DIFFUSION AND KINETIC TRANSPORT WITH VERY WEAK CONFINEMENT

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Abstract. This paper is devoted to Fokker-Planck and linear kinetic equations with very weak confinement corresponding to a potential with an at most logarithmic growth and no integrable stationary state. Our goal is to understand how to measure the decay rates when the diffusion wins over the confinement although the potential diverges at infinity. When there is no confinement potential, it is possible to rely on Fourier analysis and mode-by-mode estimates for the kinetic equations. Here we develop an alternative approach based on moment estimates and Caffarelli-Kohn-Nirenberg inequalities of Nash type for diffusion and kinetic equations.

1. **Introduction.** This paper addresses the large time behavior of the solutions to the macroscopic Fokker-Planck equation and to kinetic equations with Fokker-Planck or scattering collision operators.

The first part of this paper deals with the *macroscopic Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V}\, \nabla_x \left(e^V\, u\right)\right) \tag{1}$$

where $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$, that is, $e^{-V}\,dx$ is an *unbounded invariant measure*. There are various reasons to consider only dimensions larger than 3, among which the use of the Hardy inequality. In some cases, the dimension $d = 2$ is also covered as a limit case, while estimates in dimension $d = 1$ are of different nature and will not be considered in this paper for sake of simplicity. We shall investigate the two following examples

$$V_1(x) = \gamma\, \log|x| \quad \text{and} \quad V_2(x) = \gamma\, \log\langle x\rangle$$

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with $\gamma < d$ and $\langle x\rangle := \sqrt{1+|x|^2}$ for any $x \in \mathbb{R}^d$. These two potentials share the same asymptotic behavior as $|x| \to \infty$. The potential V_1 is invariant under scalings, whereas V_2 is smooth at the origin. In both cases, the only integrable equilibrium state is 0. Thus, if the initial datum u_0 is such that $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, we expect that the solution to (1) converges to 0 as $t \to +\infty$. When $\gamma > 0$, the potential V is *very weakly confining* in the sense that, even if it eventually slows down the decay rate, it is not strong enough to produce a stationary state of finite mass: the diffusion wins over the drift. Our goal to establish the rate of convergence in suitable norms. We shall use the notation $\|\cdot\|_p := \|\cdot\|_{\mathrm{L}^p(dx)}$ in case of Lebesgue's measure and specify the measure otherwise.

Theorem 1.1. *Assume that either $d \ge 3$, $\gamma < (d-2)/2$ and $V = V_1$ or $V = V_2$, or $d = 2$, $\gamma \le 0$ and $V = V_2$. Then any solution u of (1) with initial datum $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(\mathbb{R}^d)$ satisfies, for all $t \ge 0$,*

$$\|u(t,\cdot)\|_2^2 \le \frac{\|u_0\|_2^2}{(1+c\,t)^{\frac{d}{2}}} \quad \textit{with} \quad c := \frac{4}{d}\,\min\left\{1, 1-\tfrac{2\,\gamma}{d-2}\right\}\,\mathcal{C}_{\mathrm{Nash}}^{-1}\,\frac{\|u_0\|_2^{4/d}}{\|u_0\|_1^{4/d}}\,. \tag{2}$$

Here $\mathcal{C}_{\mathrm{Nash}}$ denotes the optimal constant in Nash's inequality [24, 11]

$$\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2 \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)\,. \tag{3}$$

Note that the rate of decay is independent of γ and we recover the classical estimate due to J. Nash when $V = 0$ (here $\gamma = 0$). The proof of Theorem 1.1 and further considerations on optimality are collected in Section 2.1. Our method involves the computation of ΔV. In dimension $d = 2$, $V = V_1$ would produce a singularity (which could be handled by an appropriate regularization procedure). In dimension $d = 1$, $V_2''(x) = (1-x^2)/(1+x^2)^2$ has no definite sign and would require new estimates, which are not covered by our result.

Theorem 1.1 does not cover the interval $(d-2)/2 < \gamma < d$. This range is covered by employing the natural setting of $\mathrm{L}^2\big(e^V\big)$ and by requiring additional moment bounds.

Theorem 1.2. *Let $d \ge 1$, $\gamma < d$, $V = V_1$ or $V = V_2$, and $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2\big(e^V\big)$. If $\gamma > 0$, let us assume that $\big\||x|^k u_0\big\|_1 < \infty$ for some $k \ge \max\{2, \gamma/2\}$. Then any solution of (1) with initial datum u_0 satisfies*

$$\forall\, t \ge 0\,, \quad \|u(t,\cdot)\|_{\mathrm{L}^2(e^V\,dx)}^2 \le \|u_0\|_{\mathrm{L}^2(e^V\,dx)}^2\,(1+c\,t)^{-\frac{d-\gamma}{2}}\,.$$

The constant c depends on d, γ, k, $\|u_0\|_{\mathrm{L}^2(e^V\,dx)}$, $\|u_0\|_1$, and $\big\||x|^k u_0\big\|_1$.

The proof of Theorem 1.2 is done in Section 2.2. Although this is a side result, let us notice that the case in which the potential contributes to the decay, *i.e.*, when $\gamma < 0$, is also covered in Theorem 1.2. The scale invariance of (1) with $V = V_1$ can be exploited to obtain intermediate asymptotics in self-similar variables. Let us define

$$u_\star(t,x) = \frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\,|x|^{-\gamma}\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)\,, \tag{4}$$

The following result on *intermediate asymptotics* allows us to identify the leading order term of the solution of (1) as $t \to +\infty$. It is the strongest of our results on (1) but initial data need to have a sufficient decay as $|x| \to \infty$.

Theorem 1.3. *Let $d\ge 1$, $\gamma\in(0,d)$ and $V=V_1$. If for some constant $K>1$, the function u_0 is such that*

$$\forall\,x\in\mathbb{R}^d\,,\quad 0\le u_0(x)\le K\,u_\star(0,x)$$

where $c_\star$ is chosen such that $\|u_\star\|_1=\|u_0\|_1$ then the solution u of (1) with initial datum u_0 satisfies

$$\forall\,t\ge0\,,\quad \|u(t,\cdot)-u_\star(t,\cdot)\|_p\le K\,c_\star^{1-\frac1p}\,\|u_0\|_1^{\frac1p}\left(\tfrac{e}{2\,|\gamma|}\right)^{\frac\gamma2\left(1-\frac1p\right)}\,(1+2\,t)^{-\zeta_p}$$

for any $p\in[1,+\infty)$, where $\zeta_p:=\frac d2\left(1-\frac1p\right)+\frac1{2\,p}\,\min\left\{2,\frac{d}{d-\gamma}\right\}$.

More detailed results will be stated in Section 2.3. Let us quote some relevant papers for (1). In the case without potential, the decay rates of the heat equation is known for more than a century and goes back to [16]. Standard techniques use the Fourier transform, Green kernel estimates and integral representations: see for instance [15]. There are many other parabolic methods which provide decay rates and will not be reviewed here like, for instance, the Maximum Principle, Harnack inequalities and the parabolic regularity theory: see for instance [28].

In his celebrated paper [24], J. Nash was able to reduce the question of the decay rates for the heat equation to (3): see [8] for detailed comments on the optimality of such a method. *Entropy methods* have raised a considerable interest in the recent years, but the most classical approach based on the so-called *carré du champ method* applies to (1) only for potentials V with some convexity properties and a sufficient growth at infinity: typically, if $V(x)=|x|^\alpha$, then $\alpha\ge1$ is required for obtaining a *Poincaré inequality* and the rate of convergence to a unique stationary solution is then exponential, when measured in the appropriate norms; see [4] for a general overview. An interesting family of *weakly confining* potentials is made of functions V with an intermediate growth, such that e^{-V} is integrable but $\lim_{|x|\to\infty}V(x)/|x|=0$: all solutions of (1) are attracted by a unique stationary solution, but the rate is expected to be algebraic rather than exponential. A typical example is $V(x)=|x|^\alpha$ with $\alpha\in(0,1)$. The underlying functional inequality is a *weak Poincaré inequality*: see [26, 21], and [3] for related Lyapunov type methods *à la* Meyn and Tweedie or [6] for recent spectral considerations. We refer to [2] and [30, 31, 32] for further considerations on, respectively, *weighted Nash inequalities* and spectral properties of the diffusion operator. This problem has also attracted attention in the physics literature (see [1] and the references therein for a list of interesting examples).

The second part of this paper is devoted to *kinetic equations* involving a degenerate diffusion operator acting only on the velocity variable or scattering operators, for *very weak potentials* like V_1 or V_2. Various *hypocoercivity* methods have been developed over the years in, *e.g.*, [17, 18, 23, 29, 13], in order to prove exponential rates in appropriate norms, in presence of a *strongly confining potential*. In that case, the growth of the potential at infinity has to be fast enough not only to guarantee the existence of a stationary solution but also to provide macroscopic coercivity properties which typically amount to a Poincaré inequality. A popular simplification is to assume that the position variable is limited to a compact set, for example a torus. Such results are the counterpart in kinetic theory of diffusions covered by the *carré du champ method*, as emphasized in [5].

Recently, hypocoercivity methods have been extended in [7] to the case without any external potential by replacing the Poincaré inequality by Nash type estimates.

The *sub-exponential* regime or the regime with *weak confinement*, *i.e.*, of a potential V such that a weak Poincaré inequality holds, has also been studied in [10, 19]. What we will study next is the range of *very weak potentials* V, which have a growth at infinity which is below the range of weak Poincaré inequalities, but are still such that $\lim_{|x|\to\infty} V(x) = +\infty$. This regime is the counterpart at kinetic level of the results of Theorems 1.1, 1.2 and 1.3. As in the case of (1) when $\gamma \geq 0$, the drift is opposed to the diffusion, but it is not strong enough to prevent that the solution locally vanishes.

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f \tag{5}$$

where $\mathsf{L} f$ is one of the two following collision operators:

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \Big(M \, \nabla_v \left(M^{-1} f \right) \Big) ,$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, M(\cdot) - f(\cdot) \, M(v') \right) dv' .$$

We consider the case of a global equilibrium of the form

$$\forall \, (x,v) \in \mathbb{R}^d \times \mathbb{R}^d , \quad \mathcal{M}(x,v) = M(v) \, e^{-V(x)} \quad \text{where} \quad M(v) = (2\pi)^{-\frac{d}{2}} \, e^{-\frac{1}{2}|v|^2} .$$

We shall say that the gaussian function $M(v)$ is the *local equilibrium* and assume that the *scattering rate* $\sigma(v, v')$ satisfies

$$(\mathbf{H1}) \quad 1 \leq \sigma(v,v') \leq \overline{\sigma} , \quad \forall \, v \, , \, v' \in \mathbb{R}^d , \quad \text{for some} \quad \overline{\sigma} \geq 1 ,$$

$$(\mathbf{H2}) \quad \int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v) \big) \, M(v') \, dv' = 0 \quad \forall \, v \in \mathbb{R}^d .$$

Notice that $\mathcal{M} \notin \mathrm{L}^1(\mathbb{R}^d \times \mathbb{R}^d)$ if $V = V_1$ or $V = V_2$, so that the space $\mathrm{L}^2\left(\mathcal{M}^{-1} dx \, dv\right)$ is defined with respect to an *unbounded measure*. As in the case of (1), the only integrable equilibrium state is 0. Thus, if the initial datum f_0 is such that $f_0 \in \mathrm{L}^1(dx \, dv)$, we expect that the solution to (5) converges to 0 locally as $t \to +\infty$ and look for the rate of convergence in suitable norms.

When $V = 0$, the optimal rate of convergence of a solution f of (5) with initial datum f_0 is known. In [7], it has been proved that there exists a constant $C > 0$ such that

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(t,\cdot,\cdot)|^2 \, d\mu \leq C \, (1+t)^{-\frac{d}{2}} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f_0|^2 \, d\mu \quad \forall \, t \geq 0 ,$$

where $d\mu = M^{-1} \, dx \, dv$ and by factorization, the result is extended with same rate for an arbitrary $\ell > d$ to the measure $\langle v \rangle^\ell \, dx \, dv$ if $f_0 \in \mathrm{L}^2\big(\mathbb{R}^d \times \mathbb{R}^d, \langle v \rangle^\ell dx \, dv\big) \cap \mathrm{L}^2_+\left(\mathbb{R}^d, \langle v \rangle^\ell dv; \mathrm{L}^1\big(\mathbb{R}^d, dx\big)\right)$. Our main result on (5) is a decay rate in the presence of a very weak potential. It is an extension of the results of Theorem 1.2 to the framework of kinetic equations.

Theorem 1.4. *Let $d \geq 1$, $V = V_2$ with $\gamma \in [0,d)$ and $k > \max\{2, \gamma/2\}$. We assume that* (H1)–(H2) *hold and consider a solution f of* (5) *with initial datum $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1} dx \, dv)$ such that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} \langle x \rangle^k \, f_0 \, dx \, dv + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^k \, f_0 \, dx \, dv < +\infty$. Then there exists $C > 0$ such that*

$$\forall \, t \geq 0 , \quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1} dx \, dv)} \leq C \, (1+t)^{-\frac{d-\gamma}{2}} .$$

Standard methods of kinetic theory can be used to establish the existence of solutions of (5) when $V = V_2$. We will not give details here. At formal level, similar results can be expected when $V = V_1$ but the singularity at $x = 0$ raises difficulties which are definitely out of the scope of this paper.

The expression of the constant C is explicit. However, due to the method, we cannot claim optimality in the estimate of Theorem 1.4, but at least the asymptotic rate is expected to be optimal by consistency with the diffusion limit, as it is the case when $V = 0$, studied in [7]. The strategy of the proof and further relevant references will be detailed in Section 3.

2. **Decay estimates for the macroscopic Fokker-Planck equation.** In this section, we establish decay rates for (1) and discuss the optimal range of the parameters.

2.1. **Decay in $\mathrm{L}^2(\mathbb{R}^d)$.** We prove Theorem 1.1. By testing (1) with u, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^2\,dx = -\,2\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} \Delta V\,|u|^2\,dx\,,$$

with either $V = V_1$ or $V = V_2$ and

$$\Delta V_1(x) = \gamma\,\frac{d-2}{|x|^2} \quad\text{and}\quad \Delta V_2(x) = \gamma\,\frac{d-2}{1+|x|^2} + \frac{2\,\gamma}{(1+|x|^2)^2}\,.$$

For $\gamma \le 0$, with the restriction that $V = V_2$ if $d = 2$, we deduce

$$\frac{d}{dt}\,\|u\|_2^2 \le -\,2\,\|\nabla u\|_2^2 \le -\,\frac{2}{\mathcal{C}_{\rm Nash}}\,\|u_0\|_1^{-4/d}\,\|u\|_2^{2+4/d}\,,$$

from Nash's inequality (3). Integration completes the proof of (2). For the case $0 < \gamma < (d-2)/2$ we use the following Hardy-Nash inequalities.

Lemma 2.1. *Let $d \ge 3$ and $\delta < (d-2)^2/4$. Then*

$$\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_\delta\left(\|\nabla u\|_2^2 - \delta\int_{\mathbb{R}^d}\frac{u^2}{|x|^2}\,dx\right)\|u\|_1^{\frac{4}{d}} \quad \forall\,u\in \mathrm{L}^1\cap\mathrm{H}^1(\mathbb{R}^d)\,, \tag{6}$$

with

$$\mathcal{C}_\delta = \mathcal{C}_{\rm Nash}\left(1-\tfrac{4\,\delta}{(d-2)^2}\right)^{-1}.$$

Let additionally $\eta < (d^2-4)/4$. Then, for any $u \in \mathrm{L}^1\cap\mathrm{H}^1(\mathbb{R}^d)$,

$$\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_{\delta,\eta}\left(\|\nabla u\|_2^2 - \delta\int_{\mathbb{R}^d}\frac{u^2}{\langle x\rangle^2}\,dx - \eta\int_{\mathbb{R}^d}\frac{u^2}{\langle x\rangle^4}\,dx\right)\|u\|_1^{\frac{4}{d}} \tag{7}$$

with

$$\mathcal{C}_{\delta,\eta} = \mathcal{C}_{\rm Nash}\left(\min\left\{1-\tfrac{4\,\delta}{(d-2)^2},\,1-\tfrac{4\,\eta}{d^2-4}\right\}\right)^{-1}.$$

The proof of Lemma 2.1 is given in Appendix C. We use Lemma 2.1 with $\delta = \gamma\,(d-2)/2$ and with $\eta = \gamma$ (for $V = V_2$), and proceed as for $\gamma \le 0$ to complete the proof of Theorem 1.1. □

Remark 2.2. The condition $\delta < (d-2)^2/4$ in Lemma 2.1 is optimal for (6) and (7). If $d \ge 3$, the restriction on γ in Theorem 1.1 is also optimal. Let $d \ge 3$, $\gamma > (d-2)/2$ and $V = V_1$ or $V = V_2$. Then there exists $u \in \mathrm{L}^1\cap\mathrm{H}^1(\mathbb{R}^d)$ such that $\|u\|_2 = 1$ and

$$-\,2\int_{\mathbb{R}^d}|\nabla u|^2\,dx + \int_{\mathbb{R}^d}\Delta V\,|u|^2\,dx > 0\,.$$

In the case $V=V_1$, it is indeed enough to observe that $(d-2)^2/4$ is the optimal constant in Hardy's inequality (see Appendix C). The case $V=V_2$ follows from the case $V=V_1$ by an appropriate scaling.

2.2. **Decay in** $\mathrm{L}^2(e^V\,dx)$**.** By testing (1) with $u\,e^V$, we obtain

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d}u^2\,e^V\,dx=-\int_{\mathbb{R}^d}e^{-V}\,\left|\nabla\left(u\,e^V\right)\right|^2\,dx\,. \tag{8}$$

In the case $V=V_1$, we have $e^V=|x|^\gamma$ and (8) takes the form

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^d}|x|^\gamma u^2\,dx=-\int_{\mathbb{R}^d}|x|^{-\gamma}\,|\nabla(|x|^\gamma u)|^2\,dx\,.$$

We first prove Theorem 1.2 for $\gamma\le0$. With $\gamma\le0$ and $a=\frac{d-\gamma}{d+2-\gamma}$, the inequality

$$\int_{\mathbb{R}^d}|x|^\gamma\,u^2\,dx\le\mathcal{C}\left(\int_{\mathbb{R}^d}|x|^{-\gamma}\,|\nabla(|x|^\gamma u)|^2\,dx\right)^a\left(\int_{\mathbb{R}^d}|u|\,dx\right)^{2(1-a)} \tag{9}$$

follows from the Caffarelli-Kohn-Nirenberg inequalities (see Appendix A, Ineq. (26) applied with $k=0$ to $v=|x|^\gamma u$). The conservation of the L^1 norm of u gives

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2\,|x|^\gamma\,dx\le-\,2\,\mathcal{C}^{-\left(1+\frac{2}{d-\gamma}\right)}\,\|u_0\|_1^{-\frac{4}{d-\gamma}}\left(\int_{\mathbb{R}^d}|u|^2\,|x|^\gamma\,dx\right)^{1+\frac{2}{d-\gamma}}.$$

The conclusion of Theorem 1.2 follows by integration. An analogous argument based on the inhomogeneous Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d}|u|^2\,\langle x\rangle^\gamma\,dx\le\mathcal{K}\left(\int_{\mathbb{R}^d}\langle x\rangle^{-\gamma}\,|\nabla(\langle x\rangle^\gamma u)|^2\,dx\right)^a\left(\int_{\mathbb{R}^d}|u|\,dx\right)^{2(1-a)}\quad\text{with}\quad a=\frac{d-\gamma}{d+2-\gamma}$$

applies to the case $\gamma\le0$, $V=V_2$ (see Appendix B, Ineq. (31) applied with $k=0$ and $v=\langle x\rangle^\gamma u$) if $\gamma\le2\,(d-2)$. A minor modification (based on Appendix B, Ineq. (30)) allows us to deal with the remaining cases.

Without additional assumptions, it is not possible to expect a similar result for $\gamma>0$. Let us explain why. In the case $V=V_1$ and with $v=|x|^\gamma u$, let us consider the quotient

$$\mathcal{Q}[v]:=\frac{\left(\int_{\mathbb{R}^d}|x|^{-\gamma}\,|\nabla v|^2\,dx\right)^a\left(\int_{\mathbb{R}^d}|x|^{-\gamma}\,|v|\,dx\right)^{2(1-a)}}{\int_{\mathbb{R}^d}|x|^{-\gamma}\,v^2\,dx}$$

As a consequence of (9), $\mathcal{Q}[v]$ is bounded from below by a positive constant if $\gamma\le0$ and $a=(d-\gamma)/(d-\gamma+2)$. Let us consider the case $\gamma>0$.

Lemma 2.3. *Let $d\ge1$, $\gamma\in(0,d)$ and $a=(d-\gamma)/(d-\gamma+2)$. Then there exists a sequence $(v_n)_{n\in\mathbb{N}}$ of smooth, compactly supported functions such that $\lim_{n\to\infty}\mathcal{Q}[v_n]=0$.*

Proof. Let us take a smooth function v and consider $v_n(x)=v(x+n\,\mathsf{e})$ for some $\mathsf{e}\in\mathbb{S}^{d-1}$. Then $\mathcal{Q}[v_n]=O\left(n^{-(1-a)\,\gamma}\right)$. With $\gamma>0$, we know that a is in the range $0<a<1$ if and only if $\gamma\in(0,d)$. □

For the proof of Theorem 1.2 in the case $0<\gamma<d$, $V=V_1$, we start by estimating the growth of the moment

$$M_k(t):=\int_{\mathbb{R}^d}|x|^k u\,dx\,,$$

which evolves according to

$$M_k' = k\left(d+k-2-\gamma\right)\int_{\mathbb{R}^d} u\,|x|^{k-2}\,dx \le k\left(d+k-2-\gamma\right) M_0^{\frac{2}{k}}\,M_k^{1-\frac{2}{k}}\,,$$

where we have used Hölder's inequality and $M_0(t) = M_0(0) = \|u_0\|_1$. Integration gives

$$M_k(t) \le \left(M_k(0)^{2/k} + 2\big(d+k-2-\gamma\big)\,M_0^{2/k}t\right)^{k/2}\,.$$

If $\gamma \in (0,d)$ and $a = \frac{d+2k-\gamma}{d+2k+2-\gamma}$, by inserting the Caffarelli-Kohn-Nirenberg inequality (26) (see Appendix A) applied to $v = |x|^\gamma u$, that is

$$\int_{\mathbb{R}^d} |x|^\gamma\,u^2\,dx \le \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^{-\gamma}\,\left|\nabla\left(|x|^\gamma u\right)\right|^2 dx\right)^a \left(\int_{\mathbb{R}^d} |x|^k\,|u|\,dx\right)^{2(1-a)}\,,$$

in (8), we observe that the function $z = \int_{\mathbb{R}^d} u^2\,|x|^\gamma\,dx$ solves

$$\frac{dz}{dt} \le -2\left(\mathcal{C}^{-1}\,z\right)^{1+\frac{2}{d+2k-\gamma}}\,M_k(t)^{-\frac{4}{d+2k-\gamma}}\,,$$

and, after integration,

$$z(t) \le z(0)\left(1 + a\left(\big(1+b\,t\big)^{1-\frac{2k}{d+2k-\gamma}} - 1\right)\right)^{-\frac{d+2k-\gamma}{2}}$$

with a and b depend only on the quantities entering into the constant c of Theorem 1.2. Let $\theta = 2k/(d+2k-\gamma)$ and observe that

$$1 + a\left(\big(1+b\,t\big)^{1-\theta} - 1\right) \ge \big(1+c\,t\big)^{1-\theta} \quad \forall\,t \ge 0\,,$$

if $c = b\,\min\left\{a, a^{1/(1-\theta)}\right\}$. Our estimate becomes

$$z(t) \le z(0)\left(1 + a\left(\big(1+b\,t\big)^{1-\theta} - 1\right)\right)^{-k/\theta}$$
$$\le z(0)\big(1+c\,t\big)^{-k\,(1-\theta)/\theta} = z(0)\big(1+c\,t\big)^{-\frac{d-\gamma}{2}}\,.$$

In the case $V = V_2$ we can adopt the same strategy, based on a moment now defined as

$$M_k(t) := \int_{\mathbb{R}^d} \langle x\rangle^k u\,dx\,,$$

and on the inhomogeneous Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d} \langle x\rangle^\gamma u^2\,dx \le \mathcal{K}\left(\int_{\mathbb{R}^d} \langle x\rangle^{-\gamma}\,|\nabla(\langle x\rangle^\gamma u)|^2\,dx\right)^a M_k^{2(1-a)}$$
$$\text{with}\quad a = \frac{d+2k-\gamma}{d+2+2k-\gamma}$$

(see Appendix B, Ineq. (31) applied to $v = \langle x\rangle^\gamma u$) if $\gamma \le 2\,(d-2)$. Again a minor modification (based on Appendix B, Ineq. (30)) allows us to deal with the remaining cases. This completes the proof of Theorem 1.2. □

2.3. **Decay in self-similar variables and intermediate asymptotics.**
We prove Theorem 1.3. With the parabolic change of variables

$$u(t,x)=(1+2\,t)^{-d/2}\,v(\tau,\xi)\,,\quad \tau=\tfrac12\,\log(1+2\,t)\,,\quad \xi=\frac{x}{\sqrt{1+2\,t}}\,,\tag{10}$$

which preserves mass and initial data, (1) is changed into

$$\frac{\partial v}{\partial \tau}=\Delta_\xi v+\nabla_\xi\cdot(v\,\nabla_\xi\Phi)\,,\tag{11}$$

where

$$\Phi(\tau,\xi)=V\left(e^\tau\,\xi\right)+\tfrac12\,|\xi|^2\,.$$

We investigate the long-time behavior of solutions of (1) by considering quasi-equilibria

$$v_\star(\tau,\xi):=M(\tau)\,e^{-\Phi(\tau,\xi)}\,,\tag{12}$$

of (11) with an appropriately chosen $M(\tau)$.

For the scale invariant case $V=V_1$, the potential $\Phi_1(\tau,\xi)=\gamma\big(\log|\xi|+\tau\big)+\frac12\,|\xi|^2$ in (11) can be replaced by the time independent potential $\phi_1(x)=\gamma\,\log|\xi|+\frac12|\xi|^2$. With $M(\tau)=c_\star\,e^{\gamma\,\tau}$, time independent equilibria

$$v_{\star,1}(\xi):=c_\star\,|\xi|^{-\gamma}\,e^{-|\xi|^2/2}\,,\tag{13}$$

are available. For the second case $V=V_2$ with potential

$$\Phi_2(\tau,\xi):=\tfrac{\gamma}{2}\,\log\left(1+e^{2\tau}\,|\xi|^2\right)+\tfrac12\,|\xi|^2\,,$$

we shall use

$$v_{\star,2}(\tau,\xi):=c_\star\left(e^{-2\tau}+|\xi|^2\right)^{-\gamma/2}\,e^{-|\xi|^2/2}\,,\tag{14}$$

so that $v_{\star,2}$ is asymptotically equivalent to $v_{\star,1}$ as $\tau\to\infty$.

If a quasi-equilibrium of the form (12) satisfies

$$\frac{\partial v_\star}{\partial\tau}\ge0\,,$$

which holds for both examples (13) and (14) if $\gamma>0$, then $v_\star$ is obviously a super-solution of (11), thus proving the following result on *uniform decay estimates.*

Proposition 2.4. *Let $\gamma\in(0,d)$ and $u(t,x)$ be a solution of (1) with initial datum such that, for some constant $c_\star>0$,*

$$0\le u(0,x)\le c_\star\left(\sigma+|x|^2\right)^{-\gamma/2}\,\exp\left(-\frac{|x|^2}{2}\right)\quad\forall\,x\in\mathbb R^d\,,$$

with $\sigma=0$ if $V=V_1$ and $\sigma=1$ if $V=V_2$. Then

$$0\le u(t,x)\le\frac{c_\star}{(1+2\,t)^{\frac{d-\gamma}{2}}}\left(\sigma+|x|^2\right)^{-\gamma/2}\,\exp\left(-\frac{|x|^2}{2\,(1+2\,t)}\right)\quad\forall\,x\in\mathbb R^d\,,\ t\ge0\,.$$

For $0<\gamma<d$, we obtain a pointwise decay: the attracting potential is too weak for confinement (no stationary state can exist, at least among $\mathrm L^1(\mathbb R^d)$ solutions) but it slows down the decay compared to solutions of the heat equation (that is, solutions corresponding to $V=0$).

The result of Proposition 2.4 is also true for $\gamma\le0$ if $V=V_1$. In that case, a repulsive potential with $\gamma<0$ accelerates the pointwise decay, but does not change the uniform decay rate as $t\to+\infty$ because

$$\forall\,t>0\,,\quad\max_{r>0}r^{-\gamma}\,\exp\left(-\frac{r^2}{4\,t}\right)=\left(\frac{e}{2\,|\gamma|\,t}\right)^{\gamma/2}\,.\tag{15}$$

In order to obtain an estimate in $\mathrm{L}^2\big(e^V dx\big)$, let us state a result on a Poincaré inequality. We introduce the notations

$$\Phi_{\gamma,\sigma}(\xi) := \tfrac12\,|\xi|^2 + \tfrac{\gamma}{2}\,\log\big(\sigma+|\xi|^2\big)\;,$$

$$Z_{\gamma,\sigma} := \int_{\mathbb{R}^d} e^{-\Phi_{\gamma,\sigma}(\xi)}\,d\xi \quad\text{and}\quad d\mu_{\gamma,\sigma} := Z_{\gamma,\sigma}^{-1}\,e^{-\Phi_{\gamma,\sigma}}\,d\xi\,.$$

Lemma 2.5. *Assume that $d\ge1$, $\gamma\in(0,d)$ and $\sigma\in\mathbb{R}^+$. With the above notations, there is a positive constant $\lambda_{\gamma,\sigma}$ such that*

$$\int_{\mathbb{R}^d}|\nabla w|^2\,d\mu_{\gamma,\sigma}\ge\lambda_{\gamma,\sigma}\int_{\mathbb{R}^d}|w-\overline{w}|^2\,d\mu_{\gamma,\sigma}$$
$$\forall\,w\in\mathrm{H}^1(\mathbb{R}^d,d\mu_{\gamma,\sigma})\ \textit{such that}\ \overline{w}=\int_{\mathbb{R}^d}w\,d\mu_{\gamma,\sigma}\,. \tag{16}$$

Moreover, for any $\gamma\in(0,d)$, $\min_{\sigma\in[0,1]}\lambda_{\gamma,\sigma}>0$.

Proof. Let us consider a potential ψ on $\mathbb{R}^d$. We assume that ψ is a measurable function such that

$$\ell=\lim_{r\to+\infty}\inf_{f\in\mathcal{D}(B_r^c)\setminus\{0\}}\frac{\int_{\mathbb{R}^d}\big(|\nabla f|^2+\psi\,|f|^2\big)\,d\xi}{\int_{\mathbb{R}^d}|f|^2\,d\xi}>0\,,$$

where $B_r^c:=\big\{x\in\mathbb{R}^d\,:\,|x|>r\big\}$ and $\mathcal{D}(B_r^c)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support in B_r^c. According to Persson's result [25, Theorem 2.1], either the lower end of the continuous spectrum of the Schrödinger operator $-\Delta+\psi$ is $\ell<+\infty$, or $\ell=+\infty$ and $-\Delta+\psi$ has pure discrete spectrum and $\lambda_{\gamma,\sigma}$ is the lowest positive eigenvalue.

With the change of unknown function $w=f\,e^{\Phi_{\gamma,\sigma}/2}$, the problem of the best constant in (16) is transformed into the Schrödinger eigenvalue problem for the potential $\psi=\frac14\,|\nabla\Phi_{\gamma,\sigma}|^2-\frac12\,\Delta\Phi_{\gamma,\sigma}$, whose kernel is generated by $e^{-\Phi_{\gamma,\sigma}/2}$, from which we deduce the existence of a constant $\lambda_{\gamma,\sigma}>0$ because $\ell=+\infty$ in that case. □

In the special case $\sigma=0$, it is possible to compute $\lambda_{\gamma,0}$ as follows.

Lemma 2.6. *If $d\ge1$ and $\gamma\in(0,d)$, then $\lambda_{\gamma,0}=\min\big\{2,\frac{d}{d-\gamma}\big\}$.*

Proof. Since $\mu_{\gamma,0}$ is radially symmetric, we can use a decomposition in spherical harmonics in order to compute $\lambda_{\gamma,0}$. The equality case is achieved either by a non-constant radial function, or by a function $w(x)=x_1\,f(|x|)$, where w solves the eigenvalue problem

$$-\,\mu_{\gamma,0}^{-1}\,\nabla\cdot(\mu_{\gamma,0}\nabla w)=\lambda\,w\,.$$

In the first case, the problem is solved by $w(x)=|x|^2-d+\gamma$ and $\lambda=2$, while in the second case the problem is solved by $f\equiv1$ and $\lambda=d/(d-\gamma)$. □

An interesting consequence of Lemma 2.6 is a result of *intermediate asymptotics*, which allows us to identify the leading order term of the solution of (1) as $t\to+\infty$.

Corollary 2.7. *Assume that $d\ge1$, $\gamma\in(0,d)$ and $V=V_1$. With the above notations, if u solves (1) with an initial datum $u_0\in\mathrm{L}^1_+(\mathbb{R}^d)$ such that $\big(u_\star(0,x)\big)^{-1}\,u_0^2\in$*

$\mathrm{L}^1_+(\mathbb{R}^d)$, *with* $u_\star$ *defined by* (4), *and if we choose* $c_\star$ *in* (4) *such that* $\|u_\star(0,\cdot)\|_1 = \|u_0\|_1$, *then*

$$\int_{\mathbb{R}^d} \frac{\big(u(t,x)-u_\star(t,x)\big)^2}{u_\star(t,x)}\,dx \le (1+2\,t)^{-\lambda_{\gamma,0}} \int_{\mathbb{R}^d} \frac{\big(u(0,x)-u_\star(0,x)\big)^2}{u_\star(0,x)}\,dx\,.$$

Proof. By definition of $u_\star$, we have

$$\int_{\mathbb{R}^d} v_{\star,1}\,d\xi = \int_{\mathbb{R}^d} v(0,\xi)\,d\xi = \int_{\mathbb{R}^d} u_0\,dx\,.$$

Then, using the Poincaré inequality (16) and Lemma 2.6, we know that

$$\frac{d}{d\tau}\int_{\mathbb{R}^d} (v-v_{1,\star})^2\,e^{\phi_1}\,d\xi = -\,2\int_{\mathbb{R}^d} \big|\nabla_\xi\big(e^{\phi_1}(v-v_{1,\star})\big)\big|^2\,e^{-\phi_1}\,d\xi$$

$$\le -\,2\,\lambda_{\gamma,0}\int_{\mathbb{R}^d} (v-v_{1,\star})^2\,e^{\phi_1}\,d\xi\,,$$

from which we deduce that

$$\int_{\mathbb{R}^d} (v-v_{1,\star})^2\,e^{\phi_1}\,d\xi \le e^{-2\,\lambda_{\gamma,0}\,\tau}\int_{\mathbb{R}^d} (u(0,x)-v_{1,\star})^2\,e^{\phi_1}\,dx\,.$$

This concludes the proof using the parabolic change of variables (10). □

Proof of Theorem 1.3. A Cauchy-Schwarz inequality shows that

$$\left(\int_{\mathbb{R}^d} \big|u(t,x)-u_\star(t,x)\big|\,dx\right)^2 \le \int_{\mathbb{R}^d} u_\star(t,x)\,dx \int_{\mathbb{R}^d} \frac{\big(u(t,x)-u_\star(t,x)\big)^2}{u_\star(t,x)}\,dx$$

$$\le (1+2\,t)^{-\lambda_{\gamma,0}}\int_{\mathbb{R}^d} u_0\,dx \int_{\mathbb{R}^d} \frac{\big(u(0,x)-u_\star(0,x)\big)^2}{u_\star(0,x)}\,dx\,.$$

The Hölder interpolation inequality

$$\|u(t,\cdot)-u_\star(t,\cdot)\|_p \le \|u(t,\cdot)-u_\star(t,\cdot)\|_1^{\frac1p}\;\|u(t,\cdot)-u_\star(t,\cdot)\|_\infty^{1-\frac1p}$$

combined with the results of Proposition 2.4 and Corollary 2.7 concludes the proof after taking (15) and the expression of $\lambda_{\gamma,0}$ stated in Lemma 2.6 into account. □

3. **Decay estimate for the kinetic equation with weak confinement.** In this section, we prove Theorem 1.4 by revisiting the L^2 approach of [13] in the spirit of [7]. However, no mode-by-mode analysis is applicable here due to the confining potential and the standard Nash inequality has to be replaced by a suitable Caffarelli-Kohn-Nirenberg inequality, which requires moment estimates. The main difference with [7] is to rely on the moments, as was already done in the proof of Theorem 1.2.

3.1. **Notations and elementary computations.** On the space $\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$, we define the scalar product

$$\langle f,g\rangle = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,g\,e^V\,M^{-1}\,dx\,dv$$

and the norm $\|f\| = \langle f,f\rangle^{1/2}$. Let Π be the orthogonal projection operator on $\mathrm{Ker}(\mathsf{L})$ given by $\Pi f := M\,\rho[f]$, where $\rho[f] := \int_{\mathbb{R}^d} f(v)\,dv$, and T be the transport operator such that $\mathsf{T}f = v\cdot\nabla_x f - \nabla_x V\cdot\nabla_v f$. We assume that

$$M(v) = (2\pi)^{-\frac d2}\,e^{-\frac12\,|v|^2}\quad\forall\,v\in\mathbb{R}^d\,.$$

Let us use the notation $u[f] := e^V \rho[f]$ and observe that

$$\mathsf{T}\,\mathsf{\Pi} f = M\, e^{-V}\, v\cdot\nabla_x u[f]\,,\quad (\mathsf{T}\,\mathsf{\Pi})^* f = -\,M\,\nabla_x\cdot\rho\,[v\,f]\;,$$
$$(\mathsf{T\Pi})^*(\mathsf{T\Pi}) f = -\,M\,\nabla_x\cdot\big(e^{-V}\,\nabla_x u[f]\big)\;,$$

where the last identity follows from $\int_{\mathbb{R}^d} M(v)\,v\otimes v\,dv = \mathsf{Id}$. To build a suitable Lyapunov functional, as in [12, 13, 7] we introduce the operator A defined by

$$\mathsf{A} := \big(\mathsf{Id} + (\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}(\mathsf{T\Pi})^*\,.$$

As in [13] we define the Lyapunov functional H by

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}f, f\rangle$$

and obtain by a direct computation that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = -\,\mathsf{D}[f]$$

with

$$\mathsf{D}[f] := -\,\langle \mathsf{L}f, f\rangle + \varepsilon\,\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle + \varepsilon\,\langle \mathsf{AT}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle$$
$$-\,\varepsilon\,\langle \mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f, f\rangle - \varepsilon\,\langle \mathsf{AL}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle\;, \tag{17}$$

where we have used that $\langle \mathsf{A}f, \mathsf{L}f\rangle = 0$ and $\mathsf{A\Pi} = 0$. The latter being a consequence of the identity $\mathsf{\Pi T\Pi} = 0$, that has been called the "diffusive macroscopic limit" in [13]. For the first term in $\mathsf{D}[f]$, we rely on the *microscopic coercivity* estimate (see [13])

$$-\,\langle \mathsf{L}f, f\rangle \ge \lambda_m\,\|(\mathsf{Id}-\mathsf{\Pi})f\|^2\,.$$

The second term $\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle$ is expected to control $\|\Pi f\|^2$. In Section 3.2, the remaining terms will be estimated to show that for ε small enough $\mathsf{D}[f]$ controls $\|(\mathsf{Id}-\mathsf{\Pi})f\|^2 + \langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle$. As in Section 2.2, estimates on moments are needed, which will be established in Section 3.3 and used in Section 3.4 to show a Nash type estimate and to complete the proof of Theorem 1.4 by relating the entropy dissipation $\mathsf{D}[f]$ to $\mathsf{H}[f]$ and by solving the resulting differential inequality.

3.2. **Proof of the Lyapunov functional property of $\mathsf{H}[f]$.** Let us define the notations

$$\langle u_1, u_2\rangle_V := \int_{\mathbb{R}^d} u_1\,u_2\,e^{-V}\,dx \quad\text{and}\quad \|u\|_V^2 := \langle u, u\rangle_V$$

associated with the norm $\mathrm{L}^2(e^{-V}\,dx)$. Unless it is specified, ∇ means ∇_x.

Lemma 3.1. *With the above notations, we have*

$$\|\mathsf{A}f\| \le \frac{1}{2}\,\|(\mathsf{Id}-\mathsf{\Pi})f\|\,,\quad \|\mathsf{TA}f\| \le \|(\mathsf{Id}-\mathsf{\Pi})f\|$$

and

$$|\langle \mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f, f\rangle| \le \|(\mathsf{Id}-\mathsf{\Pi})f\|^2\,.$$

Proof. We already know from [13, Lemma 1] that the operator TA is bounded. Let us give a short proof for completeness. The equation $\mathsf{A}f = g$ is equivalent to

$$(\mathsf{T\Pi})^* f = g + (\mathsf{T\Pi})^*\,(\mathsf{T\Pi})\,g\,. \tag{18}$$

Multiplying (18) by $g\,M^{-1}\,e^{V}$, we get that

$$\|g\|^2+\|\mathsf{T\Pi}g\|^2=\langle f,\mathsf{T\Pi}g\rangle=\langle(\mathsf{Id}-\mathsf{\Pi})f,\mathsf{T\Pi}g\rangle \leq \|(\mathsf{Id}-\mathsf{\Pi})f\|\,\|\mathsf{T\Pi}g\|\leq\frac14\,\|(\mathsf{Id}-\mathsf{\Pi})f\|^2+\|\mathsf{T\Pi}g\|^2$$

from which we deduce that $\|\mathsf{A}f\|=\|g\|\leq\frac12\,\|(\mathsf{Id}-\mathsf{\Pi})f\|$. Since $\mathsf{A}=\mathsf{\Pi A}$, because (18) can be rewritten as $g=\mathsf{\Pi T}^2\mathsf{\Pi}g-\mathsf{\Pi T}f$ using $(\mathsf{T\Pi})^*=-\,\mathsf{\Pi T}$, we also have that $\mathsf{TA}f=\mathsf{T\Pi}g$ and obtain that $\|\mathsf{TA}f\|=\|\mathsf{T\Pi}g\|\leq\|(\mathsf{Id}-\mathsf{\Pi})f\|$.

By taking into account the expression of T, Equation (18) amounts to $g=\mathsf{\Pi}g=w\,M\,e^{-V}$ where w solves

$$w-\mathcal{L}w+\nabla_x\cdot j=0$$

and $j=\int_{\mathbb{R}^d}v\,f\,dx=\int_{\mathbb{R}^d}v\,(\mathsf{Id}-\mathsf{\Pi})f\,dx$. Hence

$$\langle\mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f,h\rangle=\int_{\mathbb{R}^d}(v\cdot\nabla_xw)\,h\,dx=\langle\mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f,(\mathsf{Id}-\mathsf{\Pi})h\rangle\;.$$

This applies to $f=h$, so that

$$\langle\mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f,f\rangle=\langle\mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f,(\mathsf{Id}-\mathsf{\Pi})f\rangle\leq\|(\mathsf{Id}-\mathsf{\Pi})f\|^2\,.$$

□

The term $\langle\mathsf{AT\Pi}f,\Pi f\rangle$ is the one which gives the macroscopic decay rate. Let $w[f]$ be such that $\big(\mathsf{Id}+(\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}\mathsf{\Pi}f=w\,M\,e^{-V}$. Then w solves

$$w-\mathcal{L}w=u[f]\quad\text{where}\quad\mathcal{L}w:=e^V\,\nabla\cdot\big(e^{-V}\,\nabla w\big)\;. \tag{19}$$

Lemma 3.2. *With the above notations, if $u=u[f]$ and $w=w[f]$ solves (19), we have*

$$\langle\mathsf{AT\Pi}f,\Pi f\rangle=\|\nabla w\|_V^2+\|\mathcal{L}w\|_V^2\leq\frac54\,\|u\|_V^2\,.$$

Proof. Let w be a solution of (19). Since

$$\begin{aligned}\mathsf{AT\Pi}f&=\big(\mathsf{Id}+(\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}(\mathsf{T\Pi})^*(\mathsf{T\Pi})\,\mathsf{\Pi}f\\&=\big(\mathsf{Id}+(\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}\big(\mathsf{Id}+(\mathsf{T\Pi})^*\mathsf{T\Pi}-\mathsf{Id}\big)\,\mathsf{\Pi}f\\&=\mathsf{\Pi}f-\big(\mathsf{Id}+(\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}\,\mathsf{\Pi}f=\mathsf{\Pi}f-w\,M\,e^{-V}\,,\end{aligned}$$

we obtain that

$$\mathsf{AT\Pi}f=(u-w)\,M\,e^{-V}\,.$$

Using (19) and integrating on $\mathbb{R}^d$ after multiplying by $\mathsf{\Pi}f=u\,M\,e^{-V}$, we obtain that

$$\langle\mathsf{AT\Pi}f,\Pi f\rangle=\langle u,u-w\rangle_V=\langle w-\mathcal{L}w,-\mathcal{L}w\rangle_V=\|\nabla w\|_V^2+\|\mathcal{L}w\|_V^2\,.$$

On the other hand, we can also write that

$$\langle\mathsf{AT\Pi}f,\Pi f\rangle=\langle u,u-w\rangle_V=-\,\langle u,\mathcal{L}w\rangle_V$$

and obtain that

$$\|\nabla w\|_V^2+\|\mathcal{L}w\|_V^2=-\,\langle u,\mathcal{L}w\rangle_V\leq\|u\|_V\,\|\mathcal{L}w\|_V\leq\frac14\,\|u\|_V^2+\|\mathcal{L}w\|_V^2\,,$$

using the Cauchy-Schwarz inequality. As a consequence, we obtain that

$$\|\nabla w\|_V^2\leq\frac14\,\|u\|_V^2\quad\text{and}\quad\|\mathcal{L}w\|_V\leq\|u\|_V\,,$$

which concludes the proof. □

Lemma 3.3. *With the above notations, if $u = u[f]$ and w solves (19), we have*

$$\|\mathrm{Hess}(w)\|_V^2 \le \max\{1,\,\gamma\}\,\langle \mathsf{AT\Pi} f, \Pi f\rangle\,.$$

Proof. The operator $\mathcal{L} = \Delta - \nabla V \cdot \nabla$ is such that

$$[\mathcal{L},\nabla]\,w = \mathcal{L}(\nabla w) - \nabla(\mathcal{L}\,w) = \Big(\mathcal{L}\big(\tfrac{\partial w}{\partial x_i}\big) - \tfrac{\partial}{\partial x_i}(\mathcal{L}\,w)\Big)_{i=1}^d = \mathrm{Hess}(V)\cdot\nabla w$$

and it is self-adjoint on $\mathrm{L}^2(e^V\,dx)$ so that

$$\langle \mathcal{L}w_1, w_2\rangle_V = -\,\langle \nabla w_1, \nabla w_2\rangle_V = \langle w_1, \mathcal{L}w_2\rangle_V$$

for any w_1 and w_2. Applied first with $w_1 = w$ and $w_2 = \mathcal{L}w$ and then with $w_1 = w_2 = \nabla w$, this shows that

$$\begin{aligned}\|\mathcal{L}w\|_V^2 = -\,\langle\nabla w, \nabla\mathcal{L}w\rangle_V = -\,\langle\nabla w, \mathcal{L}\nabla w\rangle_V + \int_{\mathbb{R}^d}\nabla w\cdot[\mathcal{L},\nabla]\,w\,e^{-V}\,dx\\ = \|\mathrm{Hess}(w)\|_V^2 + \int_{\mathbb{R}^d}\mathrm{Hess}(V) : (\nabla w\otimes\nabla w)e^{-V}\,dx\end{aligned}$$

where $\|\mathrm{Hess}(w)\|_2^2 = \int_{\mathbb{R}^d}|\mathrm{Hess}(w)|^2\,e^{-V}\,dx = \sum_{i,j=1}^d\int_{\mathbb{R}^d}\big(\frac{\partial^2 w}{\partial x_i\partial x_j}\big)^2\,e^{-V}\,dx$. In the case $V = V_2$, we deduce from

$$\frac{\partial^2 V}{\partial x_i\partial x_j} = \frac{\gamma}{\langle x\rangle^2}\left(\delta_{ij} - 2\,\frac{x_i\,x_j}{\langle x\rangle^2}\right)$$

that

$$\mathrm{Hess}(V) - \gamma\,\mathsf{Id}\,.$$

Hence

$$\begin{aligned}\max\{1,\,\gamma\}\,\langle\mathsf{AT\Pi}f, \Pi f\rangle \ge \|\mathcal{L}w\|_V^2 + \max\{1,\,\gamma\}\,\|\nabla w\|_V^2\\ \ge \|\mathrm{Hess}(w)\|_V^2 - \gamma\,\|\nabla w\|_V^2 + \max\{1,\,\gamma\}\,\|\nabla w\|_V^2\,,\end{aligned}$$

which concludes the proof. □

Lemma 3.4. *With the above notations and with $m_\gamma := 3\max\{1,\,\gamma\}$, we have*

$$|\langle\mathsf{AT}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle| \le m_\gamma\,\langle\mathsf{AT\Pi}f, \Pi f\rangle^{1/2}\,\|(\mathsf{Id}-\mathsf{\Pi})f\|\,.$$

Proof. Assume that $u = u[f]$ and w solves (19). Using $g = \big(\mathsf{Id} + (\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)^{-1}f$ so that $\big(\mathsf{Id} + (\mathsf{T\Pi})^*(\mathsf{T\Pi})\big)\,g = f$ means $g - (\mathcal{L}w)\,M\,e^{-V} = f$, let us compute

$$\begin{aligned}\langle\mathsf{AT}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle = \langle\mathsf{T}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{A}^*\mathsf{\Pi}f\rangle = \langle\mathsf{T}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{T\Pi}g\rangle\\ = -\,\big\langle(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{T}^2\mathsf{\Pi}g\big\rangle = -\iint_{\mathbb{R}^d\times\mathbb{R}^d}\sqrt{M}\,v\otimes v\,\frac{(\mathsf{Id}-\mathsf{\Pi})f}{\sqrt{M}} : \mathrm{Hess}(w)\,dx\,dv\\ = -\iint_{\mathbb{R}^d\times\mathbb{R}^d}\sqrt{M}\,\big(v\otimes v - \tfrac{1}{d}\,\mathsf{Id}\big)\,\frac{(\mathsf{Id}-\mathsf{\Pi})f}{\sqrt{M}} : \mathrm{Hess}(w)\,dx\,dv\,,\end{aligned}$$

where we use that T is antisymmetric, and the fact that $\mathsf{T\Pi}g = (v\cdot\nabla_x w)\,M\,e^{-V}$ if $\mathsf{\Pi}g = w\,M\,e^{-V}$, so that $\mathsf{T}^2\mathsf{\Pi}g = \big(v\otimes v : \mathrm{Hess}(w)\big)\,M\,e^{-V}$. We conclude using a Cauchy-Schwarz inequality, Lemma 3.2 and Lemma 3.3. □

In order to have unified notations, we adopt the convention that $\overline{\sigma} = 1/\sqrt{2}$ if L is the Fokker-Planck operator.

Lemma 3.5. *With the above notations, we have*

$$\langle \mathsf{AL}(\mathsf{Id}-\Pi)f,\Pi f\rangle \le \sqrt{2}\,\overline{\sigma}\,\langle \mathsf{AT}\Pi f,\Pi f\rangle^{1/2}\,\|(\mathsf{Id}-\Pi)f\|\,.$$

Proof. We use duality to write

$$\langle \mathsf{AL}(\mathsf{Id}-\Pi)f,\Pi f\rangle = \langle \mathsf{L}(\mathsf{Id}-\Pi)f,h\rangle$$

where $h=\mathsf{A}^*f=(\mathsf{T}\Pi)g$ and $g=\big(\mathsf{Id}+(\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\big)^{-1}f$ so that

$$\big(\mathsf{Id}+(\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\big)\,g=f$$

and $h=(v\cdot\nabla w)\,M\,e^{-V}$. Here w solves (19) with $u=u[f]$.

• If L is the Fokker-Planck operator, then $\int_{\mathbb{R}^d} v\,\mathsf{L}f\,dv=-\,j$ so that

$$|\langle \mathsf{AL}(\mathsf{Id}-\Pi)f,f\rangle| = |\langle j,\nabla w\rangle_2| \le \|j\|_V\,\|\nabla w\|_V\,.$$

We know from Lemma 3.2 that $\|\nabla w\|_V\le\langle \mathsf{AT}\Pi f,\Pi f\rangle^{1/2}$. The estimate on $j=|j|\,\mathsf{e}$ where $\mathsf{e}\in\mathbb{S}^{d-1}$, goes as follows: by computing

$$\begin{aligned}|j|=\left|\int_{\mathbb{R}^d} v\,f\,dv\right| &= \left|\int_{\mathbb{R}^d} v\,(\mathsf{Id}-\Pi)f\,dv\right|\\ &\le \int_{\mathbb{R}^d}\Big((\mathsf{Id}-\Pi)f\,M^{-1/2}\Big)\Big(|v\cdot\mathsf{e}|\,M^{1/2}\Big)\,dv\\ &\le \left(\int_{\mathbb{R}^d}\big|\,\mathsf{Id}-\Pi)f\,\big|^2\,M^{-1}\,dv\int_{\mathbb{R}^d}|v\cdot\mathsf{e}|^2\,M\,dv\right)^{\frac12}\\ &= \left(\int_{\mathbb{R}^d}\big|\,\mathsf{Id}-\Pi)f\,\big|^2\,M^{-1}\,dv\right)^{\frac12}\,,\end{aligned}$$

we know that

$$\|j\,e^V\|_V^2=\int_{\mathbb{R}^d}|j|^2\,e^V\,dx\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big|\,\mathsf{Id}-\Pi)f\big|^2\,M^{-1}\,e^V\,dx\,dv=\|(\mathsf{Id}-\Pi)f\|^2\,.$$

• If L is the scattering operator, then

$$\begin{aligned}\|\mathsf{L}(\mathsf{Id}-\Pi)f\|^2 &\le \overline{\sigma}^2\int_{\mathbb{R}^d}\frac{1}{M}\left|\int_{\mathbb{R}^d}M\,M'\left|\frac{f'}{M'}-\frac{f}{M}\right|dv'\right|^2dv\\ &\le \overline{\sigma}^2\int_{\mathbb{R}^d}M\left|\int_{\mathbb{R}^d}\sqrt{M'}\,\sqrt{M'}\left|\frac{f'}{M'}-\frac{f}{M}\right|dv'\right|^2dv\\ &\le \overline{\sigma}^2\iint_{\mathbb{R}^d\times\mathbb{R}^d}M\,M'\left|\frac{f'}{M'}-\frac{f}{M}\right|^2dv\,dv'\le 4\,\overline{\sigma}^2\int_{\mathbb{R}^d}f^2\,M^{-1}\,dv\end{aligned}$$

and $\|h\|=\big\|(v\cdot\nabla w)\,M\,e^{-V}\big\|=\|\nabla w\|_2$ so that

$$\langle \mathsf{AL}(\mathsf{Id}-\Pi)f,f\rangle\le\sqrt{2}\,\overline{\sigma}\,\langle \mathsf{AT}\Pi f,\Pi f\rangle^{1/2}\,\|(\mathsf{Id}-\Pi)f\|\,.$$

Notice that for a nonnegative function f, we have the improved bounds $\|\mathsf{L}(\mathsf{Id}-\Pi)\|\le\overline{\sigma}\,\|(\mathsf{Id}-\Pi)f\|$ and $\langle \mathsf{AL}(\mathsf{Id}-\Pi)f,\Pi f\rangle\le\overline{\sigma}\,\langle \mathsf{AT}\Pi f,\Pi f\rangle^{1/2}\,\|(\mathsf{Id}-\Pi)f\|$. □

Finally, we apply the results of Lemmas 3.1, 3.4, 3.5 to the right hand side of (17):

Lemma 3.6. *With the above notations, we have*

$$\mathsf{D}[f]\ge\lambda_\varepsilon\left(\|(\mathsf{Id}-\Pi)f\|^2+\langle \mathsf{AT}\Pi f,\Pi f\rangle\right)$$

with

$$\lambda_\varepsilon := \frac{1}{2}\left(\lambda_m - \sqrt{(\lambda_m - 2\,\varepsilon)^2 + \varepsilon^2\left(m_\gamma + \sqrt{2}\,\overline{\sigma}\right)^2}\,\right)$$

and $\lambda_\varepsilon > 0$*, if* $\varepsilon > 0$ *is small enough.*

The functional $\mathsf{H}[f]$ is a Lyapunov function in the sense that $\mathsf{D}[f] \geq 0$ and the equation $\mathsf{D}[f] = 0$ has a unique solution $f = 0$.

Proof. The above mentioned Lemmas imply

$$\mathsf{D}[f] \geq (\lambda_m - \varepsilon)\,\|(\mathsf{Id} - \Pi)f\|^2 + \varepsilon\,\langle \mathsf{AT\Pi}f, \Pi f\rangle \\ - \varepsilon\left(m_\gamma + \sqrt{2}\,\overline{\sigma}\right)\|(\mathsf{Id} - \Pi)f\|\,\langle \mathsf{AT\Pi}f, \Pi f\rangle^{1/2}\,.$$

The Lyapunov function property is a consequence of (19) and Lemma 3.2. □

3.3. **Moment estimates.** Let us consider the case $V = V_2$ and define the $k^{\rm th}$ order moments in x and v by

$$J_k(t) := \|\langle x\rangle^k\,f(t,\cdot,\cdot)\|_1 \quad\text{and}\quad K_k(t) := \||v|^k\,f(t,\cdot,\cdot)\|_1.$$

Our goal is to prove estimates on J_k and K_k. Notice that $J_0 = K_0 = \|f_0\|_{\mathrm{L}^1(\mathbb{R}^d\times\mathbb{R}^d)}$ is constant if f solves (5).

Lemma 3.7. *Let* $d \geq 1$*,* $\gamma \in (0,d)$*,* $k \in \mathbb{N}$ *with* $k \geq 2$*,* $V = V_2$ *and assume that* $f \in \mathrm{C}\big(\mathbb{R}^+, \mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)\big)$ *is a nonnegative solution of* (5) *with initial datum* f_0 *such that* $\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k\,f_0\,dx\,dv < +\infty$ *and* $\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv < +\infty$*. There exist constants* $C_2, \ldots, C_k$ *such that*

$$J_\ell(t) \leq C_\ell\,(1+t)^{\ell/2} \quad\textit{and}\quad K_\ell(t) \leq C_\ell \quad \forall\, t \geq 0\,, \quad \ell = 2,\ldots,k\,. \tag{20}$$

Proof. We present the proof for a Fokker-Planck operator, the case of a scattering operator follows the same steps. A direct computation shows that

$$\frac{dK_\ell}{dt} \leq \ell\,\gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x\cdot v|}{\langle x\rangle^2}\,|v|^{\ell-2}\,f(t,x,v)\,dx\,dv + \ell\,(\ell+d-2)\,K_{\ell-2} - \ell\,K_\ell\,.$$

A bound C_ℓ for K_ℓ, $\ell \in \mathbb{N}$, follows after observing that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|x\cdot v|}{\langle x\rangle^2}\,|v|^{\ell-2}\,f(t,x,v)\,dx\,dv \leq K_{\ell-1} \leq K_0^{1/\ell}\,K_\ell^{1-1/\ell}$$

and $K_{\ell-2} \leq K_0^{2/\ell}\,K_\ell^{1-2/\ell}$ using Hölder's inequality twice.

Next, let us compute

$$\frac{dJ_\ell}{dt} = \ell\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^{\ell-2}\,x\cdot v\,f(t,x,v)\,dx\,dv =: \ell\,L_\ell\,,$$

and

$$\frac{dL_\ell}{dt} = \iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^{\ell-2}\,|v|^2\,f\,dx\,dv + (\ell-2)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^{\ell-4}\,(x\cdot v)^2\,f\,dx\,dv \\ - \gamma\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^{\ell-4}\,|x|^2\,f\,dx\,dv - L_\ell \\ \leq (\ell-1)\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^{\ell-2}\,|v|^2\,f\,dx\,dv - L_\ell\,. \tag{21}$$

Note that, again by Hölder's inequality, $|L_\ell| \leq J_\ell^{1-1/\ell}K_\ell^{1/\ell}$, $\ell = 2,\ldots,k$.

We prove the bound on $J_\ell(t)$ by induction. If $\ell = 2$, (21) implies $L_2(t) \le \max\{L_2(0), C_2\}$ and, thus, $J_2(t) \le C_2\,(1+t)$, up to a redefinition of C_2.

Now let $\ell > 2$ and assume that

$$J_{\ell-1}(t) \le C_{\ell-1}\,(1+t)^{\frac{\ell-1}{2}}\,.$$

We use Hölder's inequality once more for the right hand side of (21):

$$\frac{dL_\ell}{dt} \le (\ell-1)\,J_{\ell-1}^{\frac{\ell-2}{\ell-1}}\,K_{2(\ell-1)}^{\frac{1}{\ell-1}} - L_\ell \le (\ell-1)\,C_{\ell-1}^{\frac{\ell-2}{\ell-1}}\,C_{2(\ell-1)}^{\frac{1}{\ell-1}}\,(1+t)^{\frac{\ell}{2}-1} - L_\ell\,,$$

which implies

$$L_\ell \le C\,(1+t)^{\frac{\ell}{2}-1}\,,$$

and one more integration with respect to t establishes the estimate for J_ℓ in (20), up to an eventual redefinition of C_ℓ. □

Lemma 3.8. *Let $d \ge 1$, $\gamma \in (0,d)$, $k \in \mathbb{N}$ with $k > 2$, $V = V_2$ and assume that $f \in \mathrm{C}\big(\mathbb{R}^+, \mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)\big)$ is a nonnegative solution of (5) with initial datum f_0 such that $\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k f_0\,dx\,dv < +\infty$ and $\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k f_0\,dx\,dv < +\infty$. Let $w = w[f]$ be determined by (19) in terms of $u = u[f]$. Then there exists a positive constant C_k such that*

$$0 \le M_k(t) := \int_{\mathbb{R}^d} w\,\langle x\rangle^{k-\gamma}\,dx \le C_k\,(1+t)^{k/2} \quad \forall\,t \ge 0\,.$$

Proof. The solution w of (19) is positive by the maximum principle. In what follows we use the definition of M_ℓ for arbitrary integers ℓ and note that for $\ell \le 0$,

$$M_\ell \le M_0 = \int_{\mathbb{R}^d} w\,e^{-V}\,dx = \int_{\mathbb{R}^d} u\,e^{-V}\,dx = \|f_0\|_1. \tag{22}$$

Multiplication of (19) by $\langle x\rangle^{\ell-\gamma}$ and integration over $\mathbb{R}^d$ gives

$$M_\ell = \ell\,(\ell-2+d-\gamma)\,M_{\ell-2} - (\ell-2-\gamma)\,M_{\ell-4} + J_\ell\,, \tag{23}$$

where J_ℓ has been estimated in Lemma 3.7. Then, with $\ell = 2$ and (22), we obtain $M_2(t) \le C_2\,(1+t)$. This implies by the Hölder inequality that $M_1(t) \le \sqrt{M_0\,M_2(t)} \le C_1\,(1+t)^{1/2}$. For $2 < \ell \le k$ the estimate $M_\ell(t) \le C_\ell\,(1+t)^{\ell/2}$ follows recursively from (23). □

3.4. **Decay estimate for the kinetic equation (proof of Theorem 1.4).**

Lemma 3.9. *Let $d \ge 1$, $\gamma \in (0,d)$, $k \ge \max\{2, \gamma/2\}$, $V = V_2$ and assume that $f \in \mathrm{C}\big(\mathbb{R}^+, \mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)\big)$ is a nonnegative solution of (5) with initial datum f_0 such that $\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k f_0\,dx\,dv < +\infty$ and $\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k f_0\,dx\,dv < +\infty$. Assume the above notations, in particular with M_k defined as in Lemma 3.8, with the constant $\mathcal{K}$ from (30) (*cf.* Appendix B), and with $a = \frac{d+2k-\gamma}{d+2+2k-\gamma}$. Then*

$$\|\Pi f\|^2 \le 2\,\langle \mathsf{AT\Pi} f, \Pi f\rangle + \mathcal{K}\,M_k^{2(1-a)}\,\langle \mathsf{AT\Pi} f, \Pi f\rangle^a =: \Phi\left(\langle \mathsf{AT\Pi} f, \Pi f\rangle\,;M_k\right) \quad \forall\,t \ge 0\,.$$

Proof. If $u = u[f]$ and w solves (19), we recall that

$$\langle \mathsf{AT\Pi} f, \Pi f\rangle = \|\nabla w\|_V^2 + \|\mathcal{L} w\|_V^2$$

by Lemma 3.2. From (19), we also deduce that

$$\|u\|_V^2 = \langle u, w - \mathcal{L} w\rangle_V \le \|u\|_V\,\left(\|w\|_V^2 + 2\,\|\nabla w\|_V^2 + \|\mathcal{L} w\|_V^2\right)^{1/2}\,.$$

By inequality (31) of Appendix B, we have that

$$\|w\|_V^2 \le \mathcal{K}\,\|\nabla w\|_V^{2a}\,M_k^{2(1-a)}$$

if $\gamma \le 2\,(d-2)$ (otherwise, one has to use ineq. (30) of Appendix B: details are left to the reader). Combining these inequalities gives

$$\|u\|_V^2 \le \mathcal{K}\,\|\nabla w\|_V^{2a}\,M_k^{2(1-a)} + \|\nabla w\|_V^2 + \langle \mathsf{AT\Pi} f, \Pi f\rangle\;,$$

which, noting that $\|\Pi f\| = \|u\|_V$, implies the result. □

As a consequence of Lemmas 3.1, 3.6, 3.9 and of the properties of Φ we have

$$\begin{aligned}
\mathsf{H}[f] &= \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A} f, f\rangle\\
&\le \frac{1+\varepsilon}{2}\|f\|^2 \le \frac{1+\varepsilon}{2}\,\left(\|(\mathsf{Id}-\Pi)f\|^2 + \Phi\left(\langle \mathsf{AT\Pi} f, \Pi f\rangle\,;M_k\right)\right)\\
&\le \frac{1+\varepsilon}{2}\,\Phi\left(\|(\mathsf{Id}-\Pi)f\|^2 + \langle \mathsf{AT\Pi} f, \Pi f\rangle\,;M_k\right) \le \frac{1+\varepsilon}{2}\,\Phi\left(\frac{\mathsf{D}[f]}{\lambda_\varepsilon};M_k\right)\,,
\end{aligned}$$

implying, with Lemma 3.8,

$$\frac{d\mathsf{H}[f]}{dt} = -\,\mathsf{D}[f] \le -\,\lambda_\varepsilon\,\Phi^{-1}\left(\frac{2}{1+\varepsilon}\mathsf{H}[f];C_k(1+t)^{k/2}\right)\,.$$

The decay of $\mathsf{H}[f]$ can be estimated by the solution z of the corresponding ODE problem

$$\frac{dz}{dt} = -\,\lambda_\varepsilon\,\Phi^{-1}\left(\frac{2}{1+\varepsilon}z;C_k(1+t)^{k/2}\right)\,,\quad z(0) = \mathsf{H}[f_0]\,.$$

By the properties of Φ it is obvious that $z(t)\to 0$ monotonically as $t\to+\infty$, which implies that the same is true for $\frac{dz}{dt}$. Therefore, there exists $t_0>0$ such that, in the rewritten ODE

$$-\frac{2}{\lambda_\varepsilon}\frac{dz}{dt} + \mathcal{K}\,C_k^{2(1-a)}(1+t)^{k(1-a)}\left(-\frac{1}{\lambda_\varepsilon}\frac{dz}{dt}\right)^a = \frac{2\,z}{1+\varepsilon}\,,$$

the first term is smaller than the second for $t\ge t_0$, implying the differential inequality

$$\frac{dz}{dt} \le -\,\kappa\,z^{1/a}(1+t)^{k(1-1/a)}\quad\text{for } t\ge t_0\,,$$

with an appropriately defined positive constant κ. Integration and estimation as in Section 2.2 give

$$z(t) \le C\,(1+t)^{\frac{1+k(1-1/a)}{1-1/a}} = C\,(1+t)^{\frac{\gamma-d}{2}}\,,$$

thus completing the proof of Theorem 1.4.

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Appendix A. Homogeneous Caffarelli-Kohn-Nirenberg inequalities of Nash type.

A.1. The general Caffarelli-Kohn-Nirenberg inequalities.

The main result of [9] goes as follows. *Assume that* $d \geq 1$ *is an integer,* $p \geq 1$, $q \geq 1$, $r > 0$, $0 \leq a \leq 1$ *and*

$$\frac{1}{p} + \frac{\alpha_\star}{d} > 0\,, \quad \frac{1}{q} + \frac{\beta_\star}{d} > 0\,, \quad \frac{1}{r} + \frac{\gamma_\star}{d} > 0\,,$$

$$\frac{1}{r} + \frac{\gamma_\star}{d} = a\left(\frac{1}{p} + \frac{\alpha_\star - 1}{d}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta_\star}{d}\right)$$

and, with σ *such that* $\gamma_\star = a\,\sigma + (1-a)\beta_\star$,

$$0 \leq \alpha_\star - \sigma \quad \textit{if} \quad a > 0\,.$$

Assume moreover that

$$\alpha_\star - \sigma \leq 1 \quad \textit{if} \quad a > 0 \quad \textit{and} \quad \frac{1}{p} + \frac{\alpha_\star - 1}{d} = \frac{1}{q} + \frac{\beta_\star}{d}\,.$$

Then there exists a positive constant $\mathcal{C}$ *such that the inequality*

$$\left\||x|^{\gamma_\star}\, v\right\|_r \leq \mathcal{C}\, \left\||x|^{\alpha_\star}\, \nabla v\right\|_p^a \, \left\||x|^{\beta_\star}\, v\right\|_q^{1-a} \tag{24}$$

holds for any $v \in C_0^\infty(\mathbb{R}^d)$.

These interpolation inequalities are known in the literature as the *Caffarelli-Kohn-Nirenberg inequalities* according to [9] but were introduced earlier by V.P. Il'in in [20]. Next we specialize Ineq. (24) to various cases of Nash type corresponding to $q = 1$.

A.2. Weighted Nash type inequalities.

We consider special cases corresponding to $r = p = 2$ and $q = 1$.

• Ineq. (24) with $\alpha_\star = \beta/2$, $\beta_\star = \beta/2$, and $\gamma_\star = \beta/2$ can be written under the condition $\beta > -\,d$ as

$$\int_{\mathbb{R}^d} |x|^\beta\, v^2\, dx \leq \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^\beta\, |\nabla v|^2\, dx\right)^a \left(\int_{\mathbb{R}^d} |x|^{\beta/2}\, |v|\, dx\right)^{2(1-a)} \quad \text{with} \quad a = \frac{d}{d+2}\,. \tag{25}$$

We can indeed check that $\alpha_\star - \sigma = 0$ for any $\beta \leq 0$ and $\frac{1}{p} + \frac{\alpha_\star}{d} > 0$, $\frac{1}{q} + \frac{\beta_\star}{d} > 0$, and $\frac{1}{r} + \frac{\gamma_\star}{d} > 0$ if and only if $\beta > -\,d$.

• Ineq. (24) with $\alpha_\star = -\,\gamma/2$, $\beta_\star = k - \gamma$ and $\gamma_\star = -\,\gamma/2$ can be written as

$$\int_{\mathbb{R}^d} |x|^{-\gamma}\, v^2\, dx \leq \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^{-\gamma}\, |\nabla v|^2\, dx\right)^a \left(\int_{\mathbb{R}^d} |x|^{k-\gamma}\, |v|\, dx\right)^{2(1-a)} \quad \text{with} \quad a = \frac{d+2k-\gamma}{d+2+2k-\gamma} \tag{26}$$

under the condition $\gamma < d$ and $k \geq \gamma/2$. We can indeed check that $\alpha_\star - \sigma = \frac{2k-\gamma}{2k-\gamma+d} \geq 0$. In that case, we have $\alpha_\star - \sigma < 1$ for any $\gamma \leq 0$ and the conditions $\frac{1}{p} + \frac{\alpha_\star}{d} > 0$, $\frac{1}{q} + \frac{\beta_\star}{d} > 0$, and $\frac{1}{r} + \frac{\gamma_\star}{d} > 0$ are always satisfied.

A.3. **A weighted Nash inequality on balls.** We adapt the proof of E. Carlen and M. Loss in [11] to the case of homogeneous weights. The interest of this computation is two fold: it gives an independent proof of (26) and relates the optimal constant in the inequality with a spectral problem; it is also a preparation for a proof of a weighted Nash inequality with inhomogeneous weights, that we shall establish in Appendix B. Here we use Hardy type inequalities, or related expansions of a square which involve $|x|^{-2}$ weights, and for this reason we require that $d \geq 3$.

With $g = |x|^{-\gamma/2}\, v$, (26) is equivalent to

$$\int_{\mathbb{R}^d} g^2\,dx \leq \mathcal{C}\left(\int_{\mathbb{R}^d} |\nabla g|^2\,dx - \frac{\gamma}{4}\,(2\,d-\gamma-4)\int_{\mathbb{R}^d} \frac{g^2}{|x|^2}\,dx\right)^a \cdot\left(\int_{\mathbb{R}^d} |x|^{k-\frac{\gamma}{2}}\,|g|\,dx\right)^{2(1-a)} .$$

Without loss of generality, we can assume that the function g is nonnegative and radial, by spherically non-increasing rearrangements if $0 \leq \gamma \leq 2\,(d-2)$ and $k \geq \gamma/2$. From now on, we will only consider nonnegative, radial, non-increasing functions g and the corresponding functions $v(x) = |x|^{\gamma/2} g(x)$. For any $R > 0$, let

$$g_R := g\,\mathbb{1}_{B_R} \quad \text{and} \quad v_R(x) = |x|^{\gamma/2} g_R(x)\,.$$

We observe that $g - g_R$ is supported in $\mathbb{R}^d \setminus B_R$ and

$$g - g_R \leq g(R) \leq \overline{g}_R := \frac{\int_{\mathbb{R}^d} g_R\,|x|^{k-\gamma/2}\,dx}{\int_{B_R} |x|^{k-\gamma/2}\,dx} = \frac{\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx}{\int_{B_R} |x|^{k-\gamma/2}\,dx}$$

because g is radial non-increasing, so that

$$\int_{\mathbb{R}^d} |v - v_R|^2\,|x|^{-\gamma}\,dx = \|g - g_R\|_2^2 \leq \overline{g}_R \int_{\mathbb{R}^d} |g - g_R|\,dx \leq \overline{g}_R\,R^{\frac{\gamma}{2}-k} \int_{\mathbb{R}^d} |v - v_R|\,|x|^{k-\gamma}\,dx\,,$$

i.e.,

$$\int_{\mathbb{R}^d} |v - v_R|^2\,|x|^{-\gamma}\,dx \leq \frac{\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx}{\int_{B_R} |x|^{k-\gamma/2}\,dx}\,R^{\frac{\gamma}{2}-k} \int_{\mathbb{R}^d} |v - v_R|\,|x|^{k-\gamma}\,dx\,. \tag{27}$$

On the other hand, let us define $\widetilde{v}_R := \frac{\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx}{\int_{B_R} |x|^{2k-\gamma}\,dx}$ and observe that

$$\int_{\mathbb{R}^d} |v_R|^2\,|x|^{-\gamma}\,dx = \int_{\mathbb{R}^d} \left|v_R - \widetilde{v}_R\,|x|^k\right|^2\,|x|^{-\gamma}\,dx + \widetilde{v}_R^2 \int_{B_R} |x|^{2k-\gamma}\,dx\,.$$

From elementary variational techniques as in [14], one can prove the existence of a positive constant λ_1^R (which also depends on k) for which

$$\int_{B_R} |w|^2\,|x|^{-\gamma}\,dx \leq \frac{1}{\lambda_1^R} \int_{B_R} |\nabla w|^2\,|x|^{-\gamma}\,dx \quad \forall\, w \in \mathrm{H}^1(B_R, |x|^{-\gamma}\,dx)$$
$$\text{such that} \quad \int_{B_R} w\,|x|^{k-\gamma}\,dx = 0\,. \tag{28}$$

Note that the weight in the constraint is not $|x|^{-\gamma}$, which is unusual. We infer from the definition of $\widetilde{v}_R$ that this inequality is equivalent to

$$\int_{\mathbb{R}^d} |v_R|^2\,|x|^{-\gamma}\,dx \le \frac{1}{\lambda_1^R}\int_{B_R}\left|\nabla v-\,k\,\widetilde{v}_R\,|x|^{k-2}\,x\right|^2\,|x|^{-\gamma}\,dx+\frac{\left(\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\right)^2}{\int_{B_R}|x|^{2k-\gamma}\,dx}\,.$$

With $\lambda_1:=\lambda_1^1$, a simple scaling shows that $\lambda_1^R=\lambda_1\,R^{-2}$. On the other hand, we can use the estimate

$$\left|\nabla v-\,k\,\widetilde{v}_R\,|x|^{k-2}\,x\right|^2\le 2\left(|\nabla v|^2+k^2\,\widetilde{v}_R^2\,|x|^{2(k-1)}\right)$$

to obtain

$$\int_{\mathbb{R}^d} |v_R|^2\,|x|^{-\gamma}\,dx \le R^2\,A\int_{B_R}|\nabla v|^2\,|x|^{-\gamma}\,dx+B\,\frac{\left(\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\right)^2}{\int_{B_R}|x|^{2k-\gamma}\,dx} \tag{29}$$

with $A=\frac{2}{\lambda_1}$ and $B=1+\frac{d+2k-\gamma}{d+2k-2-\gamma}\,k^2\,A$. Notice that we need $\gamma<d+2k-2$ here, which is the case for $\gamma\le 2\,\min\{d-1,k\}$ if $d\ge3$.

Let us come back to $\int_{\mathbb{R}^d} v^2\,|x|^{-\gamma}\,dx$. By definition of v_R, we know that

$$\int_{\mathbb{R}^d} v^2\,|x|^{-\gamma}\,dx=\int_{\mathbb{R}^d}|v_R|^2\,|x|^{-\gamma}\,dx+\int_{\mathbb{R}^d}|v-v_R|^2\,|x|^{-\gamma}\,dx\,.$$

After summing (27) and (29), we arrive at

$$\begin{aligned}\int_{\mathbb{R}^d} v^2\,|x|^{-\gamma}\,dx &\le R^2\,A\int_{B_R}|\nabla v|^2\,|x|^{-\gamma}\,dx+B\,\frac{\left(\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\right)^2}{\int_{B_R}|x|^{2k-\gamma}\,dx}\\ &\quad+\frac{R^{\frac{\gamma}{2}-k}}{\int_{B_R}|x|^{k-\gamma/2}\,dx}\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\int_{\mathbb{R}^d}|v-v_R|\,|x|^{k-\gamma}\,dx\end{aligned}$$

and notice that

$$\begin{aligned}&B\,\frac{\left(\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\right)^2}{\int_{B_R}|x|^{2k-\gamma}\,dx}+\frac{R^{\frac{\gamma}{2}-k}}{\int_{B_R}|x|^{k-\gamma/2}\,dx}\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\int_{\mathbb{R}^d}|v-v_R|\,|x|^{k-\gamma}\,dx\\ &\le\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx\left[B\,\frac{\int_{\mathbb{R}^d} v_R\,|x|^{k-\gamma}\,dx}{\int_{B_R}|x|^{2k-\gamma}\,dx}+\frac{R^{\frac{\gamma}{2}-k}}{\int_{B_R}|x|^{k-\gamma/2}\,dx}\int_{\mathbb{R}^d}|v-v_R|\,|x|^{k-\gamma}\,dx\right]\\ &\le\left(\int_{\mathbb{R}^d} v\,|x|^{k-\gamma}\,dx\right)^2\,\max\left\{\frac{B}{\int_{B_R}|x|^{2k-\gamma}\,dx},\frac{R^{\frac{\gamma}{2}-k}}{\int_{B_R}|x|^{k-\gamma/2}\,dx}\right\}\\ &=\left(\int_{\mathbb{R}^d} v\,|x|^{k-\gamma}\,dx\right)^2\,C\,R^{\gamma-d-2k}\end{aligned}$$

using $k>0$ and $v_R\le v$, for some numerical constant C which depends only on d, k and γ. Collecting terms, we have found that

$$\int_{\mathbb{R}^d} v^2\,|x|^{-\gamma}\,dx\le R^2\,A\int_{B_R}|\nabla v|^2\,|x|^{-\gamma}\,dx+B\,C\,R^{\gamma-d-2k}\left(\int_{\mathbb{R}^d} v\,|x|^{k-\gamma}\,dx\right)^2\,.$$

We can summarize our observations as follows.

Proposition A.1. *Let $d\ge3$, $\gamma\in(0,d)$ and $\gamma\le2$ if $d=3$, $k\ge\gamma/2$ and $a=\frac{d+2k-\gamma}{d+2+2k-\gamma}$. If $\mathcal{C}$ denotes the optimal constant in (26), then (28) holds with a constant $\lambda_1^R=\lambda_1\,R^{-2}$ for any $R>0$, where λ_1 is a positive constant such that $\lambda_1\le\kappa\,\mathcal{C}^{-1/a}$ for some explicit positive constant κ depending only on γ and d.*

The numerical value of κ can be explicitly computed from the expression of C and from the coefficients that arise from the optimization with respect to $R>0$.

Appendix B. **Inhomogeneous Caffarelli-Kohn-Nirenberg inequalities of Nash type.** Our goal is to establish an extension of (26) adapted to the inhomogeneous case. Here weights are not singular at $x=0$, the condition $d\ge 3$ still appears, due to constraints on γ induced by symmetrization, but we can give another argument in order to cover the cases $d=1$ and $d=2$ (and also the case $d=3$ and $\gamma\in[2,3)$).

Theorem B.1. *If $d\ge 3$, $\gamma\in(0,d)$ and $k\ge\gamma/2$, then there exists a function $\mathcal H$ such that, for any $v\in\mathrm H^1(\mathbb R^d,\langle x\rangle^{-\gamma}\,dx)$ such that $\langle x\rangle^{k-\gamma}\,v\in\mathrm L^1(\mathbb R^d,dx)$, we have*

$$\int_{\mathbb R^d} v^2\,\langle x\rangle^{-\gamma}\,dx\le\left(\int_{\mathbb R^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2\mathcal H\left(\frac{\int_{\mathbb R^d}|\nabla v|^2\,\langle x\rangle^{-\gamma}\,dx}{\left(\int_{\mathbb R^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2}\right). \tag{30}$$

The function $\mathcal H$ is such that $\mathcal H(X)\le\mathcal K\,X^a$ for some optimal constant $\mathcal K>0$, where $a=\frac{d+2k-\gamma}{d+2+2k-\gamma}$ if $d\ge 3$ and $\gamma\le 2\,(d-2)$. Otherwise, if $2\,(d-2)<\gamma<d$ (which is possible only if $d=3$) or $d\le 2$, then $\mathcal H(X)\le\mathcal K\left(X^a+X^b\right)$ where $b=1-\frac{4}{\gamma+2}\,(d+2\,k+2-\gamma)^{-1}$.

Notice that, unless $d\le 2$ or $d=3$ and $\gamma\in[2,3)$, the result of Theorem B.1 is the inequality

$$\int_{\mathbb R^d}\langle x\rangle^{-\gamma}v^2\,dx\le\mathcal K\left(\int_{\mathbb R^d}\langle x\rangle^{-\gamma}|\nabla v|^2\,dx\right)^a\left(\int_{\mathbb R^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^{2(1-a)}. \tag{31}$$

Proof. As in the homogeneous case, if $\gamma\le 2\,(d-2)$, we rely on the method of E. Carlen and M. Loss in [11]. The computations are similar to the ones of Proposition A.1 except that $|x|$ has to be replaced by $\langle x\rangle$. With $g=\langle x\rangle^{-\gamma/2}v$, (31) is equivalent to

$$\begin{aligned}\int_{\mathbb R^d} g^2\,dx\le\mathcal K\Bigg(\int_{\mathbb R^d}|\nabla g|^2\,dx&-\frac{\gamma}{4}\,(2\,d-\gamma-4)\int_{\mathbb R^d} g^2\langle x\rangle^{-2}\,dx\\&-\frac{\gamma}{4}\,(\gamma+4)\int_{\mathbb R^d} g^2\langle x\rangle^{-4}\,dx\Bigg)^a\cdot\left(\int_{\mathbb R^d}\langle x\rangle^{k-\frac{\gamma}{2}}|g|\,dx\right)^{2(1-a)}.\end{aligned}$$

Without loss of generality, we assume that the function g is nonnegative, radial by spherically non-increasing rearrangements, and nonnegative, if $0<\gamma\le 2\,(d-2)$. Let $v(x)=\langle x\rangle^{\gamma/2}g(x)$ and

$$g_R:=g\,\mathbb 1_{B_R}\quad\text{and}\quad v_R(x)=\langle x\rangle^{\gamma/2}g_R(x)$$

for any $R>0$. We observe that $g-g_R$ is supported in $\mathbb R^d\setminus B_R$ and

$$g-g_R\le g(R)\le\overline g_R:=\frac{\int_{\mathbb R^d} g_R\,\langle x\rangle^{k-\gamma/2}\,dx}{\int_{B_R}\langle x\rangle^{k-\gamma/2}\,dx}=\frac{\int_{\mathbb R^d} v_R\,\langle x\rangle^{k-\gamma}\,dx}{\int_{B_R}\langle x\rangle^{k-\gamma/2}\,dx}$$

because g is radial non-increasing, so that

$$\begin{aligned}\int_{\mathbb R^d}|v-v_R|^2\,\langle x\rangle^{-\gamma}\,dx&=\|g-g_R\|_2^2\\&\le\overline g_R\int_{\mathbb R^d}|g-g_R|\,dx=\overline g_R\,\langle R\rangle^{\frac{\gamma}{2}-k}\int_{\mathbb R^d}|v-v_R|\,\langle x\rangle^{k-\gamma}\,dx\,,\end{aligned}$$

that is,

$$\int_{\mathbb{R}^d} |v - v_R|^2 \langle x \rangle^{-\gamma}\, dx \le \frac{\int_{\mathbb{R}^d} v_R \langle x \rangle^{k-\gamma}\, dx}{\int_{B_R} \langle x \rangle^{k-\gamma/2}\, dx} \langle R \rangle^{\frac{\gamma}{2}-k} \int_{\mathbb{R}^d} |v - v_R| \langle x \rangle^{k-\gamma}\, dx\,. \tag{32}$$

On the other hand, using

$$\int_{\mathbb{R}^d} |v_R|^2 \langle x \rangle^{-\gamma}\, dx = \int_{\mathbb{R}^d} \left| v_R - \widetilde{v}_R \langle x \rangle^k \right|^2 \langle x \rangle^{-\gamma}\, dx + \widetilde{v}_R^2 \int_{B_R} \langle x \rangle^{2k-\gamma}\, dx$$

$$\text{where} \quad \widetilde{v}_R := \frac{\int_{\mathbb{R}^d} v_R \langle x \rangle^{k-\gamma}\, dx}{\int_{B_R} \langle x \rangle^{2k-\gamma}\, dx}\,,$$

we deduce from the weighted Poincaré inequality

$$\int_{B_R} |w|^2 \langle x \rangle^{-\gamma}\, dx \le \frac{1}{\lambda_1^R} \int_{B_R} |\nabla w|^2 \langle x \rangle^{-\gamma}\, dx$$

$$\forall\, w \in \mathrm{H}^1(B_R) \quad \text{such that} \quad \int_{B_R} w \langle x \rangle^{k-\gamma}\, dx = 0$$

and from the definition of $\widetilde{v}_R$ that

$$\int_{\mathbb{R}^d} |v_R|^2 \langle x \rangle^{-\gamma}\, dx \le \frac{1}{\lambda_1^R} \int_{B_R} \left| \nabla v + \widetilde{v}_R \nabla \langle x \rangle^k \right|^2 \langle x \rangle^{-\gamma}\, dx + \frac{\left(\int_{\mathbb{R}^d} v_R \langle x \rangle^{k-\gamma}\, dx \right)^2}{\int_{B_R} \langle x \rangle^{2k-\gamma}\, dx}\,. \tag{33}$$

By definition of v_R, we also know that

$$\int_{\mathbb{R}^d} v^2 \langle x \rangle^{-\gamma}\, dx = \int_{\mathbb{R}^d} |v_R|^2 \langle x \rangle^{-\gamma}\, dx + \int_{\mathbb{R}^d} |v - v_R|^2 \langle x \rangle^{-\gamma}\, dx\,.$$

After summing (32) and (33), with $\mathsf{c}(R) := \frac{2}{\lambda_1^R} \frac{\int_{B_R} \left| \nabla \langle x \rangle^k \right|^2 \langle x \rangle^{-\gamma}\, dx}{\int_{B_R} \langle x \rangle^{2k-\gamma}\, dx}$, we arrive at

$$\begin{aligned}
\int_{\mathbb{R}^d} v^2 \langle x \rangle^{-\gamma}\, dx &\le \frac{2}{\lambda_1^R} \int_{B_R} |\nabla v|^2 \langle x \rangle^{-\gamma}\, dx + \mathsf{c}(R) \frac{\left(\int_{\mathbb{R}^d} v_R \langle x \rangle^{k-\gamma}\, dx \right)^2}{\int_{B_R} \langle x \rangle^{2k-\gamma}\, dx} \\
&\quad + \frac{\langle R \rangle^{\frac{\gamma}{2}-k}}{\int_{B_R} \langle x \rangle^{k-\gamma/2}\, dx} \int_{\mathbb{R}^d} v_R \langle x \rangle^{k-\gamma}\, dx \int_{\mathbb{R}^d} |v - v_R| \langle x \rangle^{k-\gamma}\, dx \\
&\le \mathsf{a}(R) \int_{B_R} |\nabla v|^2 \langle x \rangle^{-\gamma}\, dx + \mathsf{b}(R) \left(\int_{\mathbb{R}^d} v \langle x \rangle^{k-\gamma}\, dx \right)^2
\end{aligned}$$

where a and b are two positive continuous functions on $(0, +\infty)$ defined by

$$\mathsf{a}(R) := \frac{2}{\lambda_1^R} \quad \text{and} \quad \mathsf{b}(R) := \max\left\{ \frac{\mathsf{c}(R)}{\int_{B_R} \langle x \rangle^{2k-\gamma}\, dx}, \frac{\langle R \rangle^{\frac{\gamma}{2}-k}}{\int_{B_R} \langle x \rangle^{k-\gamma/2}\, dx} \right\}$$

and such that $\lim_{R\to 0_+} R^d\, \mathsf{b}(R) \in (0, +\infty)$, $\lim_{R\to+\infty} R^{d+2k-\gamma}\, \mathsf{b}(R) \in (0, +\infty)$, $\lim_{R\to+\infty} R^{-2}\, \mathsf{a}(R) = 2/\lambda_1$ where λ_1 is the optimal constant in Proposition A.1 while $\lim_{R\to 0_+} R^{-2}\, \mathsf{a}(R) = 2/\lambda$ is related with Nash's inequality as in [11] and such that

$$\int_{B_1} |w|^2\, dx \le \frac{1}{\lambda} \int_{B_1} |\nabla w|^2\, dx \quad \forall\, w \in \mathrm{H}^1(B_1) \quad \text{such that} \quad \int_{B_1} w\, dx = 0\,.$$

In order to prove (30), we can use the homogeneity of the inequality and assume that $\int_{\mathbb{R}^d} \langle x\rangle^{-\gamma} v^2\,dx = 1$. What we shown so far is that

$$\forall\, R>0\,,\quad 1 \le \left(\int_{\mathbb{R}^d} \langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2 \big(\mathsf{a}(R)\,X + \mathsf{b}(R)\big)$$

where $X = \int_{\mathbb{R}^d} \langle x\rangle^{-\gamma}\,|\nabla v|^2\,dx / \big(\int_{\mathbb{R}^d} \langle x\rangle^{k-\gamma}\,|v|\,dx\big)^2$. With the choice $R = X^{-(1-a)/2}$, we get that there exists a constant $\mathsf{K}>0$ such that $\mathsf{a}(R)\,X + \mathsf{b}(R) < \mathsf{K}\,X^a$. This proves (31) with $\mathcal{K} \le \mathsf{K}$.

To cover the cases $d \le 2$ or $d=3$ and $\gamma \in [2,3)$, another argument can be given, which relies on the IMS truncation method (see for instance [22, 27]), which goes as follows. Assume that χ is a smooth function such that $0 \le \chi \le 1$, with support in the ball B_2, such that $(1-\chi^2)$ is supported in $\mathbb{R}^d \setminus B_1$ and $\kappa := \big\||\nabla\chi|^2/\,(1-\chi^2)\big\|_\infty$ is finite. Here B_R denotes the centered ball of radius R. We define $\chi_R(x) = \chi_1(x/R)$ and to a given function v, we associate the functions

$$v_{1,R} := \chi_R\,v \quad\text{and}\quad v_{2,R} := \sqrt{1-\chi_R^2}\,v\,.$$

With these definitions, we have

$$|v|^2 = |v_{1,R}|^2 + |v_{2,R}|^2 \quad\text{and}\quad |\nabla v|^2 = |\nabla v_{1,R}|^2 + |\nabla v_{2,R}|^2 + \frac{|\nabla\chi_R|^2}{1-\chi_R^2}\,|v|^2\,.$$

We observe that $|\nabla\chi_R|^2/\,(1-\chi_R^2) = O\,(R^{-2})$. Let us denote by $\mathcal{C}_{\mathrm{P}}$ the Poincaré constant in the inequality

$$\int_{B_2} |w|^2\,dx \le \mathcal{C}_{\mathrm{P}} \int_{B_2} |\nabla w|^2\,dx \quad \forall\, w \in \mathrm{H}^1_0(B_2)$$

and apply it to $x \mapsto v_{1,R}(R\,x)$ so that

$$\int_{\mathbb{R}^d} |v_{1,R}|^2\,\langle x\rangle^{-\gamma}\,dx \le \big(1+4\,R^2\big)^{\gamma/2}\;R^2\,\mathcal{C}_{\mathrm{P}} \int_{\mathbb{R}^d} |\nabla v_{1,R}|^2\,\langle x\rangle^{-\gamma}\,dx\,,$$

using the fact that $\big(1+4\,R^2\big)^{-\gamma/2} \le \langle x\rangle^{-\gamma} \le 1$ on B_{2R}.

Now let us consider $v_{2,R}$, which is supported outside of the ball B_R. We deduce from (26) that

$$\begin{aligned}
\int_{\mathbb{R}^d} |v_{2,R}|^2\,\langle x\rangle^{-\gamma}\,dx &\le \int_{\mathbb{R}^d} |v_{2,R}|^2\,|x|^{-\gamma}\,dx\\
&\le \mathcal{C}\left(\int_{\mathbb{R}^d} |\nabla v_{2,R}|^2\,|x|^{-\gamma}\,dx\right)^a \left(\int_{\mathbb{R}^d} |x|^{k-\gamma}\,|v_{2,R}|\,dx\right)^{2(1-a)}\\
&\le S^2 \int_{\mathbb{R}^d} |\nabla v_{2,R}|^2\,|x|^{-\gamma}\,dx + C\,S^{\gamma-d-2k}\left(\int_{\mathbb{R}^d} |x|^{k-\gamma}\,|v|\,dx\right)^2
\end{aligned}$$

where the last inequality holds for any $S>0$, for some explicit constant C, and equality holds for an optimal value of $S>0$. For any $x \in \mathbb{R}^d$ such that $|x| \ge R$, we notice that

$$\begin{aligned}
&\langle x\rangle^{-\gamma} \le |x|^{-\gamma} \le \big(1+R^{-2}\big)^{\gamma/2}\;\langle x\rangle^{-\gamma}\,,\\
&|x|^{k-\gamma} \le \big(1+R^{-2}\big)^{\gamma/4}\,\langle x\rangle^{k-\gamma} \quad\text{for any}\quad k \ge \gamma/2\,.
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^d} |v_{2,R}|^2 \langle x\rangle^{-\gamma}\,dx \\ \le \left(1+R^{-2}\right)^{\gamma/2}\left(S^2\int_{\mathbb{R}^d} |\nabla v_{2,R}|^2 \langle x\rangle^{-\gamma}\,dx + C\,S^{\gamma-d-2k}\left(\int_{\mathbb{R}^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2\right).$$

Here we choose S such that

$$\left(1+4\,R^2\right)^{\gamma/2}\,R^2\,\mathcal{C}_{\rm P}=\left(1+R^{-2}\right)^{\gamma/2}\,S^2\,.$$

Notice that $S\sim R^{1+\gamma/2}$ if either $R\to 0_+$ or $R\to+\infty$. Collecting terms, we have found that

$$\begin{aligned}\int_{\mathbb{R}^d} |v|^2 \langle x\rangle^{-\gamma}\,dx &= \int_{\mathbb{R}^d} |v_{1,R}|^2 \langle x\rangle^{-\gamma}\,dx + \int_{\mathbb{R}^d} |v_{2,R}|^2 \langle x\rangle^{-\gamma}\,dx\\ &\le S^2\left(1+R^{-2}\right)^{\gamma/2}\left(\int_{\mathbb{R}^d} |\nabla v_{1,R}|^2 \langle x\rangle^{-\gamma}\,dx + \int_{\mathbb{R}^d} |\nabla v_{2,R}|^2 \langle x\rangle^{-\gamma}\,dx\right)\\ &\qquad + C\,S^{\gamma-d-2k}\left(1+R^{-2}\right)^{\gamma/2}\left(\int_{\mathbb{R}^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2\\ &\le S^2\left(1+R^{-2}\right)^{\gamma/2}\int_{\mathbb{R}^d} |\nabla v|^2 \langle x\rangle^{-\gamma}\,dx\\ &\qquad + C\,S^{\gamma-d-2k}\left(1+R^{-2}\right)^{\gamma/2}\left(\int_{\mathbb{R}^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^2.\end{aligned}$$

Altogether, we have to estimate $\int_{\mathbb{R}^d}|v|^2\langle x\rangle^{-\gamma}\,dx\left(\int_{\mathbb{R}^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^{-2}$ in terms of

$$F(S):=S^2\left(1+R^{-2}\right)^{\gamma/2}\,X+C\,S^{\gamma-d-2k}\left(1+R^{-2}\right)^{\gamma/2}$$

where $X=\int_{\mathbb{R}^d}|\nabla v|^2\langle x\rangle^{-\gamma}\,dx\left(\int_{\mathbb{R}^d}\langle x\rangle^{k-\gamma}\,|v|\,dx\right)^{-2}$, $R=f(S)\,S^{2/(\gamma+2)}$ where f is a uniformly positive, bounded function and $S>0$ that we can freely choose.

(i) In the regime as $R\to+\infty$, the optimal S is proportional to $X^{-(1-a)/2}$ and incidentally, this corresponds to $X\to 0$: $F(s)\sim X^a$.

(ii) In the intermediate regime corresponding to a finite $R>0$, we also find that $F(s)\sim X^a$ for the optimal choice of S.

(iii) In the regime as $R\to 0_+$, the optimal S is proportional to $X^{-(1-b)/2}$, $X\to\infty$ and by optimizing

$$F(S)\sim S^{\frac{4}{\gamma+2}}\,X+C\,S^{\gamma-d-2k-\frac{4}{\gamma+2}}\,,$$

we obtain that $F(S)\sim X^b$ with $b=1-\frac{4}{\gamma+2}\,(d+2\,k+2-\gamma)^{-1}$.

This completes the proof. □

Appendix C. Hardy-Nash inequalities.

C.1. **Proof of Lemma 2.1.** We start with the proof of (7) by first showing a Hardy type inequality. For some $\alpha\in\mathbb{R}$ to be fixed later, we compute

$$\begin{aligned}0&\le\int_{\mathbb{R}^d}\left|\nabla u+\frac{\alpha\,x}{1+|x|^2}\,u\right|^2dx\\ &=\int_{\mathbb{R}^d}|\nabla u|^2\,dx+\alpha^2\int_{\mathbb{R}^d}\frac{|x|^2\,u^2}{\left(1+|x|^2\right)^2}\,dx+\alpha\int_{\mathbb{R}^d}\nabla\left(u^2\right)\cdot\frac{x}{1+|x|^2}\,dx\,.\end{aligned}$$

We deduce that

$$\int_{\mathbb{R}^d} |\nabla u|^2\,dx+\alpha^2 \int_{\mathbb{R}^d} \frac{|x|^2\,u^2}{\big(1+|x|^2\big)^2}\,dx-\alpha\,d\int_{\mathbb{R}^d} \frac{u^2}{1+|x|^2}\,dx+2\,\alpha\int_{\mathbb{R}^d} \frac{|x|^2\,u^2}{\big(1+|x|^2\big)^2}\,dx \geq 0\,,$$

so that, by writing $|x|^2 = \big(1+|x|^2\big)-1$, we obtain

$$\|\nabla u\|_2^2+\alpha\,(\alpha-d+2)\int_{\mathbb{R}^d} \frac{u^2}{1+|x|^2}\,dx-\alpha\,(\alpha+2)\int_{\mathbb{R}^d} \frac{u^2}{\big(1+|x|^2\big)^2}\,dx \geq 0\,. \tag{34}$$

Concerning the second term, we choose the optimal value $\alpha = (d-2)/2$ in (34), producing the optimal upper bound for δ. It is now straightforward to show

$$\|\nabla u\|_2^2-\delta\int_{\mathbb{R}^d} \frac{u^2}{1+|x|^2}\,dx-\eta\int_{\mathbb{R}^d} \frac{u^2}{\big(1+|x|^2\big)^2}\,dx \geq \min\left\{1-\tfrac{4\,\delta}{(d-2)^2},\,1-\tfrac{4\,\eta}{d^2-4}\right\}\,\|\nabla u\|_2^2\,,$$

whence the proof of (7) is completed by an application of Nash's inequality (3).

The result (6) is shown analogously by using the standard Hardy inequality

$$\|\nabla u\|_2^2-\frac{1}{4}\,(d-2)^2\int_{\mathbb{R}^d} \frac{u^2}{|x|^2}\,dx \geq 0 \tag{35}$$

instead of (34). This completes the proof of Lemma 2.1. □

C.2. **Hardy-Nash vs. Caffarelli-Kohn-Nirenberg inequalities.** The values for $\mathcal{C}_\delta$ and $\mathcal{C}_{\delta,\eta}$ given in Lemma 2.1 cannot be expected to be optimal, since the Hardy and Nash inequalities used in the proof have different optimizing functions. Here we shall present an alternative proof of (6), showing that the optimal value for $\mathcal{C}_\delta$ can be given in terms of the optimal constant of an appropriately chosen Caffarelli-Kohn-Nirenberg inequality of Nash type.

We start by rewriting (25) with optimal constant $\mathcal{C} = \mathcal{C}_{\rm CKN}$ as

$$\left(\int_{\mathbb{R}^d} |v|^2\,|x|^\beta\,dx\right)^{1+\frac{2}{d}} \leq \mathcal{C}_{\rm CKN}^{1+\frac{2}{d}}\int_{\mathbb{R}^d} |\nabla v|^2\,|x|^\beta\,dx\left(\int_{\mathbb{R}^d} |v|\,|x|^{\beta/2}\,dx\right)^{\frac{4}{d}}\,,$$

which holds for $\beta > -d$. A straightforward computation shows that with the change of variables $v(x) = |x|^{-\beta/2}\,u(x)$, this is equivalent to (6) with $\delta = -\beta^2/4-\beta\,(d-2)/2$. Thus, the choice $\beta = 2-d+\sqrt{(d-2)^2-4\,\delta} > -d$ amounts to (6) with optimal constant $\mathcal{C}_\delta = \mathcal{C}_{\rm CKN}^{1+2/d}$.

REFERENCES

[1] E. Aghion, D. A. Kessler, and E. Barkai, *From non-normalizable Boltzmann-Gibbs statistics to infinite-ergodic theory*, Phys. Rev. Lett., 122 (2019), p. 010601.

[2] D. Bakry, F. Bolley, I. Gentil, and P. Maheux, *Weighted Nash inequalities*, Rev. Mat. Iberoam., 28 (2012), pp. 879–906.

[3] D. Bakry, P. Cattiaux, and A. Guillin, *Rate of convergence for ergodic continuous Markov processes: Lyapunov versus Poincaré*, J. Funct. Anal., 254 (2008), pp. 727–759.

[4] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, vol. 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Cham, 2014.

[5] F. Baudoin, *Bakry-Émery meet Villani*, J. Funct. Anal., 273 (2017), pp. 2275–2291.

[6] J. Ben-Artzi and A. Einav, *Weak Poincaré and Nash-type inequalities via density of states estimates*, ArXiv e-prints 1805.08557, (2018).

[7] E. Bouin, J. Dolbeault, S. Mischler, C. Mouhot, and C. Schmeiser, *Hypocoercivity without confinement*. To appear in Pure and Applied Analysis, hal-01575501, Oct. 2017.

[8] E. Bouin, J. Dolbeault, and C. Schmeiser, *A variational proof of Nash's inequality*. To appear in Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., hal-01940110.

[9] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math., 53 (1984), pp. 259–275.

[10] C. Cao, *The kinetic Fokker-Planck equation with weak confinement force*. hal-01697058, Jan. 2018.

[11] E. A. Carlen and M. Loss, *Sharp constant in Nash's inequality*, Internat. Math. Res. Notices, (1993), pp. 213–215.

[12] J. Dolbeault, C. Mouhot, and C. Schmeiser, *Hypocoercivity for kinetic equations with linear relaxation terms*, Comptes Rendus Mathématique, 347 (2009), pp. 511 – 516.

[13] ———, *Hypocoercivity for linear kinetic equations conserving mass*, Trans. Amer. Math. Soc., 367 (2015), pp. 3807–3828.

[14] J. Dolbeault, M. Muratori, and B. Nazaret, *Weighted interpolation inequalities: a perturbation approach*, Math. Ann., 369 (2017), pp. 1237–1270.

[15] L. C. Evans, *Partial differential equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2010.

[16] J. B. J. Fourier, *Théorie analytique de la chaleur*, Cambridge: Cambridge University Press, reprint of the 1822 ed. ed., 2009.

[17] F. Hérau, *Hypocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation*, Asymptot. Anal., 46 (2006), pp. 349–359.

[18] F. Hérau, *Short and long time behavior of the Fokker-Planck equation in a confining potential and applications*, J. Funct. Anal., 244 (2007), pp. 95–118.

[19] S. Hu and X. Wang, *Subexponential decay in kinetic Fokker–Planck equation: Weak hypocoercivity*, Bernoulli, 25 (2019), pp. 174–188.

[20] V. P. Il'in, *Some integral inequalities and their applications in the theory of differentiable functions of several variables*, Mat. Sb. (N.S.), 54 (96) (1961), pp. 331–380.

[21] O. Kavian and S. Mischler, *The Fokker-Planck equation with subcritical confinement force*. hal-01241680, Dec. 2015.

[22] J. D. Morgan, III, *Schrödinger operators whose potentials have separated singularities*, J. Operator Theory, 1 (1979), pp. 109–115.

[23] C. Mouhot and L. Neumann, *Quantitative perturbative study of convergence to equilibrium for collisional kinetic models in the torus*, Nonlinearity, 19 (2006), p. 969.

[24] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., 80 (1958), pp. 931–954.

[25] A. Persson, *Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator*, Math. Scand., 8 (1960), pp. 143–153.

[26] M. Röckner and F.-Y. Wang, *Weak Poincaré inequalities and* L^2*-convergence rates of Markov semigroups*, J. Funct. Anal., 185 (2001), pp. 564–603.

[27] B. Simon, *Semiclassical analysis of low lying eigenvalues. I. Nondegenerate minima: asymptotic expansions*, Ann. Inst. H. Poincaré Sect. A (N.S.), 38 (1983), pp. 295–308.

[28] J. L. Vázquez, *The mathematical theories of diffusion: nonlinear and fractional diffusion*, in Nonlocal and nonlinear diffusions and interactions: new methods and directions, vol. 2186 of Lecture Notes in Math., Springer, Cham, 2017, pp. 205–278.

[29] C. Villani, *Hypocoercivity*, Mem. Amer. Math. Soc., 202 (2009), pp. iv+141.

[30] F.-Y. Wang, *Functional inequalities for empty essential spectrum*, J. Funct. Anal., 170 (2000), pp. 219–245.

[31] ———, *Functional inequalities and spectrum estimates: the infinite measure case*, J. Funct. Anal., 194 (2002), pp. 288–310.

[32] ———, *Intrinsic ultracontractivity on Riemannian manifolds with infinite volume measures*, Sci. China Math., 53 (2010), pp. 895–904.

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