Logarithmic Sobolev inequalities: a review on stability and instability results

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Abstract

In this paper, we review recent results on stability and instability in logarithmic Sobolev inequalities, with a particular emphasis on strong norms. We consider several versions of these inequalities on the Euclidean space, for the Lebesgue and the Gaussian measures, and discuss their differences in terms of moments and stability. We give new and direct proofs, as well as examples. Although we do not cover all aspects of the topic, we hope to contribute to establishing the state of the art.

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1 Introduction

Let $d\gamma = \gamma(x)\,dx$ with $\gamma(x) = (2\,\pi)^{-d/2}\,\mathrm{e}^{-|x|^2/2}$ be the normalized Gaussian probability measure. The *Gaussian logarithmic Sobolev inequality* on $\mathbb{R}^d$ reads as

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma \ge \frac{1}{2}\int_{\mathbb{R}^d} |v|^2\,\log|v|^2\,d\gamma \tag{1}$$

for any function $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} = 1$. Moreover, by Jensen's inequality, we know that the right-hand side of (1) is nonnegative.

If v is a smooth and compactly supported function, then an elementary computation shows that (1) written for v is equivalent for $u = v\,\sqrt{\gamma}$ to the *Euclidean logarithmic Sobolev inequality* on $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \frac{1}{2}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx + \frac{d}{4}\,\log\big(2\,\pi\,\mathrm{e}^2\big) \tag{2}$$

which, by density, holds for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. However, even if u is smooth and compactly supported, it does not mean that $\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx$ is uniformly bounded from below, whatever $\|u\|_{\mathrm{H}^1(\mathbb{R}^d,dx)}$ is.

On $\mathbb{R}^d$, one can take advantage of scalings. For any $\lambda > 0$, let us consider

$$u_\lambda(x) := \lambda^{d/4}\,u\big(\sqrt{\lambda}\,x\big) \quad \forall\,x \in \mathbb{R}^d\,.$$

Inequality (2) applied to u_λ becomes

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \frac{1}{2\,\lambda}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx + \frac{d}{4\,\lambda}\,\log\big(2\,\pi\,\mathrm{e}^2\,\lambda\big) \tag{3}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. After optimizing on $\lambda > 0$, we obtain the *Euclidean logarithmic Sobolev inequality in scale invariant form*;

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \frac{\pi\,d\,e}{2}\,\exp\left(\frac{2}{d}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx\right) \tag{4}$$

for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$.

Logarithmic Sobolev inequalities have a long history. The *Gaussian logarithmic Sobolev inequality* (1) is due to L. Gross in [1] and its equivalence with (2) is well-known, while its scale invariant form (4) appeared in [2, Inequality (2.3)] in dimension $d = 1$ and in [3, Theorem 2] for any $d \ge 1$. Among earlier related results, one has to quote [4]. We refer to [5, Section 1.3.2] and also to [6–8] for further background references in information theory and to [9] for the equality case, as well as an early stability result. The equality case can also be deduced from [10]. See [11–15] and references therein for more recent results and [16–19] for related books.

The goal of this paper is to review some *stability properties* of Inequalities (1), (2), (3) and (4), mostly in strong norms. In the case of (1), the Gaussian *deficit* is defined by

$$\delta[v] := \int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma$$

and we aim either at an *improved inequality* showing that $\delta[v]$ is bounded from below by a functional evaluated on v under a constraint (otherwise (1) would not be optimal), or by a distance to the manifold of optimal functions (see Theorem 9). For Sobolev's inequality, the issue was raised by H. Brezis and E. Lieb in [20]. In [21], P.-L. Lions proved a *sequential stability* property: a normalized sequence of optimizing functions $(u_n)_{n\in\mathbb{N}}$ converges in $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$ to an Aubin-Talenti function via the concentration-compactness method. Soon after, G. Bianchi and H. Egnell proved in [22] that for some constant $\kappa_d > 0$, the deficit associated with Sobolev's inequality is bounded from below by $\kappa_d \, \mathsf{d}(v, \mathcal{M})^2$ where d is the distance induced by $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$ and $\mathcal{M}$ is the manifold of the *Aubin-Talenti functions.* A lower bound on κ_d is known from [23]. For the logarithmic Sobolev inequality, it is therefore natural to ask whether there is a *quantitative stability property* for (1), that is, whether there is some $\kappa > 0$ such that

$$\delta[v] \geq \kappa \, \mathsf{d}(v, \mathcal{M})^2 \quad \forall \, v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \,,$$

where $\mathcal{M}$ is now the manifold of optimal functions for (1) and for which distance d this stability inequality holds true. Going back to [12, 24], results are know when d is a Wasserstein distance. It is also true if d is induced by $\mathrm{L}^2(\mathbb{R}^d, d\gamma)$ according to [23] and it is not true without additional assumptions if d is based on $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$. We shall give details on known stability results in Section 3 and elaborate on examples of instabilities based on [25, 26] in Section 4. We also try to emphasize some differences between (1), (2), (3) and (4).

2 H^1 spaces and logarithmic Sobolev inequalities

Let us start by collecting some observations on the differences between the H^1 spaces with respect to Lebesgue and Gaussian measures and the consequences for the corresponding forms of the logarithmic Sobolev inequalities on $\mathbb{R}^d$.

2.1 Integrability and averages in the Euclidean case

The *Euclidean logarithmic Sobolev inequality* (2) on $\mathbb{R}^d$ can be written for any function $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$. As already noted, this is not enough to prove that $\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx$ is uniformly bounded from below as shown by the following examples.

• *Example 1.* Assume that $d = 1$ and let u be a smooth function on $\mathbb{R}$ with compact support in $(0, 1)$. Let

$$u_n := \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} u(x + k) \tag{5}$$

so that $\|u_n\|_{\mathrm{L}^2(\mathbb{R})} = \|u\|_{\mathrm{L}^2(\mathbb{R})}$ and $\|\nabla u_n\|_{\mathrm{L}^2(\mathbb{R})} = \|\nabla u\|_{\mathrm{L}^2(\mathbb{R})}$ for any $n \geq 1$, while

$$\int_{\mathbb{R}} |u_n|^2 \, \log |u_n|^2 \, dx = \int_{\mathbb{R}} |u|^2 \, \log |u|^2 \, dx - \|u\|^2_{\mathrm{L}^2(\mathbb{R})} \, \log n \to -\infty \quad \text{as} \quad n \to +\infty \,.$$

- *Example 2.* On $\mathbb{R}^d$, let us consider the function

$$u(x) = \left(1 + |x|^2\right)^{-\frac{d}{4}} \left(\log\left(2 + |x|^2\right)\right)^{-\frac{a}{2}} \quad \forall \, x \in \mathbb{R}^d$$

for some $a \in (1,2)$. This function is smooth and such that as $|x| \to +\infty$

$$\begin{aligned} &|x|^2 \, |\nabla u(x)|^2 \sim d^2 \, |u(x)|^2 = O\left(|x|^{-d} \, (\log |x|)^{-a}\right) \,, \\ &|u(x)|^2 \, \log |u(x)|^2 = O\left(|x|^{-d} \, (\log |x|)^{1-a}\right) \,. \end{aligned}$$

It is easy to check that $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ is such that $\lim\limits_{R \to +\infty} \displaystyle\int_{|x|<R} |u|^2 \, \log |u|^2 \, dx = -\infty$.

It is therefore a natural question to ask under which additional condition on $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ one can guarantee that $|u|^2 \log |u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$. If this is the case, let us observe that we can choose $\lambda > 0$ such that $\int_{\mathbb{R}^d} |u_\lambda|^2 \, \log |u_\lambda|^2 \, dx = 0$ where $u_\lambda := \lambda^{d/2} \, u(\lambda \cdot)$ because

$$\int_{\mathbb{R}^d} |u_\lambda|^2 \, \log |u_\lambda|^2 \, dx = \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx + d \, \log \lambda \, \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

uniquely determines λ. Interestingly, we have a reciprocal result that goes as follows. Let us consider the Gagliardo-Nirenberg-Sobolev inequality

$$\|\nabla u\|^\theta_{\mathrm{L}^2(\mathbb{R}^d)} \, \|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathcal{C}_{\mathrm{GNS}}(d,p) \, \|u\|_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall \, u \in \mathrm{H}^1(\mathbb{R}^d, dx) \tag{6}$$

where $\theta = d\,(p-2)/(2\,p)$ and $\mathcal{C}_{\mathrm{GNS}}(d,p) > 0$ is the optimal constant. The exponent p is larger than 2, with the additional restriction that $p \leq 2\,d/(d-2)$ if $d \geq 3$. If $d \geq 3$ and $p = 2\,d/(d-2)$, then $\theta = 1$ and (6) is the classical Sobolev inequality.

Proposition 1. *With this notation and p as above, if u is a smooth and compactly supported function such that $\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx = 0$, then*

$$\int_{\mathbb{R}^d} \left| \, |u|^2 \, \log |u|^2 \right| dx \leq 2 \, \frac{\left(\|\nabla u\|^\theta_{\mathrm{L}^2(\mathbb{R}^d)} \, \|u\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\right)^p}{(p-1)\, e \, \mathcal{C}_{\mathrm{GNS}}(d,p)^p} \,.$$

By density of smooth and compactly supported functions in $\mathrm{H}^1(\mathbb{R}^d, dx)$, the result of Proposition 1 also holds in $\mathrm{H}^1(\mathbb{R}^d, dx)$.

Proof. A simple optimization shows that

$$\inf_{t>1} \frac{t \log t}{t^p} = \frac{1}{(p-1)\, e}$$

for any $p > 2$. As a consequence with $t = |u|^2$, we have

$$-\int_{|u|\le 1} |u|^2 \log |u|^2 \, dx = \int_{|u|\ge 1} |u|^2 \log |u|^2 \, dx \le \frac{\|u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}}{(p-1)\, e} \,,$$

which completes the proof using $\int_{\mathbb{R}^d} \big| |u|^2 \log |u|^2 \big| \, dx = 2 \int_{|u|\ge 1} |u|^2 \log |u|^2 \, dx$ and (6). □

We can deduce a criterion of integrability from Proposition 1.

Corollary 2. *If $u \in \mathrm{H}^1(\mathbb{R}^d, dx) \setminus \{0\}$, then*

(i) *either for any sequence $(u_n)_{n\in\mathbb{N}}$ of smooth and compactly supported functions on $\mathbb{R}^d$ such that $\lim_{n\to+\infty} \big(\|\nabla u - \nabla u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u - u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \big) = 0$, we have*

$$\lim_{n\to+\infty} \int_{\mathbb{R}} |u_n|^2 \log |u_n|^2 \, dx = -\infty \,,$$

(ii) *or the function u is such that $|u|^2 \log |u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$.*

Proof. If (i) does not hold, then one can find a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\lambda_n = \exp\left(-\frac{1}{d} \, \frac{\int_{\mathbb{R}^d} |u_n|^2 \log |u_n|^2 \, dx}{\|u_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \right)$$

converges to some $\lambda \ge 0$. It is then clear that $\widetilde{u}_n = \lambda_n^{d/2} \, u_n\big(\sqrt{\lambda_n} \, \cdot \,\big)$ satisfies the conditions of Proposition 1: $\int_{\mathbb{R}^d} |\widetilde{u}_n|^2 \log |\widetilde{u}_n|^2 \, dx = 0$, while we notice that $\int_{\mathbb{R}^d} |\nabla \widetilde{u}_n|^2 \, dx \sim \lambda_n \int_{\mathbb{R}^d} |\nabla u_n|^2 \, dx \to 0$ as $n \to +\infty$ if $\lambda = 0$. This contradicts (2) applied to $\widetilde{u}_n$. As a consequence, we have that λ is a positive real number such that $(\widetilde{u}_n)_{n\in\mathbb{N}}$ converges to $\widetilde{u} = \lambda^{d/2} \, u\big(\sqrt{\lambda} \, \cdot \,\big)$ in $\mathrm{H}^1(\mathbb{R}^d, dx)$. Using Proposition 1 and Fatou's lemma, we conclude that $|\widetilde{u}|^2 \log |\widetilde{u}|^2 \in \mathrm{L}^1(\mathbb{R}^d)$ and as a consequence, $|u|^2 \log |u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$. □

2.2 Integrability and moments. Gaussian and Euclidean cases

If v is a smooth and compactly supported function, we already observed in Section 1 that (1) written for v is equivalent to (2) written for $u = \sqrt{\gamma}\, v$. However, using an integration by parts, we can notice that

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = \int_{\mathbb{R}^d} \left| \nabla u + \frac{x}{2} \, u \right|^2 dx = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{1}{4} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx - \frac{d}{2} \, \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \tag{7}$$

involves a second moment of $|u|^2$. In the case of the example of u_n given by (5), we have in particular that

$$\int_{\mathbb{R}^d} |x|^2\,|u_n|^2\,dx \sim \frac{1}{n}\sum_{k=0}^{n-1} k^2\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \to +\infty \quad \text{as} \quad n \to +\infty\,.$$

It turns out that a second moment condition is a sufficient condition to guarantee that $|u|^2\,\log|u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$. Here is a statement and a proof of this classical result that goes back to [27, Sec. 7].

Proposition 3. *If u is a smooth, compactly supported function, and moreover such that $\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx$ is finite and $\int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx < \infty$, then $|u|^2\,\log|u|^2 \in \mathrm{L}^1(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} \big|\,|u|^2\,\log|u|^2\big|\,dx \le \int_{\mathbb{R}^d} |u|^2\left(\log|u|^2 + |x|^2\right)dx + d\,\log(2\,\pi)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2}{e}\,.$$

Proof. Let $S := \int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx$, $S_\pm := \int_{\pm(|u|-1)\ge 0} |u|^2\,\log|u|^2\,dx$ so that $\pm S_\pm \ge 0$, and $K := \int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx$. We have $S = S_+ + S_-$ and

$$\int_{\mathbb{R}^d} \big|\,|u|^2\,\log|u|^2\big|\,dx = S_+ - S_- = S - 2\,S_-\,.$$

Using Jensen's inequality, we obtain

$$\begin{aligned} S_- + \frac{1}{2}\,K + \frac{d}{2}\,\log(2\,\pi)\int_{|u|\le 1} |u|^2\,dx &\ge \int_{|u|\le 1} \frac{|u|^2}{\gamma}\,\log\left(\frac{|u|^2}{\gamma}\right)d\gamma \\ &\ge \left(\int_{|u|\le 1} |u|^2\,dx\right)\log\left(\int_{|u|\le 1} |u|^2\,dx\right) \ge -\frac{1}{e}\,. \end{aligned}$$

As a consequence we obtain

$$\int_{\mathbb{R}^d} \big|\,|u|^2\,\log|u|^2\big|\,dx \le S + 2\left(\frac{1}{2}\,K + \frac{d}{2}\,\log(2\,\pi)\int_{|u|\le 1} |u|^2\,dx + \frac{1}{e}\right),$$

which completes the proof. □

Similar estimates can be found in [28]. By taking (2) into account, we deduce that

$$\begin{aligned} \int_{\mathbb{R}^d} \big|\,|u|^2\,\log|u|^2\big|\,dx \le \frac{2}{e} &+ \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,\log\left(\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right) \\ &+ 2\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \left(\frac{d}{2}\,\log(2\,\pi) - d\right)\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx \end{aligned}$$

for any $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx$ is finite. This second moment condition is automatically satisfied in the Gaussian case. Using (7), we obtain the following result.

Corollary 4. *If $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$, then $|v|^2 \, \log |v|^2$ is in $\mathrm{L}^1(\mathbb{R}^d, d\gamma)$ and we have*

$$\int_{\mathbb{R}^d} \big| \, |v|^2 \, \log |v|^2 \big| \, d\gamma \le \frac{2}{e} + 6 \, \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} + d \, \log\left(2\,\pi\,\mathrm{e}^2\right) \, \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \\ + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \, \log\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}\right) .$$

2.3 Improved inequalities under second moment conditions

Based on ideas of [13, 29] and [14, Proposition 1], the following result holds.

Lemma 5. *Let $d \ge 1$. With φ defined by*

$$\varphi(t) := \frac{d}{4} \left[\exp\left(\frac{2\,t}{d}\right) - 1 - \frac{2\,t}{d} \right] \quad \forall\, t \in \mathbb{R} \,, \tag{8}$$

we have

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma \ge \varphi\left(\int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma + \frac{d}{2} - \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \right) \tag{9}$$

for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma < \infty$.

Notice that (9) holds true even if $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma = \infty$, in which case the right-hand side is 0.

Proof. For completeness, let us give a short proof based on [14]. After subtracting the right-hand side of (1), that is,

$$\frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma = \frac{1}{2} \left(\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx + \frac{d}{2} \, \log\left(2\,\pi\right) + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx \right)$$

with $u = v\,\sqrt{\gamma}$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = \|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$, from both sides of (4), using (7) we obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma \\ &\quad = \int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx - \frac{d}{2} \\ &\qquad\qquad - \frac{1}{2} \left(\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx + \frac{d}{2} \, \log\left(2\,\pi\right) + \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |u|^2 \, dx \right) \\ &\quad \ge \frac{\pi\,d\,e}{2} \, \exp\left(\frac{2}{d} \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx \right) - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx - \frac{d}{4} \, \log\left(2\,\pi\,\mathrm{e}^2\right) . \end{aligned}$$

With
$$t := \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma + \frac{d}{2} - \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \,,$$
we have
$$\int_{\mathbb{R}^d} |u|^2 \, \log |u|^2 \, dx = t - \frac{d}{2} \, \log \big(2 \, \pi \, e \big)$$
which concludes the proof of (9). □

As a consequence of Lemma 9, we have the following result, which was already known from [15, Theorem 1, 1.] using a slightly different proof.

Corollary 6. *Let $d \geq 1$. Let us consider a sequence $(v_n)_{n \in \mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v_n\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ for any $n \in \mathbb{N}$. If $\limsup_{n \to +\infty} \int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, d\gamma \leq d$, then*
$$\lim_{n \to +\infty} \left(\int_{\mathbb{R}^d} |\nabla v_n|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v_n|^2 \, \log |v_n|^2 \, d\gamma \right) = 0$$
is equivalent to the convergence of $(v_n)_{n \in \mathbb{N}}$ to 1 in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$, and then we have $\lim_{n \to +\infty} \int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, d\gamma = d$.

With minimal effort, we can also recover the statement of [15, Theorem 1] which asserts that, for any sequence $(v_n)_{n \in \mathbb{N}}$, such that $\lim_{n \to +\infty} \delta[v_n] = 0$ and $\int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, d\gamma < \infty$, the two following properties are equivalent:
(i) $v_n \to 1$ in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ as $n \to +\infty$,
(ii) $\lim_{n \to +\infty} \int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, d\gamma = d$.

In [14, Proposition 1], only the case $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma = d$ was taken under consideration. One may wonder whether the condition $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma < \infty$ is restrictive. In fact, it is not so much, as we read from instance from [30, Ineq.(4)] that
$$\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \leq 2 \, (d+1) \int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma \quad \forall \, v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \quad \text{such that} \quad \int_{\mathbb{R}^d} v \, d\gamma = 0 \,.$$

With φ defined by (8), we may notice that $\varphi''(t) = (1/d) \, \exp(2 \, t/d)$ for any $t \in \mathbb{R}$ and, as a consequence, $\varphi''(t) \geq 1/d$ if $t \geq 0$. Thus,
$$\int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma \\ \geq \frac{1}{2 \, d} \left(\int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma \right)^2 + \frac{1}{8 \, d} \left(d - \int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \right)^2 \tag{10}$$
for any $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \leq d$.

A first consequence is a sequential stability result under the condition that $\limsup_{n \to +\infty} \int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, d\gamma \leq d$. Using the Csiszár-Kullback-Pinsker inequality
$$\int_{\mathbb{R}^d} |v|^2 \, \log |v|^2 \, d\gamma \geq \frac{1}{4} \left(\int_{\mathbb{R}^d} \big| v - 1 \big| \, d\gamma \right)^2 \quad \forall \, v \in \mathrm{H}^1_+(\mathbb{R}^d, d\gamma) \quad \text{s.t.} \quad \|v\|_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} = 1 \,,$$

see [31–33], and the Brezis-Lieb lemma, see [34, Theorem 2], one can then prove that the above sequence $(v_n)_{n\in\mathbb N}$ converges to 1 in $\mathrm H^1(\mathbb R^d,d\gamma)$. See [15] for further details. In fact, one can directly obtain an explicit stability estimate from (10), which goes as follows.

Corollary 7. *Let $d\ge1$. For any $v\in\mathrm H^1(\mathbb R^d,d\gamma)$ such that $\|v\|_{\mathrm L^2(\mathbb R^d,d\gamma)}=1$ and $\int_{\mathbb R^d}|x|^2\,|v|^2\,d\gamma\le d$, we have*

$$\int_{\mathbb R^d}|\nabla v|^2\,d\gamma-\frac12\int_{\mathbb R^d}|v|^2\,\log|v|^2\,d\gamma\ge\frac{8\sqrt d\,\big(\int_{\mathbb R^d}|\nabla v|^2\,d\gamma\big)^2}{\big(d+8\int_{\mathbb R^d}|\nabla v|^2\,d\gamma\big)^{3/2}}$$

Proof. With $\mathsf e:=\int_{\mathbb R^d}|v|^2\,\log|v|^2\,d\gamma$ and $\mathsf i:=\int_{\mathbb R^d}|\nabla v|^2\,d\gamma$, (10) reads as

$$e^2+d\,\mathsf e-2\,d\,\mathsf i\ge0$$

which can be inverted as $\mathsf e\ge\big(\sqrt{d\,(d+8\,\mathsf i)}-d\big)/2$ and shows the result using the convexity of $\mathsf i\mapsto4\,\mathsf i-\sqrt{d\,(d+8\,\mathsf i)}$. □

The condition $\int_{\mathbb R^d}|x|^2\,|v|^2\,d\gamma\le d$ obviously comes from the normalization of the Gaussian measure. On the Euclidean space with the Lebesgue measure, there is not such a condition. Hence if $u\in\mathrm H^1(\mathbb R^d,dx)$ is such that $\|u\|_{\mathrm L^2(\mathbb R^d)}=1$ and $\int_{\mathbb R^d}|x|^2\,|u|^2\,dx<\infty$, then

$$v(x):=\frac{\lambda^{-d/4}}{\sqrt{\gamma(x)}}\,u\left(\frac{x}{\sqrt\lambda}\right)\quad\forall\,x\in\mathbb R^d$$

is a function in $\mathrm H^1(\mathbb R^d,d\gamma)$ such that $\|v\|_{\mathrm L^2(\mathbb R^d,d\gamma)}=1$ and $\int_{\mathbb R^d}|x|^2\,|v|^2\,d\gamma=d$ if we choose

$$\lambda=\frac{d}{\int_{\mathbb R^d}|x|^2\,|u|^2\,dx}\,.\tag{11}$$

By applying Lemma 5, and undoing the above change of variables, we obtain, for (3) applied to u with λ given by (11), a stability result that goes as follows.

Corollary 8. *Let $d\ge1$. For any $u\in\mathrm H^1(\mathbb R^d,dx)$ is such that $\|u\|_{\mathrm L^2(\mathbb R^d)}=1$ and $\int_{\mathbb R^d}|x|^2\,|u|^2\,dx<\infty$*

$$\frac{d\int_{\mathbb R^d}|\nabla u|^2\,dx}{\int_{\mathbb R^d}|x|^2\,|u|^2\,dx}-\frac12\int_{\mathbb R^d}|u|^2\,\log|u|^2\,dx-\frac d4\,\log\left(\frac{2\,\pi\,d\,e^2}{\int_{\mathbb R^d}|x|^2\,|u|^2\,dx}\right)$$
$$\ge\varphi\left(\int_{\mathbb R^d}|u|^2\,\log|u|^2\,dx+\frac d2\,\log\left(\frac{2\,\pi\,e^2}{d}\int_{\mathbb R^d}|x|^2\,|u|^2\,dx\right)\right).$$

The right-hand side of the inequality is a measure (in relative entropy) of the distance of u to the Gaussian function $x\mapsto\lambda^{d/2}\,\gamma\big(\sqrt\lambda\,x\big)$. Notice that the result of Corollary 8 can be rewritten as a stability result for (3) for the special value of λ given by (11), i.e.,

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx - \frac{1}{2\,\lambda}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx - \frac{d}{4\,\lambda}\,\log\left(2\,\pi\,\mathrm{e}^2\,\lambda\right) \\ \ge \varphi\left(\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx + \frac{d}{2}\,\log\left(2\,\pi\,\mathrm{e}^2/\lambda\right)\right)\,,$$

which is a stability result for (3) for the special value of λ given by (11).

3 Stability

3.1 Optimal constants and optimal functions

Inequalities (1), (2), (3) and (4) can be rewritten for functions $u \in \mathrm{H}^1(\mathbb{R}^d, dx)$ and $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ respectively as

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma \ge \tfrac12 \int_{\mathbb{R}^d} |v|^2\,\log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}}\right)d\gamma\,, \tag{12a}$$

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \tfrac12 \int_{\mathbb{R}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx + \tfrac{d}{4}\,\log\left(2\,\pi\,\mathrm{e}^2\right)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \tag{12b}$$

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \tfrac{1}{2\,\lambda} \int_{\mathbb{R}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx + \tfrac{d}{4\,\lambda}\,\log\left(2\,\pi\,\mathrm{e}^2\,\lambda\right)\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,, \tag{12c}$$

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx \ge \tfrac{\pi\,d\,e}{2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,\exp\left(\tfrac{2}{d}\int_{\mathbb{R}^d} \frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right)dx\right)\,, \tag{12d}$$

without any normalization in either $\mathrm{L}^2(\mathbb{R}^d, dx)$ nor $\mathrm{L}^2(\mathbb{R}^d, d\gamma)$. These inequalities are written with optimal constants as can be checked using $v_\varepsilon(x) = 1 + \varepsilon\,x\cdot\nu$ in the limit as $\varepsilon \to 0$ for any given $\nu \in \mathbb{S}^{d-1}$ in case of (1) on the one hand, and $u = \sqrt{\gamma}$ in case of (2) and (4) on the other hand. The next issue is to identify all optimal functions. The first explicit result for (1) is due to E. Carlen in [9], although the *carré du champ* method of D. Bakry and M. Emery in [10] applies: we refer to [35] for more detailed explanations. Since (12a), (12b), (12c) and (12d) are equivalent for smooth and sufficiently decreasing functions as explained in Section 1, cases of equality can be reduced to optimality for any of these inequalities.

Theorem 9.

1) *A function v is optimal in (12a) if and only if $v(x) = v_{a,b}(x) := a\,\mathrm{e}^{b\cdot x}$, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$.*
2) *A function u is optimal in (12b) if and only if $u(x) = u_{a,b}(x) := a\,\mathrm{e}^{-\frac{|x-b|^2}{2}}$, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$.*
3) *For any fixed $\lambda > 0$, a function u is optimal in (12c) if and only if $u(x) = u_{a,b,\lambda}(x) := a\,\mathrm{e}^{-\frac{|x-b|^2}{2\,\lambda}}$, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$.*
4) *A function u is optimal in (12d) if and only if $u(x) = u_{a,b,\lambda}(x) = a\,\mathrm{e}^{-\frac{|x-b|^2}{2\,\lambda}}$, for any $a \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $\lambda > 0$.*

Cases 1) and 2) were explicitly established by E. Carlen in [9]. After proving that optimisers cannot change sign, one can indeed apply [9, p. 204, Theorem 4] to nonnegative optimizers. Alternatively, we give a proof based on the *carré du champ* method of [10], which directly shows Case 4) and has been used in this context only in [6]. Here we provide a proof based on the pressure variable, which is, to our knowledge, new.

Proof. Let us split the argument in three steps.

• Up to dilation and scaling, an optimiser u solves the *Euler–Lagrange equation*

$$-\Delta u = u\,\log u + \frac{d}{4}\,\log(2\,\pi\,e^2)\,u\,.$$

Here we assume that $u \ge 0$ without loss of generality.

• *The Rényi entropy power computation.* Here we work at formal level and refer to [6] for the origin of this method. Assume that $\rho = |u|^2 = e^{\mathsf{P}}$ solves the heat equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho = \nabla \cdot (\rho\,\nabla \mathsf{P}) \tag{13}$$

so that the *pressure* variable P and $u > 0$ respectively solve

$$\frac{\partial \mathsf{P}}{\partial t} = \Delta \mathsf{P} + |\nabla \mathsf{P}|^2 \quad \text{and} \quad \frac{\partial u}{\partial t} = \Delta u + \frac{|\nabla u|^2}{u}\,.$$

Further assuming that the function u is smooth and rapidly decaying as $|x| \to +\infty$, a straightforward computation shows that the entropy decays according to

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho\,\log \rho\,dx = -\int_{\mathbb{R}^d} \rho\,|\nabla \mathsf{P}|^2\,dx = -\,4 \int_{\mathbb{R}^d} |\nabla u|^2\,dx$$

while the Fisher information obeys to

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u|^2\,dx &= -\,2 \int_{\mathbb{R}^d} \Delta u \left(\Delta u + \frac{|\nabla u|^2}{u}\right) dx \\ &= -\,2 \int_{\mathbb{R}^d} \left(\|\mathrm{Hess}\,u\|^2 - 2\,\mathrm{Hess}\,u : \frac{\nabla u \otimes \nabla u}{u} + \frac{\|\nabla u \otimes \nabla u\|^2}{u^2}\right) dx\end{aligned}$$

where $A : B = \sum_{i,j=1}^d a_{ij}\,b_{ij}$ denotes the standard contraction of matrices A and B and $\|A\|^2 = A : A$. Using $\mathsf{P} = 2\,\log u$, $u\,\nabla \mathsf{P} = 2\,\nabla u$,

$$\frac{\nabla u \otimes \nabla u}{u^2} = \frac{1}{4}\,\nabla \mathsf{P} \otimes \nabla \mathsf{P} \quad \text{and} \quad \mathrm{Hess}\,u = \frac{u}{2}\left(\mathrm{Hess}\,\mathsf{P} - \frac{1}{2}\,\nabla \mathsf{P} \otimes \nabla \mathsf{P}\right),$$

we conclude that

$$\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla u|^2\,dx = -\,\frac{1}{2} \int_{\mathbb{R}^d} \rho\,\|\mathrm{Hess}\,\mathsf{P}\|^2\,dx\,.$$

By conservation of mass, we can assume that $\|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)} = \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ for any $t \ge 0$ if ρ solves (13), so that

$$\frac{d}{dt}\log\left(\int_{\mathbb{R}^d}|\nabla u|^2\,dx\,\exp\left(-\frac{2}{d}\int_{\mathbb{R}^d}\rho\,\log\rho\,dx\right)\right)$$
$$=-\frac{1}{2}\left(\int_{\mathbb{R}^d}\rho\,\|\mathrm{Hess}\,\mathsf{P}\|^2\,dx-\frac{1}{d}\left(\int_{\mathbb{R}^d}\rho\,|\nabla\mathsf{P}|^2\,dx\right)^2\right)$$
$$=-\frac{1}{2}\int_{\mathbb{R}^d}\rho\,\left\|\mathrm{Hess}\,\mathsf{P}-\frac{1}{d}\int_{\mathbb{R}^d}\rho\,|\nabla\mathsf{P}|^2\,dx\,\mathrm{Id}\right\|^2\,dx\,.$$

• *Conclusion.* These computations can be justified using approximations based on a decomposition of ρ on a finite number of Hermite functions. As an optimiser, u is such that $\int_{\mathbb{R}^d}\rho\,\left\|\mathrm{Hess}\,\mathsf{P}-\frac{1}{d}\int_{\mathbb{R}^d}\rho\,|\nabla\mathsf{P}|^2\,dx\,\mathrm{Id}\right\|^2 dx=0$, that is, $\mathsf{P}=2\,\log u=\alpha\,|x-x_0|^2+\beta$ for some constants α and β and for some $x_0\in\mathbb{R}^d$. □

3.2 Stability results in the Gaussian setting

3.2.1 Improved inequalities

Let us consider (1). A first improvement of it has been formulated in [9], in terms of the Wiener transform $\mathcal{W}$:

$$\delta[v]\ge\frac{1}{2}\int_{\mathbb{R}^d}|\mathcal{W}v|^2\,\log|\mathcal{W}v|^2\,d\gamma\,,\quad\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}=1\,,$$

where the extra term is non-negative and vanishes if and only if $v\in\mathcal{M}$.

Another direct improvement of (1) can be obtained using the *carré du champ* method of [10], which we sketch briefly. Let us define the *relative Fisher information* and the *relative entropy* functionals, respectively via $\mathcal{I}[v]=\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}$, and $\mathcal{E}[v]=\int_{\mathbb{R}^d}|v|^2\log|v|^2\,d\gamma$, for $v\in\mathrm{H}^1(\mathbb{R}^d,d\gamma)$ such that $\|v\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}=1$. Next, assume that $|w|^2$ solves the Ornstein–Uhlenbeck equation so that $w=w(t,x)$ is the solution of

$$\frac{\partial w}{\partial t}=\Delta w+\frac{|\nabla w|^2}{w}-x\cdot\nabla w\,,\quad w(t=0,\cdot)=v\,.\tag{14}$$

Then, along the flow, it holds true that

$$\frac{d}{dt}\mathcal{E}[w(t,\cdot)]=-\,4\,\mathcal{I}[w(t,\cdot)]\,,\quad\frac{d}{dt}\mathcal{I}[w(t,\cdot)]+2\,\mathcal{I}[w(t,\cdot)]=-\,2\int_{\mathbb{R}^d}\|\mathrm{Hess}\,\mathsf{P}\|^2\,|w|^2\,d\gamma\,,\tag{15}$$

where $\mathsf{P}=2\,\log w$ is the pressure variable as in Section 3.1. Integrating on the interval $(0,\infty)$, (15) implies

$$\delta[v]\ge\int_0^\infty\mathcal{R}[w(t,\cdot)]\,dt\quad\text{where}\quad\mathcal{R}[w]:=2\int_{\mathbb{R}^d}\|\mathrm{Hess}\,\mathsf{P}\|^2\,|w|^2\,d\gamma\,,\tag{16}$$

where $\mathcal{R}$ vanishes if and only if v is an optimiser of (1). Additional information can be extracted from $\mathcal{R}$, for some classes of functions v as we shall see next.

3.2.2 Functions with asymptotic exponential or Gaussian behaviour

If the measure $|v|^2\,d\gamma$ satisfies the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla\phi|^2\,|v|^2\,d\gamma \ge C_P \int_{\mathbb{R}^d} \left|\phi - \int_{\mathbb{R}^d} \phi\,|v|^2\,d\gamma\right|^2 |v|^2\,d\gamma \quad \forall\,\phi \in \mathrm{C}^\infty_c(\mathbb{R}^n) \tag{17}$$

for some positive constant C_P and if w solves (14), the same holds true for the measure $|w(t,\cdot)|^2\,d\gamma$ for all $t \ge 0$, with the constant

$$C_P(t) = \frac{C_P}{C_P + (1 - C_P)\,\mathrm{e}^{-2t}}.$$

In addition, if v is centered, i.e., $\int_{\mathbb{R}^d} x\,|v|^2\,d\gamma = 0$, then $\nabla\mathsf{P}(t,\cdot)$ is such that $\int_{\mathbb{R}^d} \nabla\mathsf{P}(t,\cdot)\,|w(t,\cdot)|^2\,d\gamma = 0$, and

$$\mathcal{R}\big[w(t,\cdot)\big] \ge C_P(t) \int_{\mathbb{R}^d} |\nabla\mathsf{P}(t,\cdot)|^2\,|w(t,\cdot)|^2\,d\gamma = C_P(t)\,\mathcal{I}[w]\,.$$

In [11], this argument allows M. Fathi, E. Indrei, and M. Ledoux to prove that

$$\delta[v] \ge \frac{C_P^2 - C_P - C_P\,\log C_P}{(1 - C_P)^2} \int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$$

for all centered functions such that they admit the Poincaré inequality (17). The authors show that the constant in the extra term is optimal in the class of functions they consider. The argument of [11] can be generalised as follows. Let us call $\mathcal{U}$ the space of centered functions v such that v admits (17) for some positive constant C_P. The flow (14) preserves $\mathcal{U}$. In addition, assume that for some $T \in (0,\infty)$, the solution $w(t,\cdot)$ to (14) with initial datum v belongs to $\mathcal{U}$ at $t = T$, hence, for any $t \ge T$. Then we obtain

$$\delta[v] \ge \mathrm{e}^{-2\,T}\,\frac{C_P^2 - C_P - C_P\,\log C_P}{(1 - C_P)^2} \int_{\mathbb{R}^d} |\nabla v|^2\,d\gamma$$

using the *backwards-in-time* estimate of [35] and [11]. The existence of such a finite T is granted if v is a compactly supported function or under the more general condition that $\int_{\mathbb{R}^d} |v|^2\,\mathrm{e}^{\theta\,|x|}\,d\gamma < \infty$ for some $\theta > 0$. This condition cannot be created along the flow (14), see [36], without additional assumptions. In [37], H.-B. Chen, S. Chewi, and J. Niles-Weed provide a sufficient condition: if for some $\varepsilon > 0$ and $\mathcal{C} > 0$,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2(x)\,|v|^2(y)\,\mathrm{e}^{\varepsilon\,|x-y|^2}\,\gamma(x)\,dx\,\gamma(y)\,dy \le \mathcal{C}\,,$$

then, the solution $w(t,\cdot)$ to (14) has the property for some finite T depending on ε and $\mathcal{C}$ but not on the dimension d. As a a result that was proved in [35], there is an explicit constant $\mathsf{c} = \mathsf{c}(\varepsilon, \mathcal{C})$ such that

$$\delta[v] \geq \mathsf{c} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma \, .$$

It is currently an open question to decide whether T is finite for a function v such that $\int_{\mathbb{R}^d} |v|^2 \, e^{\theta\,|x|} \, d\gamma < \infty$ for some $\theta > 0$.

For functions with a finite exponential moment, there are stability results based on a weaker notion of distance. See [24], [12, Theorem 1.1] and [38, ineq. (33)]. If $|v|^2$ can be written in the form $|v|^2 = e^{-h} \, d\gamma$ for h such that $1 - \varepsilon \leq \operatorname{Hess} h \leq M$, then

$$\delta[v] \geq \beta(\varepsilon, M) \, \mathrm{W}_2^2(|v|^2 \, dx, \gamma)$$

where W_2 is the 2-Wasserstein distance. For a more recent insight upon the relation between log-Hessian bounds, the Ornstein–Uhlenbeck flow, and the stability of (1), we refer to [39], where applications to statistics are also discussed.

Finally, we notice that all results in this section are optimal with respect to the exponent of the distance, which is sometimes referred in the literature as *sharp qualitative stability.*

3.2.3 Functions with finite second-order moment

Another possible way to exploit the improvement (16) is described below, for functions v such that $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \leq d$. The resulting estimate has been written in [14] via a self-improvement of (1), when the second-order moment is exactly $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma = d$. Otherwise, we attribute the result and the corresponding proof to [12], even though the key-estimate appears in [6] as well.

Going back to (16), using the Cauchy–Schwarz and the arithmetic-geometric inequalities, let us write

$$\mathcal{R}[w(t, \cdot)] \geq \frac{1}{d} \left(\int_{\mathbb{R}^d} (\Delta \mathsf{P}) \, |w|^2 \, d\gamma \right)^2 \geq \frac{4}{d} \left(\int_{\mathbb{R}^d} |\nabla w|^2 \, d\gamma \right)^2 ,$$

where the last estimate is achieved using the condition on the second-order moment. By solving the differential inequality obtained from (15) for $t \in \mathbb{R}^+$, we find

$$\delta[v] \geq \Psi \left(\int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma \right) , \quad \Psi(s) := s - \frac{d}{4} \log \left(1 + \frac{4}{d} \, s \right) \quad \forall \, s > 0 \, .$$

For $s \to 0$, we notice that $\Psi(s) = s^2 + o(s^2)$, which means that the extra term we found is in the order of $\|\nabla v\|^4_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}$ as in Corollary 7. In section 3.2.2, we found a remainder term of order 2. The difference lies in the fact that, here, we did not ask v to be centered. Then, sequences in the form, e.g., $v_\varepsilon = 1 + \varepsilon \, x_1$, which are *tangent* to the manifold of optimisers $\mathcal{M}$, are admissible, and

$$\delta[v_\varepsilon] = O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0 \, ,$$

thus proving optimality of the behaviour of $\Psi(s)$ as $s \to 0_+$. Identifying the minimal conditions for the existence of a positive such that $\delta[v] \geq \beta \int_{\mathbb{R}^d} |\nabla v|^2 \, d\gamma$, for centered functions v with $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \leq d$, is an open question.

As discussed in Section 2.3, the condition $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma \leq d$ is necessary for local stability results for (1) around constant functions, even in weaker distances such as as W_2. On the other hand, an improvement of (1) for functions $|v|^2$ with arbitrarily large but finite second order moment holds in two known cases. As found out by E. Indrei:

- in [15, Theorem 1(2)], it is shown that, for all $b > 0$, there exists a constant $\beta_b > 0$, such that, for all centered functions v such that
$$\int_{\mathbb{R}^d} |x|^4 \, |v|^2 \, d\gamma \leq b \,,$$
it holds true that
$$\big\| |v| - 1 \big\|_{\mathrm{H}^1(\mathbb{R}^d, d\gamma)} \leq \beta_b \, \Big(\delta[v] + \delta[v]^{1/2} \Big)^{1/2} \,.$$
- Stability in $\mathrm{W}^{1,1}(\mathbb{R}, d\gamma)$ is proved in [40, Theorem 1.1], in dimension $d = 1$. For all $a > 0$, there exist $\beta'_a > 0$ such that for all centered functions $v \in \mathrm{W}^{1,1}(\mathbb{R}, d\gamma)$ with $\int_{\mathbb{R}^d} |x|^2 \, |v|^2 \, d\gamma = a$, it holds
$$\Big\| |v|^2 - 1 \Big\|_{\mathrm{W}^{1,1}(\mathbb{R}, d\gamma)} \leq \beta'_a \, \Big(\delta^{1/4}[v] + \delta^{3/4}[v] \Big) \,.$$

Whether the exponents in these last two results are optimal is an open question.

3.2.4 Functions with no moment bounds: stability in L^2

Stability in L^2-norm was an open problem until recently. We refer to [25, 41] for a review on stability results in L^p-norms in the literature, which still left some points open (e.g., optimal exponents in the stability terms).

A definitive answer came with [23, 42], where J. Dolbeault, M. Esteban, A. Figalli, R. Frank, and M. Loss construct an explicit, positive, dimension-free constant β such that
$$\forall \, v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma) \,, \quad \delta[v] \geq \beta \inf_{v_{a,b} \in \mathcal{M}} \| v - v_{a,b} \|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \,, \tag{18}$$
where $\mathcal{M}$ and $v_{a,b}$ are defined in Theorem 9. The exponent in the right-hand side of (18) is optimal: see for instance [15, Theorem 2].

Even though (18) can be proved directly (see [42]), an interesting feature of this estimate is that is can be recovered as a large-dimensional limit of the constructive stability estimate of Sobolev's inequality on the sphere, according to [23]. As witnessed by the striking optimality of the constant $\frac{1}{2}$ in (1), regardless of the topological dimension d of the space, we can see (1) as an *infinite-dimensional* inequality. One explanation of this fact is that $(\mathbb{R}^d, d\gamma)$, is already *infinite dimensional*, in a sense, for any $d \geq 1$. Such an assertion can be formulated rigorously in terms of the modern theory of metric measure spaces and synthetic curvature-dimension conditions, which goes out of the scopes of the present note, and for which we refer to the work of L. Ambrosio, N. Gigli, and G. Savaré [43]. However, the heuristics of *the Gaussian measure behaving similarly to the unitary measure on a very large-dimensional sphere* is present in mathematics since the XIXth century, at least, and we refer to [44] for a complete historical account.

Let us review a few recent results for stability of functional inequalities on the sphere and Riemannian manifolds in general, and draw their connections with (1).

3.2.5 Interpolation inequalities on manifolds

One feature of (1) is being *critical*. A concept of criticality related to maximal embeddings of Orlicz spaces (which applies to the present case) has been studied by A. Cianchi and L. Pick in [45]. We specialise this notion to a particular case, i.e., the inclusions associated with Beckner's interpolation inequalities in [46]. For all $p \in (1,2)$ and all $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$, the following family of estimate holds

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right) \geq 0\,, \tag{19}$$

for which (1) represents the *critical* upper endpoint as $p \uparrow 2$. Note that for $p = 1$, we recover the Gaussian Poincaré inequality.

On the n-dimensional unit sphere $\mathbb{S}^n$, we have a similar family of interpolation inequalities, due to [47, 48], and obtained independently later in [49]. Those are the Gagliardo–Nirenberg–Sobolev family, defined by a parameter $p \in [1,2) \cup (2, 2^\star]$, where $2^\star = 2\,n/(n-2)$, for $n \geq 3$, and for any $p \in [1,2) \cup (2,+\infty)$ if $n = 1$ or 2, which interpolates between the Poincaré inequality ($p = 1$), and the critical Sobolev inequality ($p = 2^\star$) if $n \geq 3$. Under these conditions, for all $F \in \mathrm{H}^1(\mathbb{S}^n, d\mu_n)$, where $d\mu_n$ denotes the uniform probability measure on $\mathbb{S}^n$, we have

$$\int_{\mathbb{S}^n} |\nabla F|^2\, d\mu_n - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^n,d\mu_n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}\right) \geq 0 \quad \text{if} \quad p \neq 2 \tag{20}$$

and for the limit case $p = 2$, the (subcritical) logarithmic Sobolev inequality

$$\int_{\mathbb{S}^n} |\nabla F|^2\, d\mu_n - \frac{2}{d}\int_{\mathbb{S}^n} |F|^2 \,\log |F|^2\, d\mu_n \geq 0 \tag{21}$$

for all $F \in \mathrm{H}^1(\mathbb{S}^n, d\mu_n)$ such that $\|F\|_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)} = 1$. Inequality (20) can be proved via the entropy method, using (non)-linear diffusion flows. The interested reader may refer to [50–53], and [54–56], where further computations for the heat equation and the Fisher information on Riemannian manifolds are carried out.

Using $\lim_{n\to\infty} 2^\star = 2$, it turns out that for all $v \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$, there is there is a sequence of functions $(F_n)_{n\in\mathbb{N}}$ of functions of $\mathrm{H}^1(\mathbb{S}^n, d\mu_n)$ such that

$$\lim_{n\to\infty}\left(\int_{\mathbb{S}^n}|\nabla F_n|^2\,d\mu_n-\frac{d}{p-2}\left(\|F_n\|^2_{\mathrm{L}^p(\mathbb{S}^n,d\mu_n)}-\|F_n\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}\right)\right)$$
$$=\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-\frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-\|v\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right)\,. \quad (22)$$

Heuristically, the function v has to be seen as the stereographic projection of a d-marginal of F_n for any $n>d$, large enough, if we assume for instance that v is compactly supported. See [57] for a detailed statement. A similar result holds if one takes a sequence of exponents $(p_n)_{n\in\mathbb{N}}$ in $[1,2)\cup(2,2^\star)$ such that $p_n\to 2$ for $n\to\infty$, except that the right-hand side in (22) is replaced by its limit as $p\to 2$, that is, by $\delta[v]$. This procedure can be extended to stability estimates if $p<2$ (see [57] in the subcritical regime) while the case $p_n=2^\star$ is covered in [23, 42].

- For $p=2^\star$, (20) is the critical Sobolev inequality on $\mathbb{S}^n$ and the optimisers are given by the Aubin–Talenti manifold $\mathcal{M}$ made of the functions $G(x)=c\,(1+b\cdot x)^{-(n-2)/2}$ such that $c\in\mathbb{R}$ and $b\in\mathbb{R}^{n+1}$ with $|b|<1$. There is a well known stability result which follows from [22] using an inverse stereographic projection and shows that the deficit in (20) if $p=2^\star$ is bounded from below, up to a constant, by $\mathsf{d}(F,G):=\inf_{G\in\mathcal{M}}\big(\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}+\frac{d}{p-2}\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^n,d\mu_n)}\big)$. The main result of [42] is the fact that the stability constant is bounded from below by β/n, with β as in (18), and that the dimensional dependence is sharp. In fact (18) is obtained in [42] by taking the limit as $n\to+\infty$.
- For $p\in(1,2^\star)$ the stability issue for the subcritical family of inequalities (20) and (21) has been completely solved in [58, 59], with the caveat that the stability term degenerates on a n-dimensional subspace. Analogous stability estimates have been established for the subcritical family (19) in [57].

3.2.6 The Euclidean case

Let us briefly observe that (1) and (2) are equivalent, up to the issue that the two inequalities are formulated in two different spaces (and there is a cancellation of the second-order moments in proving the Euclidean form from the Gaussian form of the inequality, as already remarked in [9]). However, by density, the stability result of (18) translates into an analogous estimate for (2).

4 Examples of instability

In this last section we collect some observations on counter-examples in strong norms.

4.1 Known counter-examples

The first observation of instability of $\delta[v]$ with respect to the Wasserstein distance W_2 appears in [24]. The authors note that if such a stability estimate held for all functions, it would imply an improvement of the optimal constant in the logarithmic Sobolev inequality in the form (1), a contradiction. The first explicit counterexample was later constructed in [26] (and later in [41]), which showed the existence of a sequence $(v_n)_{n\in\mathbb{N}}$

such that

$$\lim_{n\to\infty} \delta[v_n] = 0 \text{ such that } \liminf_{n\to\infty} \mathrm{W}_2^2(|v_n|^2\,dx, d\gamma) > 0 \text{ and } \liminf_{n\to\infty} \|v_n - 1\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > 0\,.$$

The results presented in [41] and the simplified version in [60] are primarily based on the observation that one can construct minimizing sequences for (1), for which the second moment becomes arbitrarily large. Crucially, the deficit $\delta[v]$ is insensitive to the second moment, an insight made precise through a computation by E. Carlen in [9], whereas the W_2 distance is highly sensitive to it.

The H^1 instability of (1) was pointed out by E. Indrei in [15]. The author also clarified the role of moments (see Corollary 6) by constructing a sequence $(v_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \delta[v_n] = 0 \quad \text{and} \quad \liminf_{n\to\infty} \|\nabla v_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)} > 0\,.$$

4.2 A new counter-example to $\dot{\mathbf{H}}^1$ stability

Here we prove that the examples constructed in [25, 41] also provide an example of instability in the $\dot{\mathrm{H}}^1(\mathbb{R}^d, dx)$ topology.

Proposition 10. *Let $d \ge 1$. For all $a > 0$, there exists a sequence $(v_{a,n})_{n\in\mathbb{N}}$ of functions in $\mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that*

$$\int_{\mathbb{R}^d} |v_{a,n}|^2\,d\gamma = 1\,, \quad \int_{\mathbb{R}^d} x\,|v_{a,n}|^2\,d\gamma = 0\,, \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} |x|^2\,|v_{a,n}|^2\,d\gamma = d + a\,, \tag{23a}$$

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^d} |\nabla v_{a,n}|^2\,d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v_{a,n}|^2 \log\left(|v_{a,n}|^2\right) d\gamma \right) = 0\,, \tag{23b}$$

$$\liminf_{n\to\infty} \inf_{w\in\mathcal{M}} \|\nabla w - \nabla v_{a,n}\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \ge a/4 > 0\,, \tag{23c}$$

where $\mathcal{M}$ denotes the set of optimisers of (1) described in Theorem 9, 1).

Proof of Proposition 10. We notice that it is sufficient to find such a sequence in dimension $d = 1$, as in higher dimensions one can consider functions depending only on one coordinate. We follow a construction similar to [41]. Let us consider $(g_n)_{n\in\mathbb{N}}$ defined for any $x \in \mathbb{R}$ by

$$g_n(x) := \begin{cases} 1 & \text{if } |x| \le \frac{n}{2} - \frac{1}{2\,n}\,, \\ \psi_n(|x|) & \text{if } \frac{n}{2} - \frac{1}{2\,n} \le |x| \le \frac{n}{2}\,, \\ \varepsilon_n^{\frac{1}{2}}\,\mathrm{e}^{\frac{n\,x}{2} - \frac{|n|^2}{4}} & \text{if } x \ge \frac{n}{2}\,, \\ \varepsilon_n^{\frac{1}{2}}\,\mathrm{e}^{-\frac{n\,x}{2} - \frac{|n|^2}{4}} & \text{if } x \le -\frac{n}{2}\,, \end{cases} \tag{24}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence such that $2\,\varepsilon_n\,n^2 \to a$ as $n \to \infty$, and ψ_n is a cut-off function such that $\psi_n(\frac{n}{2} - \frac{1}{2\,n}) = 1$ and $\psi_n(\frac{n}{2}) = \sqrt{\varepsilon_n}$. We finally set $v_{n,a} = g_n/\|g_n\|_{\mathrm{L}^2(\mathbb{R}, d\gamma)}$. By construction, we have that $\int_{\mathbb{R}} |v_{a,n}|^2 d\gamma = 1$ and $\int_{\mathbb{R}} x\,|v_{a,n}|^2 d\gamma = 0$, since $v_{a,n}(x)=v_{a,n}(-x)$. As well, by symmetry we have that

$$\frac{1}{2}\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}=\int_0^{\frac{n}{2}-\frac{1}{2\,n}}\gamma(x)\,dx+\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}}|\psi(x)|^2\,\gamma(x)\,dx+\varepsilon_n\int_{\frac{n}{2}}^{\infty}\mathrm{e}^{n\,x-\frac{|n|^2}{2}}\,\gamma(x)\,dx\,. \tag{25}$$

By symmetry, we have that $\int_0^{\frac{n}{2}-\frac{1}{2\,n}}=\int_{-\frac{n}{2}+\frac{1}{2\,n}}^0=\frac{1}{2}-\Phi(-\frac{n}{2}+\frac{1}{2\,n})$ where Φ is the *normal cumulative function* $\Phi(x):=\int_{-\infty}^x\gamma(x)\,dx$. By completing the square, we find that

$$\int_{\frac{n}{2}}^{\infty}\mathrm{e}^{n\,x-\frac{|n|^2}{2}}\,d\gamma=\int_{\frac{n}{2}}^{\infty}\mathrm{e}^{-\frac{|x-n|^2}{2}}\frac{dx}{\sqrt{2\,\pi}}=\int_{-\frac{n}{2}}^{\infty}\mathrm{e}^{-\frac{s^2}{2}}\frac{ds}{\sqrt{2\,\pi}}=1-\Phi\left(-\tfrac{n}{2}\right)\,. \tag{26}$$

By combining (25) and (26), we find

$$\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}=1+2\,\varepsilon_n+o(\varepsilon_n^2)$$

where we have used the fact that $\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}}|\psi(x)|^2\,d\gamma\le\gamma(\frac{n}{2}-\frac{1}{2\,n})/(2\,n)=o(\varepsilon_n^2)$, which is easily deduced from the inequality $\psi^2\le 1$, and the following facts $\int_0^{\frac{n}{2}-\frac{1}{2\,n}}\gamma(x)\,dx=\int_{-\frac{n}{2}+\frac{1}{2\,n}}^0\gamma(x)\,dx=\frac{1}{2}-\Phi(-\frac{n}{2}+\frac{1}{2\,n})$, $\Phi(-\frac{n}{2}+\frac{1}{2\,n})=o(\varepsilon_n^2)$ and $\Phi(-\frac{n}{2})=o(\varepsilon_n^2)$. A similar computation also shows that

$$\int_{\mathbb{R}}|x|^2\,|v_{a,n}|^2\,d\gamma=1+2\,\varepsilon_n\,n^2+o\left(\tfrac{1}{n}\right)\to 1+a\,,$$

which completes the proof of (23a). Let us now consider (23b). From the definition (24) we find that

$$\|v'_{a,n}\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}=\frac{2}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\left(\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{2}}\left|\psi'(x)\right|^2\,d\gamma+\tfrac{1}{4}\,\varepsilon_n\,n^2\left(1-\Phi(-\tfrac{n}{2})\right)\right)$$

and

$$\mathcal{E}[v_{a,n}]=\frac{2}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\left(\int_{\frac{n}{2}-\frac{1}{2\,n}}^{\frac{n}{1}}|\psi(x)|^2\log|\psi(x)|^2\,d\gamma\right.$$
$$\left.+\,\varepsilon_n\left(\log\varepsilon_n+\tfrac{1}{2}\,n^2\right)\left(1-\Phi(-\tfrac{n}{2})\right)-n\,\varepsilon_n\,\gamma(-\tfrac{n}{2})\right)-2\,\varepsilon_n+o(\varepsilon_n^2)\,,$$

so that

$$\delta[v_{a,n}]=\frac{1}{\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\left(\varepsilon_n\,\log\varepsilon_n+o(\varepsilon_n^2)\right)+\varepsilon_n+o(\varepsilon_n^2)\,,$$

which yield (23b). Let us now prove (23c), which simply amount to establish

$$\inf_{w\in\mathcal{M}} \|v'_{a,n} - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} \geq \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1-\Phi(-\tfrac{n}{2})\right) \to \tfrac{a}{4} > 0$$

as $n \to \infty$. Let $w \in \mathcal{M}$: there exists b and $c \in \mathbb{R}$ such that $w = c\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$. Then $w'(x) = c\,\frac{b}{2}\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$, and we have three possibilities: 1) $b\,c = 0$; 2) $b\,c < 0$; or 3) $b\,c > 0$.

- In case $b\,c = 0$, then $w' = 0$, so

$$\begin{aligned}\|v'_{a,n} - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} &= \|v'_{an}\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} \\ &\geq \int_{\frac{n}{2}}^{\infty} |v'_{a,n}(x)|^2\,d\gamma = \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1-\Phi(-\tfrac{n}{2})\right) .\end{aligned}$$

- Assume now $b\,c < 0$. For $x > n/2$ we have that

$$v'_{a,n}(x) - w'(x) = \frac{n\,\varepsilon_n}{2\,\|g_n\|_{\mathrm{L}^2(\mathbb{R},d\gamma)}}\,\mathrm{e}^{\frac{n\,x}{2}-\frac{n^2}{4}} - b\,c\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}} = v'_{a,n}(x) + |b\,c|\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}} ,$$

that is, for $x > n/2$ the functions $v'_{a,n}$ and $|b\,c|\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}$ have the same sign and are both positive. This observation leads to

$$\begin{aligned}\|v'_{a,n} - w'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} &\geq \int_{\frac{n}{2}}^{\infty} |v'_{a,n}(x) + |b\,c|\,\mathrm{e}^{\frac{b\,x}{2}-\frac{b^2}{4}}|^2\,d\gamma \\ &\geq \int_{\frac{n}{2}}^{\infty} |v'_{a,n}(x)|^2\,d\gamma = \frac{\varepsilon_n\, n^2}{2\,\|g_n\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)}} \left(1-\Phi(-\tfrac{n}{2})\right) .\end{aligned}$$

As this lower bound is uniform in b and c, we take the infimum in $\|v'_{a,n} - w'\|_{\mathrm{L}^2(\mathbb{R},d\gamma)}$ and obtain the sought inequality.

- The case $b\,c < 0$ is analogous to the case $b\,c > 0$. □

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References

[1] Gross, L.: Logarithmic Sobolev inequalities. Amer. J. Math. **97**(4), 1061–1083 (1975) https://doi.org/10.2307/2373688

[2] Stam, A.J.: Some inequalities satisfied by the quantities of information of Fisher and Shannon. Information and Control **2**, 101–112 (1959) https://doi.org/10.1016/S0019-9958(59)90348-1

[3] Weissler, F.B.: Logarithmic Sobolev inequalities for the heat-diffusion semigroup. Trans. Amer. Math. Soc. **237**, 255–269 (1978) https://doi.org/10.2307/1997621

[4] Federbush, P.: Partially alternate derivation of a result of Nelson. J. Mathematical Phys. **10**, 50–52 (1969) https://doi.org/10.1063/1.1664760

[5] Villani, C.: Entropy Production and Convergence to Equilibrium, pp. 1–70. Springer, Berlin, Heidelberg (2008). https://doi.org/10.1007/978-3-540-73705-6_1

[6] Villani, C.: A short proof of the "concavity of entropy power". IEEE Trans. Inf. Theory **46**(4), 1695–1696 (2000) https://doi.org/10.1109/18.850718

[7] Toscani, G.: An information-theoretic proof of Nash's inequality. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **24**(1), 83–93 (2013) https://doi.org/10.4171/RLM/645

[8] Toscani, G.: A concavity property for the reciprocal of Fisher information and its consequences on Costa's EPI. Phys. A **432**, 35–42 (2015) https://doi.org/10.1016/j.physa.2015.03.018

[9] Carlen, E.A.: Superadditivity of Fisher's information and logarithmic Sobolev inequalities. J. Funct. Anal. **101**(1), 194–211 (1991) https://doi.org/10.1016/0022-1236(91)90155-X

[10] Bakry, D., Émery, M.: Diffusions hypercontractives. In: Séminaire de Probabilités, XIX, 1983/84. Lecture Notes in Math., vol. 1123, pp. 177–206. Springer, Berlin (1985). https://doi.org/10.1007/BFb0075847

[11] Fathi, M., Indrei, E., Ledoux, M.: Quantitative logarithmic Sobolev inequalities and stability estimates. Discrete Contin. Dyn. Syst. **36**(12), 6835–6853 (2016) https://doi.org/10.3934/dcds.2016097

[12] Bobkov, S.G., Gozlan, N., Roberto, C., Samson, P.-M.: Bounds on the deficit in the logarithmic Sobolev inequality. J. Funct. Anal. **267**(11), 4110–4138 (2014) https://doi.org/10.1016/j.jfa.2014.09.016

[13] Toscani, G.: A strengthened entropy power inequality for log-concave densities. IEEE Trans. Inf. Theory **61**(12), 6550–6559 (2015) https://doi.org/10.1109/TIT.

2015.2495302

[14] Dolbeault, J., Toscani, G.: Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. Int. Math. Res. Not. IMRN (2), 473–498 (2016) https://doi.org/10.1093/imrn/rnv131

[15] Indrei, E.: Sharp stability for LSI. Mathematics **11**(12), 2670 (2023) https://doi.org/10.3390/math11122670

[16] Ané, C., Blachère, S., Chafaï, D., Fougères, P., Gentil, I., Malrieu, F., Roberto, C., Scheffer, G.: Sur les Inégalités de Sobolev Logarithmiques. Panoramas et Synthèses [Panoramas and Syntheses], vol. 10, p. 217. Société Mathématique de France, Paris (2000). With a preface by Dominique Bakry and Michel Ledoux

[17] Guionnet, A., Zegarlinski, B.: Lectures on logarithmic Sobolev inequalities. Séminaire de probabilités de Strasbourg **36**, 1–134 (2002) https://doi.org/10.1007/978-3-540-36107-7_1

[18] Royer, G.: An Initiation to Logarithmic Sobolev Inequalities. SMF/AMS Texts and Monographs, vol. 14, p. 119. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris (2007). Translated from the 1999 French original by Donald Babbitt

[19] Bakry, D., Gentil, I., Ledoux, M.: Analysis and Geometry of Markov Diffusion Operators. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, p. 552. Springer, Cham (2014). https://doi.org/10.1007/978-3-319-00227-9

[20] Brézis, H., Lieb, E.H.: Sobolev inequalities with remainder terms. J. Funct. Anal. **62**, 73–86 (1985) https://doi.org/10.1016/0022-1236(85)90020-5

[21] Lions, P.-L.: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. H. Poincaré Anal. Non Linéaire **1**(2), 109–145 (1984)

[22] Bianchi, G., Egnell, H.: A note on the Sobolev inequality. J. Funct. Anal. **100**(1), 18–24 (1991) https://doi.org/10.1016/0022-1236(91)90099-Q

[23] Dolbeault, J., Esteban, M.J., Figalli, A., Frank, R.L., Loss, M.: Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence. Camb. J. Math. **13**(2), 359–430 (2025) https://doi.org/10.4310/cjm.250325022725

[24] Indrei, E., Marcon, D.: A quantitative log-Sobolev inequality for a two parameter family of functions. Int. Math. Res. Not. IMRN **2014**(20), 5563–5580 (2014) https://doi.org/10.1093/imrn/rnt138

[25] Indrei, E., Kim, D.: Deficit estimates for the logarithmic Sobolev inequality. Differential Integral Equations **34**(7-8), 437–466 (2021) https://doi.org/10.57262/die034-0708-437

[26] Kim, D.: Instability results for the logarithmic Sobolev inequality and its application to the Beckner–Hirschman Inequality (2018). https://arxiv.org/abs/1805.06272

[27] Carleman, T.: Sur la théorie de l'équation intégrodifférentielle de Boltzmann. Acta Math. **60**(1), 91–146 (1933) https://doi.org/10.1007/BF02398270

[28] Kubo, H., Ogawa, T., Suguro, T.: Beckner type of the logarithmic Sobolev and a new type of Shannon's inequalities and an application to the uncertainty principle. Proc. Am. Math. Soc. **147**(4), 1511–1518 (2019) https://doi.org/10.1090/proc/14350

[29] Bakry, D., Ledoux, M.: A logarithmic Sobolev form of the Li-Yau parabolic inequality. Rev. Mat. Iberoam. **22**(2), 683–702 (2006) https://doi.org/10.4171/RMI/470

[30] Dolbeault, J., Volzone, B.: Improved Poincaré inequalities. Nonlinear Analysis: Theory, Methods & Applications **75**(16), 5985–6001 (2012) https://doi.org/10.1016/j.na.2012.05.008

[31] Csiszár, I.: Information-type measures of difference of probability distributions and indirect observations. Studia Scientiarum Mathematicarum Hungarica. A Quarterly of the Hungarian Academy of Sciences **2**, 299–318 (1967)

[32] Kullback, S.: A lower bound for discrimination information in terms of variation (corresp.). IEEE Transactions on Information Theory **13**(1), 126–127 (1967) https://doi.org/10.1109/tit.1967.1053968

[33] Pinsker, M.S.: Information and Information Stability of Random Variables and Processes. Translated and edited by Amiel Feinstein, p. 243. Holden-Day Inc., San Francisco, Calif. (1964)

[34] Brézis, H., Lieb, E.H.: A relation between pointwise convergence of functions and convergence of functionals. Proc. Am. Math. Soc. **88**, 486–490 (1983) https://doi.org/10.2307/2044999

[35] Brigati, G., Dolbeault, J., Simonov, N.: Stability for the logarithmic Sobolev inequality. J. Funct. Anal. **287**(8), 110562–21 (2024) https://doi.org/10.1016/j.jfa.2024.110562

[36] Herraiz, L.: Asymptotic behaviour of solutions of some semilinear parabolic problems. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **16**(1), 49–105 (1999) https://doi.org/10.1016/S0294-1449(99)80008-0

[37] Chen, H.-B., Chewi, S., Niles-Weed, J.: Dimension-free log-Sobolev inequalities for mixture distributions. J. Funct. Anal. **281**(11), 109236–17 (2021) https://doi.org/10.1016/j.jfa.2021.109236

[38] Bolley, F., Gentil, I., Guillin, A.: Dimensional improvements of the logarithmic Sobolev, Talagrand and Brascamp–Lieb inequalities. Ann. Probability **46**(1) (2018) https://doi.org/10.1214/17-aop1184

[39] Brigati, G., Pedrotti, F.: Heat flow, log-concavity, and Lipschitz transport maps (2024). https://arxiv.org/abs/2404.15205

[40] Indrei, E.: $W^{1,1}$ stability for the LSI. Journal of Differential Equations **421**, 196–207 (2025) https://doi.org/10.1016/j.jde.2024.11.054

[41] Kim, D.: Instability results for the logarithmic Sobolev inequality and its application to related inequalities. Discrete Contin. Dyn. Syst. **42**(9), 4297–4320 (2022) https://doi.org/10.3934/dcds.2022053

[42] Dolbeault, J., Esteban, M.J., Figalli, A., Frank, R.L., Loss, M.: A short review on improvements and stability for some interpolation inequalities. https://arxiv.org/abs/2404.15205, Proceedings ICIAM 2023 (2024)

[43] Ambrosio, L., Gigli, N., Savaré, G.: Bakry-émery curvature-dimension condition and Riemannian Ricci curvature bounds. Ann. Probab. **43**(1), 339–404 (2015) https://doi.org/10.1214/14-AOP907

[44] Vershik, A.M.: Does there exist a Lebesgue measure in the infinite-dimensional space? Proceedings of the Steklov Institute of Mathematics **259**(1), 248–272 (2007) https://doi.org/10.1134/S0081543807040153

[45] Cianchi, A., Pick, L.: Optimal Gaussian Sobolev embeddings. J. Funct. Anal. **256**(11), 3588–3642 (2009) https://doi.org/10.1016/j.jfa.2009.03.001

[46] Beckner, W.: A generalized Poincaré inequality for Gaussian measures. Proc. Amer. Math. Soc. **105**(2), 397–400 (1989) https://doi.org/10.2307/2046956

[47] Bidaut-Véron, M.-F., Véron, L.: Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. Invent. Math. **106**(3), 489–539 (1991) https://doi.org/10.1007/BF01243922

[48] Bidaut-Véron, M.-F., Véron, L.: Erratum: "Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations" [Invent. Math. **106** (1991), no. 3, 489–539; MR1134481 (93a:35045)]. Invent. Math. **112**(2), 445 (1993) https://doi.org/10.1007/BF01232442

[49] Beckner, W.: Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2) **138**(1), 213–242 (1993) https://doi.org/10.2307/2946638

[50] Rothaus, O.S.: Diffusion on compact Riemannian manifolds and logarithmic Sobolev inequalities. J. Functional Analysis **42**(1), 102–109 (1981) https://doi.org/10.1016/0022-1236(81)90049-5

[51] Dolbeault, J., Esteban, M.J., Loss, M.: Nonlinear flows and rigidity results on compact manifolds. J. Funct. Anal. **267**(5), 1338–1363 (2014) https://doi.org/10.1016/j.jfa.2014.05.021

[52] Licois, J.R., Véron, L.: Un théorème d'annulation pour des équations elliptiques non linéaires sur des variétés riemanniennes compactes. C. R. Acad. Sci. Paris Sér. I Math. **320**(11), 1337–1342 (1995)

[53] Dolbeault, J., Esteban, M.J., Loss, M.: Interpolation inequalities on the sphere: linear vs. nonlinear flows (inégalités d'interpolation sur la sphère : flots non-linéaires vs. flots linéaires). Annales de la faculté des sciences de Toulouse Sér. 6 **26**(2), 351–379 (2017) https://doi.org/10.5802/afst.1536

[54] Ji, S.: Bounds for the optimal constant of the Bakry-Émery Γ_2 criterion inequality on RP^{d-1} (2024). https://arxiv.org/abs/2408.13954

[55] Ji, S.: Dissipation estimates of the Fisher information for the Landau equation (2024). https://arxiv.org/abs/2410.09035

[56] Ji, S.: Entropy dissipation estimates for the Landau equation with Coulomb potentials (2024). https://arxiv.org/abs/2305.09841

[57] Brigati, G., Dolbeault, J., Simonov, N.: On Gaussian interpolation inequalities. C. R. Math. Acad. Sci. Paris **362**, 21–44 (2024) https://doi.org/10.5802/crmath.488

[58] Frank, R.L.: Degenerate stability of some Sobolev inequalities. AIHPC (2022) https://doi.org/10.4171/aihpc/35

[59] Brigati, G., Dolbeault, J., Simonov, N.: Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results. Ann. Inst. H. Poincaré C Anal. Non Linéaire, https://doi.org/10.4171/aihpc/106 **41**(5), 1289–1321 (2024)

[60] Eldan, R., Lehec, J., Shenfeld, Y.: Stability of the logarithmic Sobolev inequality via the Föllmer process. Ann. Inst. Henri Poincaré, Probab. Stat. **56**(3), 2253–2269 (2020) https://doi.org/10.1214/19-AIHP1038