\frac{4}{\mathrm{diam}(\Omega)^2}\;\|u\|^2_{\mathrm{L}^2(B_R)}\right),$$

which completes the proof with $\mathrm{diam}(\Omega)=2\,R$ and $|\Omega|=\pi\,R^2$. □

# D A Csiszár-Kullback type inequality

The relative entropy $\mathcal{F}[v]$ with respect to the Barenblatt function of same mass as $v$ controls the $\mathrm{L}^1$ distance to the Barenblatt function. This is an extension of the historical papers [25, 62], which correspond to the limit case $p\to 1_+$. There are many variants, see for instance [83] or [19] for some classical extensions. If $v\in\mathrm{L}^1_+(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^d}|x|^2\,v\,dx=\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx$ and $\|v\|_1=\mathcal{M}$, we learn from [42] that

$$\left(\int_{\mathbb{R}^d}|v-\mathcal{B}|\left(1+|x|^2\right)dx\right)^2\le\frac{16\,\mathcal{M}}{(d+2)\,m-d}\,\mathcal{F}[v]\,.$$

In the case $m\in(m_1,1)$, the inequality that we need, without the second moment condition, appears in [21], however with no proof. For completeness, let us give a precise statement with the expression of the constant and an elementary proof.

**Lemma 20.** *For any $v\in\mathrm{L}^1_+(\mathbb{R}^d)$ such that $\mathcal{F}[v]$ is finite and $\|v\|_1=\mathcal{M}$, we have*

$$\|v-\mathcal{B}\|_1^2\le\frac{4\,\alpha}{m}\,\mathcal{M}\,\mathcal{F}[v]\,. \tag{109}$$

*Proof.* Since $\int_{\mathbb{R}^d}(v-\mathcal{B})\,dx=0$, we have that

$$\|v-\mathcal{B}\|_1=\int_{\mathbb{R}^d}|v-\mathcal{B}|\,dx=\int_{v\le\mathcal{B}}(\mathcal{B}-v)\,\mathrm{d}x+\int_{v\ge\mathcal{B}}(v-\mathcal{B})\,\mathrm{d}x=2\int_{v\le\mathcal{B}}(\mathcal{B}-v)\,\mathrm{d}x\,.$$

Let $\phi(s)=s^m/(m-1)$. If $0\le t\le s$, a Taylor expansion shows that for some $\xi\in(t,s)$ we have

$$\phi(t)-\phi(s)-\phi'(s)\,(t-s)=\frac{1}{2}\,\phi''(\xi)\,(t-s)^2\ge\tfrac{m}{2}\,s^{m-2}\,(s-t)^2\,,$$

hence

$$\sqrt{\tfrac{m}{2}}\,(s-t)\le s^{\frac{2-m}{2}}\,\sqrt{\phi(t)-\phi(s)-\phi'(s)\,(t-s)}\,.$$

Using this inequality with $s = \mathcal{B}$ and $t = v$ and the Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned}\frac{m}{2}\left(\int_{v\le\mathcal{B}}(\mathcal{B}-v)\,\mathrm{d}x\right)^2 &\le \left(\int_{v\le\mathcal{B}}\mathcal{B}^{\frac{2-m}{2}}\sqrt{\phi(v)-\phi(\mathcal{B})-\phi'(\mathcal{B})\,(v-\mathcal{B})}\,\mathrm{d}x\right)^2\\ &\le \int_{v\le\mathcal{B}}\mathcal{B}^{2-m}\,\mathrm{d}x\int_{v\le\mathcal{B}}\left(\phi(v)-\phi(\mathcal{B})-\phi'(\mathcal{B})\,(v-\mathcal{B})\right)\mathrm{d}x\\ &\le \int_{\mathbb{R}^d}\mathcal{B}^{2-m}\,dx\;\mathcal{F}[v]\end{aligned}$$

and the conclusion follows from the identity $\int_{\mathbb{R}^d}\mathcal{B}^{2-m}\,dx = \frac{\alpha}{2}\,\mathcal{M}$. See Appendix E. $\square$

## E Constants and useful identities

For the convenience of the reader, we collect some elementary identities and definitions. See [27] or [42, Appendix A] for more details. We recall that $m_1 = (d-1)/d$, $m_c = (d-2)/d$, $\alpha = d\,(m-m_c)$, $\mu^{2-d\,(1-m)} = \frac{1-m}{2\,m}$, $\mathtt{b} = \left(\frac{1-m}{2\,m\,\alpha}\right)^{1/\alpha}$ and

$$\mathcal{B}(x) = \left(1+|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d\,.$$

Using $\nabla\mathcal{B}^m = -\frac{2\,m}{1-m}\,x\,\mathcal{B}$ and an integration by parts, we obtain

$$\int_{\mathbb{R}^d}\mathcal{B}^m\,dx = -\frac{1}{d}\int_{\mathbb{R}^d}x\cdot\nabla\mathcal{B}^m\,dx = \frac{2\,m}{d\,(1-m)}\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx\,.$$

On the other hand, we deduce from $\mathcal{B}^m = \mathcal{B}^{m-1}\,\mathcal{B} = (1+|x|^2)\;\mathcal{B}$ that

$$\int_{\mathbb{R}^d}\mathcal{B}^m\,dx = \int_{\mathbb{R}^d}\left(1+|x|^2\right)\mathcal{B}\,dx = \mathcal{M}+\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx$$

where

$$\mathcal{M} = \int_{\mathbb{R}^d}\mathcal{B}\,dx = \omega_d\int_0^{+\infty}\frac{r^{d-1}}{(1+r^2)^{\frac{1}{1-m}}}\,dr = \pi^{\frac{d}{2}}\,\frac{\Gamma\!\left(\frac{1}{1-m}-\frac{d}{2}\right)}{\Gamma\!\left(\frac{1}{1-m}\right)}\,.$$

This gives the expressions

$$\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx = \frac{d\,(1-m)}{(d+2)\,m-d}\,\mathcal{M}\quad\text{and}\quad\int_{\mathbb{R}^d}\mathcal{B}^m\,dx = \frac{2\,m}{(d+2)\,m-d}\,\mathcal{M}\,.$$

With the same method, we find that

$$\mathcal{M} = \int_{\mathbb{R}^d}\mathcal{B}\,dx = -\frac{1}{d}\int_{\mathbb{R}^d}x\cdot\nabla\mathcal{B}\,dx = \frac{2}{d\,(1-m)}\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}^{2-m}\,dx$$

and $\mathcal{B} = \mathcal{B}^{m-1}\,\mathcal{B}^{2-m} = (1+|x|^2)\,\mathcal{B}^{2-m}$ so that

$$\mathcal{M} = \int_{\mathbb{R}^d}\mathcal{B}\,dx = \int_{\mathbb{R}^d}\mathcal{B}^{2-m}\,dx+\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}^{2-m}\,dx\,.$$

This amounts to

$$\int_{\mathbb{R}^d} \mathcal{B}^{2-m}\,dx = \frac{\alpha}{2}\,\mathcal{M} \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,\mathcal{B}^{2-m}\,dx = \frac{d}{2}\,(1-m)\,\mathcal{M}\,.$$

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