SPECIAL MODES AND HYPOCOERCIVITY FOR LINEAR KINETIC EQUATIONS WITH SEVERAL CONSERVATION LAWS AND A CONFINING POTENTIAL

KLEBER CARRAPATOSO, JEAN DOLBEAULT, FRÉDÉRIC HÉRAU, STÉPHANE MISCHLER, CLÉMENT MOUHOT, AND CHRISTIAN SCHMEISER

Abstract. We study linear inhomogeneous kinetic equations with an external confining potential and a collision operator with several local conservation laws (local density, momentum and energy). We exhibit all equilibria and entropy-maximizing special modes, and we prove asymptotic exponential convergence of solutions to them with quantitative rate. This is the first complete picture of hypocoercivity and quantitative H-theorem for inhomogeneous kinetic equations in this setting.

Contents

1. Introduction

1.1. **Equation and assumptions.** Consider the kinetic collisional evolution equation

$$\partial_t f = \mathscr{L} f := \mathscr{T} f + \mathscr{C} f, \quad f_{|t=0} = f_0 \,, \tag{1.1}$$

on the unknown $f = f(t, x, v)$ depending on the time variable $t \geq 0$, the spatial position variable $x \in \mathbb{R}^d$, and the velocity variable $v \in \mathbb{R}^d$. Here the *transport operator* $\mathscr{T}$ is given by

$$\mathscr{T} f := -v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f$$

with a *potential* $\phi : \mathbb{R}^d \to \mathbb{R}$. We assume that the *linear collision operator* $\mathscr{C}$ is acting only along the velocity variable $v \in \mathbb{R}^d$, is self-adjoint in $L^2(\mu^{-1})$, with

$$\mu(v) := \frac{e^{-|v|^2/2}}{(2\,\pi)^{d/2}}$$

and has the $(d+2)$-dimensional kernel of *collision invariants* given by

$$\text{Ker}\, \mathscr{C} = \text{Span}\left\{\mu, v_1\,\mu, \ldots, v_d\,\mu, |v|^2\,\mu\right\}. \tag{H0}$$

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We assume that $\mathscr{C}$ satisfies the following *spectral gap property* in velocity (this is a quantitative version of the *spatially homogeneous linearized H-theorem*)

$$-\int_{\mathbb{R}^d} (\mathscr{C} f(v))\, f(v)\, \mu(v)^{-1}\, \mathrm{d}v \geq \mathrm{c}_{\mathscr{C}}\, \|f - \Pi f\|^2_{L^2(\mu^{-1})} \tag{H1}$$

for some constant $\mathrm{c}_{\mathscr{C}} > 0$ and where Π denotes the $L^2(\mu^{-1})$-orthogonal projection onto $\mathrm{Ker}\mathscr{C}$. We suppose moreover that any polynomial function $p(v) : \mathbb{R}^d \to \mathbb{R}$ of degree less or equal than 4 is in the domain of $\mathscr{C}$, with a constant $C_p > 0$ such that for any such polynomial

$$\left| \int_{\mathbb{R}^d} (\mathscr{C} f(v))\, p(v)\, \mu(v)^{-1}\, \mathrm{d}v \right| \leq C(p)\, \|f - \Pi f\|^2_{L^2(\mu^{-1})}\,. \tag{H2}$$

We provide examples of collision operators satisfying these conditions in Appendix (Subsection B.1).

Concerning the potential $\phi : \mathbb{R}^d \to \mathbb{R}$, we shall assume throughout the paper that $\rho(x) := e^{-\phi(x)}$ is a centered density of probability

$$\int_{\mathbb{R}^d} \rho(x)\, \mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} x\rho(x)\, \mathrm{d}x = 0 \tag{H3}$$

where ϕ is of class $C^2(\mathbb{R}^d;\mathbb{R})$, and for all $\varepsilon > 0$, there exist a constant C_ε such that

$$\forall\, x \in \mathbb{R}^d, \quad |\nabla^2\phi(x)| \leq \varepsilon\, |\nabla\phi(x)|^2 + C_\varepsilon. \tag{H4}$$

We assume that the measure $\rho(x)\,\mathrm{d}x$ satisfies the Poincaré inequality with constant $c_P > 0$:

$$c_P \int_{\mathbb{R}^d} |\varphi - \langle\varphi\rangle|^2 \rho(x)\, \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 \rho(x)\, \mathrm{d}x, \tag{H5}$$

for all suitable $\varphi \in L^2(\rho)$ and where $\langle\varphi\rangle := \displaystyle\int \varphi(x)\rho(x)\,\mathrm{d}x$ is the mean of φ.

We assume the following moment bounds on ρ:

$$\int_{\mathbb{R}^d} \left(|x|^4 + |\phi|^2 + |\nabla_x\phi|^4 \right) \rho(x)\, \mathrm{d}x \leq \mathrm{C}_\phi \tag{H6}$$

for some constant $\mathrm{C}_\phi > 0$. Note that potentials $\phi(x) := (1 + |x|^2)^{\gamma/2} - Z_\phi$, with $\gamma > 1$ and a normalization constant Z_ϕ, satisfies (H3)–(H4)–(H5)–(H6) (see Subsection B.2 for more examples).

We assume the normalization

$$\langle \nabla_x^2\phi \rangle = \int_{\mathbb{R}^d} \nabla_x^2\phi(x)\, \rho(x)\, \mathrm{d}x = \mathrm{Id}_{d\times d} \tag{H7}$$

where $\nabla_x^2\phi$ denotes the Hessian matrix of ϕ and $\mathrm{Id}_{d\times d}$ the identity matrix of size d. Note that by a change of orthonormal basis, it is always possible to suppose that $\langle\nabla_x^2\phi\rangle$ is diagonal. For simplicity, we assume here that $\langle\nabla_x^2\phi\rangle$ is the identity matrix and postpone the general case to Subsection B.3.

No sign is assumed on f: one should think of f as a real fluctuation around the equilibrium in the nonlinear Boltzmann or Landau equation (see Subsection B.1). Throughout this article we shall refer to (H1) and (H5) as *spectral gap properties*, and to (H2) and (H6) as *bounded moment properties*. They are the structural assumptions on $\mathscr{C}$ and ϕ for our main result.

Let us denote by $\mathcal{M}$ the *Maxwellian* stationary solution to (1.1) defined by

$$\forall\, (x,v) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \mathcal{M}(x,v) := \rho(x)\, \mu(v) = \frac{e^{-\frac{1}{2}|v|^2 - \phi(x)}}{(2\,\pi)^{d/2}}.$$

Finally, since we are concerned with asymptotic behavior, we assume the evolution equation (1.1) to be well-posed (which is always satisfied for standard kinetic equations):

(H8) $\quad t \mapsto e^{(\mathscr{T}+\mathscr{C})\,t}$ is a strongly continuous semi-group in the Hilbert space $L^2(\mathcal{M}^{-1})$.

1.2. **The main theorem.** To state the main theorem, we have to introduce the equilibria and special modes $F(t,x,v)$ which satisfy (1.1) and are thermalized at every x:

$$\mathscr{C}F = 0, \qquad \partial_t F = \mathscr{T} F. \tag{1.2}$$

We introduce the classical Hamiltonian (microscopic energy)

$$\mathcal{H}(x,v) := \left(\frac{|v|^2 - d}{2} + \phi(x) - \langle \phi \rangle \right),$$

and note that the energy mode $\mathcal{H}(x,v)\mathcal{M}(x,v)$ is also an obvious equilibrium since $\mathscr{T}\mathcal{H} = 0$.

We introduce then the set of *infinitesimal rotations compatible with* ϕ:

$$\mathcal{R}_\phi := \left\{ x \mapsto Ax \mid A \in \mathfrak{M}_d^{\text{skew}}(\mathbb{R}) \text{ s.t. } \forall\, x \in \mathbb{R}^d, \quad \nabla_x \phi(x) \cdot Ax = 0 \right\}, \tag{1.3}$$

which we identify with a subset of skew-symmetric matrices $\mathfrak{M}_d^{\text{skew}}(\mathbb{R}^d) := \{A \in \mathfrak{M}_d; {}^T A = -A\}$, and which gives rise to the corresponding set of infinitesimal rotation modes compatible with ϕ

$$\mathfrak{R}_\phi = \{ (x,v) \longmapsto (Ax \cdot v)\, \mathcal{M}(x,v), \quad A \in \mathcal{R}_\phi \}.$$

Functions in $\mathfrak{R}_\phi$ are equilibria of (1.1) associated to the possible rotational invariances of ϕ.

Some additional stationary solutions exist when ϕ has *harmonic directions.* To present them, we consider three different cases depending on the potential ϕ. Let us define

$$E_\phi := \text{Span}\left\{ \nabla_x \phi(x) - x, \quad x \in \mathbb{R}^d \right\}, \quad d_\phi := \dim E_\phi. \tag{1.4}$$

The case $d_\phi = d$ is denoted *fully non harmonic*, since ϕ has no harmonic direction. The case $1 \le d_\phi \le d-1$ is denoted *partially harmonic*, and corresponds to the situation where ϕ is not the harmonic potential but there are $d - d_\phi$ directions in which ϕ is harmonic, more precisely $\partial_{x_i}\phi = x_i$ in those directions. In such case we work with a coordinate system where $\{d_\phi + 1, \dots, d\}$ denotes the harmonic coordinates. Associated to this set of harmonic directions, we call *harmonic directional modes* functions in the following set

$$\mathfrak{D}_\phi := \text{Span}\Big\{ (x_i \cos t - v_i \sin t),\ (x_i \sin t + v_i \cos t),\ i \in \{d_\phi + 1, \dots, d\} \Big\} \mathcal{M}.$$

We also define $\mathfrak{D}_\phi := \{0\}$ when $d_\phi = d$. These modes correspond to an inertia-driven oscillation of the particles in a potential well along a direction in $E_\phi^\perp$ with period 1. These harmonic oscillations arise when in a certain direction, the spatial equilibrium de-couples from the other directions and is Gaussian in such a way that it "resonates" with the Gaussian behavior of the velocity equilibrium. These modes can be superposed independently along different harmonic coordinates, and independently of the previous energy mode and infinitesimal rotation modes. Finally, the case $d_\phi = 0$ is denoted *fully harmonic*. Due to the normalization we have then $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$. We call *harmonic pulsating modes* functions in the following set

$$\mathfrak{P}_\phi = \text{Span}\left\{ \left(\frac{|x|^2 - |v|^2}{2} \cos(2t) - x \cdot v \sin(2t) \right), \left(\frac{|x|^2 - |v|^2}{2} \sin(2t) + x \cdot v \cos(2t) \right) \right\} \mathcal{M},$$

and we denote $\mathfrak{P}_\phi = \{0\}$ when $d_\phi \neq 0$. These modes correspond to a radially symmetric pulsation of the particles in the potential well with period 2 when the full Gaussian behavior of the spatial equilibrium "resonates" with that of the velocity equilibrium.

We can now state the main theorem of this paper.

Theorem 1.1 (Entropy maximizers and hypocoercivity)**.** *Assume that the potential ϕ and the collision kernel $\mathscr{C}$ satisfy the assumptions* (H0)–(H1)–(H2)–(H3)–(H4)–(H5)–(H6)–(H7)–(H8). *Then (modulo the multiplication factors) the Maxwellian, energy mode, infinitesimal rotation modes compatible with ϕ, harmonic directional or pulsating modes (when ϕ has the appropriate harmonicity) listed above are the only solutions to* (1.1) *that maximize the entropy. In addition, there is a constant $C, \kappa > 0$ (with quantitative estimate from the proof) such that given $f_0 \in L^2(\mathcal{M}^{-1})$ and the solution f to* (1.1) *in*

$\mathcal{C}(\mathbb{R}^+, L^2(\mathcal{M}^{-1}))$ with initial data f_0, there are unique α, $\beta \in \mathbb{R}$, $x \mapsto Ax \in \mathcal{R}_\phi$, $F_{\rm dir} \in \mathfrak{D}_\phi$ and $F_{\rm pul} \in \mathfrak{P}_\phi$ such that

$$\|f(t) - (\alpha\mathcal{M} + \beta\mathcal{H}\mathcal{M} + Ax\cdot v\mathcal{M} + F_{\rm dir}(t) + F_{\rm pul}(t))\|_{L^2(\mathcal{M}^{-1})} \leq Ce^{-\kappa t}\,\|f_0\|_{L^2(\mathcal{M}^{-1})}\,.$$

Here the constants κ, C only depend on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant defined in (1.5).

1.3. **Comments and context.** In the two last decades there was a great interest for so-called hypocoercive methods in the study of inhomogeneous kinetic equations. Different linear or non-linear models were tackled, including Fokker-Planck-type models, Boltzmann or Landau type models in various geometries, from bounded domains to the whole space with or without potentials.

Hypocoercivity essentially refers to the study of the quantitative trend to the equilibrium for these equations, in the spirit of the celebrated H-Theorem by Boltzmann about the decay of entropy, but with explicit constants. Regarding the initial developments we refer to the memoir [29] by Villani. Among the first articles devoted to this field, mention the article [6] devoted to polynomial trend to the equilibrium the Boltzmann equation in a box with a priori bounded moments. A series of works by Guo [14, 13, 15] for the linearized Boltzmann equation with no external potential using micro-macro methods inspired from Grad's 13 moments method [11]. On the other hand the study of linear or non-linear perturbative inhomogeneous kinetic equations with single conservation laws –such as linear Boltzmann of Fokker-Planck models– has benefited from ideas coming from the theory of hypoellipticity [21] and gave rise to robust Hilbertian hypocoercive methods in [19, 25, 18, 29] or [7] in exponential weighted spaces. As a last breakthrough, we also mention the theory of enlargement of spaces [12] that extend hypocoercivity results to larger physical polynomially weighted spaces.

The justification of a need of explicit and constructive estimates –typically not obtained via a compactness argument– is multiple (see for example [28]). It comes first from the need of range of validity of a perturbation study for the non linear problem. It is mainly associated to the study of the so-called diffusive or hydrodynamical limits, when small parameters (linked to the Knudsen number) are in front of the collision operator and or in front of the transport part. In order to have a good control of the limiting process leading to hydrodynamical equations, in pertinent time/parameter ranges, the need of constructive estimates is crucial and we refer for example to [2] and more recently among a huge literature [20] or [3] in polynomially weighted spaces.

In this article, we focus on a important old problem in the field, namely when there is an external confining potential ϕ, together with several local conservation laws in the collision process. We restrict our analysis to the linear framework, in $\mathbb{R}^d$ and in exponentially weighted spaces but with rather general confining potentials and linear or linearized collision operators with a spectral gap. The aim is to state and prove hypocoercivity in this case, taking profit from the natural Hilbertian functional structure. Our study is motivated by the futur study of the nonlinear Boltzmann and Landau equations with confining potentials. The linear problem was solved quantitatively in the case of a single local conservation law for the Fokker-Planck equation (see [19]) and for linear scattering operators [7, 10, 8], but no quantitative results was so far available in the case of several local conservation laws case without strong symmetry assumptions. A non-constructive argument was obtained in [9].

When the domain is bounded and $\phi = 0$ in the series of work by Guo, under convexity asumptions in the domain. The quantitative case with ϕ and when there is no a priori symmetry or no quadratic hypothesis was open and we propose here a complete result. Difficulties already arising for example in [6] in the study of the bounded domain case are present in this case. As we saw in the preceding section, special modes naturally appear in the analysis, due to the partial symmetries of the potential. These special modes which were not known to our knowledge and some of them (or their equivalent in our geometry) do not exist neither in the case with boundary in [6] nor in the one-dimension collision kernel case e.g.in [19, 17]. We mention that they also provide solutions to the full inhomogeneous non-linear Boltzmann case (see Subsection B.5).

To estimate the decay rate, a key difficulty is to quantify "how far" the potential ϕ is from having certain partial symmetries. To solve this, and inspired by [5, 6], we shall use a variant of the so-called

Korn inequalities [23, 24] that is proven in our companion paper [4]. Such inequality is used to control the full differential of the local momentum by its symmetric part. It relies Schwartz Lemma, low-order Poincar-type inequalities, and the following rigidity constant

$$0 < c_{\mathrm{K}} = \min\left\{ \int |\nabla\phi \cdot Ax|^2 \rho(x)\,\mathrm{d}x \mid x \mapsto Ax \in \mathcal{R}_\phi^\perp \text{ s.t. } \int |Ax|^2 \rho(x)\,\mathrm{d}x = 1 \right\}, \tag{1.5}$$

where $\mathcal{R}_\phi^\perp$ is the orthogonal complement in $L^2(\rho)$ of the set $\mathcal{R}_\phi$ of infinitesimal rotations compatible with ϕ, i.e. when ϕ is invariant by the rotation group $t \mapsto e^{\theta A}$ (see the next section):

$$(x \mapsto Ax) \in \mathcal{R}_\phi \qquad \Longleftrightarrow \qquad \forall\, x \in \mathbb{R}^d,\ \theta \in \mathbb{R}, \quad \phi(e^{\theta A}x) = \phi(x).$$

Regarding the other special modes in $\mathfrak{D}_\phi$ and $\mathfrak{P}_\phi$, they are related to the (possible partial) harmonicity of ϕ and another difficulty is to quantify "how far" the potential ϕ is from being (partially) harmonic. Such analysis relies on the finite dimensional space E_ϕ defined in (1.4) and various explicit associated quantities.

Regarding the spectral gap assumptions (H1) in v and (H5) in x, they reflect the corresponding confining properties in respectively velocity and space. Note the link between Poincaré inequality and the natural Hodge Laplacian associated to the geometry, sometimes called the *Witten Laplacian*. Denote ∇_x^* the adjoint of ∇_x in $L^2(\rho)$, defined on vector fields φ by $\nabla_x^*\varphi = (-\nabla_x + \nabla_x\phi)\cdot\varphi$. Then the Witten Laplace operator $\nabla_x^* \cdot \nabla_x$ is self-adjoint in $L^2(\rho)$ (see Section 4.2) and we denote

$$\Omega := \nabla_x^* \cdot \nabla_x + 1 = (-\nabla\phi(x) + \nabla_x) \cdot \nabla_x + 1. \tag{1.6}$$

Note that the kernel of $\nabla_x^* \cdot \nabla_x$ is made of constant functions and the Poincaré constant c_P is the first non zero eigenvalue of $\nabla_x^* \cdot \nabla_x$. We use the operator Ω to contain loss of derivative and keep inequalities in $L^2(\rho)$, see for instance using for example the following "0-order" Poincaré inequality (see the Poincaré-Lions inequality (4.10) later): there is $c_{P,1} > 0$ depends on c_P and (H4) so that

$$c_{P,1}\,\|\varphi - \langle\varphi\rangle\|^2 \le \left\|\Omega^{-\frac{1}{2}}\nabla_x\varphi\right\|^2.$$

We propose two proofs of the main theorem. The first one follows a micro-macro method, close to the ones in the works by Guo and the approach in [10], with an analysis here of high complexity due to the interaction between the local conservation laws and the potential and the lack of a priori symmetry assumptions. Note that this method has conceptual similarities with that used for hyperbolic equations with damping in [16, 26, 27] following the seminal paper of Kawashima and Shizuta [22]. The second method is given under slightly more restrictive hypotheses –namely that the collision operator $\mathcal{C}$ is bounded and ϕ has bounded derivatives of order two and more– and is based on commutator estimates in the spirit of the seminal studies [19, 25] or [29], where ideas coming from the study of hypoellipticity [21] were crucial. In practice, an elegant triple cascade of commutators based on the equality $[\nabla_v, v\cdot\nabla_x] = \nabla_x$ is needed to control all microscopic quantities.

The plan of the article is the following. In the next Section 2, we review all possible (cycles of) conservation laws and their relations with the special modes. Then we present the so-called macroscopic equations associated to our main equation (1.1) and state the result in an adapted Hilbertian framework via a conjugation with the Maxwellian. In Section 3, we prove the first part of the main Theorem 1.1 concerning relative entropy minimizers and exhibit the corresponding equilibria or stationary modes. Note that already at this stage, partial entropy-dissipation arguments are used. In Section 4, we prove the hypocoercive part of Theorem 1.1 using the micro-macro method. In Section 5 we give the alternative proof using the commutator's method under the more restrictive assumptions recalled above. Next in Section A of the Appendix, we give some intermediate lemmas and computations needed in the proofs. To conclude, we give in Section B some examples and remarks in relation with the main result of this paper, including a spectral interpretation of the result, the exhibition of special solutions to the full Boltzmann equation, a remark on the normalization and examples of collision operators and potentials.

2. Conservation laws and macroscopic equations

In this section we identify the conservation laws and macroscopic equations associated to (1.1).

2.1. **Conservation laws and identification of special modes.**

The conservation of mass writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0 \tag{2.1}$$

We denote $\alpha := \int f_0\,\mathrm{d}x\,\mathrm{d}v$ and note that $\alpha\mathcal{M}$ is a solution with same (conserved) mass than f.

The conservation of energy writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} \mathcal{H}(x,v) f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0.$$

Then we define

$$\beta := \frac{1}{\mathrm{Var}(\mathcal{H})}\int_{\mathbb{R}^{2d}} f_0\mathcal{H}\,\mathrm{d}x\,\mathrm{d}v \quad \text{ with } \quad \mathrm{Var}(\mathcal{H}) := \int_{\mathbb{R}^{2d}} \mathcal{H}^2\mathcal{M}\,\mathrm{d}x\,\mathrm{d}v. \tag{2.2}$$

and we note that $\beta\mathcal{H}\mathcal{M}$ is a solution with same (conserved) energy than f.

When there is a rotation invariance for ϕ infinitesimal rotational modes appear: consider a rotation group $(R_\theta)_{\theta\in\mathbb{R}}$ defined by $R_\theta := e^{\theta A}$ for a skew-symmetric matrix $A \in \mathfrak{M}^{\mathrm{skew}}_{d\times d}$ and $x_0 \in \mathbb{R}^d$ so that $\phi(R_\theta(x - x_0)) + x_0) = \phi(x)$, then by differentiation in θ there is a vector $u \in \mathbb{R}^d$ such that

$$\forall\, x \in \mathbb{R}^d, \quad (Ax + u)\cdot\nabla_x\phi(x) = 0.$$

Now integrating this equality against $(u\cdot x)\rho$ yields $u = 0$ by integration by part using that $\rho(x)\,\mathrm{d}x$ is centered. A law of conservation of total *angular momentum* along this rotation then appears

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2d}} (Ax\cdot v)\, f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0.$$

The set of such centered infinitesimal rotations $x \mapsto Ax$ is exactly $\mathcal{R}_\phi$ defined in (1.3). Associated with f_0, we introduce the initial momentum $m_0(x) := (\int v f_0(x,v)\,\mathrm{d}v)e^{\phi(x)}$. We introduce then the infinitesimal rotation $x \mapsto A_0x := \mathbb{P}_\phi(m_0)(x)$ where $\mathbb{P}_\phi$ is the orthogonal projection onto the set $\mathcal{R}_\phi$ in $L^2(\rho)$. We can then check that the function ("infinitesimal rotational mode")

$$F_{\mathrm{rig}} : (x,v) \mapsto (A_0x\cdot v)\,\mathcal{M} \tag{2.3}$$

is in $\mathfrak{R}_\phi$ and solution to (1.1), with same as conserved total angular momentum as f. Denoting

$$m_f(t,x) := \left(\int_{\mathbb{R}^d} v f(t,x,v)\,\mathrm{d}v\right) e^{\phi(x)}$$

the momentum of f, the associated conservation law reads then

$$\mathbb{P}_\phi(m_f) = \mathbb{P}_\phi(m_0) \quad \text{ or equivalently } \quad \mathbb{P}(m_f - m_0) \in \mathcal{R}_\phi^\perp \tag{2.4}$$

where $\mathbb{P}$ is the projection onto all infinitesimal rotations, and $\mathcal{R}_\phi^\perp$ is the orthogonal of $\mathcal{R}_\phi$ in $L^2(\rho)$. We refer to Subsection A.1 for a short verification of this fact.

Now we deal with harmonic directional modes, which are associated to *cycles of "conservation" laws of global quantities* and appear when $d_\phi \le d-1$. In the basis we choose at the end of Subsection 1.1 and from (H7), we have for all $i \in \{d_\phi+1,\dots,d\}$ that $\partial_{x_i}\phi(x) = x_i$. In that case the confinement is *harmonic* in these direction x_i, and there is an additional 2-*cycle (almost) conservation law*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^{2d}} x_i f\,\mathrm{d}x\,\mathrm{d}v\right) = \left(\int_{\mathbb{R}^{2d}} v_i f\,\mathrm{d}x\,\mathrm{d}v\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^{2d}} v_i f\,\mathrm{d}x\,\mathrm{d}v\right) = -\left(\int_{\mathbb{R}^{2d}} x_i f\,\mathrm{d}x\,\mathrm{d}v\right)$$

which implies that these two global quantities evolve as a scalar harmonic oscillator with period equal to 1. In this case, some harmonic directional modes may appear. Precisely let us define for $i \in \{d_\phi+1,\dots,d\}$

$$\gamma_i := \int_{\mathbb{R}^{2d}} x_i f_0\,\mathrm{d}x\,\mathrm{d}v, \qquad \bar\gamma_i := \int_{\mathbb{R}^{2d}} v_i f_0\,\mathrm{d}x\,\mathrm{d}v$$

where we recall that here $x_i = \partial_i \phi$. We introduce then the functions

$$F_{\mathrm{dir},i} := \gamma_i \big[x \cos(t) + v \sin(t) \big] + \bar{\gamma}_i \big[v_i \cos(t) - x_i \sin(t) \big] \quad \text{and} \quad F_{\mathrm{dir}} := \sum_{i=d_\phi+1}^{d} F_{\mathrm{dir},i}. \tag{2.5}$$

This function is a solution to (1.1) and $f - F_{\mathrm{dir}}$ satisfies the additional conservation law

$$\forall t \geq 0, \quad \int_{\mathbb{R}^{2d}} x_i \big[f(t,x,v) - F_{\mathrm{dir}}(t,x,v) \big] \,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^{2d}} v_i \big[f(t,x,v) - F_{\mathrm{dir}}(t,x,v) \big] \,\mathrm{d}x\,\mathrm{d}v = 0. \tag{2.6}$$

When *all* coordinates are harmonic ($d_\phi = 0$), then $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$ due to the normalization (H7) and there is an additional (and final) 2-cycle:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} (x \cdot v) f \,\mathrm{d}x\,\mathrm{d}v = -2 \int_{\mathbb{R}^{2d}} \frac{|x|^2 - |v|^2}{2} f \,\mathrm{d}x\,\mathrm{d}v,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} \frac{|x|^2 - |v|^2}{2} f \,\mathrm{d}x\,\mathrm{d}v = 2 \int_{\mathbb{R}^{2d}} (x \cdot v) f \,\mathrm{d}x\,\mathrm{d}v,$$

which implies that these two global quantities evolves as a scalar harmonic oscillator (with period 2). For all $t \geq 0$, we introduce then the following constants

$$\delta := \frac{1}{d} \int_{\mathbb{R}^{2d}} (x \cdot v) f_0 \,\mathrm{d}x\,\mathrm{d}v, \qquad \bar{\delta} := \frac{1}{d} \int_{\mathbb{R}^{2d}} \left(\frac{|x|^2 - |v|^2}{2} \right) f_0 \,\mathrm{d}x\,\mathrm{d}v$$

and define

$$F_{\mathrm{pul}}(t,x,v) = \delta \left(x \cdot v \cos(2t) + \frac{|x|^2 - |v|^2}{2} \sin(2t) \right) \mathcal{M} + \bar{\delta} \left(\frac{|x|^2 - |v|^2}{2} \cos(2t) - x \cdot v \sin(2t) \right) \mathcal{M}. \tag{2.7}$$

This function solves (1.1) and $f - F_{\mathrm{pul}}$ satisfies the additional conservation law

$$\begin{aligned} \forall t \geq 0, \quad & \int_{\mathbb{R}^{2d}} (x \cdot v) \Big[f(t,x,v) - F_{\mathrm{pul}}(t,x,v) \Big] \,\mathrm{d}x\,\mathrm{d}v \\ & = \int_{\mathbb{R}^{2d}} \left(\frac{|x|^2 - |v|^2}{2} \right) \Big[f(t,x,v) - F_{\mathrm{pul}}(t,x,v) \Big] \,\mathrm{d}x\,\mathrm{d}v = 0. \end{aligned} \tag{2.8}$$

The property of orthogonality of these modes that we implicitly used before is:

Lemma 2.1. *The functions $\mathcal{M}$, $(\mathrm{Var}(\mathcal{H}))^{-1}\mathcal{H}\mathcal{M}$, $C_A^{-1}(Ax \cdot v)\mathcal{M}$ (where $A \in \mathcal{R}_\phi$ and C_A, $x_i\mathcal{M}$ and $v_i\mathcal{M}$ (when $i \geq d_\phi + 1$) and $\frac{(x \cdot v)}{\sqrt{d}}$, $\frac{(|x|^2 - |v|^2)}{2\sqrt{d}}$ (when $d_\phi = 0$) are orthonormal in $L^2(\mathcal{M})$.*

Proof of Lemma 2.1. This can be made by direct computation, using many times that $\langle x \rangle = 0$, $\langle \nabla_x^2 \phi \rangle = \mathrm{Id}_{d \times d}$ and standard properties of Hermite functions. □

As a fundamental consequence of this Lemma, we get that all functions $\alpha\mathcal{M}$, $\beta\mathcal{H}\mathcal{M}$, F_{rig}, $F_{\mathrm{dir},i}$ and F_{pul} have all conserved quantities equal to zero apart from the one which coincides with the one of f_0. This orthogonality property will allow to simplify the statement of Theorem 1.1 in the next section. In other words, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^{2d}} \Big(f - \alpha\mathcal{M} - \beta\mathcal{H}\mathcal{M} - F_{\mathrm{rig}} - F_{\mathrm{dir}} - F_{\mathrm{pul}} \Big) \omega \,\mathrm{d}x\,\mathrm{d}v = 0,$$

for any ω in the set of admissible multiplicators defined in Lemma 2.1.

2.2. **Rescaling and macroscopic quantities.** To make calculations cleaner we remove the modes built in (2.1)–(2.2)–(2.3)–(2.5)–(2.7) and change the unknown to work in $L^2(\mathcal{M})$: given f solution to (1.1) in $L^2(\mathcal{M}^{-1})$, we define

$$h := \frac{f - \alpha\mathcal{M} - \beta\mathcal{H}\mathcal{M} - F_{\rm rig} - F_{\rm dir} - F_{\rm pul}}{\mathcal{M}} \in L^2(\mathcal{M}). \tag{2.9}$$

Then h satisfies the new main equation

$$\partial_t h = \mathcal{L}h := \mathcal{T}h + \mathcal{C}h, \quad h|_{t=0} = h_0 \tag{2.10}$$

with

$$\mathcal{T}h := \mathscr{T}h = \nabla_x\phi\cdot\nabla_v h - v\cdot\nabla_x h \quad \text{and} \quad \mathcal{C}h := \mu^{-1}\mathscr{C}\,(\mu h)\,.$$

When the index is omitted $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ refer to $L^2(\mathcal{M})$ from now on. Note that when considering a function of x only, respectively v only, the norms $L^2(\rho)$, respectively $L^2(\mu)$, coincide with $L^2(\mathcal{M})$ via a unitary embedding. By now $\langle\cdot\rangle$ stands for the mean in $L^2(\rho)$. Operator $\mathcal{C}$ is selfadjoint in $L^2(\mathcal{M})$ and $L^2(\mu)$, acting only and velocity with kernel

$$\mathrm{Ker}_{L^2(\mu)}\mathcal{C} = \mathrm{Span}\left\{1, v_1, \ldots, v_d, |v|^2\right\}. \tag{2.11}$$

Denoting $\mathfrak{E}(v) = \frac{(|v|^2-d)}{\sqrt{2d}}$, we decompose h in the following orthogonal way in $L^2(\mathcal{M})$

$$h = \pi h + h^\perp, \quad \pi h := r + m\cdot v + e\,\mathfrak{E}(v), \tag{2.12}$$

where $h^\perp = h^\perp(t,x,v)$ is the *microscopic* part and the *macroscopic quantities* are

$$r(t,x) := \int_{\mathbb{R}^d} h(t,x,v)\mu(v)\,\mathrm{d}v, \qquad \text{(local) density}$$
$$m(t,x) := \int_{\mathbb{R}^d} v h(t,x,v)\mu(v)\,\mathrm{d}v, \qquad \text{(local) momentum}$$
$$e(t,x) := \int_{\mathbb{R}^d} \mathfrak{E}(v)\,h(t,x,v)\mu(v)\,\mathrm{d}v \qquad \text{(local) kinetic energy.}$$

With these notations, (H1) reads then $-\langle\mathcal{C}h, h\rangle \geq \mathrm{c}_{\mathscr{C}}\|h^\perp\|^2$. According to the definition of h and the properties of all special modes listed in the preceding section, h as multiple conservation laws. We list them below. The conservation of total mass and energy writes

$$\langle r\rangle = 0 \quad \text{and} \quad \sqrt{\frac{d}{2}}\langle e\rangle + \langle(\phi - \langle\phi\rangle)r\rangle = 0. \tag{2.13}$$

Then the possible conservation law associated to rotation symmetry of ϕ reads

$$\mathbb{P}(m) \in \mathcal{R}_\phi^\perp, \tag{2.14}$$

where we recall $\mathbb{P}$ is the $L^2(\rho)$-projection onto infinitesimal rotations, $\mathcal{R}_\phi$ the infinitesimal rotations compatible with ϕ and $\mathcal{R}_\phi^\perp$ its orthogonal complement in $L^2(\rho)$. Note that $\mathbb{P}_\phi(m) = 0$ because we have deducted $F_{\rm rig}$ to f in (2.9), and have thus removed all possible non-zero infinitesimal rotational modes. Along the harmonic directions (when present), there is an additional invariance by centered rotation in the plane (x_i, v_i), which leads to the additional conservation law

$$\forall\, i \in \{d_\phi + 1, \ldots, d\}\,, \qquad \langle r x_i\rangle = 0 \;\text{ and }\; \langle m_i\rangle = 0. \tag{2.15}$$

In the full harmonic case $d_\phi = 0$ (with $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$), there is an additional invariance by planar rotation of the radial variables $(|x|, |v|)$, which leads to the final conservation law

$$\langle m\cdot x\rangle = 0 \;\text{ and }\; \sqrt{\frac{d}{2}}\langle e\rangle - \langle(\phi - \langle\phi\rangle)r\rangle = 0. \tag{2.16}$$

2.3. **Macroscopic equations.** In this subsection, we identify the evolution equations for the macroscopic part of h solution to (2.10). For this we first notice that for any polynomial function $p = p(v) : \mathbb{R}^d \to R$ we can easily compute $J_p[h] = \int_{\mathbb{R}^d} p(v) h\mu \, \mathrm{d}v$ using standard properties of Hermite functions in velocity, and a straightforward computation gives

$$J_p[h] = r \left(\int_{\mathbb{R}^d} p(v)\mu \, \mathrm{d}v \right) + m \cdot \left(\int_{\mathbb{R}^d} v p(v)\mu \, \mathrm{d}v \right) + e \left(\int_{\mathbb{R}^d} \frac{(|v|^2 - d)}{\sqrt{2d}} p(v)\mu \, \mathrm{d}v \right) + J_p[h^\perp].$$

and using (2.10) we get

$$\begin{aligned} \partial_t J_p[h] = &- \nabla_x r \cdot \left(\int_{\mathbb{R}^d} v p(v)\mu \, \mathrm{d}v \right) - \nabla_x m : \left(\int_{\mathbb{R}^d} v \otimes v p(v)\mu \, \mathrm{d}v \right) + m \cdot \nabla_x \phi \left(\int_{\mathbb{R}^d} p(v)\mu \, \mathrm{d}v \right) \\ &- \nabla_x e \cdot \left(\int_{\mathbb{R}^d} v \, \frac{(|v|^2 - d)}{\sqrt{2d}} p(v)\mu \, \mathrm{d}v \right) + e \nabla \phi \cdot \left(\int_{\mathbb{R}^d} v p(v)\mu \, \mathrm{d}v \right) \sqrt{\frac{2}{d}} + \int_{\mathbb{R}^d} (\mathcal{L} h^\perp) p(v)\mu \, \mathrm{d}v. \end{aligned}$$

In particular the evolution of the (local) density, momentum, kinetic energy and some suitable high-order moments of h is given by

$$\begin{cases} \partial_t r = \nabla_x^* \cdot m \\ \partial_t m = -\nabla_x r + \sqrt{\frac{2}{d}} \nabla_x^* e + \nabla_x^* \cdot E[h^\perp] \\ \partial_t e = -\sqrt{\frac{2}{d}} \nabla_x \cdot m + \nabla_x^* \cdot \Theta[h^\perp] \\ \partial_t E[h] = -2 \nabla_x^{\mathrm{sym}} m + E[\mathcal{L} h^\perp] \\ \partial_t \Theta[h] = -\left(1 + \frac{2}{d}\right) \nabla_x e + \Theta[\mathcal{L} h^\perp], \end{cases} \tag{2.17}$$

where the matrix $E[h]$ and the vector $\Theta[h]$ are defined as

$$\begin{cases} E[h] = \int_{\mathbb{R}^d} (v \otimes v - \mathrm{Id}_{d\times d}) \, h\mu \, \mathrm{d}v = \sqrt{\frac{2}{d}} \, e \, \mathrm{Id}_{d\times d} + E[h^\perp] \\ \Theta[h] = \Theta[h^\perp] := \int_{\mathbb{R}^d} v \, \frac{\left(|v|^2 - d - 2\right)}{\sqrt{2d}} \, h\mu \, \mathrm{d}v. \end{cases} \tag{2.18}$$

3. Macroscopic solutions

In this section, we prove the first part of Theorem 1.1 concerning the special modes. Using the reformulation given in Subsection 2.2, we are reduced to find solutions in $L^2(\mathcal{M})$ of the form $\pi = r + m \cdot v + e\mathfrak{E}(v)$, with π_0 of the same form, to the transport/hyperbolic equation $\partial_t \pi = \mathcal{L}\pi = \mathcal{T}\pi$ and satisfying the conservation laws (2.13)–(2.14) and (when present) (2.15)–(2.16). Note that $\partial_t \pi = \mathcal{L}\pi$ since $\mathcal{C}\pi =$ by (2.11). The following Proposition directly implies the first part of Theorem 1.1, and its proof is the aim of this section.

Proposition 3.1 (Macroscopic solutions)**.** *Under the above hypothesis* $\pi \equiv 0$.

We proceed by cooking up step by step a suitable entropy, and more precisely a positive (but not symmetric) quadratic form $\mathcal{F} : L^2(\mathcal{M}) \to \mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}(\pi(t)) = -D_{\mathcal{F}}(\pi(t)) \quad \text{and} \quad \|\pi\|^2 \lesssim \mathcal{D}(\pi) \lesssim \mathcal{F}(\pi) \lesssim \|\pi\|^2.$$

As a first step, since the equation is hyperbolic, we directly get that the L^2 norm is conserved.

Lemma 3.2. *There holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\pi\|^2 = 0.$$

Proof of Lemma 3.2. The results follows from $\mathcal{C}\pi = 0$, $\mathcal{T}^* = -\mathcal{T}$ and $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\pi\|^2 = (\pi, \mathcal{T}\pi) = 0$. □

3.1. **Main differential equation.** Using that $h = \pi = r + m \cdot v + e\mathfrak{E}(v)$, which means $h^\perp = 0$ in (2.12), we get that the macroscopic equations (2.17) reduces to

$$\begin{cases} \partial_t r = \nabla_x^* \cdot m, \quad \partial_t m = -\nabla_x r + \sqrt{\frac{2}{d}}\nabla_x^* e, \quad \partial_t e = -\sqrt{\frac{2}{d}}\nabla_x \cdot m \\ \sqrt{\frac{2}{d}}\,(\partial_t e)\,\mathrm{Id}_{d\times d} = -2\nabla_x^{\mathrm{sym}} m, \quad 0 = -\left(1 + \frac{2}{d}\right)\nabla_x e. \end{cases} \tag{3.1}$$

As a first step, we establish thanks to the macroscopic equations (3.1) that the macroscopic quantities have a remarkable and simple structure in the position variable so that the problem reduces to the control of only possibly time dependent quantities. Note that in all what follows we shall use without explicit mention assumption (H6).

Lemma 3.3. *There hold*

$$r(t,x) = -x \cdot b'(t) + \frac{1}{2\sqrt{2d}}\xi_2(x)c''(t) + \sqrt{\frac{2}{d}}\xi_\phi c(t), \tag{3.2}$$

$$m(t,x) = Ax + b(t) - \frac{1}{\sqrt{2d}} x c'(t), \tag{3.3}$$

$$e(t,x) = c(t), \tag{3.4}$$

with

$$b(t) := \langle m \rangle, \quad A := \langle \nabla_x^{\mathrm{skew}} m \rangle, \quad c(t) := \langle e \rangle,$$

where A is a constant matrix and we have denoted $\xi_2(x) := |x|^2 - \langle |x|^2 \rangle$ and $\xi_\phi(x) := \phi - \langle \phi \rangle$.

Proof of Lemma 3.3. From the last equation in (3.1), we directly get that $e = \langle e \rangle$ does not depend on the space variable, and therefore we obtain (3.4). Plugging this information in the the fourth equation of (3.1), this one simplifies into

$$\sqrt{\frac{2}{d}}\, c'\,\mathrm{Id}_{d\times d} = -2\nabla_x^{\mathrm{sym}} m. \tag{3.5}$$

Differentiating that equation and using (**??**), we get that $\nabla_x^2 m = 0$, so that in particular $\nabla_x^{\mathrm{skew}} m$ is constant in the x-variable and equal to its mean. Together with (3.5), we therefore deduce that

$$m(t,x) = \langle \nabla m \rangle x + \langle m \rangle = A(t)x + b(t) - \frac{1}{\sqrt{2d}} c'(t)x, \tag{3.6}$$

with the above definitions of A and b. In order to show that A does not depend on time, we plug the expressions (3.4) of e and (3.6) of m in the second equation in (3.1), and we deduce

$$\nabla_x r = -\partial_t m + \sqrt{\frac{2}{d}}\nabla_x^* e = -A'x - b' + \frac{1}{\sqrt{2d}}c''x + \sqrt{\frac{2}{d}}c\nabla\phi.$$

Taking the skew-symmetric gradient of this equation gives then $0 = -A'$ so that A is indeed a constant matrix, and we have established (3.3). Taking into account (3.4), (3.3) and that $\langle r \rangle = 0$, we can then take the primitive in space of the second equation in (3.1) and we immediately deduce the expression (3.2) for the macroscopic density. Using the preceding expressions of m and r yields the following crucial differential equation satisfied by the quantities A, $b(t)$ and $c(t)$:

$$\frac{1}{\sqrt{2d}}\left(2\xi_\phi + \nabla_x\phi \cdot x - d\right)c' + \frac{1}{2\sqrt{2d}}\xi_2 c''' - \nabla_x\phi \cdot b - x \cdot b'' - \nabla_x\phi \cdot Ax = 0 \tag{3.7}$$

where we have used the first macroscopic equation in (3.1) with the two expressions of m and r obtained in (3.2) and (3.3). (This equation suggests that partial harmonicity of the potential ϕ affects the estimates, as we shall indeed see.) □

3.2. **Control of A.** In order to control A, we first use the equation (3.7): multiplying (3.7) by x_k for $k = 1, \dots, d$, then integrating against $\rho(x)$, performing some integrations by part, using that $\rho(x)\,\mathrm{d}x$ is centered and that the terms involving $\nabla\phi$ vanish, we get the vectorial equation

$$\frac{1}{\sqrt{2d}}\langle 2\phi x\rangle c' + \frac{1}{2\sqrt{2d}}\langle |x|^2 x\rangle c''' - b - \langle x\otimes x\rangle b'' = 0. \tag{3.8}$$

Let us now define

$$X = X(b', c', c'') := \frac{1}{\sqrt{2d}}\left[2\xi_\phi + \nabla_x\phi\cdot x - d\right]c + \frac{1}{2\sqrt{2d}}\xi_2 c'' - x\cdot b' \tag{3.9}$$

and

$$Y = Y(b', c', c'') := \frac{1}{\sqrt{2d}}\langle 2\phi x\rangle c + \frac{1}{2\sqrt{2d}}\langle |x|^2 x\rangle c'' - \langle x\otimes x\rangle b', \tag{3.10}$$

so that previous identities (3.7)-(3.8) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(X - Y\cdot\nabla_x\phi\right) = \nabla_x\phi\cdot Ax.$$

Therefore since A is a constant of time, we deduce the

Lemma 3.4. *The infinitesimal rotation matrix A satisfies*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle (X - Y\cdot\nabla_x\phi), \nabla_x\phi\cdot Ax\rangle = -\|\nabla_x\phi\cdot Ax\|^2. \tag{3.11}$$

At this place we note that $A = \mathbb{P}(m)$, so that A is in the orthogonal of the compatible infinitesimal rotations $\mathcal{R}_\phi$. We then use the conservation law (2.14) and the inequality (1.5) to get

Lemma 3.5. *The infinitesimal rotation matrix A satisfies the following Korn-type inequality*

$$\|\nabla_x\phi\cdot Ax\|^2 \geq c_{\mathrm{K}}|A|^2. \tag{3.12}$$

In the preceding statement on from now on in this article, we denote

$$|A|^2 := \int_{\mathbb{R}^d}|Ax|^2\rho(x)\,\mathrm{d}x$$

and recall that since $\mathfrak{M}^{\mathrm{skew}}(\mathbb{R}^d)$ is of finite dimension, all norms on this vectorial space are equivalent. This will be convenient to choose this one for the coming analysis.

3.3. **Control of b, b'', c' and c'''.** Let us now control b, b'', c' and c''':

Lemma 3.6. *The following estimates on the functions b, b'', c' and c''' hold*

$$|b| + |b''| + |c'| + |c'''| \lesssim |A|. \tag{3.13}$$

Proof of Lemma 3.6. For this we will have to split later the study into different cases depending on the harmonicity properties of ϕ. We rewrite equation (3.7) as

$$\frac{1}{\sqrt{2d}}\Big[2\xi_\phi + \nabla_x\phi\cdot x - d\Big]c' + \frac{1}{2\sqrt{2d}}\xi_2 c''' - \nabla_x\phi\cdot b - x\cdot b'' = R_0 \tag{3.14}$$

with $R_0 := \nabla_x\phi Ax$. Multiplying (3.14) by $\nabla_x\phi$ and integrating against $\rho(x)$, it follows after integration by part using that ρ is centered and observing that the term involving $2\xi_\phi - d$ and c''' vanish that

$$b'' = -\langle\nabla_x^2\phi\rangle b + \frac{1}{\sqrt{2d}}\langle\nabla_x^2\phi x\rangle c' + R_1 = -b + \frac{1}{\sqrt{2d}}\langle\nabla_x^2\phi x\rangle c' + R_1 \tag{3.15}$$

with

$$R_1 := \langle R_0, \nabla_x\phi\rangle = \mathcal{O}(|A|).$$

Putting this expression back to (3.14) one gets

$$4\Psi_1(x)c' + \Psi_2(x)c''' - \Phi(x)\cdot b = R_2 \tag{3.16}$$

with $R_2 := R_0 + R_1 \cdot x$ and where we define

$$\Phi(x) := \nabla_x \phi - \langle \nabla_x^2 \phi \rangle x = \nabla_x \phi - x$$

and

$$\Psi_1(x) := \frac{1}{2\sqrt{2d}} \left(\xi_\phi + \frac{\nabla_x \phi \cdot x}{2} - \frac{d}{2} - \frac{1}{2} \langle \nabla_x^2 \phi x \rangle x \right), \qquad \Psi_2(x) := \frac{1}{2\sqrt{2d}} \xi_2.$$

Defining

$$M_\phi := \langle \Phi \otimes \Phi \rangle \in \mathfrak{M}^{\rm sym}_{d\times d}(\mathbb{R}), \quad \alpha_2 := \langle \Psi_2 \Phi \rangle \in \mathbb{R}^d, \quad \alpha_1 := \langle \Psi_1 \Phi \rangle \in \mathbb{R}^d, \tag{3.17}$$

we obtain after multiplication by Φ and integration in $L^2(\rho)$ that

$$M_\phi b = 4\alpha_1 c' + \alpha_2 c''' + R_3. \tag{3.18}$$

where $R_3 := -\langle R_2 \Phi \rangle = \mathcal{O}(|A|)$ thanks to the bounded moment properties of ϕ in (H6).

The main question at this stage is to be able to (partially) invert the matrix M_ϕ in order to get an expression of b and exhibit a differential equation satisfied by c (up to the error term $\mathcal{O}(A)$). This will be possible when taking into account the cycle of conservation laws. From now on we split the proof into three different cases depending on harmonic properties of the potential ϕ. Recalling $\Phi(x) = \nabla_x \phi - x$ we have $E_\phi = \mathrm{Span}\{\Phi(x) : x \in \mathbb{R}^d\}$ and recall also that $d_\phi = \dim E_\phi$. We then make the distinction between the fully non-harmonic potential where $d_\phi = d$, the partially harmonic potential where $1 \le d_\phi \le d-1$ and the fully harmonic case $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$, $d_\phi = 0$.

Fully non-harmonic case ($d_\phi = d$). One assumes here that ϕ is fully non harmonic, that is $d_\phi = d$. Then from Lemma A.2 proven in the appendix, the matrix M_ϕ is invertible, and (3.18) yields

$$b = 4 M_\phi^{-1} \alpha_1 c' + M_\phi^{-1} \alpha_2 c''' + M_\phi^{-1} R_3 \tag{3.19}$$

and hence, together with (3.16), it follows that

$$4\widetilde{\Psi}_1(x) c' + \widetilde{\Psi}_2(x) c''' = R_4, \tag{3.20}$$

with $R_4 := R_2 + \Phi(x) \cdot M_\phi^{-1} R_3$ and

$$\widetilde{\Psi}_1(x) := \Psi_1(x) - \frac{1}{4}\Phi(x) \cdot M_\phi^{-1} \alpha_1, \qquad \widetilde{\Psi}_2(x) := \Psi_2(x) - \Phi(x) \cdot M_\phi^{-1} \alpha_2. \tag{3.21}$$

Lemma A.3 proven in the appendix then states that $\mathrm{Rank}(\widetilde{\Psi}_1, \widetilde{\Psi}_2) = 2$, and we immediately deduce from (3.20) that $c' = \mathcal{O}(A)$ and $c''' = \mathcal{O}(A)$. Using then (3.19) and (3.15), we also deduce $b = \mathcal{O}(A)$ and $b'' = \mathcal{O}(A)$, and the proof of this case is complete.

Partially harmonic case ($1 \le d_\phi \le d-1$). In this case we have to consider the adapted basis: recall that $\{e_1, \dots, e_{d_\phi}\}$ be a basis of E_ϕ, and let $\{e_1, \dots, e_{d_\phi}, e_{d_\phi+1}, \dots, e_d\}$ be a basis of $\mathbb{R}^d$. For any vector $x \in \mathbb{R}^d$ write $x = (\hat{x}, \check{x})$ with $\hat{x} \in \mathbb{R}^{d_\phi}$ and $\check{x} \in \mathbb{R}^{d-d_\phi}$. For any vector-field $\xi : \mathbb{R}^d \to \mathbb{R}^d$, write $\xi(x) = (\hat{\xi}(x), \check{\xi}(x))$. In particular one has $\Phi = (\hat{\Phi}, 0)$ and we also recall that from the (cycle) conservation laws we have $\check{b} = 0$ and therefore $b = (\hat{b}, 0)$. This implies that identity (3.16) becomes

$$4\Psi_1 c' + \Psi_2 c''' - \hat{\Phi} \cdot \hat{b} = R_2. \tag{3.22}$$

The matrix M_ϕ defined in (3.17) is given by

$$M_\phi = \begin{pmatrix} \hat{M}_\phi & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\hat{M}_\phi := \langle \hat{\Phi} \otimes \hat{\Phi} \rangle \in \mathfrak{M}^{\rm sym}_{d_\phi \times d_\phi}. \tag{3.23}$$

Following the same procedure as after (3.17) we obtain after multiplication by $\hat{\Phi}$ and integration in $L^2(\rho)$ that

$$\hat{M}_\phi \hat{b} = 4\hat{\alpha}_1 c' + \hat{\alpha}_2 c''' + \hat{R}_3, \tag{3.24}$$

with $R_3 := -\langle R_2 \hat{\Phi} \rangle = \mathcal{O}(A)$, $\hat{\alpha}_1 = \langle \Psi_1 \hat{\Phi} \rangle$ and $\hat{\alpha}_2 = \langle \Psi_2 \hat{\Phi} \rangle$. Again Lemma A.2 proven in the appendix implies that the matrix $\hat{M}_\phi$ is invertible so that from (3.24) one obtains

$$\hat{b} = 4\hat{M}_\phi^{-1}\hat{\alpha}_1 c' + \hat{M}_\phi^{-1}\hat{\alpha}_2 c''' + \hat{M}_\phi^{-1}\hat{R}_3. \tag{3.25}$$

Hence, together with (3.22), it follows

$$4\widehat{\Psi}_1(x)c' + \widehat{\Psi}_2(x)c''' = \hat{R}_4$$

with $\hat{R}_4 := R_2 + \hat{\Phi}(x) \cdot \hat{M}_\phi^{-1}\hat{R}_3 = \mathcal{O}(A)$ and

$$\widehat{\Psi}_1(x) := \Psi_1(x) - \frac{1}{4}\hat{\Phi}(x) \cdot \hat{M}_\phi^{-1}\hat{\alpha}_1, \qquad \widehat{\Psi}_2(x) := \Psi_2(x) - \hat{\Phi}(x) \cdot \hat{M}_\phi^{-1}\hat{\alpha}_2. \tag{3.26}$$

As in the full rank case, we postpone to the appendix Lemma A.3 insuring that $\mathrm{Rank}(\widehat{\Psi}_1, \widehat{\Psi}_2) = 2$, and similarly, this directly yields that $c' = \mathcal{O}(A)$ and $c''' = \mathcal{O}(A)$(3.20). From (3.25) and (3.15), we also get $\hat{b} = \mathcal{O}(A)$ and $\hat{b}'' = \mathcal{O}(A)$, and since $\check{b} = 0$ we eventually get $b = \mathcal{O}(A)$ and $b'' = \mathcal{O}(A)$. The proof of this case partially harmonic case is complete.

Fully harmonic case ($d_\phi = 0$). We use the cycles of conservation laws (2.13)–(2.15): first the two second equations in (2.13) and (2.16) implies that $c = 0$, and thus $c'' = c' = 0$; second, the second equation in (2.15) implies that $b = 0$, and thus $b' = 0$. The result of Proposition 3.6 then follows.

This concludes the proof of Lemma 3.6. □

3.4. **Control of b', c'' and c.**

We complete the picture by the following differential inequality:

Lemma 3.7. *There hold*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b' \rangle \le -|b'|^2 + \mathcal{O}(|A|^2), \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle -c', c'' \rangle \le -|c''|^2 + \mathcal{O}(|A|^2). \tag{3.27}$$

Proof of Lemma 3.7. We write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b' \rangle = \langle -b', b' \rangle + \langle -b, b'' \rangle,$$

from which and the estimates (3.13) on b and b'' the first differential inequality in (3.27) follows. The second differential inequality in (3.27) can be established similarly. □

We finally control the function c.

Lemma 3.8. *There holds*

$$|c| \lesssim |b'| + |c''|. \tag{3.28}$$

Proof of Lemma 3.8. We recall the expression of r given in (3.2) which writes

$$\sqrt{\frac{2}{d}}\xi_\phi c - r = x \cdot b' - \frac{1}{2\sqrt{2d}}\xi_2 c''. \tag{3.29}$$

Multiplying that equation by ξ_ϕ and using these expressions in the second conservation law in (2.13) give then

$$c\left[\sqrt{\frac{2}{d}}\langle \xi_\phi^2 \rangle + \sqrt{\frac{d}{2}}\right] = \left\langle \xi_\phi \Big(x \cdot b' - \frac{1}{2\sqrt{2d}}\xi_2 c'' \Big) \right\rangle,$$

from what (3.28) immediately follows. □

3.5. **Proof of Proposition 3.1.** We define the Lyapunov function

$$\mathcal{F}(\pi) := \|\pi\|^2 - \varepsilon_A \langle (X - Y \cdot \nabla_x \phi), \nabla_x \phi \cdot Ax \rangle - \varepsilon_b \langle b, b' \rangle - \varepsilon_c \langle c', c'' \rangle,$$

for some convenient $\varepsilon_i > 0$. On the one hand, we observe that

$$\mathcal{F}(\pi) \simeq \|\pi\|^2 \simeq |A|^2 + |b'|^2 + |c''|^2, \tag{3.30}$$

for $\varepsilon_i > 0$ small enough. Indeed, by simple orthonormalization argument and using the expressions (3.3) and (3.4) of m and e, we have

$$\|\pi\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2, \qquad \|m\|^2 = |b|^2 + |A|^2 + |c'|, \qquad \|e\|^2 = |c|^2.$$

We also notice that

$$b' = \langle -\nabla_x r \rangle = \langle -\nabla_x \phi r \rangle = \mathcal{O}(\|r\|)$$

by using the second equation in (3.1), performing one integration by parts and using the Cauchy-Schwarz inequality. Coming back to the expressions (3.2) and (3.4) of r and e and using some orthonormalization argument again, we get

$$\|r\|^2 + \|e\|^2 \simeq |b'|^2 + |c''|^2 + |c|^2.$$

We conclude to the second equivalence in (3.30) by using Lemma 3.6 and Lemma 3.8. We prove one side of the first equivalence in (3.30) by first observing that because of the dependence of X and Y in b', c' and c'' and the Young inequality, we have

$$\mathcal{F}(\pi) \gtrsim \|\pi\|^2 - \varepsilon_A(|b'|^2 + |c'|^2 + |c''|^2 + |A|^2) - \varepsilon_b(|b|^2 + |b'|^2) - \varepsilon_c(|c'|^2 + |c''|^2).$$

Because of the same estimates as used in the proof of the second equivalence in (3.30), we have

$$\mathcal{F}(\pi) \gtrsim \|\pi\|^2 - (\varepsilon_A + \varepsilon_b + \varepsilon_c)\|\pi\|^2 \geq \frac{1}{2}\|\pi\|^2,$$

by choosing $\varepsilon_i > 0$ small enough (with respect to constants). The proof of the other side of the first equivalence in (3.30) is similar (but does not require ant smallness condition on the ε_i).

On the other hand, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\pi) = -\mathcal{D}(\pi),$$

with

$$\mathcal{D}(\pi) = \varepsilon_A \|\nabla_x \phi \cdot Ax\|^2 + \varepsilon_b |b'|^2 - \varepsilon_b \mathcal{O}(|A|^2) + \varepsilon_c |c''|^2 - \varepsilon_c \mathcal{O}(|A|^2) \geq \varepsilon_A' |A|^2 + \varepsilon_b |b'|^2 + \varepsilon_c |c''|^2,$$

first by using Lemma 3.4 and Lemma 3.7, next by choosing ε_b and ε_c small enough comparatively to ε_A and using Lemma 3.5. As a consequence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\pi) \leq -\lambda\, \mathcal{F}(\pi),$$

with $\lambda > 0$, by using the equivalence (3.30). Thanks to the Gronwall lemma and the equivalence (3.30) again, we deduce

$$\|\pi(t)\| \lesssim \mathcal{F}(\pi(t)) \leq e^{-\lambda t} \mathcal{F}(\pi_{|t=0}).$$

Since we already know that A is a constant, we have $|A|^2 \lesssim \|\pi(t)\| \to 0$ and thus $A = 0$. Using then Lemma 3.6 we also get that b, b'', c' and c''' are identically zero and therefore that c is a constant of time. From Lemma 3.8 and $|b'|^2 + |c''|^2 \lesssim \|\pi(t)\| \to 0$, we deduce that $c = 0$. As a consequence of (3.30), we conclude that $\pi = 0$.

4. Proof of hypocoercivity by the micro-macro method

In this section we prove the hypocoercivity part of Theorem 1.1 using the "micro-macro method" based on decomposing the solution between microscopic and macroscopic parts. The proof of Proposition 3.1 serves as a blueprint of the cascade of estimates to perform. The analysis is however more intricate because of the presence of microscopic terms. Using the notation of Subsection 2.2, the following Proposition directly implies the hypocoercivity part of Theorem 1.1.

Proposition 4.1. *Consider h solution to (2.10) in $L^2(\mathcal{M})$, then there exists $\kappa > 0$ such that*

$$\|h(t)\| \leq Ce^{-\kappa t} \|h_0\|,$$

where C and κ depends only on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant defined below in (1.5).

Recall that in all what follows $h = r + m\cdot v + e\mathfrak{E}(v) + h^\perp$ and that r, m and e satisfy the conservation laws stated at the end of subsection 2.2. By construction, we also have

$$\|h\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2 + \|h^\perp\|^2. \tag{4.1}$$

We use the macroscopic equations in (2.17), and define the following *error* quantities:

$$e_s := e - \langle e\rangle \tag{4.2}$$

$$m_s := m - \langle\nabla_x^{\text{skew}} m\rangle x - \frac{1}{d}\langle\nabla_x\cdot m\rangle x - \langle m\rangle \tag{4.3}$$

$$r_s := r - \langle\nabla_x r\rangle\cdot x - \frac{1}{2d}\langle\Delta_x r\rangle\xi_2 \tag{4.4}$$

$$w := r - \sqrt{\frac{2}{d}}\langle e\rangle\phi \tag{4.5}$$

$$w_s := r_s - \sqrt{\frac{2}{d}}\langle e\rangle\phi_s \quad \text{with} \quad \phi_s = \xi_\phi - \frac{1}{2d}\langle\Delta_x\phi\rangle\xi_2. \tag{4.6}$$

In particular, and for further reference, we have

$$w_s = w - \langle\nabla_x w\rangle x - \frac{1}{2d}\langle\Delta w\rangle\xi_2 + \sqrt{\frac{2}{d}}\langle e\rangle\langle\phi\rangle. \tag{4.7}$$

In the coming proof of Proposition 4.1 we split naturally the analysis into two parts: we consider first consider microscopic quantities in Subsection 4.2, and second macroscopic quantities in Subsection 4.3; in the latter the analysis shall be very close to the previous Section 3.

4.1. **Toolbox about the Witten-Hodge operator and the Korn inequality.** In this section, we gather several estimates that will be used to control the macroscopic quantities. Some of these estimates are classical and we refer to [4] for references and details of constructive proofs. Note that the assumptions (H3)–(H4)–(H5) are exactly the hypotheses needed the results in [4, Subsection 1.2]. We denote $\lfloor\nabla\phi\rceil := \sqrt{1+|\nabla\phi|^2}$ in the sequel. Among the results recalled in [4], we first shall use the *strong Poincaré inéquality*

$$\int_{\mathbb{R}^d} |\varphi - \langle\varphi\rangle|^2\lfloor\nabla\phi\rceil^2\,\rho\,\mathrm{d}x \lesssim \int_{\mathbb{R}^d} |\nabla_x\varphi|^2\,\rho\,\mathrm{d}x. \tag{4.8}$$

proven in [4, Proposition 5]. In order to work in $L^2(\rho)$, we shall use operator Ω introduced in (1.6) using the same later Ω when it acts on (coefficients) of vectors or matrices. As a consequence of (4.8), for any $\varphi\in L^2(\rho)$, there holds

$$\|\Omega^{-1}[\nabla^2\varphi]\| + \|\Omega^{-1}[\lfloor\nabla\phi\rceil\nabla\varphi]\| + \|\Omega^{-1}[\lfloor\nabla\phi\rceil^2\varphi]\| \lesssim \|\varphi\|, \tag{4.9}$$

see [4, Proposition 8]. Slightly more involved consequences are the following zeroth order Poincaré inequality, sometimes called the Poincaré-Lions inequality

$$\|\varphi - \langle\varphi\rangle\| \lesssim \left\|\Omega^{-\frac{1}{2}}\nabla_x\varphi\right\| \lesssim \|\varphi - \langle\varphi\rangle\|, \tag{4.10}$$

see [4, Proposition 5], the -1th order Poincaré-Lions inequality

$$\left\|\Omega^{-\frac{1}{2}}(\varphi - \langle\varphi\rangle)\right\| \lesssim \|\Omega^{-1}\nabla_x\varphi\| \lesssim \left\|\Omega^{-\frac{1}{2}}(\varphi - \langle\varphi\rangle)\right\|, \tag{4.11}$$

proven in [4, Lemma 10] and the variants

$$\|\varphi - \langle\varphi\rangle\| \lesssim \left\|\nabla_x\Omega^{-\frac{1}{2}}\varphi\right\| + \left\|\Omega^{-\frac{1}{2}}\nabla_x\varphi\right\| \lesssim \|\varphi - \langle\varphi\rangle\|. \tag{4.12}$$

Another key estimate we will use is the following Korn inequality: for any vector field $u : \mathbb{R}^d \to \mathbb{R}^d$ such that $\langle u\rangle = 0$ and $\langle\nabla_x^{\text{skew}} u\rangle = 0$, there holds

$$\|u\| \lesssim \left\|\Omega^{-\frac{1}{2}}\nabla_x^{\text{sym}} u\right\|, \tag{4.13}$$

which is established in [4, Theorem 1] using (**??**).

4.2. **Control of infinite-dimensional quantities.** We now build a global entropy function by constructing dissipative functionals step by step on each part of the unknown h.

4.2.1. *Control of $h^\perp$.* We first control the dissipation of the microscopic part of the unknown.

Lemma 4.2. *There exists some constant $\kappa_0 > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 \le -\kappa_0\|h^\perp\|^2.$$

Proof of Lemma 4.2. Since $\mathcal{C}^* = \mathcal{C}$ and $\mathcal{T}^* = -\mathcal{T}$, there holds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 = (\mathcal{C}h, h). \tag{4.14}$$

Thanks to the spectral gap assumption (H1), we conclude that (4.14) holds with $\kappa_0 := \mathrm{c}_{\mathscr{C}}$. ☐

We next control the macroscopic part of h by modifying the Lyapunov function built during the proof of Proposition 3.1.

4.2.2. *Control of e_s.* We next turn to the space inhomogeneous part e_s of the energy defined in (4.2):

Lemma 4.3. *There are some constants $\kappa_1, C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\Omega^{-1}\nabla_x e, \Theta[h]\right\rangle \le -\kappa_1\|e_s\|^2 + C\|h^\perp\|\|h\|. \tag{4.15}$$

Proof of Lemma 4.3. Recall that $\Theta[h] = \Theta[h^\perp]$ from (2.18). Compute then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\Omega^{-1}\nabla_x e, \Theta[h]\right\rangle &= \left\langle\Omega^{-1}\nabla_x e, -\left(1+\frac{2}{d}\right)\nabla_x e + \Theta[\mathcal{L}h^\perp]\right\rangle + \left\langle\Omega^{-1}\nabla_x(\partial_t e), \Theta[h]\right\rangle \\ &\le -\frac{1}{2}\left(1+\frac{2}{d}\right)\|\Omega^{-\frac{1}{2}}\nabla_x e\|^2 + C\|\Omega^{-\frac{1}{2}}\Theta[\mathcal{L}h^\perp]\|^2 + C\|\Omega^{-1}\nabla_x(\partial_t e)\|\|h^\perp\|.\end{aligned}$$

by the Cauchy-Schwarz and Young inequalities. Then (2.17) and (H2) and (4.9)–(4.12) imply

$$\begin{cases}\Omega^{-1}\nabla_x(\partial_t e) = -\sqrt{\dfrac{2}{d}}\Omega^{-1}\nabla_x\nabla_x\cdot m + \Omega^{-1}\nabla_x\nabla_x^*\cdot\Theta[h^\perp] = \mathcal{O}(\|h\|), \\ \Omega^{-\frac{1}{2}}\Theta[\mathcal{L}h^\perp] = \mathcal{O}(\|h^\perp\|).\end{cases}$$

Together with (4.10) it concludes the proof. ☐

4.2.3. *Control of m_s.* We now turn to the irrotational part m_s of the momentum defined in (4.3):

Lemma 4.4. *There are some constants $\kappa_2, C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}} m_s, E[h] - \sqrt{\frac{2}{d}}\langle e\rangle Id_{d\times d}\right\rangle \leq -\kappa_2\|m_s\|^2 + C\left(\|e_s\| + \|h^\perp\|\right)\|h\|. \tag{4.16}$$

Proof of Lemma 4.4. Remark that from (4.3), one has

$$\nabla_x^{\mathrm{sym}} m = \nabla_x^{\mathrm{sym}} m_s + \frac{1}{d}\langle \nabla_x \cdot m\rangle \,\mathrm{Id}_{d\times d},$$

and from (2.18), one has

$$E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \,\mathrm{Id}_{d\times d} = \sqrt{\frac{2}{d}} e_s \,\mathrm{Id}_{d\times d} + E[h^\perp].$$

Moreover from (2.17), one has

$$\frac{\mathrm{d}}{\mathrm{dt}}\langle e\rangle = -\sqrt{\frac{2}{d}}\langle \nabla_x \cdot m\rangle.$$

As a consequence, from the fourth equation in (2.17), one obtains

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}} m_s, E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \,\mathrm{Id}_{d\times d}\right\rangle \\
&= \left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}} m_s, -2\nabla_x^{\mathrm{sym}} m + E[\mathcal{L}h^\perp] + \frac{2}{d}\langle \nabla_x\cdot m\rangle \,\mathrm{Id}_{d\times d}\right\rangle + \left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}}(\partial_t m_s), E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \,\mathrm{Id}_{d\times d}\right\rangle \\
&= -2\|\Omega^{-\frac12}\nabla_x^{\mathrm{sym}} m_s\|^2 + \left\langle \Omega^{-\frac12}\nabla_x^{\mathrm{sym}} m_s, \Omega^{-\frac12}E[\mathcal{L}h^\perp]\right\rangle + \left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}}(\partial_t m_s), \sqrt{\frac{2}{d}} e_s \,\mathrm{Id}_{d\times d} + E[h^\perp]\right\rangle.
\end{aligned}$$

Using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}} m_s, E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \mathrm{Id}_{d\times d}\right\rangle \\
&\leq -\left\|\Omega^{-\frac12}\nabla_x^{\mathrm{sym}} m_s\right\|^2 + C\left\|\Omega^{-\frac12}E[\mathcal{L}h^\perp]\right\|^2 + C\left\|\Omega^{-1}\nabla_x^{\mathrm{sym}}(\partial_t m_s)\right\|\left\|\sqrt{\frac{2}{d}} e_s \mathrm{Id}_{d\times d} + E[h^\perp]\right\|.
\end{aligned}$$

Using the Korn inequality (4.13) and observing that

$$\left\|\Omega^{-1}\nabla_x^{\mathrm{sym}}(\partial_t m_s)\right\| = \mathcal{O}(\|h\|), \quad \left\|\Omega^{-\frac12}E[\mathcal{L}h^\perp]\right\| = \mathcal{O}\left(\left\|h^\perp\right\|\right)$$

from (4.9) and (4.12) as in the proof of Lemma 4.3, we immediately conclude to (4.16). □

4.2.4. *Control of w_s.* We now control the term w_s (part of the spatial density) defined in (4.6):

Lemma 4.5. *There are some constants $\kappa_3, C > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{dt}}\left\langle \Omega^{-1}\nabla_x w_s, m_s\right\rangle \leq -\kappa_3\|w_s\|^2 + C\|e_s\|^2 + C\|h^\perp\|^2 + C\|m_s\|\|h\| \tag{4.17}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}}\left\langle -\Omega^{-1}\partial_t w_s, w_s\right\rangle \leq -\left\|\Omega^{-\frac12}\partial_t w_s\right\|^2 + C\|w_s\|\|h\|. \tag{4.18}$$

Proof of Lemma 4.5. Observe that (2.17), definitions (4.5)–(4.2) and (4.7) imply

$$\partial_t m = -\nabla_x w + \sqrt{\frac{2}{d}}\nabla_x^* e_s + \nabla_x^*\cdot E[h^\perp] = -\nabla_x w_s - \langle \nabla_x w\rangle - \frac{1}{d}\langle \Delta_x w\rangle x + \sqrt{\frac{2}{d}}\nabla_x^* e_s + \nabla_x^*\cdot E[h^\perp].$$

Integrating the second equation in (2.17) and using again the definitions (4.5)–(4.2), one gets

$$\partial_t\langle m\rangle = -\langle \nabla_x r\rangle = -\langle \nabla_x w\rangle,$$

$$\partial_t\langle \nabla_x\cdot m\rangle = -\langle \Delta_x r\rangle + \sqrt{\frac{2}{d}}\langle e\Delta_x\phi\rangle + \langle \mathrm{Tr}(E[h^\perp]\nabla_x^2\phi)\rangle = -\langle \Delta_x w\rangle + \sqrt{\frac{2}{d}}\langle e_s\Delta_x\phi\rangle + \langle \mathrm{Tr}(E[h^\perp]\nabla_x^2\phi)\rangle.$$

Finally, by differentiating the second equation in (2.17), one has

$$\partial_t\nabla_x m = -\nabla_x^2 r + \sqrt{\frac{2}{d}}\nabla_x^*\nabla_x e + \sqrt{\frac{2}{d}}\nabla_x^2\phi e + \nabla_x^*\cdot(\nabla\otimes E[h^\perp]) + E[h^\perp]\nabla_x^2\phi,$$

and after integration of the skew-symmetric part, it yields

$$\partial_t\langle \nabla_x^{\mathrm{skew}} m\rangle = \left\langle \left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle. \tag{4.19}$$

As a consequence, from the very definition (4.3) of m_s and gathering these identities, one gets

$$\partial_t m_s = -\nabla_x w_s + \sqrt{\frac{2}{d}}\left[\nabla_x^* e_s - \frac{1}{d}\langle e_s\Delta_x\phi\rangle x\right] + \left\{-\left\langle\left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle + \frac{1}{d}\left\langle \mathrm{Tr}\left(E\left[h^\perp\right]\nabla_x^2\phi\right)\right\rangle\right\} x.$$

On the other hand, from the definition (4.6) of w_s and the equations (2.17), we have

$$\begin{aligned}\partial_t w_s &= \partial_t r - \langle \nabla\partial_t r\rangle\cdot x - \frac{1}{2d}\langle \Delta\partial_t r\rangle\xi_2, - \sqrt{\frac{2}{d}}\langle \partial_t e\rangle\phi_s,\\ &= \nabla^* m - \langle \nabla\nabla^*\cdot m\rangle\cdot x - \frac{1}{2d}\langle \Delta\nabla^*\cdot m\rangle\xi_2 - \frac{2}{d}\langle \nabla\cdot m\rangle\phi_s,\end{aligned} \tag{4.20}$$

and finally

$$\Omega^{-1}\nabla\partial_t w_s = \mathcal{O}(\|m\|), \tag{4.21}$$

where we have used (4.9) in order to estimate the first term and we perform several integration by part and use the boundedness assumption (H6) on ϕ in order to estimate the three last terms.

Combining these estimates and using the Cauchy Schwartz and Young inequalities, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega^{-1}\nabla_x w_s, m_s\right\rangle = \\ &\left\langle \Omega^{-1}\nabla_x w_s, -\nabla_x w_s + \sqrt{\frac{2}{d}}\left[\nabla_x^* e_s - \frac{1}{d}\langle e_s\Delta_x\phi\rangle x\right] + \left\{-\left\langle\left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle + \frac{1}{d}\left\langle \mathrm{Tr}\left(E\left[h^\perp\right]\nabla_x^2\phi\right)\right\rangle\right\} x\right\rangle\\ &\quad + \left\langle \Omega^{-1}\nabla_x(\partial_t w_s), m_s\right\rangle \le -\frac{1}{2}\|\Omega^{-\frac12}\nabla_x w_s\|^2 + C\|\Omega^{-\frac12}\nabla_x e_s\|^2 + C\|e_s\|^2 + C\|h^\perp\|^2 + C\|m\|\|m_s\|^2.\end{aligned}$$

We conclude to (4.17) thanks to the zeroth order Poincaré inequality (4.10).

In order to control the time-derivative of w_s, we write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -\Omega^{-1}\partial_t w_s, w_s\rangle = -\left\|\Omega^{-\frac12}\partial_t w_s\right\|^2 - \left\langle \Omega^{-1}\partial_{tt}^2 w_s, w_s\right\rangle. \tag{4.22}$$

Differentiating the equation (4.20) on w_s, we have

$$\partial_{tt}^2 w_s = \nabla^*\cdot(\partial_t m) - \langle \nabla\nabla^*\cdot(\partial_t m)\rangle\cdot x - \frac{1}{2d}\langle \Delta\nabla^*\cdot(\partial_t m)\rangle\xi_2 - \frac{2}{d}\langle \nabla\cdot(\partial_t m)\rangle\phi_s,$$

where the first term is

$$\nabla^*(\partial_t m) = -\nabla_x^*\cdot\nabla_x w + \sqrt{\frac{2}{d}}\nabla_x^*\cdot\nabla_x^* e_s + \nabla_x^*\cdot\nabla_x^*\cdot E\left[h^\perp\right],$$

and similar expressions hold for the three next terms. Arguing similarly as for (4.21), we have

$$\Omega^{-1}\partial_{tt}^2 w_s = \mathcal{O}(\|h\|).$$

Together with (4.22) it proves (4.18). □

We end this section by constructing a first partial Lyapunov functional:

$$\mathcal{F}_1(h) := \|h\|^2 + \varepsilon_1 \left\langle \Omega^{-1}\nabla_x e, \Theta[h]\right\rangle + \varepsilon_2\left\langle \Omega^{-1}\nabla_x^{\mathrm{sym}} m_s, E[h] - \sqrt{\frac{2}{d}}\langle e\rangle \, \mathrm{Id}_{d\times d}\right\rangle + \varepsilon_3 \left\langle \Omega^{-1}\nabla_x w_s, m_s\right\rangle + \varepsilon_4\langle -\Omega^{-1}\partial_t w_s, w_s\rangle, \tag{4.23}$$

for some $0 < \varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 < 1$ to be specified later and the associated dissipation term

$$\mathcal{D}_1(h) := \|h^\perp\|^2 + \|e_s\|^2 + \|m_s\|^2 + \|w_s\|^2 + \left\|\Omega^{-\frac{1}{2}}\partial_t w_s\right\|^2. \tag{4.24}$$

Lemma 4.6. *There exist some constants $\kappa_0', C > 0$ such that for any $\varepsilon_1 > 0$ small enough, there exists a convenient choice of ε_i, $i = 2, 3, 4$, $\varepsilon_1^2 \ll \varepsilon_4 \ll \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1$, such that there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(h) \le -\kappa_0'\|h^\perp\|^2 - \varepsilon_4\mathcal{D}_1(h) + \varepsilon_1^2 C\|h\|^2. \tag{4.25}$$

Proof of Lemma 4.6. Gathering Lemma 4.2, Lemma 4.3, Lemma 4.4 and Lemma 4.5, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(h) \le &- \kappa_0\|h^\perp\|^2 - \varepsilon_1\kappa_1\|e_s\|^2 + \varepsilon_1 C\|h^\perp\|\,\|h\| - \varepsilon_2\kappa_2\|m_s\|^2 + \varepsilon_2 C\|h^\perp\|\,\|h\| + \varepsilon_2 C\|e_s\|\,\|h\| \\ &- \varepsilon_3\kappa_3\|w_s\|^2 + \varepsilon_3 C\|e_s\|^2 + \varepsilon_3 C\|h^\perp\|^2 + \varepsilon_3 C\|m_s\|\,\|h\| - \varepsilon_4\|\Omega^{-\frac{1}{2}}\partial_t w_s\|^2 + \varepsilon_4 C\|w_s\|\,\|h\|,\end{aligned} \tag{4.26}$$

for any $\varepsilon_i \in (0,1)$, $i = 1,2,3,4$. Using repeatedly the Young inequality, we have

$$\begin{aligned}&\varepsilon_1 C\|h^\perp\|\,\|h\| \le \frac{\kappa_0}{6}\|h^\perp\|^2 + \varepsilon_1^2 C\|h\|^2, \quad \varepsilon_2 C\|h^\perp\|\,\|h\| \le \frac{\kappa_0}{6}\|h^\perp\|^2 + \varepsilon_2^2 C\|h\|^2,\\ &\varepsilon_2 C\|e_s\|\,\|h\| \le \frac{\varepsilon_1\kappa_1}{2}\|e_s\|^2 + \frac{\varepsilon_2^2}{\varepsilon_1}C\|h\|^2, \quad \varepsilon_3 C\|m_s\|\,\|h\| \le \frac{\varepsilon_2\kappa_2}{2}\|m_s\|^2 + \frac{\varepsilon_3^2}{\varepsilon_2}C\|h\|^2,\\ &\varepsilon_4 C\|w_s\|\,\|h\| \le \frac{\varepsilon_3\kappa_3}{2}\|w_s\|^2 + \frac{\varepsilon_4^2}{\varepsilon_3}C\|h\|^2,\end{aligned}$$

and therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(h) \le &- \kappa_0\left(\frac{2}{3} - \varepsilon_3 C\right)\|h^\perp\|^2 - \frac{\varepsilon_1\kappa_1}{2}\left(1 - \frac{\varepsilon_3}{\varepsilon_1}C\right)\|e_s\|^2 - \frac{\varepsilon_2\kappa_2}{2}\|m_s\|^2\\ &- \frac{\varepsilon_3\kappa_3}{2}\|w_s\|^2 - \varepsilon_4\|\Omega^{-\frac{1}{2}}\partial_t w_s\|^2 + C\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{\varepsilon_1} + \frac{\varepsilon_3^2}{\varepsilon_2} + \frac{\varepsilon_4^2}{\varepsilon_3}\right)\|h\|^2,\end{aligned} \tag{4.27}$$

where we have used the ordered condition $\varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ in order to simplify the last term.

In order to conclude, we have to choose the $(\varepsilon_i)_{i=1}^4$ so that (1) ε_3 and $\frac{\varepsilon_3}{\varepsilon_1}$ are small enough, and (2) the last term $\|h\|^2$ is negligible with respect to the previous ones; this is satisfied with $\varepsilon_2 := {\varepsilon_1}^{\frac{3}{2}}$, $\varepsilon_3 := {\varepsilon_1}^{7/4}$, $\varepsilon_4 := {\varepsilon_1}^{15/8}$. We then obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1(h) \le &- \frac{\kappa_0}{2}\left(1 - \varepsilon_1^{7/4}C\right)\|h^\perp\|^2 - \frac{\varepsilon_1\kappa_1}{2}\left(1 - \varepsilon_1^{3/4}C\right)\|e_s\|^2 - \frac{\varepsilon_1^{\frac{3}{2}}\kappa_2}{2}\|m_s\|^2\\ &- \frac{\varepsilon_1^{7/4}\kappa_3}{2}\|w_s\|^2 - \varepsilon_1^{15/8}\|\Omega^{-\frac{1}{2}}\partial_t w_s\|^2 + C\,\varepsilon_1^2\,\|h\|^2.\end{aligned} \tag{4.28}$$

In particular, for $\varepsilon_1 > 0$ small enough, it implies (4.25). □

4.3. **Control of finite-dimensional quantities and proof of Proposition 4.1.** In order to complete the proof, we need to control the time depending *global scalar* quantities $\langle e\rangle$, $\langle\nabla_x^{\mathrm{skew}} m\rangle$, $\langle\nabla_x\cdot m\rangle$, $\langle m\rangle$, $\langle\nabla_x r\rangle$, $\langle\Delta_x r\rangle$ involved in the definition of e_s, m_s and w_s in (4.2)–(4.6). We proceed similarly as in the proof of Proposition 3.1. To simplify the notation, we introduce

$$A(t) := \langle\nabla_x^{\mathrm{skew}} m\rangle, \quad b(t) := \langle m\rangle, \quad c(t) := \langle e\rangle, \tag{4.29}$$

and we rewrite the expression of the macroscopic quantity in terms of these new functions, which thus only dependent of the time variable.

Lemma 4.7. *We have the following expressions*

$$r(t,x) = -b'(t)\cdot x + c''(t)\frac{1}{2\sqrt{2d}}\xi_2 + c(t)\sqrt{\frac{2}{d}}\xi_\phi + z(t,x), \tag{4.30}$$

$$m(t,x) = A(t)x + b(t) - c'(t)\frac{1}{\sqrt{2d}}x + m_s(t,x), \tag{4.31}$$

$$e(t,x) = c(t) + e_s(t,x), \tag{4.32}$$

where the quantity z is controlled by the already controlled macroscopic quantities and more precisely

$$\|z\|^2 \lesssim \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2, \tag{4.33}$$

$$\|\Omega^{-\frac{1}{2}}\partial_t z\|^2 \lesssim \|\Omega^{-\frac{1}{2}}\partial_t w_s\|^2 + \|m_s\|^2 + \|h^\perp\|^2. \tag{4.34}$$

Proof of Lemma 4.7. The expression (4.30) is nothing but (4.2). From (2.17) one observes that

$$c' = \frac{\mathrm{d}}{\mathrm{d}t}\langle e\rangle = -\sqrt{\frac{2}{d}}\langle\nabla_x\cdot m\rangle, \tag{4.35}$$

so that (3.3) follows from the definition (4.3) of m_s. Note here that there is no reason for $A(t)$ to be the orthogonal projection of m onto infinitesimal rotation matrices. We shall have to take into account this fact later. Inspired by (3.2), we take (4.30) as the definition of the new macroscopic quantity z. Thanks to (4.4) and (4.6) and using that from (2.17), we have

$$b' = \frac{\mathrm{d}}{\mathrm{d}t}\langle m\rangle = -\langle\nabla_x r\rangle, \tag{4.36}$$

we write

$$r = w_s + \sqrt{\frac{2}{d}}\langle e\rangle\phi_s + \langle\nabla_x r\rangle\cdot x + \frac{1}{2d}\langle\Delta_x r\rangle\xi_2 = w_s + \sqrt{\frac{2}{d}}c\big(\xi_\phi - \frac{1}{2d}\langle\Delta_x\phi\rangle\xi_2\big) - b'\cdot x + \frac{1}{2d}\langle\Delta_x r\rangle\xi_2.$$

Together with the definition (4.30) of z, we deduce

$$z = w_s + \Big[\frac{1}{2d}\langle\Delta_x r\rangle - \frac{1}{2\sqrt{2d}}c'' - \frac{1}{d\sqrt{2d}}\langle\Delta_x\phi\rangle c\Big]\xi_2.$$

Finally, thanks to (2.17), we compute

$$c'' = -\sqrt{\frac{2}{d}}\,\langle\nabla_x\cdot\partial_t m\rangle = \sqrt{\frac{2}{d}}\,\langle\Delta_x r\rangle - \frac{2}{d}\,\langle\nabla_x\cdot\nabla_x^* e\rangle - \sqrt{\frac{2}{d}}\,\Big\langle\nabla_x\cdot\Big(\nabla_x^*\cdot E\left[h^\perp\right]\Big)\Big\rangle\,,$$

and thus

$$c'' = \sqrt{\frac{2}{d}}\,\langle\Delta_x r\rangle - \frac{2}{d}\langle e\Delta\phi\rangle - \sqrt{\frac{2}{d}}\,\Big\langle\mathrm{Tr}\,\Big(E\left[h^\perp\right]\nabla_x^2\phi\Big)\Big\rangle\,. \tag{4.37}$$

The two last identities together imply

$$z = w_s + \frac{1}{2d}\,\Big\langle E\left[h^\perp\right] : \nabla_x^2\phi\Big\rangle\,\xi_2 + \frac{1}{d\sqrt{2d}}\,\langle e_s\Delta_x\phi\rangle\,\xi_2,$$

from what (4.33) follows. From (2.18)–(2.17)–(4.3) and (4.35), we have

$$\partial_t E[h^\perp] = -2\nabla_x^{\mathrm{sym}} m_s + E[\mathcal{L}h^\perp] - \sqrt{\frac{2}{d}}\,(\partial_t e_s)\,\mathrm{Id}_{d\times d}.$$

Differentiating the above expression of z and using that last identity, we then get

$$\partial_t z = \partial_t w_s - \frac{1}{d}\,\langle\nabla_x^{\mathrm{sym}} m_s : \nabla_x^2\phi\rangle\,\xi_2 + \frac{1}{2d}\,\Big\langle E\left[\mathcal{L}h^\perp\right] : \nabla_x^2\phi\Big\rangle\,\xi_2.$$

The estimate (4.34) immediately follows from integrations by part and (H6). □

Using the expressions (4.30) and (4.31) in the first macroscopic equation in (2.17), we write

$$\partial_t r = \sqrt{\frac{2}{d}}\xi_\phi c' - x\cdot b'' + \frac{1}{2\sqrt{2d}}\xi_2 c''' + \partial_t z = \nabla_x^*\cdot m_s + \nabla_x\phi A x - \frac{1}{\sqrt{2d}}(\nabla_x\phi\cdot x - d)c' + \nabla_x\phi\cdot b.$$

We deduce then the main differential equation satisfied by A, b and c, which is very similar to (3.7) up to additional controlled terms involving m_s and z, namely

$$\frac{1}{\sqrt{2d}}\left[2\xi_\phi + \nabla_x\phi\cdot x - d\right]c' + \frac{1}{2\sqrt{2d}}\xi_2 c''' - \nabla_x\phi\cdot b - x\cdot b'' - \nabla_x\phi A x = \nabla_x^*\cdot m_s - \partial_t z. \tag{4.38}$$

We start controlling the skew-symmetric matrix A in the following way.

Lemma 4.8. *There are some constants $\kappa_5, C > 0$ such that the following differential inequation controlling the matrix A holds true*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \left(X - Y\cdot\nabla_x\phi\right), \nabla_x\phi\cdot A x\Big\rangle \le -\kappa_5|A|^2 + C\mathcal{D}_1(h) + C\|h^\perp\|\,\|h\|, \tag{4.39}$$

where $X = X(b', c', c'')$ and $Y = Y(b', c', c'')$ are defined in (3.9) and (3.10).

Proof of Lemma 4.8. We take up again a similar argument as in the proof of Lemma 3.4. We multiply (4.38) by x_k for $k = 1, \dots, d$ and after integration, we get

$$\frac{1}{\sqrt{2d}}\langle 2(\phi - \langle\phi\rangle)x_k\rangle c' + \frac{1}{2\sqrt{2d}}\langle \xi_2 x_k\rangle c''' - b_k - \langle x_k x_\ell\rangle b_\ell'' = \langle m_{s,k}\rangle - \langle x_k\partial_t z\rangle. \tag{4.40}$$

Using the definitions of X and Y into equations (4.38) and (4.40) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}(X - Y\cdot\nabla_x\phi) = \nabla_x\phi A x + \nabla_x^*\cdot m_s - \langle m_s\rangle\cdot\nabla_x\phi - \partial_t z + \langle x\partial_t z\rangle\cdot\nabla_x\phi.$$

Recalling (4.29) and (4.19), we therefore have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle -(X - Y\cdot\nabla_x\phi), \nabla_x\phi A x\rangle &= -\|\nabla_x\phi A x\|^2 - \langle\nabla_x^*\cdot m_s, \nabla_x\phi A x\rangle + \Big\langle \langle m_s\rangle\cdot\nabla_x\phi, \nabla_x\phi A x\Big\rangle \\ &+ \langle\partial_t z, \nabla_x\phi A x\rangle - \Big\langle\langle x\partial_t z\rangle\cdot\nabla_x\phi, \nabla_x\phi A x\Big\rangle - \left\langle \left(X - Y\cdot\nabla_x\phi\right), \nabla_x\phi\left[\left\langle\left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle\right]x\right\rangle.\end{aligned}$$

For the first term and thanks to the conservation law (2.14), we note that

$$\mathcal{R}_\phi^\perp \ni \mathbb{P}(m) = Ax + \mathbb{P}(m_s),$$

so that we can apply inequality (1.5) to $V = A + \mathbb{P}(m_s)$ to get

$$c_{\mathrm{K}}\|Ax + \mathbb{P}(m_s)\|^2 \le \|\nabla_x\phi A x + \nabla_x\phi\cdot\mathbb{P}(m_s)\|^2,$$

which yields the following control of A:

$$c_{\mathrm{K}}|A|^2 = c_{\mathrm{K}}\|Ax\|^2 \le 4\|\nabla_x\phi A x\|^2 + C\left\|m_s\right\|^2. \tag{4.41}$$

In order to estimate the other terms, we use the rough estimates

$$\begin{aligned}&\left\langle \left(X - Y\cdot\nabla_x\phi\right), \nabla_x\phi\left[\left\langle\left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle\right]x\right\rangle \\ &\lesssim \left\|\Omega^{-1}(X - Y\cdot\nabla_x\phi)\right\|\left\|\Omega^1\left(\nabla_x\phi x\right)\right\|\left|\left\langle\left(E\left[h^\perp\right]\nabla_x^2\phi\right)^{\mathrm{skew}}\right\rangle\right| \lesssim \left(|b'| + |c'| + |c''|\right)\|h^\perp\|,\end{aligned}$$

and

$$\Big\langle\langle m_s\rangle\cdot\nabla_x\phi, \nabla_x\phi\cdot A x\Big\rangle \lesssim \|m_s\|\|\nabla_x\phi\cdot Ax\| \lesssim \|m_s\||A|.$$

Thanks to the zeroth order Poincaré inequality (4.10), we also have

$$\langle\nabla_x^*\cdot m_s, \nabla_x\phi A x\rangle = \left\langle\Omega^{-\frac{1}{2}}\left(\nabla_x^*\cdot m_s\right), \Omega^{\frac{1}{2}}\left(\nabla_x\phi A x\right)\right\rangle \lesssim \|m_s\||A|$$

as well as similar estimates for the terms in $\partial_t z$ and $\langle x\partial_t z\rangle\cdot\nabla_x\phi$. Using these estimates and the control in (4.41) we eventually get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle -\left(X - Y\cdot\nabla_x\phi\right), \nabla_x\phi A x\right\rangle \le -c_{\mathrm{K}}^{-1}|A|^2 + C\left(\|m_s\| + \|\Omega^{-\frac12}\partial_t z\|\right)|A| + \left(|b'| + |c'| + |c''|\right)\|h^\perp\|,$$

and we conclude to (4.39) thanks to the Young inequality and (4.34). □

We now have to control the time dependent quantities b, b', b'' and c, c', c'', c''': we proceed in three steps. We first deal with b, b'', c' and c''' in the

Lemma 4.9. *The following estimates on the functions b, b'', c' and c''' hold*

$$|b| + |b''| + |c'| + |c'''| \lesssim |A| + \left\|\Omega^{-\frac12}\partial_t w_s\right\| + \|m_s\| + \|h^\perp\|. \tag{4.42}$$

Proof of Lemma 4.9. Now we deal with b, b'', c' and c'''. Coming back to (4.38), one can write

$$\frac{1}{\sqrt{2d}}[2\xi_\phi + \nabla_x\phi\cdot x - d]c' + \frac{1}{2\sqrt{2d}}\xi_2 c''' - \nabla_x\phi\cdot b - x\cdot b'' = R_0 \tag{4.43}$$

with $R_0 := \nabla_x\phi A x + \nabla_x^*\cdot m_s - \partial_t z$. Arguing as in the proof of Lemma 3.6 with this new definition of R_0, we obtain that (3.18) holds with

$$R_3 := -\langle R_0\widetilde{\Phi}\rangle, \quad \widetilde{\Phi} := \nabla\phi - x - \langle(\nabla\phi - x)\otimes x\rangle\nabla\phi.$$

Observing that

$$R_3 = -\left\langle\nabla_x\phi A x\widetilde{\Phi}\right\rangle - \left\langle {}^T\!\left(D\widetilde{\Phi}\right)m_s\right\rangle + \left\langle\Omega^{\frac12}\widetilde{\Phi}, \Omega^{-\frac12}\partial_t z\right\rangle = \mathcal{O}\left(|A| + \|m_s\| + \left\|\Omega^{-\frac12}\partial_t z\right\|\right)$$

and using (4.34), we may argue exactly as in the proof of Lemma 3.6 and conclude to (4.42). □

Similarly as in Lemma 3.7, we may control b' and c' in the following way.

Lemma 4.10. *There exists a constant $C > 0$ so that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle -b, b'\right\rangle \le -|b'|^2 + C|A|^2 + C\mathcal{D}_1(h), \quad \frac{\mathrm{d}}{\mathrm{d}t}\left\langle -c', c''\right\rangle \le -|c''|^2 + C|A|^2 + C\mathcal{D}_1(h). \tag{4.44}$$

In order to complete the scheme we finally need to control c and r.

Lemma 4.11. *The following controls hold*

$$|c| \lesssim |b'| + |c''| + \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2, \tag{4.45}$$

$$\|r\| \lesssim |b'| + |c''| + \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2. \tag{4.46}$$

Proof of Lemma 4.11. We write the expression of r given in (3.2) as

$$r = \sqrt{\frac{2}{d}}\xi_\phi c + R_5 \tag{4.47}$$

where $R_5 := z - x\cdot b' + \frac{1}{2\sqrt{2d}}\xi_2 c'' = \mathcal{O}(|b'| + |c''| + \mathcal{D}_1(h))$ because of (4.33). Using these expression in the conservation law (2.13) and recalling that $\langle e\rangle = c$ give then

$$\sqrt{\frac{d}{2}}c\left[1 + \frac{2}{d}\left\langle\xi_\phi^2\right\rangle\right] = -\left\langle\xi_\phi R_5\right\rangle,$$

from what (4.45) follows. Coming back to (4.47), we conclude to (4.46). □

We now introduce the Lyapunov function

$$\mathcal{F}_2(h) := \mathcal{F}_1(h) - \varepsilon_5\langle(X - Y\cdot\nabla_x\phi), \nabla_x\phi\cdot A x\rangle - \varepsilon_6\langle b, b'\rangle - \varepsilon_6\langle c', c''\rangle \tag{4.48}$$

for some additional small parameters $0 < \varepsilon_6 \ll \varepsilon_5 \ll \varepsilon_4$ and the associated dissipation functional

$$\mathcal{D}_2(h) := \mathcal{D}_1(h) + |A|^2 + |b'|^2 + |c''|^2. \tag{4.49}$$

Lemma 4.12. *For any* $0 < \varepsilon_6 < \varepsilon_5 < \varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ *with* ε_1 *small enough, there holds*

$$\|h\|^2 \lesssim \mathcal{F}_2(h) \lesssim \mathcal{D}_2(h) \lesssim \|h\|^2. \tag{4.50}$$

Proof of Lemma 4.12. On the one hand, we may control all quantities involved in the definitions of $\mathcal{F}_2$ and $\mathcal{D}_2$ by $\|h\|^2$. Indeed, from (4.1) and next the very definition (4.29), we have

$$\|r\| + \|m\| + \|e\| + \|h^\perp\| + |b| + |c| \lesssim \|h\| \tag{4.51}$$

and thus also $\|e_s\| \lesssim \|e\| + |c| \lesssim \|h\|$ from (4.32). Next, we observe from (4.35) that

$$|c'| = \sqrt{\frac{2}{d}}|\langle m \cdot \nabla\phi\rangle| \lesssim \|m\| \le \|h\|,$$

from (4.29) that

$$|A| = |\langle m\nabla_x^{\rm skew}\phi\rangle| \lesssim \|m\| \le \|h\|,$$

and thus also $\|m_s\| \lesssim \|m\| + |A| + |b| + |c'| \lesssim \|h\|$ from (4.31). Similarly, we observe from (4.36) that

$$|b'| = |\langle r\nabla_x\phi\rangle| \lesssim \|r\| \le \|h\|.$$

Coming back to the definition of w_s and using (4.36) we get

$$w_s = r - \sqrt{\frac{2}{d}}c\big(\xi_\phi - \frac{1}{2d}\langle\Delta_x\phi\rangle\xi_2\big) + b'\cdot x - \frac{1}{2d}\left\langle r(|\nabla\phi|^2 - \Delta\phi)\right\rangle\xi_2,$$

and deduce $\|w_s\| \lesssim \|r\| + |c| + |b'| \lesssim \|h\|$. Similarly, from (4.37), we also have $|c''| \lesssim \|r\| + \|e\| + \|h^\perp\| \le \|h\|$. Summing up, we have proved

$$\|e_s\| + \|m_s\| + \|w_s\| + |A| + |c'| + |b'| + |c''| \lesssim \|h\|. \tag{4.52}$$

We finally have to control the terms $\|\Omega^{-\frac{1}{2}}\partial_t w_s\|$.

From (4.6), (4.4) and (2.17), we have

$$\partial_t w_s = \nabla_x^* \cdot m - \langle\nabla_x\nabla_x^*\cdot m\rangle\cdot x - \frac{1}{2d}\left\langle\Delta_x\nabla_x^*\cdot m\right\rangle\xi_2 - \sqrt{\frac{2}{d}}c'\phi_s,$$

from what we get performing several integration by parts

$$\left\|\Omega^{-\frac{1}{2}}\partial_t w_s\right\| \lesssim \|m\| + |c'| \lesssim \|h\|. \tag{4.53}$$

As a consequence of the estimates (4.51), (4.52), (4.53) and of the very definition (4.23)-(4.48) of $\mathcal{F}_2$, we have

$$\big|\|h\|^2 - \mathcal{F}_2(h)\big| \le C\varepsilon_1\|h\|^2,$$

and that concludes the proof of the first equivalence in (4.50). For the same reason, we have $\mathcal{D}_2(h) \lesssim \|h\|^2$. On the other way round, from (4.30) and (4.45), we have

$$\|r\| \lesssim |b'| + |c''| + |c| + \|w_s\| + \|e_s\| + \|h^\perp\| \lesssim |b'| + |c''| + \|w_s\| + \|e_s\| + \|h^\perp\|$$

and similarly from (4.31) and (4.42), we have

$$\|m\| \lesssim |A| + |b| + |c'| + \|m_s\| \lesssim |A| + |c'| + \|m_s\| + \|\Omega^{-\frac{1}{2}}\partial_t w_s\| + \|h^\perp\|.$$

Combining the last two estimates, (4.32)–(4.1) and the definition (4.24)–(4.50) of $\mathcal{D}_2$, we deduce the reverse inequality $\|h\|^2 \lesssim \mathcal{D}_2(h)$, which ends the proof of the second equivalence in (4.50). □

Proof of Proposition 4.1. We differentiate in time the new Lyapunov function $\mathcal{F}_2(h)$ by taking advantage of the already established estimate (4.27) about the partial Lyapunov function $\mathcal{F}_1(h)$ as well as the new estimates (4.39) and (4.44) on the additional terms, and we then have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le &-\kappa_0'\|h^\perp\|^2 - \varepsilon_4\mathcal{D}_1(h) + C\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{\varepsilon_1} + \frac{\varepsilon_3^2}{\varepsilon_2} + \frac{\varepsilon_4^2}{\varepsilon_3}\right)\|h\|^2 - \varepsilon_5\kappa_5|A|^2\\ &+ \varepsilon_5 C\mathcal{D}_1(h) + \varepsilon_5 C\|h^\perp\|\,\|h\| - \varepsilon_6|b'|^2 - \varepsilon_6|c''|^2 + \varepsilon_6 C|A|^2 + \varepsilon_6 C\mathcal{D}_1(h),\end{aligned}$$

which holds true for any $0 < \varepsilon_6 < \varepsilon_5 < \varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1 < 1$ such that ε_1 and $\frac{\varepsilon_3}{\varepsilon_1}$ are small enough. Using the Young inequality

$$\varepsilon_5 C \|h^\perp\| \, \|h\| \le \kappa_0' \|h^\perp\|^2 + \varepsilon_5^2 C \|h\|^2,$$

we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le & -\varepsilon_4 \left(1 - \varepsilon_5 C - \varepsilon_6 C\right) \mathcal{D}_1(h) - \varepsilon_5 \left(\kappa_5 - C\frac{\varepsilon_6}{\varepsilon_5}\right) |A|^2 \\ & - \varepsilon_6 |b'|^2 - \varepsilon_6 |c''|^2 + C\left(\varepsilon_1^2 + \frac{\varepsilon_2^2}{\varepsilon_1} + \frac{\varepsilon_3^2}{\varepsilon_2} + \frac{\varepsilon_4^2}{\varepsilon_3} + \varepsilon_5^2\right) \|h\|^2.\end{aligned}$$

As in the proof of Lemma 4.6 we shall choose appropriately the small parameters ε_i such that the quantities ε_5, ε_6 and $\frac{\varepsilon_6}{\varepsilon_5}$ are small enough and the last term $\|h\|^2$ is negligible with respect to the previous ones. We recall $\varepsilon_2 := {\varepsilon_1}^{\frac{3}{2}}$, $\varepsilon_3 := {\varepsilon_1}^{7/4}$, $\varepsilon_4 := {\varepsilon_1}^{15/8}$ and furthermore take $\varepsilon_5 := \varepsilon_1^{61/32}$, $\varepsilon_6 := \varepsilon_1^{62/32}$. We get then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le & -\varepsilon_1^{15/8}\left(\kappa_4' - \varepsilon_1^{61/32} C\right) \mathcal{D}_1(h) - \varepsilon_1^{61/32}\left(\kappa_5 - C\varepsilon_1^{1/32}\right) |A|^2 \\ & - \varepsilon_1^{62/32} |b'|^2 - \varepsilon_1^{62/32} |c''|^2 + C\left(\varepsilon_1^2 + \varepsilon_1^{61/16}\right) \|h\|^2.\end{aligned}$$

Choosing ε_1 small enough, the differential inequality simplifies into

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le -\varepsilon_1^{62/32} \mathcal{D}_2(h) + C\varepsilon_1^2 \|h\|^2.$$

Because of the equivalences established in Lemma 4.12, there are two constants $K_i > 0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le -\varepsilon_1^{62/32}\left(K_1 - K_2 \varepsilon_1^{2/32}\right) \mathcal{F}_2(h).$$

Choosing $\varepsilon_1 > 0$ smaller if necessary, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2(h) \le -\kappa \, \mathcal{F}_2(h),$$

for some $\kappa > 0$, which implies $\mathcal{F}_2(h(t)) \le e^{-\kappa t} \mathcal{F}_2(h(0))$. We conclude the proof of Proposition 4.1 by using once again the equivalences established in Lemma 4.12. ☐

5. Proof of hypocoercivity by the commutator method

In this section we give an alternative elegant proof of our main result in Theorem 1.1 using a commutator's method, under the additional hypotheses that the linear collision operator $\mathscr{C}$ is bounded in $L^2(\mu^{-1})$ and ϕ has bounded second derivative and is super-linear. Using the notation of Subsection 2.2 and under these additional boundedness and growth conditions, the following Proposition recovers Theorem 1.1 under these additional assumptions. We include this alternative commutator approach because of its interesting algebraic properties, in spite of the additional hypotheses (that could most likely be relaxed with further work).

Proposition 5.1. *In addition to* (H0)–(H1)–(H2)–(H3)–(H4)–(H5)–(H6)–(H7)–(H8) *assume that the operator $\mathscr{C}$ is bounded on $L^2(\mu)$, that $\lim_{|x|\to\infty} |\nabla\phi(x)| = \infty$ and that derivatives of order two or more of ϕ are bounded. Consider then h solution to* (2.10) *in $L^2(\mathcal{M})$. Then there exists $\kappa > 0$ such that*

$$\|h(t)\| \le 2e^{-\kappa t} \|h_0\|$$

where κ depends only on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant defined in (1.5).

Recall that in all what follows $h = r + m\cdot v + e\mathfrak{E}(v) + h^\perp$ and that r, m and e satisfy the conservation laws stated at the end of subsection 2.2. We define π the orthogonal projection onto fluid functions, that is here $\pi h = r + m\cdot v + e\mathfrak{E}(v)$. recall that ∇_x and ∇_v map scalar functions to vectorial functions, and their $L^2(\mathcal{M})$-adjoints are $\nabla_x^* = -\nabla_x \cdot + \nabla_x\phi$ and $\nabla_v^* = -\nabla_v \cdot + v\cdot$ and map vectorial functions back to scalar functions. The operators ∇_x and ∇_v commute; *however* because of the vectorial aspect, when contracting back with their adjoints the order matters and such contraction do not always commute. The key commutator property is

$$[\nabla_v, \mathcal{T}] = -\nabla_x, \quad [\nabla_x, \mathcal{T}] = H_\phi \nabla_v, \quad [\nabla_v^*, \mathcal{T}] = -\nabla_x^*, \quad [\nabla_x^*, \mathcal{T}] = \left(\nabla_x^2 \nabla_v\right)^* \tag{5.1}$$

where we denote for convenience $H_\phi = (\partial^2_{x_i x_j}\phi)_{i,j}$ the Hessian matrix of ϕ.

In addition to the operator $\Omega = \nabla_x^* \cdot \nabla_x + 1$ defined in (1.6), we also introduce the scalar operators

$$\Gamma = \nabla_v^* \cdot \nabla_v + 1, \qquad \Lambda = \nabla_v^* \cdot \nabla_v + \nabla_x^* \cdot \nabla_x + 1, \tag{5.2}$$

and we shall denote them by the same letter when acting coordinate by coordinate on tensors. From e.g. [4] and Subsection 4.3 (see also [17], [19]), these operators are self-adjoint in $L^2(\mathcal{M})$. As in the micro-macro method presented in the preceding section, the core of the analysis is done through a cascade of estimates, but here directly on the original equation and not on macroscopic quantities.

5.1. **Cascade of infinite-dimensional correctors.** The three following operators are in the core of the cascade method and will play the role of *kinetic correctors*

$$\begin{cases} A_0 := \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^* \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}} \nabla_v \otimes \nabla_x \otimes \nabla_x, \\[2mm] A_1 := C_1\, \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac{1}{2}} \nabla_v \otimes \nabla_v \otimes \nabla_x, \\[2mm] A_2 := C_2\, \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_v \otimes \nabla_v, \end{cases} \tag{5.3}$$

From [17, 19] or simply by standard pseudo-differential calculus arguments (see Lemma A.4), A_0, A_1 and A_2 are bounded operators in $L^2(\mathcal{M})$. The reason of their introduction is the following. Denoting $\mathbb{R}_n[V]$ the space of real polynomials of v with degree less or equal to $n \in \mathbb{N}$, we have

$$\begin{cases} \nabla_v^{\otimes 3}\left(\mathbb{R}_2[V]\right) \subset \{0\}, \quad \nabla_v^{\otimes 2}\left(\mathbb{R}_1[V]\right) \subset \{0\}, \quad \nabla_v\left(\mathbb{R}_0[V]\right) \subset \{0\}, \\[2mm] A_2\left(\mathbb{R}_2[V]\right) \subset \{0\}, \quad A_1\left(\mathbb{R}_1[V]\right) \subset \{0\}, \quad A_0\left(\mathbb{R}_0[V]\right) \subset \{0\}. \end{cases}$$

This means that the A_is provide a cascading system to study the low-order moments, remembering that higher moments are already controlled by the damping from the linear collision kernel.

Remark 5.2. *Let us mention here that A_0 is very close to the fundamental corrector $\nabla_x^* \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \nabla_v$ used in [19] for Fokker-Planck type models, with here the addition of two more derivatives ∇_x or ∇_x^* at each extremities, and a larger power of Λ in the center to induce boundedness.*

Let us now introduce the three following self-adjoint non negative operators, which will appear after commutation between the A_i's and the transport $\mathcal{T}$.

$$\begin{cases} \Lambda_0 := \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^* \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}} \nabla_x \otimes \nabla_x \otimes \nabla_x, \\ \Lambda_1 := \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac{1}{2}} \nabla_x \otimes \nabla_v \otimes \nabla_x \\ \qquad\quad + \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac{1}{2}} \nabla_v \otimes \nabla_x \otimes \nabla_x, \\ \Lambda_2 := \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_x \otimes \nabla_v \otimes \nabla_v \\ \qquad\quad + \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_x \otimes \nabla_v \\ \qquad\quad + \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_v \otimes \nabla_x. \end{cases}$$

We postpone to the Subsection A.3 in the Appendix the verification that they indeed are self-adjoint, and note that they are bounded, where the proof of the boundedness follows exactly the same lines as the one of the A_i's, see Lemma A.4. For convenience we introduce for all study the following macroscopic error quantities (note that they are different from the one introduced in (4.2)-(4.4)).

$$\tilde e := e - \langle e\rangle \tag{5.4}$$

$$\tilde m := m - \langle \nabla_x m\rangle x - \langle m\rangle \tag{5.5}$$

$$\tilde r := r - \frac12 \langle \nabla_x^{\otimes 2} r\rangle : \big(x\otimes x - \langle x\otimes x\rangle\big) - \langle \nabla_x r\rangle\cdot x - \langle r\rangle \tag{5.6}$$

The following Lemma is a first step in the commutator approach.

Proposition 5.3. *For all $i\in\{0,1,2\}$ we have*

$$\frac{\mathrm d}{\mathrm dt}\langle A_i h, h\rangle = -\langle \Lambda_i h, h\rangle + \langle B_i h, h\rangle \tag{5.7}$$

where the B_i's are bounded and satisfy

$$\begin{cases} \langle B_0 h, h\rangle \lesssim \|h\|\left(\|h^\perp\| + \|\tilde e\| + \|\tilde m\|\right) \\ \langle B_1 h, h\rangle \lesssim \|h\|\left(\|h^\perp\| + \|\tilde e\|\right) \\ \langle B_2 h, h\rangle \lesssim \|h\|\|h^\perp\|. \end{cases} \tag{5.8}$$

Proof of Proposition 5.3. Since $\mathcal C$ is self-adjoint and $\mathcal T$ is skew-adjoint, we get for $i=0,1,2$

$$\frac{\mathrm d}{\mathrm dt}\langle A_i h, h\rangle = \langle \left([A_i,\mathcal T] + A_i\mathcal C + \mathcal C A_i\right) h, h\rangle\,.$$

We can therefore write $[A_i,\mathcal T] + A_i\mathcal C + \mathcal C A_i = B_i - \Lambda_i$ where by using (5.1) we have the following explicit formulas

$$\begin{aligned} B_0 :=\ & A_0\,\mathcal C + \mathcal C\, A_0 \\ &+ \nabla_x^*\otimes\nabla_x^*\otimes\nabla_x^*[\Lambda^{-\frac32},\mathcal T] : \Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes\nabla_x \\ &+ \nabla_x^*\otimes\nabla_x^*\otimes\nabla_x^*\Lambda^{-\frac32} : [\Lambda^{-\frac32},\mathcal T]\nabla_v\otimes\nabla_x\otimes\nabla_x \\ &+ \nabla_x^*\otimes\nabla_x^*\otimes\nabla_x^*\Lambda^{-\frac32} : \Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes(H_\phi\nabla_v) \\ &+ \nabla_x^*\otimes\nabla_x^*\otimes\nabla_x^*\Lambda^{-\frac32} : \Lambda^{-\frac32}\nabla_v\otimes(H_\phi\nabla_v)\otimes\nabla_x \\ &+ (H_\phi\nabla_v)^*\otimes\nabla_x^*\otimes\nabla_x^*\Lambda^{-\frac32} : \Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes\nabla_x \\ &+ \nabla_x^*\otimes\nabla_x^*\otimes(H_\phi\nabla_v)^*\Lambda^{-\frac32} : \Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes\nabla_x, \\ &+ \nabla_x^*\otimes(H_\phi\nabla_v)^*\otimes\nabla_x^*\Lambda^{-\frac32} : \Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes\nabla_x, \end{aligned} \tag{5.9}$$

$$\begin{aligned} B_1 :=\ & A_1\mathcal C + \mathcal C A_1 \\ &+ \nabla_x^*\otimes\nabla_v^*\otimes\nabla_x^*[\Lambda^{-1}\Gamma^{-\frac12},\mathcal T] : \Lambda^{-\frac32}\nabla_v\otimes\nabla_v\otimes\nabla_x \\ &+ \nabla_x^*\otimes\nabla_v^*\otimes\nabla_x^*\Lambda^{-\frac32} : [\Lambda^{-1}\Gamma^{-\frac12},\mathcal T]\nabla_v\otimes\nabla_v\otimes\nabla_x \\ &+ \nabla_x^*\otimes\nabla_v^*\otimes\nabla_x^*\Gamma^{-\frac12}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac12}\nabla_v\otimes\nabla_v\otimes(H_\phi\nabla_v) \\ &+ \nabla_x^*\otimes\nabla_v^*\otimes(H_\phi\nabla_v)^*\Gamma^{-\frac12}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_x \\ &- \nabla_x^*\otimes\nabla_x^*\otimes\nabla_x^*\Gamma^{-\frac12}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_x \\ &+ (H_\phi\nabla_v)^*\otimes\nabla_v^*\otimes\nabla_x^*\Gamma^{-\frac12}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_x, \end{aligned} \tag{5.10}$$

$$\begin{aligned} B_2 :=\ & A_2\mathcal C + \mathcal C A_2 \\ &+ \nabla_v^*\otimes\nabla_v^*\otimes\nabla_x^*[\Lambda^{-\frac12}\Gamma^{-1},\mathcal T] : \Gamma^{-1}\Lambda^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_v \\ &+ \nabla_v^*\otimes\nabla_v^*\otimes\nabla_x^*\Lambda^{-\frac12}\Gamma^{-1} : [\Gamma^{-1}\Lambda^{-\frac12},\mathcal T]\nabla_v\otimes\nabla_v\otimes\nabla_v \end{aligned} \tag{5.11}$$

$$+\nabla_v^*\otimes\nabla_v^*\otimes(H_\phi\nabla_v)^*\Gamma^{-1}\Lambda^{-\frac12}:\Lambda^{-\frac12}\Gamma^{-1}\nabla_v\otimes\nabla_v\otimes\nabla_v$$
$$-\nabla_v^*\otimes\nabla_x^*\otimes\nabla_x^*\Gamma^{-1}\Lambda^{-\frac12}:\Lambda^{-\frac12}\Gamma^{-1}$$
$$-\nabla_x^*\otimes\nabla_v^*\otimes\nabla_x^*\Gamma^{-1}\Lambda^{-\frac12}:\Lambda^{-\frac12}\Gamma^{-1}:\nabla_v\otimes\nabla_v\otimes\nabla_v,$$

We first deal with B_2. We note first that $|\langle A_2\mathcal{C}h,h\rangle| = |\langle A_2\mathcal{C}h^\perp,h\rangle| \lesssim \|h^\perp\|\|h\|$ and that $|\langle \mathcal{C}A_2h,h\rangle| = |\langle A_2h,\mathcal{C}h\rangle| = |\langle A_2h,\mathcal{C}h^\perp\rangle| \lesssim \|h^\perp\|\|h\|$ since we assumed in this section that $\mathcal{C}$ is bounded and self-adjoint. Next we deal with the second line in the definition of B_2 : we freely use that $\Lambda^{\frac12}\Gamma[\Lambda^{-\frac12}\Gamma^{-1},\mathcal{T}]$ is bounded by standard pseudo-differential calculus (see [17]) to get that

$$B_{2,2}:=\nabla_v^*\otimes\nabla_v^*\otimes\nabla_x^*[\Lambda^{-\frac12}\Gamma^{-1},\mathcal{T}]$$

is bounded. Now the second line writes

$$\left|\left\langle B_{2,2}:\Gamma^{-1}\Lambda^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_v h,h\right\rangle\right| = \left|\left\langle B_{2,2}:\Gamma^{-1}\Lambda^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_v\left(r+mv+e\mathfrak{E}+h^\perp\right),h\right\rangle\right|$$
$$\lesssim \left|\left\langle B_{2,2}:\Gamma^{-1}\Lambda^{-\frac12}\nabla_v\otimes\nabla_v\otimes\nabla_v h^\perp,h\right\rangle\right| \lesssim \|h^\perp\|\|h\|$$

since three derivatives on the right cancel all macroscopic quantities. For the other terms, using that H_ϕ is bounded and similar arguments and get eventually that $|\langle B_2h,h\rangle| \lesssim \|h\|\|h^\perp\|$.

Now we deal with the term B_1. The treatment of $A_1\mathcal{C}+\mathcal{C}A_1$ is the same as done before. Let us have a look at the second line in the definition of B_1 : we again freely use that $\Lambda\Gamma^{\frac12}[\Lambda^{-1}\Gamma^{-\frac12},\mathcal{T}]$ is bounded by standard pseudo-differential calculus to get that

$$B_{1,2}:=\nabla_x^*\otimes\nabla_v^*\otimes\nabla_x^*[\Lambda^{-1}\Gamma^{-\frac12},\mathcal{T}]$$

is bounded. Now the second line writes

$$\left|\left\langle B_{1,2}:\Gamma^{-\frac12}\Lambda^{-1}\nabla_v\otimes\nabla_v\otimes\nabla_x h,h\right\rangle\right| = \left|\left\langle B_{1,2}:\Gamma^{-\frac12}\Lambda^{-1}\nabla_v\otimes\nabla_v\otimes\nabla_x(r+mv+e\mathfrak{E}+h^\perp),h\right\rangle\right|$$
$$\lesssim \left|\left\langle B_{1,2}:\Gamma^{-\frac12}\Lambda^{-1}\nabla_v\otimes\nabla_v\otimes\nabla_x(e\mathfrak{E}+h^\perp),h\right\rangle\right| \lesssim \left|\left\langle B_{1,2}:\Omega^{-1}\nabla_x e\otimes \mathrm{Id}_{d\times d},h\right\rangle\right| + \|h^\perp\|\|h\|,$$

where we used that

$$\nabla_v\otimes\nabla_v\otimes\nabla_x e\mathfrak{E} = \nabla_x e\otimes\mathrm{Id}_{d\times d}$$

and that the two other macroscopic quantities are canceled since two derivatives in velocity are involved. Now we refer to the appendix for the proof of the following bound

$$\|\Omega^{-1}\nabla_x e\| \le \|\Omega^{-\frac12}\nabla_x e\| \le \|\tilde e\|$$

where for the first inequality we used that $\Omega \ge \mathrm{Id}_{d\times d}$. Eventually we get

$$\left|\left\langle B_{1,2}:\Gamma^{-\frac12}\Lambda^{-1}\nabla_v\otimes\nabla_v\otimes\nabla_x h,h\right\rangle\right| \le \left(\|\tilde e\|+\|h^\perp\|\right)\|h\|.$$

The same proof can be adapted to all other terms in B_2 so that they can be bounded similarly.

Eventually we only sketch the proof for the term B_0. The treatment of $A_0\mathcal{C}+\mathcal{C}A_0$ is similar. We focus on the second line of B_0 again and denote

$$B_{0,2}:=\nabla_v^*\otimes\nabla_v^*\otimes\nabla_x^*\left[\Lambda^{-\frac32},\mathcal{T}\right],$$

which is bounded by the same type of argument. We now similarly show that the second line writes

$$\left|\left\langle B_{0,2}:\Lambda^{-\frac32}\nabla_v\otimes\nabla_x\otimes\nabla_x h,h\right\rangle\right| \le \left(\|\tilde e\|+\|\tilde m\|+\|h^\perp\|\right)\|h\|. \tag{5.12}$$

Indeed a direct computation gives

$$\Gamma^{-1}\Lambda^{-\frac12}\nabla_v\otimes\nabla_x\otimes\nabla_x h = \Omega^{-\frac32}\nabla_x\otimes\nabla_x m + \tilde\Omega^{-\frac32}\nabla_x\otimes\nabla_x\otimes ve$$

where we introduced $\tilde\Omega = (\nabla_x^*\nabla_x+2)$. This factors 2 comes form the fact that for all $\alpha\in\mathbb{R}$,

$$\Lambda^\alpha(ve) = (\nabla_x^*\nabla_x+\nabla_v^*\nabla_v+1)^\alpha(ve) = (\nabla_x^*\nabla_x+1+1)^\alpha ve$$

since v is an eigen-function of $\nabla_v^*\nabla_v$ with eigenvalue 1. To conclude to the proof of (5.12), this is sufficient to notice that

$$\left\|\Omega^{-\frac{3}{2}}\nabla_x^2 e\right\| \le \left\|\Omega^{-1}\nabla_x^2 e\right\| \le \|\tilde{e}\|$$

and (see Subsection A.4 in the Appendix)

$$\left\|\Omega^{-\frac{3}{2}}\nabla_x^2 m\right\| \le \left\|\Omega^{-1}\nabla_x^2 m\right\| \le \|\tilde{m}\|$$

The other terms in B_0 can be treated similarly. This ends the proof. □

Now we deal with the main non negative terms of type $\langle \Lambda_i h, h\rangle$ appearing in (5.7) for which spectral gap properties will arise. The second key step in the commutator approach is the following.

Proposition 5.4. *There exists positive constructive constants C_0, C_1, C_2 and $\bar\lambda_0$, $\bar\lambda_1$ and $\bar\lambda_2$ such that*

$$\begin{cases} -\langle \Lambda_0 h, h\rangle \le -\bar\lambda_0\,\|\tilde r\|^2 + C_0\left(\|\tilde m\| + \|\tilde e\| + \|h^\perp\|\right)\|h\| \\ -\langle \Lambda_1 h, h\rangle \le -\bar\lambda_1\,\|\tilde m\|^2 + C_1\left(\|\tilde e\| + \|h^\perp\|\right)\|h\| \\ -\langle \Lambda_2 h, h\rangle \le -\bar\lambda_2\,\|\tilde e\|^2 + C_2\|h^\perp\|\|h\| \end{cases} \tag{5.13}$$

Proof of Proposition 5.4. In the first part of the proof we compute the effect of the Λ_0, Λ_1 and Λ_2 on respectively r, m and e. We first notice that due to the number of derivatives in velocity appearing in the right-hand side of the expressions giving the Λ_i's we have $\Lambda_i(\mathbb{R}_{i-1}[V]) \subset \{0\}$ for $i = 1, 2$ and $\Lambda_i(\mathbb{R}_i[V]) \subset \mathbb{R}_i[V]$ for $i = 0, 1, 2$, therefore one obtains

$$\begin{cases} \langle \Lambda_0 r, r\rangle = \displaystyle\int_{\mathbb{R}^{2d}} \left|\Omega^{-\frac{3}{2}}\nabla_x^3 r\right|^2 \mathcal{M}\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d} \left|\Omega^{-\frac{3}{2}}\nabla_x^3 r\right|^2 \rho\,\mathrm{d}x \\ \langle \Lambda_1(m(x)\cdot v), (m(x)\cdot v)\rangle \\ \quad = \displaystyle\int_{\mathbb{R}^{2d}} \Big((\Omega^{-1}\partial^2_{x_i x_k} m_j)(\Omega^{-1}\partial^2_{x_i x_k} m_j) + (\Omega^{-1}\partial^2_{x_i x_k} m_j)(\Omega^{-1}\partial^2_{x_j x_k} m_i)\Big)\,\mathcal{M}\,\mathrm{d}x\,\mathrm{d}v \\ \quad = 2\displaystyle\int_{\mathbb{R}^d} \left|\Omega^{-1}\nabla_x\nabla_x^{\mathrm{sym}} m\right|^2 \rho\,\mathrm{d}x \\ \langle \Lambda_2\left(e(x)\mathfrak{E}(v)\right), \left(e(x)\mathfrak{E}(v)\right)\rangle \\ \quad = \displaystyle\int_{\mathbb{R}^{2d}} \big((\Omega^{-\frac{1}{2}}\partial_{x_k} e)(\partial^2_{v_i v_j}\mathfrak{E})(\Omega^{-\frac{1}{2}}\partial_{x_k} e)(\partial^2_{v_i v_j}\mathfrak{E}) \\ \qquad\qquad + (\Omega^{-\frac{1}{2}}\partial_{x_j} e)(\partial^2_{v_i v_k}\mathfrak{E})(\Omega^{-\frac{1}{2}}\partial_{x_k} e)(\partial^2_{v_i v_j}\mathfrak{E}) \\ \qquad\qquad + (\Omega^{-\frac{1}{2}}\partial_{x_i} e)(\partial^2_{v_j v_k}\mathfrak{E})(\Omega^{-\frac{1}{2}}\partial_{x_k} e)(\partial^2_{v_i v_j}\mathfrak{E})\big)\,\mathcal{M}\,\mathrm{d}x\,\mathrm{d}v \\ = \displaystyle\int_{\mathbb{R}^d} \left(2(\Omega^{-\frac{1}{2}}\partial_{x_k} e)^2 + \frac{2}{d}(\Omega^{-\frac{1}{2}}\partial_{x_i} e)^2 + \frac{2}{d}(\Omega^{-\frac{1}{2}}\partial_{x_j} e)^2\right)\rho\,\mathrm{d}x \\ = \left(\dfrac{4}{d} + 2\right)\displaystyle\int_{\mathbb{R}^d} \left|\Omega^{-\frac{1}{2}}\nabla_x e\right|^2 \rho\,\mathrm{d}x \end{cases} \tag{5.14}$$

Now we can use the cascade of Poincaré inequalities as stated in Lemma A.6. For the density r, this implies that there exists a constant $\overline{\lambda}_0$, depending only on the Poincaré constant and semi-norms of ϕ such that

(5.15)

$$\langle \Lambda_0 r, r\rangle = \int_{\mathbb{R}^d} \left|\Omega^{-\frac{3}{2}}\nabla_x^3 r\right|^2 \rho\,\mathrm{d}x \ge 2\overline{\lambda}_0\left\|r - \langle r\rangle - \langle \nabla_x r\rangle\cdot x - \frac{1}{2}\langle \nabla_x^2 r\rangle : [x\otimes x - \langle x\otimes x\rangle]\right\|^2 = 2\overline{\lambda}_0\,\|\tilde r\|^2$$

Concerning the momentum m, one first observes that $|\Omega^{-1}\nabla_x\nabla_x^{\mathrm{sym}}m|^2 \geq \frac{1}{9}|\Omega^{-1}\nabla_x^2 m|^2$ thanks to the Schwartz lemma in the following version

$$\forall\, i,j,k \in \{1,\dots,d\}, \quad \partial^2_{ij} m_k = \partial_i \left(\nabla^{\mathrm{sym}} m\right)_{jk} + \partial_j \left(\nabla^{\mathrm{sym}} m\right)_{ik} - \partial_k \left(\nabla^{\mathrm{sym}} m\right)_{ij}.$$

The cascade of Poincaré inequalities at order 2 stated in Lemma A.6 then implies from (5.14) that there exists a constant $\overline{\lambda_1}$ such that

$$\langle \Lambda_1 m, m\rangle \geq \frac{2}{9}\|\nabla_x^2 m\|^2 \geq \frac{2}{9}\lambda_0^2 \left\|m - \langle m\rangle\langle\nabla_x m\rangle x - \langle m\rangle\right\|^2 = 2\overline{\lambda_1}\|\tilde m\|^2, \tag{5.16}$$

Eventually for the energy e, we can use the standard the Poincaré inequality in $L^2(\rho)$ (the order 1 inequality Lemma A.6) and one obtains that there exists a constant $\overline{\lambda}_2$ such that

$$\langle \bar\Lambda_2 e, e\rangle = \left(\frac{4}{d}+2\right)\|\nabla_x e\|^2 \geq \left(\frac{4}{d}+2\right) c_1 \|e - \langle e\rangle\|^2 = 2\overline{\lambda}_2\|\tilde e\|^2 \tag{5.17}$$

Thanks to the above estimates, we can now investigate all terms appearing in $\langle \Lambda_i h, h\rangle$. We get according to the number of velocity and space gradients appearing in Λ_2 that

$$\langle \Lambda_2 h, h\rangle = \Big\langle \Lambda_2(e\mathfrak{E}(v)), e\mathfrak{E}(v)\Big\rangle + \Big\langle \Lambda_2(e\mathfrak{E}(v)), h^\perp\Big\rangle + \Big\langle \Lambda_2 h^\perp, h\Big\rangle,$$

from which one obtains with (5.17) that

$$-\langle \Lambda_2 h, h\rangle \leq -\bar\lambda_2\|\tilde e\|^2 + \mathcal{O}\left(\|h^\perp\|\|h\|\right). \tag{5.18}$$

Similarly for m, one has using in addition that Λ_1 is skewadjoint and that $\Lambda_1 e\mathfrak{E}(v) = \Lambda_1 \tilde e\mathfrak{E}(v)$

$$\langle \Lambda_1 h, h\rangle = \langle \Lambda_1(m\cdot v), m\cdot v\rangle + \Big\langle \Lambda_1(m\cdot v), (e-\langle e\rangle)\mathfrak{E}(v) + h^\perp\Big\rangle + \Big\langle \Lambda_1\left((e-\langle e\rangle)\,\mathfrak{E}(v) + h^\perp\right), h\Big\rangle,$$

which implies using (5.16) that

$$-\langle C_1\Lambda_1 h, h\rangle \leq -\bar\lambda_1\|\tilde m\|^2 + \mathcal{O}(\|\tilde e\|\|h\|) + \mathcal{O}(\|h^\perp\|\|h\|). \tag{5.19}$$

Finally, for the last term r, using again that Λ_0 is self-adjoint and cancels polynomials in space of order less than 1, one has similarly

$$\langle \Lambda_0 h, h\rangle = \langle \Lambda_0 r, r\rangle + \Big\langle \Lambda_0 r, \Big(\tilde m\cdot v + \tilde e\mathfrak{E}(v) + h^\perp\rangle + \langle \Lambda_0(\tilde m\cdot v + \tilde e\mathfrak{E}(v) + h^\perp\Big), h\Big\rangle$$

and it follows from (5.15) that

$$-\langle \Lambda_0 h, h\rangle \leq -\bar\lambda_0\|\tilde r\|^2 + \mathcal{O}\left(\|\tilde m\|\|h\|\right) + \mathcal{O}\left(\|\tilde e\|\|h\|\right) + \mathcal{O}\left(\|h^\perp\|\|h\|\right). \tag{5.20}$$

The proof of Proposition 5.4 is complete. □

Remark 5.5. *Note here that global averages are all controlled by the base norm thanks to the moments assumed on ϕ, and precisely we have (after possible integration by parts)*

$$\langle \nabla_x r\rangle = \langle r\nabla_x\phi\rangle \lesssim \|h\|, \qquad \langle \nabla_x^2 r\rangle = \langle r(\nabla_x\phi\otimes\nabla_x\phi - \nabla_x^2\phi)\rangle \lesssim \|h\|$$

$$\langle \nabla_x m\rangle = \langle m\otimes\nabla_x\phi\rangle \lesssim \|h\|, \qquad \langle e\rangle \lesssim \|h\|.$$

so that $\tilde r$, $\tilde m$ and $\tilde e$ are themselves bounded in norm by $\|h\|$.

We now in position to build the first part of our hypocoercivity norm as

$$\|h\|^2_{\mathcal{H}_1} := \|h\|^2 + \varepsilon_0\langle A_0 h, h\rangle + \varepsilon_1\langle A_1 h, h\rangle + \varepsilon_2\langle A_2 h, h\rangle$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$ will be chosen small later. Then Propositions 5.3 and 5.4 yield

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_1} \leq{}& -\nu\|h^\perp\|^2 - \varepsilon_2\bar\lambda_2\left\|\widetilde{e}\right\|^2 - \varepsilon_1\bar\lambda_1\|\widetilde{m}\|^2 - \varepsilon_0\bar\lambda_0\left\|\widetilde{r}\right\|^2 \\ &+ C\varepsilon_2\|h\|\|h^\perp\| + C\varepsilon_1\|h\|\left(\|h^\perp\| + \|\widetilde{e}\|\right) + C\varepsilon_0\|h\|\left(\|h^\perp\| + \|\widetilde{m}\| + \|\widetilde{e}\|\right),\end{aligned}$$

for some constant $C > 0$. We can simplify the latter estimate by optimizing the small constants and we get the following lemma, which concludes that first step of the commutator method.

Lemma 5.6. *Choosing $\nu \gg \varepsilon_2 \gg \varepsilon_1 \gg \varepsilon_0$, we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_1} \le -\frac{\nu}{2}\|h^\perp\|^2 - \frac{\varepsilon_2}{2}\bar\lambda_2\,\|\widetilde{e}\|^2 - \frac{\varepsilon_1}{2}\bar\lambda_1\|\widetilde{m}\|^2 - \varepsilon_0\bar\lambda_0\,\|\widetilde{r}\|^2 + \eta_1\|h\|^2, \tag{5.21}$$

for some $0 < \eta_1 \ll \varepsilon_0$.

5.2. **Modifications of the first hypocoercivity norm.** In the preceding subsection, we were able to build a partial Lyapunov functional allowing to control all microscopic parts $\tilde e$, $\tilde m$ and $\tilde r$ of h. There only remains to get a control and build associated additional Lyapunov terms for the macroscopic/finite dimensional parts

$$\langle r\rangle, \qquad \langle\nabla_x m\rangle\cdot x + \langle m\rangle \quad \text{ and } \quad \frac{1}{2}\langle\nabla_x^{\otimes 2} r\rangle : \big(x\otimes x - \langle x\otimes x\rangle\big) - \langle\nabla_x r\rangle\cdot x - \langle r\rangle.$$

This is done using the conservation laws as in Subsection 4.3. We only sketch the proof.

5.2.1. *Control of derivatives of micro quantities.* For further use, we note first that the *time-variation* of the density, the momentum and the energy controlled by the right-hand side in the above estimate can also be controlled as follows. Indeed we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\partial_t\widetilde{e},\Omega^{-1}\widetilde{e}\big\rangle \ge \|\Omega^{-\frac12}\partial_t\widetilde{e}\|^2 - \mathcal{O}(\|h\|\|\widetilde{e}\|)\\
&\frac{\mathrm{d}}{\mathrm{d}t}\big\langle\partial_t\widetilde{m},\Omega^{-1}\widetilde{m}\big\rangle \ge \|\Omega^{-\frac12}\partial_t\widetilde{m}\|^2 - \mathcal{O}(\|h\|\|\widetilde{m}\|)\\
&\frac{\mathrm{d}}{\mathrm{d}t}\,\big\langle\partial_t\widetilde{r},\Omega^{-1}\widetilde{r}\big\rangle \ge \Big\|\Omega^{-\frac12}\partial_t\widetilde{r}\Big\|^2 - \mathcal{O}\left(\|h\|\,\|\widetilde{r}\|\right),
\end{aligned}$$

remarking that we have

$$\big|\big\langle\partial_t\widetilde{e},\Omega^{-1}\widetilde{e}\big\rangle\big| + \big|\big\langle\partial_t\widetilde{m},\Omega^{-1}\widetilde{m}\big\rangle\big| + \big|\big\langle\partial_t\widetilde{r},\Omega^{-1}\widetilde{r}\big\rangle\big| \lesssim \|h\|^2.$$

and that second-order time-derivatives of macroscopic quantities can be controlled by Ω (as was done also in the micro-macro approach in the preceding section). This allows to update the hypocoercivity norm to

$$\|h\|^2_{\mathcal{H}_2} := \|h\|^2_{\mathcal{H}_1} - \varepsilon_2'\big\langle\partial_t\widetilde{e},\Omega^{-1}\widetilde{e}\big\rangle - \varepsilon_1'\big\langle\partial_t\widetilde{m},\Omega^{-1}\widetilde{m}\big\rangle - \varepsilon_0'\,\big\langle\partial_t\widetilde{r},\Omega^{-1}\widetilde{r}\big\rangle\,. \tag{5.22}$$

We get then the following second partial Lyapunov functional updating (5.21)

Lemma 5.7. *For $1 \gg \varepsilon_2 \gg \varepsilon_2' \gg \varepsilon_1 \gg \varepsilon_1' \gg \varepsilon_0 \gg \varepsilon_0' \gg \eta_1$, we have*

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|h\|^2_{\mathcal{H}_2} \le -\frac{\nu}{4}\|h^\perp\|^2 - \frac{\varepsilon_2}{4}\bar\lambda_2\,\|\widetilde{e}\|^2 - \frac{\varepsilon_1}{4}\bar\lambda_1\|\widetilde{m}\|^2 - \frac{\varepsilon_0}{2}\bar\lambda_0\,\|\widetilde{r}\|^2&\\
- \varepsilon_2'\bar\lambda_2\,\Big\|\Omega^{-\frac12}\partial_t\widetilde{e}\Big\|^2 - \varepsilon_1'\bar\lambda_1\,\Big\|\Omega^{-\frac12}\partial_t\widetilde{m}\Big\|^2 - \varepsilon_0'\bar\lambda_0\,\Big\|\Omega^{-\frac12}\partial_t\widetilde{r}\Big\|^2 + \eta_2\|h\|^2&
\end{aligned} \tag{5.23}$$

for some $0 < \eta_2 \ll \varepsilon_0'$.

5.2.2. *Control of means of moment.* We focus now on means of the moment m and its derivatives. Given $i \ne j$, v_iv_j is orthogonal to $1, v, |v|^2$, and hence one easily computes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v_iv_jh\rangle = 2\left\langle(\nabla_x^{\mathrm{sym}}m)_{ij}\right\rangle + \left\langle[\mathcal{T}(v_iv_j)+\mathcal{C}(v_iv_j)]h^\perp\right\rangle,$$

therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle v_iv_jh\rangle\right)\langle v_iv_jh\rangle\right] \ge 2\left\langle(\nabla_x^{\mathrm{sym}}m)_{i,j}\right\rangle^2 - \left\langle[\mathcal{T}(v_iv_j)+\mathcal{C}(v_iv_j)]h^\perp\right\rangle^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}^2t}\langle v_iv_jh\rangle\right)\langle v_iv_jh\rangle$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle v_iv_jh\rangle\right)(\langle v_iv_jh\rangle) \lesssim \|h\|\|h^\perp\|$$

$$\left|\left\langle [\mathcal{T}(v_iv_j)+\mathcal{C}(v_iv_j)]h^\perp\right\rangle^2\right| + \left(\frac{\mathrm{d}^2}{\mathrm{d}^2t}\langle v_iv_jh\rangle\right)(\langle v_iv_jh\rangle) \lesssim \|h\|\|h^\perp\|.$$

Define for each $i=1,\dots,d$

$$\psi_i(v) := 1 + \sqrt{\frac{d}{2}}\left(1+\frac{4}{d}\right)\mathfrak{E}(v) - \sqrt{\frac{d}{2}}|v_i|^2\mathfrak{E}(v) \tag{5.24}$$

which is orthogonal to $1, v, |v|^2$ and compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi_i h\rangle = 4\left\langle \frac{\nabla_x\cdot m}{d} - \partial_{x_i}m_i\right\rangle + \left\langle\left[\mathcal{T}\left(\psi_i\right)+\mathcal{C}\left(\psi_i\right)\right]h^\perp\right\rangle,$$

hence we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_ih\rangle\right)\langle\psi_ih\rangle\right] \geq 8\left\langle\frac{\nabla_x\cdot m}{d}-\partial_{x_i}m_i\right\rangle^2 - \left\langle[\mathcal{T}(\psi_i(v))+\mathcal{C}(\psi_i(v))]h^\perp\right\rangle^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}^2t}\langle\psi_ih\rangle\right)\langle\psi_ih\rangle$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_ih\rangle\right)(\langle\psi_ih\rangle) \lesssim \|h\|^2$$

$$\left|\left\langle[\mathcal{T}(\psi_i(v))+\mathcal{C}(\psi_i(v))]h^\perp\right\rangle\right|^2 + \left(\frac{\mathrm{d}^2}{\mathrm{d}^2t}\langle\psi_ih\rangle\right)(\langle\psi_ih\rangle) \lesssim \|h\|\|h^\perp\|.$$

This allows to update the hypocoercivity norm (5.22) into

$$\begin{aligned}\|h\|^2_{\mathcal{H}_3} := \|h\|^2_{\mathcal{H}_2} &- \varepsilon_3\sum_{i\neq j}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle v_iv_jh\rangle\right)(\langle v_iv_jh\rangle) - \varepsilon_3\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi_ih\rangle\right)(\langle\psi_ih\rangle)\\ &- \varepsilon_3'\sum_{i\neq j}\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\left(\nabla_x^{\mathrm{sym}}m\right)_{i,j}\right\rangle\left\langle\left(\nabla_x^{\mathrm{sym}}m\right)_{i,j}\right\rangle - \varepsilon_3'\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\nabla_x\cdot m}{d}-\partial_{x_i}m_i\right)\right\rangle\left\langle\frac{\nabla_x\cdot m}{d}-\partial_{x_i}m_i\right\rangle\end{aligned} \tag{5.25}$$

for $1\gg\varepsilon_3\gg\varepsilon_3'\gg\varepsilon_2\gg\varepsilon_2'\gg\varepsilon_1\gg\varepsilon_1'\gg\varepsilon_0\gg\varepsilon_0'$. Therefore, defining as in (4.3) and (4.2)

$$m_s := m - \langle\nabla_x^{\mathrm{skew}}m\rangle x - \frac{1}{d}\langle\nabla_x\cdot m\rangle x - \langle m\rangle \quad\text{and}\quad e_s := \tilde{e} = e - \langle e\rangle, \tag{5.26}$$

we update (5.23) into

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|h\|^2_{\mathcal{H}_3} \leq &-\frac{\nu}{8}\|h^\perp\|^2 - \frac{\varepsilon_2}{4}\bar{\lambda}_2\|e_s\|^2 - \frac{\varepsilon_1}{8}\bar{\lambda}_1\|\widetilde{m}\|^2 - \frac{\varepsilon_1}{8}\bar{\lambda}_1\|m_s\|^2 - \frac{\varepsilon_0}{2}\bar{\lambda}_0\|\widetilde{r}\|^2 - \frac{\varepsilon_2'}{2}\bar{\lambda}_2\left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{e}\right\|^2\\ &-\frac{\varepsilon_1'}{2}\bar{\lambda}_1\left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{m}\right\|^2 - \frac{\varepsilon_1'}{2}\bar{\lambda}_1\left\|\Omega^{-\frac{1}{2}}\partial_tm_s\right\|^2 - \varepsilon_0'\bar{\lambda}_0\left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{r}\right\|^2 + \eta_3\|h\|^2\end{aligned} \tag{5.27}$$

for another $0<\eta_3\ll\varepsilon_0'$. Let us denote by $\mathcal{C}[h]$ the semi-norm of the controlled quantities

$$\begin{aligned}\mathcal{C}_3[h] := \Bigg[&\|h^\perp\|^2 + \|\widetilde{e}\|^2 + \|\widetilde{m}\|^2 + \|m_s\|^2 + \|\widetilde{r}\|^2\\ &+\left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{e}\right\|^2 + \left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{m}\right\|^2 + \left\|\Omega^{-\frac{1}{2}}\partial_tm_s\right\|^2 + \left\|\Omega^{-\frac{1}{2}}\partial_t\widetilde{r}\right\|^2\Bigg]^{\frac{1}{2}}\end{aligned}$$

and

$$\varepsilon := \min\left\{\frac{\nu}{8}, \frac{\varepsilon_2\bar{\lambda}_2}{4}, \frac{\varepsilon_1\bar{\lambda}_1}{8}, \frac{\varepsilon_0\bar{\lambda}_0}{2}, \frac{\varepsilon_2'\bar{\lambda}_2}{2}, \frac{\varepsilon_1'\bar{\lambda}_1}{2}, \varepsilon_0'\bar{\lambda}_0\right\} \quad\text{and}\quad \eta := \max\{\eta_1,\eta_2,\eta_3\}\ll\varepsilon.$$

The estimate (5.27) then rewrites, for some $0<\eta\ll\varepsilon\ll1$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_3} \leq -\varepsilon\,\mathcal{C}_3[h]^2 + \eta\|h\|^2. \tag{5.28}$$

5.2.3. *Control of the moments of the density.* We finally estimate averages of the local density r and of its derivatives in order to control the finite dimensional quantities

$$\langle \nabla_x^{\otimes 2} r\rangle : \big(x\otimes x - \langle x\otimes x\rangle\big) - \langle \nabla_x r\rangle\cdot x - \langle r\rangle. \tag{5.29}$$

To keep this section concise, we rather directly use Lemma 4.5 where we introduced in (4.6)

$$w_s := r - \sqrt{\frac{2}{d}}\langle e\rangle\phi - \langle\nabla_x w\rangle x - \frac{1}{2d}\langle\Delta w\rangle\xi_2 + \sqrt{\frac{2}{d}}\langle e\rangle\langle\phi\rangle.$$

This allows to update the hypocoercivity norm (5.25) into

$$\|h\|^2_{\mathcal{H}_4} := \|h\|^2_{\mathcal{H}_3} + \varepsilon_w\left\langle \Omega^{-1}\nabla_x w_s, m_s\right\rangle + \varepsilon_w'\left\langle -\Omega^{-1}\partial_t w_s, w_s\right\rangle \tag{5.30}$$

for $1\gg\varepsilon_3\gg\varepsilon_3'\gg\varepsilon_2\gg\varepsilon_2'\gg\varepsilon_1\gg\varepsilon_1'\gg\varepsilon_0\gg\varepsilon_0'\gg\varepsilon_w\gg\varepsilon_w'$. We then update (5.28) into

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_4} \le & -\frac{\nu}{16}\|h^\perp\|^2 - \frac{\varepsilon_2}{8}\bar\lambda_2\,\|e_s\|^2 - \frac{\varepsilon_1}{8}\bar\lambda_1\,\|\widetilde m\|^2 - \frac{\varepsilon_1}{16}\bar\lambda_1\,\|m_s\|^2 - \frac{\varepsilon_0}{2}\bar\lambda_0\,\|\widetilde r\|^2 - \frac{\varepsilon_w}{2}\,\|w_s\|^2\\ & - \frac{\varepsilon_2'}{2}\bar\lambda_2\left\|\Omega^{-\frac12}\partial_t\widetilde e\right\|^2 - \frac{\varepsilon_1'}{2}\bar\lambda_1\left\|\Omega^{-\frac12}\partial_t\widetilde m\right\|^2 - \frac{\varepsilon_1'}{2}\bar\lambda_1\left\|\Omega^{-\frac12}\partial_t m_s\right\|^2 - \varepsilon_0'\bar\lambda_0\left\|\Omega^{-\frac12}\partial_t\widetilde r\right\|^2\\ & - \varepsilon_w'\left\|\Omega^{-\frac12}\partial_t w_s\right\|^2 + \eta_4\|h\|^2\end{aligned} \tag{5.31}$$

for another $0<\eta_4\ll\varepsilon_w'$. Let us denote by $\mathcal{C}_4[h]$ the semi-norm of the controlled quantities

$$\begin{aligned}\mathcal{C}_4[h] := \Bigg[& \|h^\perp\|^2 + \|\widetilde e\|^2 + \|\widetilde m\|^2 + \|m_s\|^2 + \|\widetilde r\|^2 + \|w_s\|^2\\ & + \left\|\Omega^{-\frac12}\partial_t e_s\right\|^2 + \left\|\Omega^{-\frac12}\partial_t \widetilde m\right\|^2 + \left\|\Omega^{-\frac12}\partial_t m_s\right\|^2 + \left\|\Omega^{-\frac12}\partial_t \widetilde r\right\|^2 + \left\|\Omega^{-\frac12}\partial_t w_s\right\|^2\Bigg]^{\frac12}\end{aligned}$$

and update the values of ε and η by posing

$$\varepsilon := \min\left\{\frac{\nu}{16},\frac{\varepsilon_2\bar\lambda_2}{8},\frac{\varepsilon_1\bar\lambda_1}{16},\frac{\varepsilon_0\bar\lambda_0}{2},\frac{\varepsilon_2'\bar\lambda_2}{2},\frac{\varepsilon_1'\bar\lambda_1}{2},\varepsilon_0'\bar\lambda_0,\frac{\varepsilon_w}{2},\varepsilon_w'\right\} \quad\text{and}\quad \eta := \max\{\eta_1,\eta_2,\eta_3,\eta_4\}\ll\varepsilon.$$

After this *cascade of commutator estimates*, estimate (5.31) yields, for some $0<\eta\ll\varepsilon\ll1$,

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_4} \le -\varepsilon\,\mathcal{C}_4[h]^2 + \eta\|h\|^2. \tag{5.32}$$

Remark 5.8. *Note that in the very last part of the analysis concerning the study of the means of r, we did not really used the decay in $\tilde r$ and $\Omega^{-\frac12}\partial_t\tilde r$. This was replaced by the corresponding micro Lyapunov functional involving w_s. This could be interesting to study the control of terms in (5.29) with an another method. We chose not to develop this point of view in this first preprint.*

5.3. **Control of finite-dimensional quantities and the proof of Proposition 5.1.** Estimate (5.32) control the infinite-dimensional (microscopic and macroscopic) parts of the solution similarly as Lemma 4.6 in the micro-macro method. The remaining finite-dimensional quantities can then be treated exactly as presented in Section 4.3. This completes the proof of Proposition 5.1.

Appendix A. Technical lemmas

A.1. **Momentum conservation vs. infinitesimal rotations.** In this subsection we prove the formula (2.4) used in Subsection 2.1. Given f solution to (1.1) in $L^2(\mathcal{M}^{-1})$ we introduce

$$m_0(x) := \left(\int_{\mathbb{R}^d} v f_0(x,v)\,\mathrm{d}v\right)e^{\phi(x)}, \qquad m_f(x) := \left(\int_{\mathbb{R}^d} v f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)}$$

$$r_f(t,x) := \left(\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)}, \qquad e_f(t,x) := \left(\int_{\mathbb{R}^d} \mathfrak{E}(v)\,f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)}$$

and write $f := r_f\mathcal{M} + m_f\cdot v\mathcal{M} + e_f\mathfrak{E}\mathcal{M} + h^\perp\mathcal{M}$. We denote then $x\mapsto A_0x := \mathbb{P}_\phi(m_0)(x)$.

Lemma A.1. *We have* $\mathbb{P}(m_f - m_0) \in \mathcal{R}_\phi^\perp$.

Proof of Lemma A.1. The result is clear at time $t = 0$ since $\mathbb{P}(m_0 - m_0) = 0$. Let us now consider $x \mapsto B \cdot x$ in $\mathcal{R}_\phi$. We want to prove that $\mathbb{P}(m_f - m_0)$ is orthogonal to $x \mapsto toB \cdot x$ and for this it is sufficient to prove that

$$\forall\, t \geq 0, \quad \langle m_f(t) - m_0, Bx\rangle = 0.$$

For this we differentiate the preceding quantity (omitting the space and time variables for readability) and freely use the macroscopic equations (2.17) also valid for r_f, m_f and e_f by direct computation. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle m_f - m_0, Bx\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle m_f, Bx\rangle = \left\langle \frac{\mathrm{d}m_f}{\mathrm{d}t}, Bx\right\rangle = \left\langle -\nabla_x r_f + \sqrt{\frac{2}{d}}\nabla_x^* e_f + \nabla_x^* \cdot E\left(h^\perp\right), Bx\right\rangle$$

where we have used equation (2.17) and its notation. By integration by part then

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle m_f - m_0, Bx\rangle = \langle r, \nabla_x^* \cdot Bx\rangle + \sqrt{\frac{2}{d}}\,\langle e, \nabla_x \cdot Bx\rangle + \left\langle E\left(h^\perp\right) : \nabla \otimes Bx\right\rangle.$$

The first term in the right hand side vanishes $\nabla_x^* \cdot Bx = -\nabla_x \cdot Bx + \nabla\phi \cdot Bx = 0$ since B is skew-symmetric and $Bx \in \mathcal{R}_\phi$. The the second one vanishes as well since $\nabla_x \cdot Bx = 0$. The third one also vanishes since $E(h^\perp) : \nabla \otimes Bx = E(h^\perp) : B = 0$ because $E(h^\perp)$ is symmetric and B is skew-symmetric. This proves $\frac{\mathrm{d}}{\mathrm{d}t}\langle m(t) - m_0, Bx\rangle = 0$ and concludes the proof. ☐

A.2. **Intermediate results used in Section 3.3.** In this subsection, we state and prove two intermediate results used in Subsection 3.3 when exhibiting minimizers of the entropy, and implicitly used in Subsection 4.3 when showing hypocoercivity. The first result is about the invertibility of the matrices M_ϕ and $\hat{M}_\phi$ defined respectively in (3.17) and (3.23).

Lemma A.2. *When $d_\phi = d$ the matrix M_ϕ is invertible and when $1 \leq d_\phi \leq d-1$ the matrix $\hat{M}_\phi$ is invertible.*

Proof of Lemma A.2. First look at the case when $d_\phi = d$. We shall prove that $\ker M_\phi = \{0\}$, so let $u \in \mathbb{R}^d$ be such that $M_\phi u = 0$. Then $M_\phi u \cdot u = \langle |\Phi \cdot u|^2\rangle = 0$, which implies that $\Phi(x) \cdot u = 0$ for any $x \in \mathbb{R}^d$ hence the result. The proof in the case $d_\phi \leq d-1$ follows exaclty the same scheme. ☐

The second result concerns the linear independence of the two functions $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ defined in (3.21) (and similarly for $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ defined in (3.21)).

Lemma A.3. *In the case when $d_\phi = d$, we have* $\mathrm{Rank}(\widetilde{\Psi}_1, \widetilde{\Psi}_2) = 2$ *and when $1 \leq d_\phi \leq d-1$ we also have* $\mathrm{Rank}(\widehat{\Psi}_1, \widehat{\Psi}_2) = 2$

Proof of Lemma A.3. We first give a complete proof in the case $d_\phi = d$, with

$$\widetilde{\Psi}_1 = \Psi_1 - \frac{1}{4}\Phi \cdot M_\phi^{-1}\alpha_1, \qquad \widetilde{\Psi}_2 = \Psi_2 - \Phi \cdot M_\phi^{-1}\alpha_2.$$

We argue by contradiction. Assume that $\widetilde{\Psi}_1 = \lambda\widetilde{\Psi}_2$ for some $\lambda \in \mathbb{R}^*$, that is, there are constants $\alpha, \beta \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$ such that

$$\phi - \frac{1}{2}\nabla_x\phi \cdot x + \alpha \cdot \nabla_x\phi = \lambda|x|^2 + \beta \cdot x + \gamma. \tag{A.1}$$

We first look for quadratic solutions to (A.1) of the form $\phi_0 = x \cdot M_0 x + b_0 \cdot x + c_0$ with $M_0 \in \mathscr{M}_d(\mathbb{R})$, $b_0 \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$. Plugging this into (A.1) one obtains $M_0 = \lambda\mathrm{Id}_{d\times d}$, $b_0 = \frac{2}{3}(\beta - \lambda\alpha)$ and $c_0 = \gamma - \frac{2}{3}\alpha \cdot (\beta - \lambda\alpha)$. Now let ϕ be a solution to (A.1). Define $\psi_0(x) = \phi(x) - \phi_0(x)$ and then $\psi(y) = \psi(y + 2\alpha)$, which hence verifies

$$\psi(y) - \frac{1}{2}\nabla_y\psi(y) \cdot y = 0. \tag{A.2}$$

Let $\zeta(y) = |y|^2\psi(y)$ so that $\nabla_y\zeta(y)\cdot y = 2|y|^2(\psi(y) - \frac{1}{2}\nabla_y\psi(y)\cdot y) = 0$ for any $y \in \mathbb{R}^d$. In polar coordinates (r,θ), this implies that $\zeta(y) = \zeta(\theta)$ and hence

$$\psi(r,\theta) = \frac{\zeta(\theta)}{r^2}, \quad \forall\, r > 0.$$

But ψ is well defined at the origin, therefore $\lim_{r\to 0}\psi(r,\theta)$ is finite, which in turn implies that $\psi(r,\theta) = 0$. Finally one gets $\phi = \phi_0$ which means that the solution ϕ to (A.1) is quadratic, that is

$$\phi(x) = \frac{1}{2}x\cdot Mx + b\cdot x + c = \lambda|x|^2 + b\cdot x + c.$$

Thanks to the normalization (H7), one gets $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log 2\pi$ and hence

$$E_\phi = \mathrm{Span}\big\{\nabla_x\phi(x) - x : x\in\mathbb{R}^d\big\} = \{0\},$$

which contradicts the hypothesis $d_\phi = d$. This ends the proof when $d_\phi = d$. When $d_\phi \le d-1$, we argue similarly on $\widehat\Psi_1$ and $\widehat\Psi_2$. □

A.3. **Some computations about the commutator method.** In this subsection, we prove technical claims used in the commutator Section 5. The first Lemma concerns boundedness of operators involved in the computations. Recall that in that section, we assumed that both L and the derivatives of order two or more of ϕ are bounded (in particular H_ϕ).

Lemma A.4. *The operators Λ_i, A_i and B_i, where $i\in\{1,\dots,3\}$, are bounded.*

Proof of Lemma A.4. As a typical example, we focus first on Λ_1 for which this is sufficient to show that $\Lambda^{-1}\Gamma^{-\frac{1}{2}}\partial_{x_i}\partial_{v_j}\partial_{v_k}$ is bounded in $L^2(\mathcal{M})$. Adopting the point of view of [17, Proposition A.7], we first conjugate with $\mathcal{M}^{\frac{1}{2}}$ and we just have to check that $\tilde\Gamma^{-\frac{1}{2}}(\partial_{v_k} + \frac{v_k}{2})$ and $\tilde\Lambda^{1}(\partial_{x_i} + \frac{x_i}{2})(\partial_{v_j} + \frac{v_j}{2})$ are bounded in L^2 flat, where

$$\begin{cases} \tilde\Lambda = \displaystyle\sum_i \left(-\partial_{x_i} + \frac{x_i}{2}\right)\left(\partial_{x_i} + \frac{x_i}{2}\right) + \left(-\partial_{v_i} + \frac{v_i}{2}\right)\left(\partial_{v_i} + \frac{v_i}{2}\right), \\ \tilde\Gamma = \displaystyle\sum_i \left(-\partial_{v_i} + \frac{v_i}{2}\right)\left(\partial_{v_i} + \frac{v_i}{2}\right) \end{cases}$$

For $\tilde\Gamma^{-\frac{1}{2}}(\partial_{v_k} + \frac{v_k}{2})$ this is due to the fact that $\tilde\Gamma^{-\frac{1}{2}}$ is of order -1 and $\partial_{v_k} + \frac{v_k}{2}$ of order 1 in the pseudo-differential calculus associated to the metric $\frac{(\mathrm{d}v^2+\mathrm{d}\eta^2)}{(1+|v|^2+|\eta|^2)}$, η being the dual variable of v. The composition is then of order 0 and the Calderon-Vaillancourt Theorem implies the boundedness. For $\tilde\Lambda^{1}(\partial_{x_i} + \frac{x_i}{2})(\partial_{v_j} + \frac{v_j}{2})$, which is true since Λ^{-1} is of order -2 and $(\partial_{x_i} + \frac{x_i}{2})(\partial_{v_j} + \frac{v_j}{2})$ is of order 2 in the pseudo-differential calculus associated to the metric $\frac{(\mathrm{d}x^2+\mathrm{d}v^2+\mathrm{d}\xi^2+\mathrm{d}\eta^2)}{(1+|\eta|^2+|v|^2+|\nabla\phi|^2+|\xi|^2)}$, ξ being the dual variable of v. This implies the desired boundedness. Such calculus with two levels (involving Λ in all variable and Γ only in velocity variables) is also in the core of the boundedness of terms like $\Lambda\Gamma^{\frac{1}{2}}[\Lambda^{-1}\Gamma^{-\frac{1}{2}}, \mathcal{T}]$ where we use that Λ and Γ commute and that the commutation decrease the order by 1, so that this operator is of order 0 and therefore bounded by the Calderon-Vaillancourt Theorem. Note in addition that H_ϕ (appearing e.g. in the B_i's) is of order 0 which greatly simplifies the proofs. For all other terms Λ_i, A_i and B_i, similar computations give the result. □

The second lemma concerns symmetry and non-negativity of operator Λ_1.

Lemma A.5. *Operator Λ_1 is symmetric and non-negative.*

Proof of Lemma A.5. First we check that $\nabla_x^*\nabla_v^*\nabla_x^*\Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}}\nabla_v\nabla_x\nabla_x$ is symmetric, since for the other part of Λ_1 this is obvious:

$$\langle\nabla_x^*\nabla_v^*\nabla_x^*\Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}}\nabla_v\nabla_x\nabla_x f, g\rangle = \sum_{i,j,k}\langle\partial_{x_i}^*\partial_{v_j}^*\partial_{x_k}^*\Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}}\partial_{v_k}\partial_{x_j}\partial_{x_i} f, g\rangle$$

$$
\begin{aligned}
&= \sum_{i,j,k} \langle \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{v_k}\partial_{x_j}\partial_{x_i} f, \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{x_k}\partial_{v_j}\partial_{x_i} g\rangle \\
&= \sum_{i,j,k} \langle \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{x_j}\partial_{v_k}\partial_{x_i} f, \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{v_j}\partial_{x_k}\partial_{x_i} g\rangle \\
&= \sum_{i,j,k} \langle f, \partial_{x_i}^*\partial_{v_k}^*\partial_{x_j}^*\partial_{v_j}\partial_{x_k}\partial_{x_i} g\rangle \\
&= \langle f, \nabla_x^*\nabla_v^*\nabla_x^*\Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}}\nabla_v\nabla_x\nabla_x g\rangle.
\end{aligned}
$$

Next check that Λ_1 is indeed a non-negative operator:

$$
\begin{aligned}
\langle \Lambda_1 f, f\rangle &= \langle \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}} \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* f, f\rangle \\
&\quad + \langle \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}}\Lambda^{-1} : \Lambda^{-1}\Gamma^{-\frac{1}{2}} \nabla_v^* \otimes \nabla_x^* \otimes \nabla_x^* f, f\rangle \\
&= \sum_{i,j,k} \langle \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{x_k}\partial_{v_j}\partial_{x_i} f, \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{x_k}\partial_{v_j}\partial_{x_i} f\rangle \\
&\quad + \langle \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{v_k}\partial_{x_j}\partial_{x_i} f, \Lambda^{-1}\Gamma^{-\frac{1}{2}} \partial_{x_k}\partial_{v_j}\partial_{x_i} f\rangle \\
&= \sum_{i,j,k} \frac{1}{2} \left\| \Lambda^{-1}\Gamma^{-\frac{1}{2}} (\partial_{x_k}\partial_{v_j} + \partial_{v_k}\partial_{x_j})\partial_{x_i} f \right\|^2
\end{aligned}
$$

This concludes the proof. □

A.4. **A cascade of Poincaré-Lions inequalities.** We prove in this subsection several inequalities used in the commutator method. We assume that ϕ has bounded second order derivative and that $\nabla\phi$ goes to infinity at infinity. Let φ a smooth function in $L^2(\rho)$ with compact support and

$$
\begin{cases}
P_0(\varphi) := \langle \varphi\rangle, \\
P_1(\varphi) := \langle \varphi\rangle + \langle \nabla_x \varphi\rangle \\
P_2(\varphi) := \langle \varphi\rangle + \langle \nabla_x \varphi\rangle x + \dfrac{1}{2} \langle \nabla_x \otimes \nabla_x \varphi\rangle : \big(x\otimes x - \langle x\otimes x\rangle\big).
\end{cases}
\tag{A.3}
$$

Lemma A.6. *Let $n \in \{1,2,3\}$. Then there exists a constant $c_{P,n} > 0$ such that for all smooth φ with compact support we have*

$$
c_{P,n} \left\| \varphi - P_{n-1}(\varphi)\right\|^2 \le \left\| \Omega^{-n/2}\nabla_x^{\otimes n}\right\|^2
$$

Proof of Lemma A.6. For $n = 1$ this is exactly the Poincaré-Lions Theorem as stated for example in [4, Proposition 5] and here in (4.10). Let us prove the result for $n = 2$. Let φ be smooth and with compact support. We notice then that

$$
\langle \varphi - P_1(\varphi)\rangle = \langle \varphi\rangle - \langle \varphi\rangle - \langle \nabla_x\varphi\rangle\langle x\rangle = 0
$$

since $\langle x\rangle = 0$. We can therefore apply the Poincaré-Lions inequality (i.e. the case $n = 1$) to $\varphi - P_1(\varphi)$, which gives

$$
\left\|\varphi - P_1(\varphi)\right\|^2 \le c_{P,1}^{-1} \left\| \Omega^{-\frac{1}{2}} \nabla_x \left(\varphi - P_1(\varphi)\right)\right\|^2 = c_{P,1}^{-1} \left\| \Omega^{-\frac{1}{2}} \big(\nabla_x\varphi - \langle \nabla_x\varphi\rangle\big)\right\|^2
$$

We can then apply the Poincaré Lions inequality of order -1 as stated in [4, Lemma 10] and (4.11) here to $\nabla\varphi$ which directly gives that

$$
\left\| \Omega^{-\frac{1}{2}} \big(\nabla_x\varphi - \langle \nabla_x\varphi\rangle\big)\right\|^2 \le C_{LPL} \left\| \Omega^{-1}(\nabla_x^2\varphi)\right\|^2
$$

where $C_{LPL} > 0$ depends only on semi-norms of ϕ. This concludes the proof of the case $n = 2$ with $c_{P,2} = \frac{c_{P,1}}{C_{LPL}}$.

Regarding the case $n = 3$, we define $\psi := \varphi - \frac{1}{2}\langle\nabla^2\phi\rangle : x \otimes x$ and we check that

$$\begin{aligned}\psi - P_1(\psi) &= \psi - \langle\psi\rangle - \langle\nabla_x\psi\rangle\cdot x\\ &= \varphi - \frac{1}{2}\langle\nabla^2\varphi\rangle : x\otimes x - \langle\varphi\rangle + \frac{1}{2}\langle\nabla^2\varphi\rangle : \langle x\otimes x\rangle + \langle\nabla_x\varphi\rangle\cdot x - \langle\nabla_x^2\varphi\rangle : \langle x\rangle\otimes x\\ &= \varphi - P_2(\varphi)\end{aligned}$$

since $\langle x\rangle = 0$. We therefore can apply the preceding inequality for $n = 2$ which gives

$$\|\varphi - P_2(\varphi)\|^2 = \|\psi - P_1(\psi)\|^2 \le c_{P,2}^{-1}\left\|\Omega^{-1}\nabla_x^2\psi\right\|^2 = c_{P,2}^{-1}\left\|\Omega^{-1}(\nabla_x^2\varphi - \langle\nabla_x^2\varphi\rangle\right\|^2 \tag{A.4}$$

where we used again that Ω is identity on constants. Arguing exactly as for the proof of the "(-1)-order" Poincaré-Lions inequality as stated in [4, Lemma 10] we could prove the "(-2)-order" Poincaré-Lions inequality with constant $C_{\rm LPL} > 0$ which reads for any f smooth with compact support

$$\left\|\Omega^{-1}(f - \langle f\rangle\right\|^2 \le C_{\rm LPL}\left\|\Omega^{-\frac{3}{2}}\nabla_x f\right\|^2.$$

Applying this in (A.4) to $\nabla_x^2\varphi$ gives then

$$\|\varphi - P_2(\varphi)\|^2 \le c_{P,2}^{-1}C_{\rm LPL}\left\|\Omega^{-\frac{3}{2}}\nabla_x^3\varphi\right\|^2$$

This proves the case $n = 3$ with $c_{P,3} = \frac{c_{P,2}}{C_{\rm LPL}}$. This concludes the proof. □

Appendix B. Examples and remarks

B.1. **Examples of collision kernels.** In this short subsection, we present examples of linear collision operators $\mathscr{C}$ satisfying the hypotheses of Theorem 1.1, namely the spectral gap property (H1) and the boundedness property (H2).

Example B.1 (The full linear Boltzmann operator)**.** *Consider*

$$\mathscr{C}f := -\left(f - r_f\mathcal{M} - m_f\cdot v\mathcal{M} - e_f\mathfrak{E}\mathcal{M}\right)$$

where r_f, m_f and e_f are defined by

$$\begin{aligned} r_f(t,x) &:= \left(\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)}, && \text{(local) density}\\ m_f(t,x) &:= \left(\int_{\mathbb{R}^d} v f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)}, && \text{(local) momentum}\\ e_f(t,x) &:= \left(\int_{\mathbb{R}^d} \mathfrak{E}(v)\,f(t,x,v)\,\mathrm{d}v\right)e^{\phi(x)} && \text{(local) kinetic energy.}\end{aligned}$$

By construction, $\mathscr{C}$ satisfies the spectral gap condition (H1) *and since it is bounded, it satisfies also the boundedness property* (H2)*.*

Example B.2 (The linearized Boltzmann collision operator for hard spheres)**.** *Consider*

$$\mathscr{C}f := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\left[f'\mu'_* + f'_*\mu' - f\mu_* - f_*\mu\right]|v - v_*|^\gamma\,\mathrm{d}\sigma\,\mathrm{d}v_*$$

with the usual notations $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$, and where $\gamma > 0$. Assuming that $\gamma > 0$, this operator satisfies the spectral gap property (H1) *(see e.g. [1]) but it is not bounded on $L^2(\mu^{-1})$. Polynomials are however in the domain of $\mathscr{C}$ and therefore it satisfies also the bounded property* (H2)*. Other type of interactions, including collision kernels without angular cutoff models or inverse power laws, satisfy also both hypotheses and we refer to e.g. [29] for such examples.*

Example B.3 (The linearized Landau collision operator)**.** *Consider*

$$\mathscr{C} f := \nabla_v \cdot \left(\int_{\mathbb{R}^d} |v - v_*|^{\gamma+2} \left[\mathrm{Id} - \frac{(v - v_*)}{|v - v_*|} \otimes \frac{(v - v_*)}{|v - v_*|} \right] \left(\nabla \frac{f}{\mu}(v) - \nabla \frac{f}{\mu}(v_*) \right) \mu \mu_* \, \mathrm{d}v \, \mathrm{d}v_* \right)$$

with the collision parameter $\gamma \in (0, 1]$. This operator is non local, of order 2 in velocity (of diffusive *type) and therefore not bounded. It satisfies the spectral gap condition* (H1) *when $\gamma > 0$ (see e.g.* [1] *for constructive estimates) and again all polynomials in velocity are in its domain.*

Remark B.4. *It is worth mentioning here that the last two preceding linear operators were obtained after a linearization of their bilinear original form around the Gaussian μ and not around the Maxwellian $\mathcal{M}$: when linearizing full non linear inhomogeneous kinetic models around a Maxwellian, one gets an additional term $\rho(x)$ in front of the collision operator that goes to zero at infinity. We have not considered this degeneracy in the present paper: it is likely to create significant difficulties since there is then no uniform-in-x spectral gap for $\rho\mathscr{C}$.*

B.2. **Examples of potentials.** We comment here on the hypotheses (H5) and (H6) on the potential ϕ. The bounded moment hypothesis (H6) is not restrictive. Functions like $\phi(x) = \frac{d+1}{2} \ln(1+x^2) - C_\phi$ which are very slowly increasing satisfy this hypothesis, as well as fast-increasing ones like $\phi(x) = e^{|x|^4} - C_\phi$ (here C_ϕ is a constant of normalization of $e^{-\phi}$ in L^1). Regarding the Poincaré inequality (H5), many works have been devoted to the study of sufficient conditions in order to get the result. We mention some examples below.

Example B.5. *The harmonic potential $\Phi(x) = \frac{|x|^2}{2} + \frac{d}{2} \ln 2\pi$ satisfies the Poincaré inequality with constant $c_\phi = 1$. In the flat L^2 space, the change of unknown $u = \varphi e^{-\phi/2}$ show that the Poincaré inequality is equivalent to the spectral gap inequality for the harmonic oscillator operator*

$$P = -\Delta_x + \frac{|x|^2}{2} - d,$$

which is nothing but Ω defined in (1.6) *up to a constant.*

Example B.6. *For a general ϕ, the change of unknown $u = \varphi e^{-\phi/2}$ yields the operator*

$$P_\phi = -\Delta_x + \frac{|\nabla_x \phi|^2}{4} - \Delta_x \phi$$

and the celebrated Bakry-Émery theory shows that there is a spectral gap

$$c_\phi \geq \min \left\{ \lambda \mid \lambda \text{ is an eigenvalue of } \nabla_x^2 \phi \right\}$$

as soon as the Hessian $\nabla_x^2 \phi$ of ϕ is uniformly positive.

Example B.7. *Note that all ϕ such that P_ϕ has compact resolvent satisfy the Poincaré inequality* (H5)*. This happens in particular when*

$$\lim_{|x| \to \infty} \frac{|\nabla_x \phi|^2}{4} - \Delta_x \phi = +\infty$$

due to standard results on Schrdinger operators, and in this case 0 is a single eigenvalue. This in turn is implied by the stronger assumption

$$\lim_{|x| \to \infty} |\nabla_x \phi| = +\infty, \qquad \text{and} \qquad \lim_{|x| \to \infty} \frac{\Delta_x \phi(x)}{|\nabla_x \phi(x)|^2} = 0 \tag{B.1}$$

This argument however does not give an explicit estimate on c_ϕ.

Example B.8. *Here is an exotic example of potential that does not satisfy* (B.1) *or the Bakry-Émery criterium (uniform convexity of ϕ) and for which the Poincaré inequality holds. Considering in $\mathbb{R}^2$*

$$\phi(x, y) = x^2 \left(1 + y^2\right)^2 - C_\phi.$$

One can check that $\rho = e^{-\phi} \in L^1(\mathbb{R}^2)$ (and C_ϕ is then the normalization constant), and that P_ϕ has a spectral gap, even though ϕ is constant on the unbounded set $\{x = 0\}$.

B.3. **Change of coordinates.** We give details in this subsection on the reduction to the normalization H7. We first note that the formulas for $\mathrm{Ker}(\mathscr{L})$ are invariant by change of orthonormal velocity variables. By orthonormal change of velocity and space variables, this is then easy to reduce the problem to the case when ϕ satisfies

$$\langle \nabla_x^2 \phi \rangle = \begin{pmatrix} p_1^2 & 0 & 0 & \cdots & 0 \\ 0 & p_2^2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & p_d^2 \end{pmatrix},$$

where we suppose without restriction that all p_j's are positive. The analysis of the present paper can be adapted to this case, including the main Theorem 1.1, with the following changes. We define the set of adapted centered infinitesimal rotational modes compatible with ϕ as in (1.3):

$$\mathfrak{R}_\phi^{(p)} = \{(Ax \cdot v) \ \mid A \in \mathcal{R}_\phi\}\, \mathcal{M}. \tag{B.2}$$

Then denoting $E_\phi^{(p)} := \mathrm{Span}\{\nabla_x \phi(x) - px : x \in \mathbb{R}^d\}$, and $d_\phi := \dim E_\phi$, where $px := (p_1 x_1, \dots, p_d x_d)$, we define the *harmonic directional modes* when $d_\phi \le d-1$ by

$$\mathfrak{D}_\phi^{(p)} = \mathrm{Span}\Big\{ (p_j x_j \cos p_j t - v_j \sin p_j t)\,,\ (p_j x_i \sin p_j t + v_j \cos p_j t)\,,\ \ j \in \{d_\phi + 1, \dots, d\} \Big\}\mathcal{M}, \tag{B.3}$$

and the *harmonic pulsating modes* when $d_\phi = 0$, and in which case necessarily all p_j are equal to a common value denoted p, by

$$\mathfrak{P}_\phi^{(p)} = \mathrm{Span}\Big\{ \left(\frac{|px|^2 - |v|^2}{2} \cos(2pt) - px \cdot v \sin(2pt) \right),\ \left(\frac{|px|^2 - |v|^2}{2} \sin(2pt) + px \cdot v \cos(2pt) \right) \Big\}\mathcal{M}.$$

One checks then that functions in $\mathfrak{R}_\phi^{(p)}$, $\mathfrak{D}_\phi^{(p)}$ and $\mathfrak{P}_\phi^{(p)}$ indeed are entropy minimizing solutions to (1.1), and it is straightforward to adapt our proof to show hypocoercivity as stated in Theorem 1.1 with these new sets of functions.

B.4. **Spectral interpretation.** We have focused so far on *real* solutions to (1.1), which is natural since physical solutions (densities of probability) are real. However when considering complex solutions we can interpret the results in terms of the *complex* spectrum of the non-negative operator

$$-\mathscr{L} = v \cdot \nabla_x - \nabla_x \phi \cdot \nabla_v - \mathscr{C}$$

in $L^2_{\mathbb{C}}(\mathcal{M}^{-1})$, the complexification of $L^2(\mathcal{M}^{-1})$. We consider ϕ as in the previous subsection with

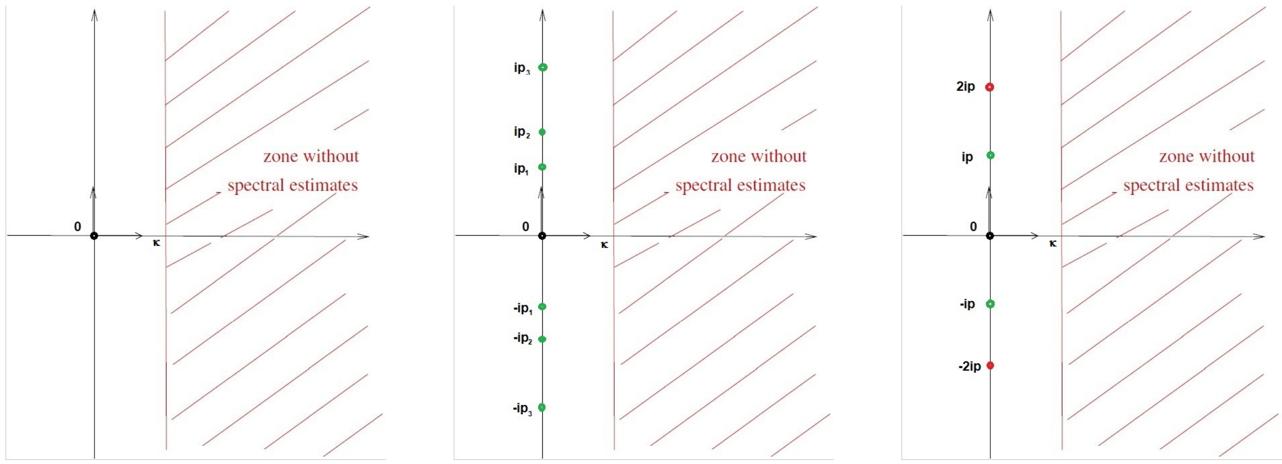
$$\langle \nabla_x^2 \phi \rangle = \begin{pmatrix} p_1^2 & 0 & 0 & \cdots & 0 \\ 0 & p_2^2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & p_d^2 \end{pmatrix}.$$

We can then describe precisely the spectrum of $-\mathscr{L}$ and obtain resolvent estimates in half-plane that includes the imaginary axis. First 0 is in the spectrum of $-\mathscr{L}$ with associated eigen-space

$$\mathrm{Span}_{\mathbb{C}}(\mathcal{M}) \oplus \mathrm{Span}_{\mathbb{C}}(\mathcal{H}\mathcal{M}) \oplus \mathfrak{R}_{\phi,\mathbb{C}}$$

where $\mathfrak{R}_{\phi,\mathbb{C}}$ is the set of infinitesimal rotational modes as defined in (B.2) but extended to the corresponding $\mathbb{C}$-vectorial space. This set is then of (complex) dimension $2 + \dim(\mathfrak{R}_\phi)$. Then depending on the harmonicity of ϕ we have three cases summarized in Figure 1.

(a) Case with no harmonic modes ($d_\phi = d$). In this case ϕ has no harmonic directions and there no additional eigenvalues on the imaginary axis.



(a) No harmonic modes (b) Harmonic directional modes (c) Harmonic directional and pulsating modes

FIGURE 1. Complex spectrum of $-\mathscr{L}$

(b) Case with harmonic directional modes but no pulsating modes ($1 \le d_\phi \le d-1$). In this case, the *real* vectorial space of functions $\mathfrak{D}^{(p)}_\phi$ in (B.3) yields the complex set

$$\mathfrak{D}^{(p)}_{\phi,\mathbb{C}} = \mathrm{Span}_{\mathbb{C}}\Big\{ (p_j x_j - iv_j)\, e^{-ip_j t},\ (p_j x_j + iv_j)\, e^{ip_j t}, \quad j \in \{d_\phi+1,\dots,d\}\Big\}\mathcal{M},$$

to which we can associate the eigen-functions of $-\mathscr{L}$ (for the eigenvalue $\mp i p_j$)

$$(x,v) \mapsto f^\pm_j(x,v) = (p_j x_j \pm i v_j)\mathcal{M}(x,v).$$

(c) Case with harmonic directional and pulsating modes ($d_\phi = 0$). In this last case necessarily all p_j's are equal to a common value $p > 0$ and $\phi(x) = \frac{1}{2}|px|^2 + \frac{d}{2}\log 2\pi - d\log(p)$. All possible harmonic directional modes exist, as well as all possible infinitesimal rotational modes $\mathfrak{R}_{\phi,\mathbb{C}}$ with $\mathcal{R}_\phi = \mathfrak{M}^{\mathrm{skew}}_d(\mathbb{C})$. The complexification of the set $\mathfrak{P}^{(p)}_\phi$ defined in (B.3) is then

$$\mathfrak{P}^{(p)}_{\phi,\mathbb{C}} = \mathrm{Span}\Big\{ \Big(px\cdot v - i\frac{|px|^2-|v|^2}{2}\Big)\, e^{-2ipt},\ \Big(px\cdot v + i\frac{|px|^2-|v|^2}{2}\Big)\, e^{2ipt}, \Big\}\mathcal{M}.$$

to which we can associate the eigen-functions of $-\mathscr{L}$ (for the eigenvalue $\mp 2ip$)

$$(x,v) \mapsto g^\pm(x,v) = \Big(px\cdot v \pm i\frac{|px|^2-|v|^2}{2}\Big)\mathcal{M}(x,v).$$

The analysis of the paper can be exported to the complex Hilbertian space $L^2_{\mathbb{C}}(\mathcal{M}^{-1})$: denote

$$\mathcal{S} = \mathrm{Span}_{\mathbb{C}}(\mathcal{M}) \oplus \mathrm{Span}_{\mathbb{C}}(\mathcal{H}\mathcal{M}) \oplus \mathfrak{R}_{\phi,\mathbb{C}} \oplus \mathrm{Span}\left\{f^\pm_j \mid j\in\{d_\phi+1,\dots,d\}\right\} \oplus \mathrm{Span}\left\{g^\pm\right\}$$

where the f_j's and the $g^\pm$'s are defined above (when ϕ has the relevant harmonicity), and consider $\mathcal{S}^\perp$ the orthogonal of $\mathcal{S}$ in $L^2_{\mathbb{C}}(\mathcal{M}^{-1})$. We note that since $\mathscr{L}$ is a real operator, both $\mathcal{S}$ and $\mathcal{S}^\perp$ are stable by conjugation and therefore stable by $\mathscr{L}$ and $\mathscr{L}^*$. Using then the Laplace transform, we get then from Theorem 1.1 the following resolvent estimate for $-\mathscr{L}|_{\mathcal{S}^\perp}$:

$$\forall\, z \in \mathbb{C} \text{ with } \Re(z) < \kappa, \quad \left\|(z\mathrm{Id} + \mathscr{L}|_{\mathcal{S}^\perp})^{-1}\right\|_{\mathcal{B}(\mathcal{S}^\perp)} \le \frac{\tilde C}{\kappa - \Re(z)}$$

where $\tilde{C}$ is an explicit constant depending on κ and C in Theorem 1.1 and $\|\cdot\|_{\mathcal{B}(\mathcal{S}^\perp)}$ stands for the operator norm on $\mathcal{S}^\perp$. The provides the resolvent estimates in the left half-planes in Figure 1.

B.5. **Special solutions of the full non-linear Boltzmann equation.** In this last subsection, we mention how to build special modes minimizing entropy for the full non-linear Boltzmann equation, which are full nonlinear counterpart to the linearized special modes studied in the present paper. The full nonlinear Boltzmann equation writes

$$\partial_t F + v\cdot\nabla_x F - \nabla_x\phi\cdot\nabla_v F = \Gamma(F,F), \tag{B.4}$$

with

$$\Gamma(F,F) =:= \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left[F' F'_* - F F_*\right] B(v-v_*,\sigma)\,\mathrm{d}\sigma\,\mathrm{d}v_*,$$

where $B \geq 0$ is the collision kernel and we assume the normalization (H7) on ϕ. We consider then functions h in the space spanned by (1) $\{(Ax\cdot v)\,,\ A\in\mathcal{R}_\phi\}$ if ϕ has rotational invariances, and

$$(2)\quad \Big\{ (x_i\cos t + v_i\sin t)\,,\ (v_i\cos t - x_i \sin t)\,,\ i\in\{1,\dots,d_\phi\}\Big\},$$

$$(3)\quad \left\{ \left(\frac{|x|^2-|v|^2}{2}\cos(2t) - x\cdot v\sin(2t)\right),\ \left(\frac{|x|^2-|v|^2}{2}\sin(2t) + x\cdot v\cos(2t)\right)\right\}$$

if ϕ has harmonic directions or is fully harmonic. Then $F(t,x,v) := e^{h(t,x,v)}\mathcal{M}(x,v)$ is a time-dependent periodic solution to (B.4) since the microscopic conservation of momentum $v'+v'_* = v+v_*$ and energy $|v'|^2+|v'_*|^2 = |v|^2+|v_*|^2$ imply

$$h(t,x,v') + h(t,x,v'_*) = h(t,x,v) + h(t,x,v_*).$$

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(K. Carrapatoso) Centre de mathmatiques Laurent Schwartz, École Polytechnique
Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France

(J. Dolbeault) Centre de Recherche en Mathmatiques de la Dcision (CEREMADE, CNRS UMR n° 7534)
Universits PSL & Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

(F. Hérau) Laboratoire de Mathématiques Jean Leray, Université de Nantes
2, rue de la Houssinière BP 92208 F-44322 Nantes Cedex 3, France

(S. Mischler) Centre de Recherche en Mathmatiques de la Dcision (CEREMADE, CNRS UMR n° 7534)
Universits PSL & Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

(C. Mouhot) Department of Pure Mathematics and Mathematical Statistics, University of Cambridge
Wilberforce Road, Cambridge CB3 0WA, UK

(C. Schmeiser) Fakultät für Mathematik, Universität Wien
Oskar-Morgenstern-Platz 1, 1090 Wien, Austria