

# A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities

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## Abstract

This note is devoted to several inequalities deduced from a special form of the logarithmic Hardy-Littlewood-Sobolev, which is well adapted to the characterization of stationary solutions of a Keller-Segel system written in self-similar variables, in case of a subcritical mass. For the corresponding evolution problem, such functional inequalities play an important role for identifying the rate of convergence of the solutions towards the stationary solution with same mass.

**Keywords.** Keller-Segel model; chemotaxis; large time asymptotics; subcritical mass; self-similar solutions; relative entropy; free energy; Lyapunov functional; spectral gap; logarithmic Hardy-Littlewood-Sobolev inequality; Onofri's inequality; Legendre duality; best constants — MSC (2010): Primary: 26D10; 92C17. Secondary: 35B40

**Un cadre fonctionnel pour le système de Keller-Segel : inégalité logarithmique de Hardy-Littlewood-Sobolev et inégalités de trou spectral reliées**

**Résumé** Cette note est consacrée à plusieurs inégalités fonctionnelles déduites d'une forme particulière de l'inégalité logarithmique de Hardy-Littlewood-Sobolev, qui est bien adaptée à la caractérisation des solutions stationnaires d'un système de Keller-Segel écrit en variables auto-similaires, dans le cas d'une masse sous-critique. Pour le problème d'évolution correspondant, ces inégalités fonctionnelles jouent un rôle important dans l'identification des taux de convergence des solutions vers la solution stationnaire de même masse.

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## Version française abrégée

Dans  $\mathbb{R}^2$ , l'inégalité logarithmique de Hardy-Littlewood-Sobolev a été établie avec des constantes optimales dans [7,1]. On peut l'écrire sous la forme

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{M \mu} \right) dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (n(x) - M \mu(x)) \log |x - y| (n(y) - M \mu(y)) dx dy \geq 0$$

où  $M = \int_{\mathbb{R}^2} n dx$  et  $1/\mu(x) = \pi(1 + |x|^2)^2$  pour tout  $x \in \mathbb{R}^2$ . De plus, par dualité de Legendre, elle est équivalente à l'inégalité d'Onofri euclidienne (voir [5,9] et [12] pour une forme équivalente sur la sphère).

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Pour étudier le système parabolique-elliptique de Keller-Segel écrit en variables auto-similaires

$$\frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n x) - \nabla \cdot (n \nabla c), \quad c = (-\Delta)^{-1} n, \quad x \in \mathbb{R}^2, \quad t > 0, \quad (1)$$

on est amené à considérer une forme de l'inégalité logarithmique de Hardy-Littlewood-Sobolev qui s'écrit, sous réserve que  $M < 8\pi$ , sous la forme

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{n_M} \right) dx + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (n(x) - n_M(x)) \log |x - y| (n(y) - n_M(y)) dx dy \geq 0 \quad (2)$$

et où  $(n_M, c_M)$  est l'unique solution stationnaire, régulière, à symétrie radiale, de (1), donnée par

$$-\Delta c_M = M \frac{e^{-\frac{1}{2}|x|^2 + c_M}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2 + c} dx} =: n_M, \quad x \in \mathbb{R}^2.$$

Exactement comme dans [7,1,5,9], on montre par dualité de Legendre qu'à (2) correspond une nouvelle inégalité de type Onofri.

**Théorème 1** *Pour tout  $M \in (0, 8\pi)$ , pour toute fonction  $\phi$  régulière à support compact, on a*

$$\log \left( \int_{\mathbb{R}^2} e^{\phi} d\mu_M \right) - \int_{\mathbb{R}^2} \phi d\mu_M \leq \frac{1}{2M} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx.$$

Ici,  $d\mu_M := \frac{1}{M} n_M dx$  est une mesure de probabilité et comme dans [8], on montre une inégalité de trou spectral en effectuant un développement autour de  $\phi \equiv 1$ . Par densité, il est par ailleurs possible d'étendre l'inégalité à l'espace fonctionnel obtenu par complétion, pour la norme  $\|\phi\|^2 = \int_{\mathbb{R}^2} |\nabla \phi|^2 dx + (\int_{\mathbb{R}^2} \phi d\mu_M)^2$ , de l'ensemble des fonctions régulières à support compact.

Dans sa forme linéarisée, le système de Keller-Segel s'écrit

$$\frac{\partial f}{\partial t} = \frac{1}{n_M} \nabla \cdot [n_M \nabla (f - g c_M)] =: \mathcal{L} f \quad \text{où} \quad g c_M = (-\Delta)^{-1} (f n_M). \quad (3)$$

On montre que le noyau de  $\mathcal{L}$  est engendré par une fonction  $f_{0,0}$  déterminée par  $-\Delta f_{0,0} = f_{0,0} n_M$ . En effectuant un développement limité à l'ordre deux autour de  $n_M$ , il est aisé de voir que

$$\mathbf{Q}_1[f] := \int_{\mathbb{R}^2} |f|^2 d\mu_M + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \log |x - y| f(y) d\mu_M(x) d\mu_M(y) \geq 0.$$

De plus  $\mathbf{Q}_1[f] = 0$  si et seulement si  $f$  est proportionnelle à  $f_{0,0}$ . On montre alors le résultat suivant.

**Théorème 2** *Il existe  $\kappa > 1$  tel que, pour tout  $f \in L^2(\mathbb{R}^2, d\mu_M)$ , si  $\int_{\mathbb{R}^2} f f_{0,0} d\mu_M = 0$ , alors on a*

$$\int_{\mathbb{R}^2} f^2 d\mu_M \leq \kappa \mathbf{Q}_1[f].$$

Si l'on définit maintenant  $\mathbf{Q}_2[f] := \langle f, \mathcal{L} f \rangle$ , on montre une dernière inégalité de trou spectral.

**Théorème 3** *Pour toute fonction  $f \in L^2(\mathbb{R}^2, f\mu_M)$  vérifiant  $\int_{\mathbb{R}^2} f f_{0,0} d\mu_M = 0$ , on a*

$$\mathbf{Q}_1[f] \leq \mathbf{Q}_2[f].$$

Il est alors facile d'en déduire que si  $f$  est une solution de (3), alors  $\mathbf{Q}_1[f(t, \cdot)] \leq \mathbf{Q}_1[f(0, \cdot)] e^{-2t}$  pour tout  $t \geq 0$ . Pour une preuve détaillée des Théorèmes 2 et 3, on renverra à [6]. Au prix d'une estimation un peu plus compliquée basée sur la formule de Duhamel, on montre que cette estimation en temps grand s'applique aussi à  $f := (n - n_M)/n_M$ , où  $n$  est la solution de (1).

## 1. Introduction

In  $\mathbb{R}^2$ , the logarithmic Hardy-Littlewood-Sobolev has been established with optimal constants in [7] (also see [1]) and can be written as

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{M} \right) dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log |x - y| dx dy + M (1 + \log \pi) \geq 0 \quad (1)$$

for any function  $n \in L^1_+(\mathbb{R}^2)$  with  $M = \int_{\mathbb{R}^2} n dx$ . As a consequence (see [10]), the *free energy* functional

$$F[n] := \int_{\mathbb{R}^2} n \log n dx + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 n dx - \frac{1}{2} \int_{\mathbb{R}^2} n c dx + K \quad \text{with} \quad c = (-\Delta)^{-1} n := -\frac{1}{2\pi} \log |\cdot| * n$$

is bounded from below if  $M \in (0, 8\pi]$ . Here  $K = K(M)$  is a constant to be fixed later. We may observe that  $F$  is not bounded from below if  $M > 8\pi$ , for instance by considering  $\lambda \mapsto F[n_\lambda]$  where  $n_\lambda(x) = \lambda^2 n(\lambda x)$  for some given function  $n$ , and by taking the limit  $\lambda \rightarrow \infty$ . See [11] for more details. Equality in (1) is achieved by

$$\mu(x) := \frac{1}{\pi(1+|x|^2)^2} \quad \forall x \in \mathbb{R}^2,$$

which solves  $-\Delta \log \mu = 8\pi \mu$  and can be inverted as  $(-\Delta)^{-1} \mu = \frac{1}{8\pi} \log \mu + \frac{1}{8\pi} \log \pi$ .

Consider the probability measure  $d\mu := \mu dx$ . Written in Euclidean form, Onofri's inequality (see [12] for the equivalent version on the sphere)

$$\log \left( \int_{\mathbb{R}^2} e^\phi d\mu \right) - \int_{\mathbb{R}^2} \phi d\mu \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx \quad (2)$$

plays in dimension  $d = 2$  the role of Sobolev's inequality in higher dimensions. The inequality holds for any smooth function with compact support and, by density, for any function  $\phi$  in the space obtained by completion with respect to the norm given by:  $\|\phi\|^2 = \int_{\mathbb{R}^2} |\nabla \phi|^2 dx + (\int_{\mathbb{R}^2} \phi d\mu)^2$ . Onofri's inequality can be seen as the *dual* inequality of the logarithmic Hardy-Littlewood-Sobolev, cf [7,1,5,9].

The rescaled parabolic-elliptic Keller-Segel system reads

$$\frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n x) - \nabla \cdot (n \nabla c), \quad c = (-\Delta)^{-1} n, \quad x \in \mathbb{R}^2, \quad t > 0 \quad (3)$$

Assume that the initial datum is  $n(0, \cdot) = n_0$ . If  $M = \int_{\mathbb{R}^2} n_0 dx > 8\pi$ , solutions blow-up in finite time. If  $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2) dx)$ ,  $n_0 |\log n_0| \in L^1(\mathbb{R}^2)$  and  $M < 8\pi$ , solutions globally exists and it has been shown in [3, Theorem 1.2] that

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_M\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_M\|_{L^2(\mathbb{R}^2)} = 0,$$

where  $(n_M, c_M)$  is the unique, smooth and radially symmetric solution of

$$-\Delta c_M = M \frac{e^{-\frac{1}{2}|x|^2 + c_M}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2 + c} dx} =: n_M, \quad x \in \mathbb{R}^2. \quad (4)$$

Notice that  $n_M = M e^{c_M - |x|^2/2} / \int_{\mathbb{R}^2} e^{c_M - |x|^2/2} dx$  with  $c_M = (-\Delta)^{-1} n_M$ . The case  $M = 8\pi$  has also been extensively studied, but is out of the scope of this note.

Ineq. (2) and the Moser-Trudinger inequality have been repeatedly used to study the Keller-Segel system in bounded domains. In the whole space case, Ineq. (1) turns out to be very convenient, at least for existence issues. Ineq. (2) and Ineq. (1) correspond to the  $M = 8\pi$  case. For  $M < 8\pi$ , we will establish a new inequality of Onofri type, which is our first main result: see Theorem 2.1.

An important issue in the study of (3) is to characterize the rate of convergence of  $n$  towards  $n_M$ . See [2,4]. For this purpose, it is convenient to linearize the Keller-Segel system (3) by considering

$$n(t, x) = n_M(x) (1 + \varepsilon f(t, x)) \quad \text{and} \quad c(t, x) = c_M(x) (1 + \varepsilon g(t, x))$$

and formally take the limit as  $\varepsilon \rightarrow 0$ . At order  $O(\varepsilon)$ ,  $(f, g)$  solves

$$\frac{\partial f}{\partial t} = \frac{1}{n_M} \nabla \cdot [n_M \nabla (f - g c_M)] =: \mathcal{L} f \quad \text{and} \quad g c_M = (-\Delta)^{-1} (f n_M). \quad (5)$$

As we shall see in Section 3, several spectral gap inequalities (see Theorems 3.1 and 3.2) are related with (1) and involve the linear operator  $\mathcal{L}$ . Detailed proofs and applications to the full Keller-Segel system (3) will be given in a forthcoming paper, [6], whose main result is that  $\|n(t, \cdot) - n_M\|_{L^1(\mathbb{R}^2)} = O(e^{-t})$  as  $t \rightarrow \infty$ .

## 2. Duality and stationary solutions of the Keller-Segel model in self-similar variables

For any  $M \in (0, 8\pi)$ , the function  $c_M$  given by (4) can be characterized either as a minimizer of

$$G[c] := \frac{1}{2} \int_{\mathbb{R}^2} n c \, dx - M \log \left( \int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2+c} \, dx \right)$$

where  $n$  and  $c$  are related through the Poisson equation,  $-\Delta c = n$ , or in terms of  $n$ , seen as a minimizer of the functional  $n \mapsto F[n]$ . Inspired by [1,5,7,9], we can characterize the corresponding functional inequalities and observe that they are *dual* of each other. Let us give some details.

Consider the *free energy* functional  $n \mapsto F[n] = F_1[n] - F_2[n]$  (for an appropriate choice of the constant  $K$ ) on the set  $\mathcal{X}_M$  of all nonnegative integrable functions with mass  $M > 0$ , where

$$F_1[n] = \int_{\mathbb{R}^2} n \log \left( \frac{n}{n_M} \right) \, dx \quad \text{and} \quad F_2[n] = \frac{1}{2} \int_{\mathbb{R}^2} (n - n_M) (-\Delta)^{-1} (n - n_M) \, dx.$$

The free energy  $F$  is bounded from below by (1). Since  $n_M$  is a minimizer for  $F$  and  $F[n_M] = 0$ , we actually have the functional inequality  $F_1[n] \geq F_2[n]$  for any  $n \in \mathcal{X}_M$ . This inequality can be rewritten as

$$\int_{\mathbb{R}^2} n \log \left( \frac{n}{n_M} \right) \, dx + \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (n(x) - n_M(x)) \log |x - y| (n(y) - n_M(y)) \, dx \, dy \geq 0$$

for any  $n \in \mathcal{X}_M$  with  $M < 8\pi$ .

By Legendre's duality, we have:  $F_1^*[\phi] \leq F_2^*[\phi]$  where  $F_i^*[\phi] := \sup_{n \in \mathcal{X}_M} \left( \int_{\mathbb{R}^2} \phi n \, dx - F_i[n] \right)$ ,  $i = 1, 2$ , is defined on  $L^\infty(\mathbb{R}^2)$ . A straightforward computation shows that  $F_1^*[\phi] = \int_{\mathbb{R}^2} \phi n \, dx - F_1[n]$  if and only if  $\log\left(\frac{n}{n_M}\right) = \phi - \log\left(\int_{\mathbb{R}^2} e^\phi \, d\mu_M\right) + \log M$ , so that

$$F_1^*[\phi] = M \log \left( \int_{\mathbb{R}^2} e^\phi \, d\mu_M \right) - M \log M.$$

Here  $d\mu_M$  is the probability measure

$$d\mu_M := \mu_M \, dx, \quad \text{with} \quad \mu_M := \frac{1}{M} n_M.$$

It is clear that we can impose at no cost that  $\int_{\mathbb{R}^2} \phi \, d\mu_M = 0$ . It is also standard to observe that  $F_2^*[\phi] = \int_{\mathbb{R}^2} \phi n \, dx - F_2[n]$  if and only if  $\phi = (-\Delta)^{-1} (n - n_M)$ , so that

$$F_2^*[\phi] = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx.$$

Notice that  $\int_{\mathbb{R}^2} |\nabla \phi|^2 dx$  is well defined as  $-\Delta \phi = n - n_M$  is integrable and such that  $\int_{\mathbb{R}^2} (n - n_M) dx = 0$ . With  $c_M = (-\Delta)^{-1} n_M$  and  $\phi = c - c_M$ , we recover that  $G[\phi + c_M]$  is equal to  $F_2^*[\phi] - F_1^*[\phi]$  up to a constant. Replacing  $\phi$  by  $\phi - \int_{\mathbb{R}^2} \phi d\mu_M$ , we arrive at the following result in the space  $\mathcal{H}_M$  obtained by completion with respect to the norm given by:  $\|\phi\|^2 = \int_{\mathbb{R}^2} |\nabla \phi|^2 dx + (\int_{\mathbb{R}^2} \phi d\mu_M)^2$ .

**Theorem 2.1** *For any  $M \in (0, 8\pi)$ , with  $n_M$  defined as the unique minimizer of  $F$ , i.e. the unique solution  $n_M$  given by (4), and  $n_M dx = d\mu_M$ , with  $c_M = (-\Delta)^{-1} n_M$ , we have the following inequality:*

$$\log \left( \int_{\mathbb{R}^2} e^\phi d\mu_M \right) - \int_{\mathbb{R}^2} \phi d\mu_M \leq \frac{1}{2M} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx \quad \forall \phi \in \mathcal{H}_M. \quad (6)$$

As a consequence, if we consider the special case  $\phi = 1 + \varepsilon \psi$  and consider the limit  $\varepsilon \rightarrow 0$  in (6), as in [8], we get an interesting spectral gap inequality.

**Corollary 2.2** *With the above notations, for any  $\psi \in \mathcal{H}_M$ , the following inequality holds*

$$\int_{\mathbb{R}^2} |\psi - \bar{\psi}|^2 n_M dx \leq \int_{\mathbb{R}^2} |\nabla \psi|^2 dx \quad \text{where} \quad \bar{\psi} = \int_{\mathbb{R}^2} \psi d\mu_M.$$

### 3. Linearized Keller-Segel model, spectral gap inequalities and consequences

Exactly as for Ineq. (6), we observe that

$$\mathbf{Q}_1[f] := \int_{\mathbb{R}^2} |f|^2 d\mu_M + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) \log |x - y| f(y) d\mu_M(x) d\mu_M(y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} F[n_M(1 + \varepsilon f)] \geq 0$$

Notice that  $\mathbf{Q}_1[f] = \int_{\mathbb{R}^2} |f|^2 n_M dx - \int_{\mathbb{R}^2} |\nabla(g c_M)|^2 dx$  if  $\int_{\mathbb{R}^2} f d\mu_M = 0$ . We also notice that  $f_{0,0} := \partial_M \log n_M$  generates the kernel  $\text{Ker}(\mathcal{L})$  considered as an operator on  $L^2(\mathbb{R}^2, d\mu_M)$  and the functions  $f_{1,i} := \partial_{x_i} \log n_M$  with  $i = 1, 2$  and  $f_{0,1} := x \cdot \nabla \log n_M$  are eigenfunctions of  $\mathcal{L}$  with eigenvalues 1 and 2 respectively; moreover they generate the corresponding eigenspaces (see [6] for details). It is remarkable that  $\mathbf{Q}_1[f] = 0$  if and only if  $f \in \text{Ker}(\mathcal{L})$  and this allows to establish a first spectral gap inequality.

**Theorem 3.1** *There exists  $\kappa > 1$  such that*

$$\int_{\mathbb{R}^2} f^2 d\mu_M \leq \kappa \mathbf{Q}_1[f] \quad \forall f \in L^2(\mathbb{R}^2, f\mu_M) \quad \text{such that} \quad \int_{\mathbb{R}^2} f f_{0,0} d\mu_M = 0.$$

The proof of Theorem 3.1 relies on spectral properties of Schrödinger operators. See [6] for details. Since  $\mathbf{Q}_1[f] = 0$  if and only if  $f \in \text{Ker}(\mathcal{L})$ , that is if  $f$  is proportional to  $f_{0,0}$ , we can define the scalar product  $\langle \cdot, \cdot \rangle$  induced by the quadratic form  $\mathbf{Q}_1$  on the space  $\mathcal{D}_M$  orthogonal of  $f_{0,0}$  in  $L^2(\mathbb{R}^2, d\mu_M)$ . With this definition, we have  $\mathbf{Q}_1[f] = \langle f, f \rangle$ . On the space  $\mathcal{D}_M$  with scalar product  $\langle \cdot, \cdot \rangle$ , the operator  $\mathcal{L}$  is self-adjoint. Let

$$\mathbf{Q}_2[f] := \langle f, \mathcal{L} f \rangle.$$

Then we have a second spectral gap inequality.

**Theorem 3.2** *For any function  $f \in \mathcal{D}_M$ , we have*

$$\mathbf{Q}_1[f] \leq \mathbf{Q}_2[f].$$

Moreover, if  $f$  is a radial function, then we have  $2\mathbf{Q}_1[f] \leq \mathbf{Q}_2[f]$ . The operator  $\mathcal{L}$  has only discrete spectrum as a consequence of Persson's lemma, or as can be shown by direct investigation using the tools of the concentration-compactness method and the Sturm-Liouville theory. By rewriting the spectral problem for  $\mathcal{L}$  in terms of *cumulated densities*, it is possible to prove that the eigenspace corresponding

to the lowest non-zero eigenvalue is generated by  $f_{1,i}$  with  $i = 1, 2$ , which completes the proof. See [6] for details.

As a simple consequence, if  $f$  is a solution to (5), then

$$\frac{d}{dt} \langle f, f \rangle = - \langle f, \mathcal{L} f \rangle \leq - 2 \langle f, f \rangle ,$$

which shows the exponential convergence of  $f$  towards 0. The nonlinear Keller-Segel model (3) can be rewritten in terms of  $f := (n - n_M)/n_M$  and  $g := (c - c_M)/c_M$  as

$$\frac{\partial f}{\partial t} - \mathcal{L} f = - \frac{1}{n_M} \nabla \cdot [f n_M (\nabla (g c_M))] .$$

Estimates based on Duhamel's formula allow to prove that  $t \mapsto \mathbf{Q}_1[f(t, \cdot)]$  is bounded uniformly with respect to  $t > 0$  and

$$\frac{d}{dt} \mathbf{Q}_1[f(t, \cdot)] \leq - \mathbf{Q}_1[f(t, \cdot)] \left[ 2 - \delta(t, \varepsilon) \left( \mathbf{Q}_1[f(t, \cdot)]^{\frac{1-\varepsilon}{2-\varepsilon}} + \mathbf{Q}_1[f(t, \cdot)]^{\frac{1}{2+\varepsilon}} \right) \right] .$$

for any  $\varepsilon > 0$  small enough, for some continuous  $\delta$  such that  $\lim_{t \rightarrow \infty} \delta(t, \varepsilon) = 0$ . This proves that  $\lim_{t \rightarrow \infty} e^{2t} \mathbf{Q}_1[f(t, \cdot)]$  is finite. Details will be given in [6].

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