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Special macroscopic modes and hypocoercivity

Abstract. We study linear inhomogeneous kinetic equations with an external confining potential and a collision operator admitting several local conservation laws (local density, momentum and energy). We classify all special macroscopic modes (stationary solutions and time-periodic solutions). We also prove the convergence of all solutions of the evolution equation to such non-trivial modes, with a quantitative exponential rate. This is the first hypocoercivity result with multiple special macroscopic modes with constructive estimates depending on the geometry of the potential.

Keywords. Hypocoercivity; linear kinetic equations; collision operator; transport operator; collision invariant; local conservation laws; micro/macro decomposition; commutators; hypoellipticity; confinement potential; rotations; rotational invariance; symmetries; global conservation laws; spectral gap; Poincaré-Korn inequality; Witten-Laplace operator; partially harmonic potential; time-periodic solutions; classification of steady states; special macroscopic modes

1. Introduction

Since the publication of Boltzmann's paper [5] in 1876, the existence of time-periodic steady states of the *inhomogeneous Boltzmann equation* in the whole Euclidean space

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is known, in presence of an external harmonic potential. As explained in [10, p. 147], "equilibrium is not necessarily achieved in an harmonic field. [...] [D]ensity, velocity and temperature oscillate with the natural frequency of the field or with twice such a frequency." Beyond such remarks, the classification of the steady states according to the symmetries of the domain or the symmetries of the external potential remained untouched for more than a century, although some special solutions were known [10,44]. When symmetry partially or completely breaks, this turns out to be a difficult issue. With symmetry, special modes have to be taken into account in some configurations and local collision laws of the collision operator add significant difficulties to the understanding of the convergence in asymptotic regimes in all cases, even if there is no particular symmetry.

Without external potential and for a bounded domain, the problem has been studied in [12]. In presence of a given external potential, the question was so far open and our first result is to classify all steady solutions for linear kinetic equations with collision operators satisfying the local conservation laws of physics. Even more difficult is the problem of the stability of the (possibly time-periodic) steady states and the proof of the convergence to such states, with an exponential rate, for inhomogeneous kinetic equations. The question goes back to the celebrated H-theorem of Boltzmann, but became quantitative only recently with the theory of hypocoercivity. All results involving an external potential deal with collision operators admitting only one collision invariant, up to a few attempts like [15,16] which discard special modes, with non-constructive methods. Our second result gives the very first answer to the question of the convergence rate in the whole space for an external potential without any *a priori* symmetry, using an entirely new scheme made of a cascade of several hypocoercive estimates. Alternatively we also propose a commutator method in the spirit of [27,46].

Even when the potential has no specific symmetry, which forbids the existence of any special mode other than the standard stationary solution, the fact that the collision operator admits several *collision invariants* is a source of difficulties: when the potential is almost symmetric, convergence rates get deteriorated and the geometric properties of the potential have therefore to be taken into account. The notion of steady states, defined as the set of attractors in large time asymptotics, is widely used in physics, and corresponds in our case to minimizers of the mathematical entropy (that is, the physical entropy, up to the sign). In this paper we shall speak of *special macroscopic modes* in relation with *special* symmetries of the potential.

1.1. Equation and assumptions

Consider the kinetic equation

$$\partial_t f = \mathscr{L} f := \mathscr{T} f + \mathscr{C} f\,, \quad f_{|t=0} = f_0\,, \tag{1.1}$$

for the unknown distribution function $f = f(t,x,v)$ depending on the time variable $t \geq 0$, the position variable $x \in \mathbb{R}^d$, and the velocity variable $v \in \mathbb{R}^d$, where $d \geq 1$ is an arbitrary dimension. The *transport operator* $\mathscr{T}$ is given by

$$\mathscr{T} f := -v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f$$

with a stationary, position dependent *potential* $\phi : \mathbb{R}^d \to \mathbb{R}$. We assume that the *linear collision operator* $\mathscr{C}$ is acting only along the velocity variable $v \in \mathbb{R}^d$, is self-adjoint in $\mathrm{L}^2(\mu^{-1})$, with weight given by the *local Maxwellian* function

$$\forall\, v \in \mathbb{R}^d\,, \quad \mu(v) := \frac{e^{-|v|^2/2}}{(2\,\pi)^{d/2}}\,, \tag{1.2}$$

and has the $(d+2)$-dimensional kernel of *collision invariants* given by

$$\operatorname{Ker}\mathscr{C} = \operatorname{Span}\left\{\mu, v_1\,\mu, \ldots, v_d\,\mu, |v|^2\,\mu\right\}\,, \tag{H0}$$

corresponding to the local conservation of mass, momentum and energy. Here $\mathrm{L}^2(\mu^{-1})$ is the subspace of $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}v)$ of the functions f such that

$$\|f\|_{\mathrm{L}^2(\mu^{-1})} := \left(\int_{\mathbb{R}^d} \frac{|f|^2}{\mu(v)}\,\mathrm{d}v\right)^{1/2}$$

is finite.

We assume that $\mathscr{C}$ satisfies the following *spectral gap property* (which is a quantitative version of the *spatially homogeneous linearized H-theorem*)

$$-\int_{\mathbb{R}^d} \big(\mathscr{C} f(v)\big)\, f(v)\, \mu(v)^{-1}\,\mathrm{d}v \ge \mathrm{c}_{\mathscr{C}}\, \|f - \Pi f\|^2_{\mathrm{L}^2(\mu^{-1})} \tag{H1}$$

for some constant $\mathrm{c}_{\mathscr{C}} > 0$ and all f in the domain of $\mathscr{C}$, where Π denotes the $\mathrm{L}^2(\mu^{-1})$-orthogonal projection onto $\operatorname{Ker}\mathscr{C}$. Moreover, we suppose that for any polynomial function $p(v) : \mathbb{R}^d \to \mathbb{R}$ of degree at most 4, the function $p\,\mu$ is in the domain of $\mathscr{C}$ and

$$C(p) := \|\mathscr{C}(p\,\mu)\|_{\mathrm{L}^2(\mu^{-1})} < \infty\,. \tag{H2}$$

We provide examples of collision operators satisfying these conditions in Appendix C.1, including the linearized Boltzmann and Landau operators.

Throughout the paper, we assume that the potential $\phi : \mathbb{R}^d \to \mathbb{R}$ is such that $\rho(x) := e^{-\phi(x)}$ is a centred probability density, *i.e.*,

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} x\,\rho(x)\,\mathrm{d}x = 0\,. \tag{H3}$$

We also assume that ϕ is of class $C^2(\mathbb{R}^d;\mathbb{R})$, and for all $\varepsilon > 0$, there exists a constant C_ε such that

$$\forall\, x \in \mathbb{R}^d\,, \quad |\nabla_x^2 \phi(x)| \le \varepsilon\,|\nabla_x \phi(x)|^2 + C_\varepsilon\,, \tag{H4}$$

where $\nabla_x^2 \phi$ denotes the Hessian matrix of ϕ. We further assume that the measure $\rho(x)\,\mathrm{d}x$ satisfies the Poincaré inequality with a constant $c_{\mathrm{P}} > 0$,

$$c_{\mathrm{P}} \int_{\mathbb{R}^d} |\varphi - \langle\varphi\rangle|^2\,\rho\;\mathrm{d}x \le \int_{\mathbb{R}^d} |\nabla_x \varphi|^2\,\rho\;\mathrm{d}x\,, \tag{H5}$$

for all $\varphi \in \mathrm{L}^2(\rho)$, where

$$\langle \varphi \rangle := \int_{\mathbb{R}^d} \varphi\, \rho \, \mathrm{d}x$$

is the average of φ. Here $\mathrm{L}^2(\rho)$ is the subspace of $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$ of the functions φ such that $\|\varphi\|^2_{\mathrm{L}^2(\rho)} = \int_{\mathbb{R}^d} |\varphi|^2 \, \rho \, \mathrm{d}x$ is finite.

We assume moment bounds on ρ, namely

$$\int_{\mathbb{R}^d} \left(|x|^4 + |\phi|^2 + |\nabla_x \phi|^4 \right) \rho \, \mathrm{d}x \le \mathrm{C}_\phi \tag{H6}$$

for some constant $\mathrm{C}_\phi > 0$. We also introduce the normalization

$$\left\langle \nabla^2_x \phi \right\rangle = \int_{\mathbb{R}^d} \nabla^2_x \phi \, \rho \, \mathrm{d}x = \mathrm{Id}_{d\times d} \,, \tag{H7}$$

where $\mathrm{Id}_{d\times d}$ the identity matrix of size d. The assumption that $\langle \nabla^2_x \phi \rangle$ is diagonal is not a restriction since it can be obtained through a rotation in position space. Note that the same rotation in velocity space leaves the kinetic equation invariant and all assumptions made so far remain valid. The stronger assumption (H7) is made for notational simplicity, and a discussion of the general case is given in Appendix C.3.

The potential

$$\phi(x) := (1 + |a_\gamma \, x|^2)^{\gamma/2} - Z_\gamma \,,$$

with $\gamma > 1$ and real normalization constants a_γ, Z_γ, satisfies (H3)–(H4)–(H5)–(H6)–(H7). See Appendix C.2 for other examples. *No sign* is assumed on f: one should think of f as a real valued fluctuation around the equilibrium in the nonlinear Boltzmann or Landau equation (see Appendix C.1). Throughout this article we shall refer to (H1) and (H5) as *spectral gap properties*, and to (H2) and (H6) as *bounded moment properties*. These are the structural assumptions on $\mathscr{C}$ and ϕ for our theory.

Finally, since we are concerned with large time asymptotic behaviour, we require that the evolution equation (1.1) is well-posed, a condition which is satisfied by our standard examples of application, and assume that

$$t \mapsto e^{t\mathscr{L}} \textit{ is a strongly continuous semi-group on the space } \mathrm{L}^2(\mathcal{M}^{-1}), \tag{H8}$$

where $\mathcal{M}$ is the *global Maxwellian equilibrium* function given by

$$\forall\, (x, v) \in \mathbb{R}^d \times \mathbb{R}^d \,, \quad \mathcal{M}(x, v) := \rho(x)\, \mu(v) = \frac{e^{-\frac{1}{2}|v|^2 - \phi(x)}}{(2\,\pi)^{d/2}} \,, \tag{1.3}$$

and the space $\mathrm{L}^2(\mathcal{M}^{-1})$ is the subspace of $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d, \mathrm{d}x\, \mathrm{d}v)$ of the functions f such that

$$\|f\|_{\mathrm{L}^2(\mathcal{M}^{-1})} := \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f|^2}{\mathcal{M}} \, \mathrm{d}x \, \mathrm{d}v \right)^{1/2}$$

is finite.

1.2. The main result

From here on, we assume the normalization conditions (H3)–(H7). The function $\mathcal{M}$ defined by (1.3) is a stationary solution of (1.1) but it is not the unique attractor of the time-dependent solutions of (1.1), even up to a mass normalization. Let us introduce a larger class of steady states. *Special macroscopic modes* of (1.1) are the solutions $F = F(t, x, v)$ to the system

$$\mathscr{C} F = 0\,, \quad \partial_t F = \mathscr{T} F\,. \tag{1.4}$$

Of course we read from (H0) that $F = \alpha\,\mathcal{M}$, $\alpha \in \mathbb{R}$, is a special macroscopic mode but we also look for solutions to (1.4) that can be written as

$$F = \big(r(t,x) + m(t,x)\cdot v + e(t,x)\,\mathfrak{E}(v)\big)\,\mathcal{M}\,, \tag{1.5}$$

for some functions r, m and e with values respectively in $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{R}$, with

$$\mathfrak{E}(v) := \frac{|v|^2 - d}{\sqrt{2\,d}}\,. \tag{1.6}$$

The *energy mode* $F = \beta\,\mathcal{H}\,\mathcal{M}$, $\beta \in \mathbb{R}$, is another stationary solution to (1.4) where $\mathcal{H}$ defined by

$$\mathcal{H}(x,v) := \tfrac{1}{2}\left(|v|^2 - d\right) + \phi(x) - \langle\phi\rangle \tag{1.7}$$

is the Hamiltonian energy associated with the characteristics of the transport equation $\partial_t f = \mathscr{T} f$. As we shall see in Section 2.1, it turns out that the linear combination of global Maxwellian equilibrium functions and energy modes are the only special macroscopic modes for "generic potentials". Other special macroscopic modes are available under additional symmetry properties of ϕ as observed by L. Boltzmann in [5]. These modes deserve some explanations.

The set of *infinitesimal rotations compatible with* ϕ defined as

$$\mathcal{R}_\phi := \left\{x \mapsto A\,x \;:\; A \in \mathfrak{M}^{\rm skew}_{d\times d}(\mathbb{R}) \text{ s.t. } \forall\,x \in \mathbb{R}^d\,,\ \nabla_x\phi(x)\cdot A\,x = 0\right\} \tag{1.8}$$

is identified with a subset of the space of skew-symmetric matrices

$$\mathfrak{M}^{\rm skew}_{d\times d}(\mathbb{R}) := \left\{A \in \mathfrak{M}_{d\times d}(\mathbb{R}) \;:\; {}^T\!A = -\,A\right\}.$$

In other words, $A \in \mathcal{R}_\phi$ if and only if ϕ is invariant by the rotation group $\theta \mapsto e^{\theta A}$, *i.e.*,

$$\forall\,(\theta,x) \in \mathbb{R}\times\mathbb{R}^d\,, \quad \phi\big(e^{\theta A}x\big) = \phi(x)\,.$$

The set $\mathcal{R}_\phi$ gives rise to the set of *rotation modes compatible with* ϕ defined by

$$\mathfrak{R}_\phi := \left\{(x,v) \mapsto (A\,x\cdot v)\,\mathcal{M}(x,v) : A \in \mathcal{R}_\phi\right\}.$$

Functions in $\mathfrak{R}_\phi$ are stationary solutions of (1.4) associated to all invariances of ϕ under rotation.

There are also some time-periodic special macroscopic modes when ϕ has *harmonic directions*. Let us define

$$E_\phi := \mathrm{Span}_{\mathbb{R}^d}\Big(\{\nabla_x\phi(x) - x\}_{x\in\mathbb{R}^d}\Big)\,,\quad d_\phi := \dim E_\phi\,. \tag{1.9}$$

Notice that E_ϕ is a subspace of $\mathbb{R}^d$ and $d_\phi \le d$. Alternatively, we can characterize d_ϕ by

$$d_\phi = \dim\,\mathrm{Span}_{\mathrm{L}^2(\rho)}\Big(\,\big\{\partial_{x_i}\phi\big\}_{i=1,\ldots,d}\cup\{x_i\}_{i=1,\ldots,d}\,\Big) - d$$

and choose cartesian coordinates $(x_1, x_2, \ldots, x_d)$ such that $\partial_{x_i}\phi = x_i$ if and only if $i \in I_\phi := \{d_\phi + 1, \ldots, d\}$. We face three different cases.

▷ The case $d_\phi = d$ is called *fully non-harmonic*: $E_\phi = \mathbb{R}^d$ and ϕ has no harmonic direction and it that case, as we shall see below, there are no time-periodic solutions.

▷ In the case $1 \le d_\phi \le d-1$, the potential is called *partially harmonic*. In the harmonic coordinates $x_{d_\phi+1}, \ldots, x_d$, we have $\partial_{x_i}\phi = x_i$ and define *harmonic directional modes* by

$$\mathfrak{D}_\phi := \mathrm{Span}\Big\{(x_i\cos t - v_i\sin t)\,\mathcal{M}\,,\ (x_i\sin t + v_i\cos t)\,\mathcal{M}\ \ :\ i\in I_\phi\Big\}\,. \tag{1.10}$$

Harmonic directional modes are also defined if $d_\phi = 0$. By convention, we set $\mathfrak{D}_\phi := \{0\}$ if $d_\phi = d$. All functions in $\mathfrak{D}_\phi$ are solutions to (1.4) which correspond to an inertia-driven oscillation of period 1 of particles in a potential well along a direction in $E_\phi^\perp$. These modes are independent of each other.

▷ In the case $d_\phi = 0$, the potential is called *fully harmonic* and $\phi(x) = \frac12\,|x|^2 + \frac d2\,\log(2\pi)$. In addition to the harmonic directional modes, the set of *harmonic pulsating modes*

$$\mathfrak{P}_\phi := \mathrm{Span}\bigg\{\Big(\tfrac12\Big(|x|^2 - |v|^2\Big)\cos(2\,t) - x\cdot v\,\sin(2\,t)\Big)\,\mathcal{M}\,,\qquad \Big(\tfrac12\Big(|x|^2 - |v|^2\Big)\sin(2\,t) + x\cdot v\,\cos(2\,t)\Big)\,\mathcal{M}\bigg\} \tag{1.11}$$

is also made of solutions to (1.4). By convention, we set $\mathfrak{P}_\phi = \{0\}$ if $d_\phi \ge 1$. These macroscopic modes correspond to a radially symmetric pulsation of period $1/2$ of particles in the potential well.

Summing up the above observations, we have obtained *special macroscopic modes* of the form

$$F = \alpha\,\mathcal{M} + \beta\,\mathcal{H}\,\mathcal{M} + A\,x\cdot v\,\mathcal{M} + F_{\mathrm{dir}} + F_{\mathrm{pul}} \tag{1.12}$$

where $\alpha, \beta \in \mathbb{R}$, $(x \mapsto A\,x) \in \mathcal{R}_\phi$, $F_{\mathrm{dir}} \in \mathfrak{D}_\phi$ and $F_{\mathrm{pul}} \in \mathfrak{P}_\phi$. With these definitions at hand, we can now state the main result of this paper.

Theorem 1.1 (Special macroscopic modes and hypocoercivity). *Assume that the potential ϕ and the collision operator $\mathscr{C}$ satisfy the assumptions* (H0)–(H1)–(H2)–(H3)–(H4)–(H5)–(H6)–(H7)–(H8). *Then*

(1) *All special macroscopic modes of* (1.4) *are given by* (1.12), i.e., *are linear combinations of the Maxwellian, the energy mode, rotation modes compatible with ϕ, and harmonic directional or pulsating modes if allowed by ϕ.*

(2) *There are explicit constants* $C > 0$ *and* $\kappa > 0$ *such that, for any* $f \in C\big(\mathbb{R}^+; \mathrm{L}^2(\mathcal{M}^{-1})\big)$ *solving* (1.1) *with initial datum* $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$*, there exists a unique special macroscopic mode* F *such that*

$$\forall\, t \geq 0\,, \quad \|f(t) - F(t)\|_{\mathrm{L}^2(\mathcal{M}^{-1})} \leq C\, e^{-\kappa\, t}\, \|f_0 - F(0)\|_{\mathrm{L}^2(\mathcal{M}^{-1})}\;.$$

The constants in the decay estimate being explicit means that the proof is constructive and provides a finite algorithm for computing C and κ.

In the following, the norm and scalar product without subscript, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, refer to the space $\mathrm{L}^2(\mathcal{M})$, so that

$$\|h\|^2 = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |h|^2\, \mathcal{M}\, \mathrm{d}x\, \mathrm{d}v = \langle h, h\rangle\,. \tag{1.13}$$

With $h := f/\mathcal{M}$, $h_0 := f_0/\mathcal{M}$ and $h^{\parallel} := F/\mathcal{M}$, Part (2) of Theorem 1.1 amounts to

$$\forall\, t \geq 0\,, \quad \left\|h(t) - h^{\parallel}(t)\right\| \leq C\, e^{-\kappa\, t}\left\|h_0 - h^{\parallel}(0)\right\|\;.$$

When considering functions of x only, the $\mathrm{L}^2(\rho)$-norm coincides with the $\mathrm{L}^2(\mathcal{M})$-norm. We recall that $\langle\cdot\rangle$ stands for the average with respect to $\rho\,\mathrm{d}x$.

In Theorem 1.1, the constants C and κ depend only on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant (to be defined later) which admits a quantitative estimate. Moreover, the special macroscopic mode F can be explicitly computed in terms of the initial data f_0 (see Section 2.2).

1.3. Framework, comments and methods

During the last two decades, new *hypocoercive methods* were developed for the study of spatially inhomogeneous kinetic equations. Many linear or nonlinear models were tackled, including Fokker-Planck, Boltzmann and Landau equations in various geometries, ranging from bounded domains to the whole Euclidean space, with or without confining potentials. The central issue is the trend to equilibrium for these equations, in the spirit of the celebrated H-Theorem by Boltzmann on the decay of the entropy, but with constructive estimates which measure the rate of convergence towards asymptotic regimes described by steady states. The set of steady states is not fully characterized by the entropy dissipation, but also depends on the transport operator and the geometric setting governed either by boundary conditions or by properties of the potential. The goal of this paper is to make the notion of steady states explicit by classifying all special macroscopic modes, and to derive quantitative estimates on the rate of convergence, with explicit constants.

Let us give a brief account of the literature. In a series of papers [20–22] on Landau, Boltzmann and Vlasov-Boltzmann equations in a periodic box, Y. Guo used micro-macro methods inspired from Grad's 13 moments method introduced in [17]. The approach of [12] relies on the derivation of a suitable set of ordinary differential inequalities. It provides an algebraic rate of convergence to equilibrium under strong smoothness assumptions on the solution. The study of linear inhomogeneous kinetic equations with single conservation laws, such as the linear Boltzmann or Fokker-Planck equations, and nonlinear equations in a nonlinear but perturbative regime, took advantage of various ideas of the theory of hypoellipticity, for instance of [29], and gave rise to robust Hilbertian hypocoercive methods. T. Gallay coined the word *hypocoercivity*, by analogy with hypoellipticity, when coercivity is degenerate in the ambient space but recovered using commutators, in the context of convergence to steady states. Hypocoercivity is well adapted to kinetic equations with general collision operators. We refer to the memoir [46] by C. Villani for an overview of the initial developments of this theory and to [13, 14, 26, 27, 36] for various other contributions in exponentially weighted spaces. The theory of *enlargement of spaces* of [18] allows to extend convergence rates to larger, and physically more relevant, polynomially weighted spaces.

Usually, explicit and constructive estimates cannot be obtained via compactness arguments. Such estimates are essential for applications in physics (typical time-scale for relaxation) but also for a wide array of mathematical questions: range of validity of perturbation methods applied to nonlinear kinetic equations, conditions of convergence in the study of diffusive or macroscopic limits, control of the limiting processes leading to hydrodynamical equations when the Knudsen number tends to zero, control of the range of parameters, time and length scales in the corresponding asymptotic regimes, *etc.* Among a huge literature, we can refer for instance to [3, 45] and to [7, 23, 28] in polynomially weighted spaces.

In this article, we focus on an important and old problem. We study kinetic equations involving an external confining potential as well as several local conservation laws in the collision process. The linear problem was solved for a fully harmonic potential in [15] and under full asymmetry assumptions on the potential in [16], both with nonconstructive arguments and for well-prepared initial data so that, in particular, there are no special macroscopic modes beyond the Maxwellian stationary solution. Such an assumption destroys the rich structure of special macroscopic modes and bypasses the non-trivial consequences of the geometric properties of the potential on convergence rates. Our contribution is precisely the study of these consequences, which requires new methods, by classifying all *special macroscopic modes* and proving hypocoercivity results with constructive convergence rates in a natural Hilbertian structure. As in [15, 16], we restrict our analysis to the linear framework and, for simplicity, to exponentially weighted spaces, but cover rather general confining potentials and discuss the consequences of their geometric properties in terms of symmetry, partial symmetry or lack of symmetry under rotations. On the one hand, the extension of our results to polynomially weighted spaces in the spirit of [18] is probably doable. On the other hand, nonlinear stability for Boltzmann and Landau equations with confining potentials, close to special macroscopic modes, presents additional difficulties.

The special macroscopic modes other than the global Maxwellian stationary solutions and the energy modes are consequences of the symmetries of the potential. Some of these modes are known in the literature, although no systematic study seems to have been done. From the point of view of mechanics, any function $F(x,v) = G\big(\mathcal{H}(x,v), A\,x\cdot v\big)$ is a stationary solution of the transport equation where $A\,x\cdot v$ is known as the *angular momentum* generated by $A\in\mathfrak{M}_{d\times d}(\mathbb{R})$. The property that there is no other stationary solution, under appropriate conditions, is known in astrophysics as *Jeans' theorem* (usually considered with a potential induced by a mean field coupling) and has to do with Noether's theorem: see [4] and references therein. Of course the only profile G compatible with (H0) is $G(\mathsf{h},a)=p(\mathsf{h},a)\,e^{-\mathsf{h}}$ where p is a polynomial of order at most two. Proving that any stationary solution of (1.1) has to solve (1.4) is also known as a *factorization* result.

The existence of time-periodic steady states for the fully harmonic potential was shown by L. Boltzmann in [5] and is mentioned in some references: see for instance [6, 10, 19, 44]. In [19], time-periodic modes are called *breathing modes*. The consideration of partially harmonic potentials and their corresponding harmonic directional modes seems to be new. The fact that special macroscopic modes also exist for the nonlinear Boltzmann equation is discussed in Appendix C.5.

Now let us review some of the tools which are used in our paper. To estimate the convergence rate, a major difficulty is to quantify "how far" the potential ϕ is from having certain partial symmetries. Inspired by [11, 12], we use some Korn inequalities for bounded domains which go back to [32, 33] and adapt them to the Euclidean space, in presence of a confining potential: see [9]. A typical quantity involved in our approach is the *rigidity constant*

$$c_{\mathrm{K}}:=\min\left\{\int_{\mathbb{R}^d}|\nabla\phi(x)\cdot A\,x|^2\,\rho(x)\,\mathrm{d}x\;:\;A\in\mathcal{R}_\phi^\perp\text{ such that }\int_{\mathbb{R}^d}|A\,x|^2\,\rho(x)\,\mathrm{d}x=1\right\}>0\,,\quad(1.14)$$

where $\mathcal{R}_\phi^\perp$ is the orthogonal complement in $\mathrm{L}^2(\rho)$ of the set $\mathcal{R}_\phi$ defined in (1.8). The time-periodic special macroscopic modes in $\mathfrak{D}_\phi$ and $\mathfrak{P}_\phi$, defined in (1.10) and (1.11) respectively, are related to the (partial) harmonicity of ϕ and another difficulty is to quantify "how far" the potential ϕ is from being (partially) harmonic. As for Korn type inequalities, the analysis relies on the finite dimension of the space E_ϕ defined in (1.9).

The spectral gap assumptions (H1) in v and (H5) in x reflect the corresponding confining properties respectively in velocity and space. The Poincaré inequality introduced in (H5) is linked with the natural Hodge Laplacian associated to the geometry, sometimes called the *Witten Laplacian*. Denote by ∇_x^* the adjoint of ∇_x in $\mathrm{L}^2(\rho)$ acting on vector fields $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ according to $\nabla_x^*\cdot\varphi=(-\nabla_x+\nabla_x\phi)\cdot\varphi$. The Witten-Laplace operator $\nabla_x^*\cdot\nabla_x$ is self-adjoint in $\mathrm{L}^2(\rho)$, with kernel spanned by constant functions and its first non-zero eigenvalue determines the optimal Poincaré constant c_{P}. The operator

$$\Omega:=\nabla_x^*\cdot\nabla_x+1=-\Delta_x+\nabla_x\phi\cdot\nabla_x+1\quad(1.15)$$

is used in the 0*th-order Poincaré inequality*

$$c_{\mathrm{P},1}\,\|\varphi-\langle\varphi\rangle\|^2\le\left\|\Omega^{-\frac12}\,\nabla_x\varphi\right\|^2$$

which holds for some constant $c_{\mathrm{P},1} > 0$ under assumptions (H5) and (H4), in the spirit of Poincaré-Lions inequalities studied in [9].

We provide two proofs of Theorem 1.1. The first proof follows a micro-macro decomposition as in [20–22] and [16]. Due to the lack of *a priori* symmetry assumptions and the delicate interaction of local conservation laws corresponding to the collision invariants with the potential, the complexity is significantly increased. There are also deep similarities with the analysis of hyperbolic equations with damping studied in [24, 39, 40] after the seminal paper [30] by S. Kawashima and Y. Shizuta. The second proof is given under slightly more restrictive hypotheses, namely that the collision operator $\mathscr{C}$ is bounded and ϕ has bounded derivatives of order two and more. The method is based on commutator estimates as in [27, 36, 46] in the spirit of the hypoellipticity theory of [29]. In practice, an elegant triple cascade of commutators based on the equality $[\nabla_v, v \cdot \nabla_x] = \nabla_x$ is needed to control all macroscopic quantities.

The plan of the article is the following. In Section 2, we review all possible conservation laws and their relations with the special macroscopic modes. Then we present the so-called macroscopic equations associated to the evolution equation (1.1) and perform a change of unknown in order to work in a simplified Hilbertian framework. In Section 3, we classify all steady states of (1.1) and prove that they correspond to the special macroscopic modes. At this stage, we already use entropy-dissipation arguments in order to prove that factorization occurs and reduce the problem to (1.4). In Section 4, we prove the remaining part of Theorem 1.1, that is, the hypocoercivity result, using the micro-macro method. In Section 5 we expose the second proof based on the commutator's method. A number of technical results are collected in two appendices. Appendix A collects some computations and intermediate lemmata needed in the proofs. For completeness, an extension to weakly coercive collision operators is given in Appendix B. Appendix C is devoted to examples and remarks, for instance on the normalization, including a spectral interpretation of Theorem 1.1, the extension of our special macroscopic modes to the fully nonlinear Boltzmann equation, and various examples of collision operators and potentials.

2. Conservation laws and macroscopic equations

In this section we characterize the special macroscopic modes, as defined by (1.4), for generic potentials. We also identify the global conservation laws and the macroscopic equations associated to (1.1). From here on, we assume that (H6) holds. This assumption is needed to justify the computations, which are given below only at formal level, for sake of simplicity.

2.1. The equations for the special macroscopic modes

We recall that by (1.5), any special macroscopic mode F can be written as

$$F = h^{\parallel} \mathcal{M}, \quad h^{\parallel} = r + m \cdot v + e\, \mathfrak{E}(v) .$$

By (1.4), we know that $\partial_t F = \mathscr{T} F$. By integrating in v the evolution equation against 1, v, $v \otimes v$ and $v\,\mathfrak{E}$, we obtain *macroscopic equations* on the macroscopic quantities $r = r(t,x)$, $m = m(t,x)$ and $e = e(t,x)$:

$$\partial_t r = \nabla_x^* \cdot m\,, \tag{2.1a}$$

$$\partial_t m = -\nabla_x r + \sqrt{\tfrac{2}{d}}\, e\, \nabla_x \phi \tag{2.1b}$$

$$\partial_t e = -\sqrt{\tfrac{2}{d}}\, \nabla_x \cdot m\,, \tag{2.1c}$$

$$\tfrac{1}{\sqrt{2}\,d}\,(\partial_t e)\,\mathrm{Id}_{d\times d} = -\nabla_x^{\mathrm{sym}}\, m\,, \tag{2.1d}$$

$$0 = \nabla_x e\,, \tag{2.1e}$$

where the *symmetric gradient* is defined by

$$\forall\, i,\ j = 1,\ldots,d\,, \quad (\nabla_x^{\mathrm{sym}}\, m)_{ij} := \frac{1}{2}\left(\partial_j m_i + \partial_i m_j\right). \tag{2.2}$$

From (2.1e), we deduce that e does not depend on the space variable, and therefore

$$e = \langle e \rangle =: c \tag{2.3}$$

is a function $t \mapsto c(t)$ depending on t only. We recall that the average is defined as $\langle \varphi \rangle := \int_{\mathbb{R}^d} \varphi\, \rho\, \mathrm{d}x$. Then we read from (2.1d) that

$$\tfrac{c'}{\sqrt{2}\,d}\,\mathrm{Id}_{d\times d} = -\nabla_x^{\mathrm{sym}}\, m\,. \tag{2.4}$$

By the Schwarz Lemma applied to $m = (m_k)_{k=1}^d$, we have

$$(\nabla_x^2 m_k)_{i,j} = \partial^2_{x_i x_j} m_k = \partial_{x_i}\,(\nabla^{\mathrm{sym}} m)_{j,k} + \partial_{x_j}\,(\nabla^{\mathrm{sym}} m)_{i,k} - \partial_{x_k}\,(\nabla^{\mathrm{sym}} m)_{i,j} \tag{2.5}$$

for any $i, j, k = 1, \ldots, d$. By differentiating (2.4) with respect to x_i, x_j and x_k, we get that $\nabla_x^2 m = 0$, so that in particular the *skew-symmetric gradient*, defined by

$$\forall\, i,\ j = 1,\ldots,d\,, \quad (\nabla_x^{\mathrm{skew}} m)_{ij} := \frac{1}{2}\left(\partial_j m_i - \partial_i m_j\right), \tag{2.6}$$

is constant in the x-variable and equal to its average. Together with (2.4), using $\nabla_x m = \nabla_x^{\mathrm{skew}} m + \nabla_x^{\mathrm{sym}}\, m$, we deduce that

$$m(t,x) = \langle \nabla m \rangle\, x + \langle m \rangle = A(t)\, x + b(t) - \tfrac{1}{\sqrt{2}\,d}\, c'(t)\, x\,, \tag{2.7}$$

with $A(t) := \langle \nabla_x^{\mathrm{skew}} m \rangle$ and $b(t) := \langle m \rangle$. Taking (2.3) and (2.7) into account in (2.1b) implies

$$\nabla_x r = -\partial_t m + \sqrt{\tfrac{2}{d}}\, e\, \nabla_x \phi = -A'\, x - b' + \tfrac{1}{\sqrt{2}\,d}\, c''\, x + \sqrt{\tfrac{2}{d}}\, c\, \nabla \phi\,.$$

Taking the skew-symmetric gradient of this equation gives $0 = -A'$. Hence A is a skew-symmetric matrix which does not depend on t and

$$m(t,x) = A\, x + b(t) - \tfrac{1}{\sqrt{2}\,d}\, c'(t)\, x\,. \tag{2.8}$$

Taking (2.3) and (2.8) into account, we can then take the primitive in space of (2.1b) and we immediately deduce that the macroscopic density satisfies

$$r(t,x) = r_0 - b'(t)\cdot x + \frac{1}{2\sqrt{2\,d}}\,c''(t)\,\xi_2(x) + \sqrt{\tfrac{2}{d}}\,c(t)\,\xi_\phi(x) \tag{2.9}$$

where r_0 is an integration constant,

$$\xi_2(x) := |x|^2 - \langle |x|^2\rangle \tag{2.10}$$

and

$$\xi_\phi(x) := \phi - \langle\phi\rangle\,. \tag{2.11}$$

An integration against ρ shows that $r_0 = \langle r(t,\cdot)\rangle$ is in fact independent of t. Inserting the expressions of r and m given by (2.8) and (2.9) into (2.1a) yields a differential equation satisfied by A, $b(t)$ and $c(t)$:

Proposition 2.1. *Assume that r, m and e solve (2.1). With the above notations, $A(t) := \langle\nabla_x^{skew} m\rangle$, $b(t) := \langle m\rangle$ and $c(t) := \langle e\rangle$ solve*

$$\frac{2\,\xi_\phi(x)+\nabla_x\phi\cdot x-d}{\sqrt{2\,d}}\,c' + \frac{\xi_2(x)}{2\sqrt{2\,d}}\,c''' - \nabla_x\phi\cdot b - b''\cdot x - \nabla_x\phi\cdot A\,x = 0\,. \tag{2.12}$$

Equation (2.12) suggests, on the one hand, that (partial) harmonicity of the potential ϕ allows for non-trivial choices of b and c, as we shall indeed see later. On the other hand, for a generic potential ϕ in the sense that the functions 1, x, ϕ, $\nabla_x\phi\cdot x$, $\nabla\phi$, $|x|^2$ and $\nabla_x\phi\cdot A\,x$ (if $A\neq 0$) are linearly independent, equation (2.12) implies that

$$c' = 0\,,\quad b = 0\quad\text{and}\quad A = 0\,,$$

so that $r = r_0 + c\,\sqrt{\frac{2}{d}}\,\xi_\phi$, $m = 0$ and $e = c$, for two constants r_0 and $c\in\mathbb{R}$. In other words, we have

$$h^{\parallel} = r_0 + \sqrt{\tfrac{2}{d}}\,c\,\mathcal{H} - \sqrt{\tfrac{2}{d}}\,c\,\big(\langle\phi\rangle - \tfrac{d}{2}\big)$$

if ϕ is not a (partially) harmonic potential: any special macroscopic mode is then a linear combination of a Maxwellian function and an energy mode.

2.2. *Global conservation laws*

Consider a solution $f\in C\big(\mathbb{R}^+;\mathrm{L}^2(\mathcal{M}^{-1})\big)$ to (1.1) with initial datum $f_0\in\mathrm{L}^2(\mathcal{M}^{-1})$. Associated with the symmetries of the equation, there are local conservations which, after integration on the phase space $\mathbb{R}^d\times\mathbb{R}^d$, give rise to global conservation laws. These laws allow us to identify the special macroscopic modes compatible with ϕ which, as we shall see later, attract the solutions to the Cauchy problem.

The *conservation of mass* writes

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0\,.$$

Hence $\alpha\,\mathcal{M}$ is a solution to (1.4) with same mass

$$\alpha := \iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0\,\mathrm{d}x\,\mathrm{d}v \tag{2.13}$$

as f. With $\mathcal{H}$ defined by (1.7), the *conservation of energy* amounts to

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{H}(x,v)\,f(t,x,v)\,\mathrm{d}x\,\mathrm{d}v = 0\,.$$

The distribution function $\beta\,\mathcal{H}\,\mathcal{M}$, with

$$\beta := \frac{\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{H}\,f_0\,\mathrm{d}x\,\mathrm{d}v}{\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{H}^2\,\mathcal{M}\,\mathrm{d}x\,\mathrm{d}v}\,, \tag{2.14}$$

is a solution to (1.4) with same energy as the conserved energy of f. With $\mathsf{f}_1 := \mathcal{M}$ and $\mathsf{f}_{\mathcal{H}} := \mathcal{H}\,\mathcal{M}/\|\mathcal{H}\,\mathcal{M}\|_{\mathrm{L}^2(\mathcal{M}^{-1})}$, we have that

$$\alpha\,\mathcal{M} = \langle f_0, \mathsf{f}_1\rangle_{\mathrm{L}^2(\mathcal{M}^{-1})}\,\mathsf{f}_1 \quad \text{and} \quad \beta\,\mathcal{H}\,\mathcal{M} = \langle f_0, \mathsf{f}_{\mathcal{H}}\rangle_{\mathrm{L}^2(\mathcal{M}^{-1})}\,\mathsf{f}_{\mathcal{H}}\,.$$

Moreover, the global conservations of mass and energy write

$$\forall\,t\ge 0,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} \big(f(t,x,v)-\alpha\,\mathcal{M}(x,v)\big)\,\mathrm{d}x\,\mathrm{d}v = 0\,,$$
$$\forall\,t\ge 0,\quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{H}(x,v)\,\big(f(t,x,v)-\beta\,\mathcal{H}(x,v)\,\mathcal{M}(x,v)\big)\,\mathrm{d}x\,\mathrm{d}v = 0\,.$$

The transport operator can be written as $\mathcal{T} f = X\cdot\nabla_{x,v} f = \nabla_{x,v}\cdot(X\,f)$, where $X = (-v, \nabla_x\phi) = (-\nabla_v\mathcal{H}, \nabla_x\mathcal{H})$ is a divergence free vector field, in the sense that

$$\nabla_{x,v}\cdot X = 0\,. \tag{2.15}$$

As a consequence, the volume conservation in the phase space under the action of the flow induces the local mass conservation and the (2.15) symmetry gives rise to the global mass conservation. Another symmetry is associated with the fact that $\mathcal{H}$ is conserved along the characteristics of Newton's equations $\dot{x} = v$ and $\dot{v} = -\nabla_x\phi$. This is reflected by the Poisson brackets: a stationary solution F satisfies

$$0 = \{F,\mathcal{H}\} := \nabla_v\mathcal{H}\cdot\nabla_x F - \nabla_x\mathcal{H}\cdot\nabla_v F\,, \tag{2.16}$$

but by replacing F by $\mathcal{H}\,F$, it is also clear that

$$\{\mathcal{H}\,F,\mathcal{H}\} = \mathcal{H}\,\{F,\mathcal{H}\} + \{\mathcal{H},\mathcal{H}\}\,\mathcal{H} = 0\,.$$

The underlying reason is that the transport dynamics involving a time-independent potential is invariant under a translation in time, which gives rise to the global conservation of energy. These considerations can be generalized. To any continuous group of transformations which leaves $\mathscr{T}$ invariant, we can associate an infinitesimal transformation $\mathcal{G}(x,v)$

such that $\{\mathcal{G}, \mathcal{H}\} = 0$ and as a consequence, if f solves the transport equation $\partial_t f = \mathscr{T} f$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{G}(x,v)\, f(t,x,v)\, \mathrm{d}x\, \mathrm{d}v = 0\,. \tag{2.17}$$

Additionally, if $v \mapsto \mathcal{G}(x,v)\,\mu(v)$ is in the kernel of the collision operator $\mathscr{C}$ for any $x \in \mathbb{R}^d$, then $\gamma\, \mathcal{G}\, \mathcal{M}$ is a solution of (1.4), *i.e.*, a *special macroscopic mode*, for any $\gamma \in \mathbb{R}$. Then (2.17) holds for any solution f of (1.1) with initial datum f_0 and there is a unique $\gamma \in \mathbb{R}$ such that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{G}(x,v)\,\big(f_0(x,v) - \gamma\, \mathcal{G}(x,v)\, \mathcal{M}(x,v)\big)\, \mathrm{d}x\, \mathrm{d}v = 0\,.$$

More considerations on symmetries, local and global conservation laws, and Noether's theorem can be found in textbooks on classical mechanics like, for instance, [31, 42]. The case of rotational symmetries enters this framework:

When ϕ is invariant under a rotation, stationary *rotation modes* appear. Let $A \in \mathcal{R}_\phi$ as defined in (1.8) and consider the rotation group $(R_\theta)_{\theta\in\mathbb{R}}$ defined by $R_\theta := e^{\theta A}$ and a point $x_0 \in \mathbb{R}^d$ so that $\phi\big(R_\theta(x - x_0) + x_0\big) = \phi(x)$ for any $\theta \in \mathbb{R}$. By differentiation with respect to θ, we get

$$\forall\, x \in \mathbb{R}^d\,, \quad (A\, x + u)\cdot\nabla_x\phi(x) = 0$$

with $u = -A\, x_0$. Integrating the above identity against $(u\cdot x)\,\rho$ yields $u = 0$ after an integration by parts because $\rho(x)\,\mathrm{d}x$ is centred according to Assumption (H3). *Rotation modes compatible with* ϕ are therefore restricted to $x_0 = 0$: see [9] for similar computations. As a consequence, if we compute the Poisson bracket as defined in (2.16), we find that

$$\{\mathcal{G}, \mathcal{H}\} = 0 \quad \text{if} \quad \mathcal{G}(x,v) = (A\, x\cdot v)$$

and the conservation of the *total angular momentum* associated with this rotation writes

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (A\, x\cdot v)\, f(t,x,v)\, \mathrm{d}x\, \mathrm{d}v = 0\,.$$

Given f_0, let us identify the corresponding special macroscopic mode.

Associated with f_0, we introduce the initial momentum

$$m_0(x) := \left(\int_{\mathbb{R}^d} v\, f_0(x,v)\, \mathrm{d}v\right) e^{\phi(x)}$$

and the infinitesimal rotation $x \mapsto A\, x := \mathbb{P}_\phi m_0(x)$, where $\mathbb{P}_\phi$ is the orthogonal projection onto the vector space $\mathcal{R}_\phi$ in $\mathrm{L}^2(\rho)$. We can then check that $x \mapsto A\, x$ belongs to $\mathcal{R}_\phi$ and thus the function (rotational mode compatible with ϕ)

$$F_{\mathrm{rot}}(x,v) := (A\, x\cdot v)\, \mathcal{M} \tag{2.18}$$

belongs to $\mathfrak{R}_\phi$, so that F_{rot} is a solution to (1.4) with same conserved total angular momentum as f. Denoting

$$m_f(t,x) := \left(\int_{\mathbb{R}^d} v\, f(t,x,v)\, \mathrm{d}v\right) e^{\phi(x)}$$

the momentum of f, the associated conservation law then reads

$$\mathbb{P}_\phi m_f(t) = \mathbb{P}_\phi m_0 \quad \text{or equivalently} \quad \mathbb{P}(m_f - m_0) \in \mathcal{R}_\phi^\perp\,, \tag{2.19}$$

where $\mathbb{P}$ is the orthogonal projection onto the vector space of all infinitesimal rotations $\mathcal{R} := \{x \mapsto A\,x \;:\; A \in \mathfrak{M}^{\text{skew}}_{d\times d}(\mathbb{R})\}$, identified with $\mathfrak{M}^{\text{skew}}_{d\times d}(\mathbb{R})$, in $\mathrm{L}^2(\rho)$, and $\mathcal{R}_\phi^\perp$ is the orthogonal of $\mathcal{R}_\phi$, seen as a subspace of $\mathcal{R}$, for the scalar product induced by $\mathrm{L}^2(\rho)$. We refer to Lemma A.1 in Appendix A.1 for a precise statement and a short proof.

If we denote by $(A_j)_{j\in J_\phi} = (x \mapsto A_j\,x)_{j\in J_\phi}$ a basis of $\mathcal{R}_\phi$ normalized by the condition $\left\|\mathsf{f}_{\text{rot},j}\right\|_{\mathrm{L}^2(\mathcal{M}^{-1})} = 1$ for any $j \in J_\phi$, where $\mathsf{f}_{\text{rot},j}(x,v) := \left(A_j\,x \cdot v\right)\mathcal{M}(x,v)$, then the conservation of the total angular momentum implies for all $j \in J_\phi$:

$$\forall\, t \geq 0, \quad \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left[f(t,x,v) - \langle f_0, \mathsf{f}_{\text{rot},j}\rangle_{\mathrm{L}^2(\mathcal{M}^{-1})}\,\mathsf{f}_{\text{rot},j}(x,v)\right]\,\mathrm{d}x\,\mathrm{d}v = 0.$$

Now let us turn our attention to the *time-periodic special macroscopic modes* and start with the harmonic directional modes, which appear when $d_\phi \leq d-1$. We choose a coordinate system such that $\partial_{x_i}\phi = x_i$ for $i \in I_\phi = \{d_\phi+1,\ldots,d\}$. In that case, the potential ϕ is such that $x \mapsto \phi(x) - \frac12\sum_{i\in I_\phi} x_i^2$ depends only on $(x_1,\ldots,x_{d_\phi})$ and for any $i \in I_\phi$ the *harmonic directional modes*, defined for all $(t,x,v) \in \mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d$ by

$$\mathsf{f}^+_{\text{dir},i}(t,x,v) := \left(x_i\,\cos t + v_i\,\sin t\right)\mathcal{M}(x,v)\,,$$
$$\mathsf{f}^-_{\text{dir},i}(t,x,v) := \left(v_i\,\cos t - x_i\,\sin t\right)\mathcal{M}(x,v)\,,$$

solve (1.4). A direct computation of the solution of (1.1) with initial datum $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d} x_i\,f\,\mathrm{d}x\,\mathrm{d}v = \iint_{\mathbb{R}^d\times\mathbb{R}^d} v_i\,f\,\mathrm{d}x\,\mathrm{d}v\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d} v_i\,f\,\mathrm{d}x\,\mathrm{d}v = -\iint_{\mathbb{R}^d\times\mathbb{R}^d} x_i\,f\,\mathrm{d}x\,\mathrm{d}v\,,$$

which implies that these two global quantities evolve as an harmonic oscillator with period equal to 1. For any $i \in I_\phi$, let us define

$$\gamma_i := \iint_{\mathbb{R}^d\times\mathbb{R}^d} x_i\,f_0\,\mathrm{d}x\,\mathrm{d}v\,, \quad \bar\gamma_i := \iint_{\mathbb{R}^d\times\mathbb{R}^d} v_i\,f_0\,\mathrm{d}x\,\mathrm{d}v\,.$$

The function

$$F_{\text{dir}} := \sum_{i=d_\phi+1}^{d}\left(\gamma_i\,\mathsf{f}^+_{\text{dir},i} + \bar\gamma_i\,\mathsf{f}^-_{\text{dir},i}\right) \tag{2.20}$$

solves (1.4) and belongs to $\mathfrak{D}_\phi$ as defined by (1.10). Moreover $(f - F_{\text{dir}})$ satisfies the following two global conservation laws: for any $t \geq 0$,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} x_i\left(f(t,x,v) - F_{\text{dir}}(t,x,v)\right)\mathrm{d}x\,\mathrm{d}v = 0\,,$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} v_i\left(f(t,x,v) - F_{\text{dir}}(t,x,v)\right)\mathrm{d}x\,\mathrm{d}v = 0\,.$$

When *all* coordinates are harmonic ($d_\phi = 0$), then $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log(2\pi)$ due to the normalization (H7) and the *harmonic pulsating modes*, defined for all $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\mathsf{f}^+_{\rm pul}(t,x,v) := \tfrac{1}{\sqrt{d}}\left(x\cdot v\cos(2t) + \tfrac{1}{2}\left(|x|^2-|v|^2\right)\sin(2t)\right)\mathcal{M}(x,v)\,,$$
$$\mathsf{f}^-_{\rm pul}(t,x,v) := \tfrac{1}{\sqrt{d}}\left(\tfrac{1}{2}\left(|x|^2-|v|^2\right)\cos(2t) - x\cdot v\sin(2t)\right)\mathcal{M}(x,v)\,,$$

solve (1.4). A direct computation of the solution of (1.1) with initial datum $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(x\cdot v)\,f\,\mathrm{d}x\,\mathrm{d}v = -\,2\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\left(|x|^2-|v|^2\right)f\,\mathrm{d}x\,\mathrm{d}v\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\left(|x|^2-|v|^2\right)f\,\mathrm{d}x\,\mathrm{d}v = 2\iint_{\mathbb{R}^d\times\mathbb{R}^d}(x\cdot v)\,f\,\mathrm{d}x\,\mathrm{d}v\,,$$

which implies that these two global quantities evolve as an harmonic oscillator with period equal to $1/2$. With

$$\delta := \frac{1}{\sqrt{d}}\iint_{\mathbb{R}^d\times\mathbb{R}^d}(x\cdot v)\,f_0\,\mathrm{d}x\,\mathrm{d}v\,,\quad \bar\delta := \frac{1}{\sqrt{d}}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\left(|x|^2-|v|^2\right)f_0\,\mathrm{d}x\,\mathrm{d}v\,,$$

the function

$$F_{\rm pul} := \delta\,\mathsf{f}^+_{\rm pul} + \bar\delta\,\mathsf{f}^-_{\rm pul} \tag{2.21}$$

solves (1.4) and belongs to $\mathfrak{P}_\phi$ as defined by (1.11). Moreover $(f - F_{\rm pul})$ satisfies the following two global conservation laws: for any $t \ge 0$,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}(x\cdot v)\left(f(t,x,v) - F_{\rm pul}(t,x,v)\right)\mathrm{d}x\,\mathrm{d}v = 0\,,$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{1}{2}\left(|x|^2-|v|^2\right)\left(f(t,x,v) - F_{\rm pul}(t,x,v)\right)\mathrm{d}x\,\mathrm{d}v = 0\,.$$

Let us consider the set of the generators of all above special macroscopic modes

$$\widehat{\mathcal{S}} := \{\mathsf{f}_1, \mathsf{f}_{\mathcal{H}}\} \cup \{\mathsf{f}_{{\rm rot},j}\}_{j\in J_\phi} \cup \{\mathsf{f}^\pm_{{\rm dir},i}\}_{i\in I_\phi,\pm} \cup \{\mathsf{f}^\pm_{\rm pul}\}_\pm\,.$$

We have the following orthogonality property.

Lemma 2.2. *The functions of* $\widehat{\mathcal{S}}$ *are orthonormal in* $\mathrm{L}^2(\mathcal{M})$.

Proof of Lemma 2.2. This follows from direct computation using standard properties of Hermite functions. ∎

As a straightforward consequence of Lemma 2.2, we obtain

Corollary 2.3. *Assume that* $f \in C\big(\mathbb{R}^+;\mathrm{L}^2(\mathcal{M}^{-1})\big)$ *is a solution to* (1.1) *with initial datum* $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$. *With the above notations, for any* $\mathsf{f} \in \widehat{\mathcal{S}}$ *and any* $t \ge 0$, *we have*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(f(t) - \alpha\,\mathcal{M} - \beta\,\mathcal{H}\,\mathcal{M} - F_{\rm rot} - F_{\rm dir}(t) - F_{\rm pul}(t)\Big)\frac{\mathsf{f}}{\mathcal{M}}\,\mathrm{d}x\,\mathrm{d}v = 0\,.$$

2.3. A micro-macro decomposition

Let us consider a solution $f \in C\big(\mathbb{R}^+; \mathrm{L}^2(\mathcal{M}^{-1})\big)$ to (1.1). We get rid of the special macroscopic modes built in (2.13)–(2.14)–(2.18)–(2.20)–(2.21) and rewrite the evolution problem in $\mathrm{L}^2(\mathcal{M})$ in terms of

$$h := \frac{f - \alpha\,\mathcal{M} - \beta\,\mathcal{H}\,\mathcal{M} - F_{\mathrm{rot}} - F_{\mathrm{dir}} - F_{\mathrm{pul}}}{\mathcal{M}} \tag{2.22}$$

for all (t, x, v), where only h, f, F_{dir} and F_{pul} depend on t. Then h satisfies

$$\partial_t h = \mathcal{L}\, h := \mathscr{T} h + C h\,, \quad h|_{t=0} = h_0 \tag{2.23}$$

with $\mathscr{T}$ defined as before by $\mathscr{T} h = \nabla_x \phi \cdot \nabla_v h - v \cdot \nabla_x h$ and the new collision operator

$$C h := \mu^{-1}\,\mathscr{C}\,(\mu\, h) \tag{2.24}$$

where μ is defined in (1.2).

The operator C acts only on the velocity variable, is self-adjoint in $\mathrm{L}^2(\mathcal{M})$ (when integrating in x, v) and $\mathrm{L}^2(\mu)$ (when integrating in v) and

$$\operatorname{Ker} C = \operatorname{Span}\left\{1, v_1, \ldots, v_d, |v|^2\right\}\,.$$

Let us consider the *micro-macro decomposition*

$$h = h^{\parallel} + h^{\perp}\,, \quad h^{\parallel} := r + m \cdot v + e\,\mathfrak{E}(v)\,,$$

where $h^{\parallel}$ is the $\mathrm{L}^2(\mu)$-orthogonal projection of h on $\operatorname{Ker} C$ and $\mathfrak{E}$ defined by (1.6) is a normalized Hermite polynomial of degree 2. In other words,

$$\begin{aligned} r(t,x) &:= \int_{\mathbb{R}^d} h(t,x,v)\,\mu(v)\,\mathrm{d}v \\ m(t,x) &:= \int_{\mathbb{R}^d} v\,h(t,x,v)\,\mu(v)\,\mathrm{d}v \\ e(t,x) &:= \int_{\mathbb{R}^d} \mathfrak{E}(v)\,h(t,x,v)\,\mu(v)\,\mathrm{d}v \end{aligned} \tag{2.25}$$

are the *macroscopic quantities* corresponding to the spatial density, the local flux and thermal energy, while $h^{\perp}$ is the *microscopic* part. The definition (2.25) coincides with the definition (1.5) used in Section 2.1 to define the special macroscopic modes.

With these notations (H1) reads

$$-\,\langle C h, h\rangle \geq \mathrm{c}_{\mathscr{C}}\,\|h^{\perp}\|^2\,.$$

According to Corollary 2.3, h has *multiple global conservation laws.*

Corollary 2.4. *Assume that* $f \in C\big(\mathbb{R}^+; \mathrm{L}^2(\mathcal{M}^{-1})\big)$ *is a solution to* (1.1) *with initial datum* $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$*. With* h *and* (r, m, e) *respectively defined by* (2.22)–(2.23) *and* (2.25)*, we have the following properties.*

▷ *Conservation of total mass and total energy*

$$\langle r \rangle = 0 \quad \textit{and} \quad \sqrt{\tfrac{d}{2}}\,\langle e \rangle + \langle \phi\, r \rangle = 0\,. \tag{2.26}$$

▷ *Global conservation laws associated to rotational symmetries of* ϕ

$$\mathbb{P}_\phi m = 0\,. \tag{2.27}$$

This also means $\mathbb{P}(m) \in \mathcal{R}_\phi^\perp$ *as in* (2.19).

▷ *Global conservation laws corresponding to the* harmonic directional modes

$$\forall\, i \in I_\phi\,, \quad \langle r\, x_i \rangle = 0 \quad \textit{and} \quad \langle m_i \rangle = 0\,. \tag{2.28}$$

▷ *In the fully harmonic case* $d_\phi = 0$*, global conservation laws corresponding to the* harmonic pulsating modes

$$\langle m \cdot x \rangle = 0 \quad \textit{and} \quad \sqrt{\tfrac{d}{2}}\,\langle e \rangle - \langle \phi\, r \rangle = 0\,. \tag{2.29}$$

2.4. *The equations for the macroscopic modes*

We write the evolution equations for r, m and e defined in (2.25). In the mathematical literature, such equations are sometimes called *local conservation laws* but as this might introduce confusions with the *local conservation law of the collision operator*, which we call here *collision invariants*, and the *global conservation laws* studied in Section 2.2, we shall simply refer to these equations as the *equations for the macroscopic modes* or simply the *macroscopic equations*.

Assume that h solves (2.23). For any Hermite polynomial $p : \mathbb{R}^d \to \mathbb{R}$ considered as a function of the velocity variable, we compute $J_p[h] = \int_{\mathbb{R}^d} p\, h\, \mu\, \mathrm{d}v$ using standard properties of Hermite functions:

$$J_p[h] = r \int_{\mathbb{R}^d} p\, \mu\, \mathrm{d}v + m \cdot \int_{\mathbb{R}^d} v\, p\, \mu\, \mathrm{d}v + e \int_{\mathbb{R}^d} \mathfrak{E}\, p\, \mu\, \mathrm{d}v + J_p[h^\perp]$$

and, using (2.23), we also get

$$\begin{aligned} \partial_t J_p[h] = &- \nabla_x r \cdot \int_{\mathbb{R}^d} v\, p\, \mu\, \mathrm{d}v - \nabla_x m : \int_{\mathbb{R}^d} v \otimes v\, p\, \mu\, \mathrm{d}v + m \cdot \nabla_x \phi \int_{\mathbb{R}^d} p\, \mu\, \mathrm{d}v \\ &- \nabla_x e \cdot \int_{\mathbb{R}^d} v\, \mathfrak{E}\, p\, \mu\, \mathrm{d}v + \sqrt{\tfrac{2}{d}}\, e\, \nabla \phi \cdot \int_{\mathbb{R}^d} v\, p\, \mu\, \mathrm{d}v + \int_{\mathbb{R}^d} (\mathcal{L}\, h^\perp)\, p\, \mu\, \mathrm{d}v\,. \end{aligned}$$

Plugging successively $p = 1$, v, $\mathfrak{E}$, $v \otimes v - \mathrm{Id}_{d\times d}$ and $v(\mathfrak{E} - \sqrt{\tfrac{2}{d}})$ we get

$$\partial_t r = \nabla_x^* \cdot m\,, \tag{2.30a}$$

$$\partial_t m = -\nabla_x r + \sqrt{\tfrac{2}{d}}\,\nabla_x^* e + \nabla_x^* \cdot E[h^\perp]\,, \tag{2.30b}$$

$$\partial_t e = -\sqrt{\tfrac{2}{d}}\,\nabla_x \cdot m + \nabla_x^* \cdot \Theta[h^\perp]\,, \tag{2.30c}$$

$$\partial_t E[h] = -2\,\nabla_x^{\rm sym} m + E[\mathcal{L}\,h^\perp]\,, \tag{2.30d}$$

$$\partial_t \Theta[h] = -\left(1 + \tfrac{2}{d}\right)\nabla_x e + \Theta[\mathcal{L}\,h^\perp]\,, \tag{2.30e}$$

where $\nabla_x^{\rm sym} m$ is defined by (2.2) and the matrix valued function $E[h]$ and the vector valued function $\Theta[h]$ are higher-order moments of h defined by

$$E[h] := \int_{\mathbb{R}^d} (v \otimes v - \mathrm{Id}_{d\times d})\, h\,\mu\,\mathrm{d}v = \sqrt{\tfrac{2}{d}}\, e\,\mathrm{Id}_{d\times d} + E[h^\perp]\,, \tag{2.31a}$$

$$\Theta[h] := \int_{\mathbb{R}^d} v\left(\mathfrak{E}(v) - \sqrt{\tfrac{2}{d}}\right) h\,\mu\,\mathrm{d}v = \Theta[h^\perp]\,. \tag{2.31b}$$

If f is a special macroscopic mode, then (2.30) is reduced to (2.1) because, in that case, $h^\perp = 0$ and $\Theta[h^\parallel] = 0$.

3. Classification of the special macroscopic modes

In this section, we prove Part (1) of Theorem 1.1. We write $\mathsf{a} \lesssim \mathsf{b}$ if there is some positive constant c such that $\mathsf{a} \le \mathsf{b}\,\mathsf{c}$ and $\mathsf{a} \simeq \mathsf{b}$ if and only if $\mathsf{a} \lesssim \mathsf{b} \lesssim \mathsf{a}$. Throughout this section, we assume that (H0)–(H1)–(H2)–(H3)–(H4)–(H5)–(H6)–(H7)–(H8) hold, without further notice.

3.1. Statement and preliminary results

Theorem 1.1–(1) writes:

Proposition 3.1 (Special macroscopic modes). *If f is a solution of* (1.4)*, then h given by* (2.22) *is such that $h = 0$.*

We recall that f is a special macroscopic mode if and only if, by definition (1.4), $\mathscr{C} f = 0$ and $\partial_t f = \mathscr{T} f$. With the definitions of (2.13)–(2.14)–(2.18)–(2.20)–(2.21), the function

$$F := f - \alpha\,\mathcal{M} - \beta\,\mathcal{H}\,\mathcal{M} - F_{\rm rot} - F_{\rm dir} - F_{\rm pul}$$

is also a special macroscopic mode and (1.5) implies that h defined in (2.22) satisfies $h = h^\parallel = r + m \cdot v + e\,\mathfrak{E}(v)$. According to (2.23), h solves the transport equation $\partial_t h = \mathscr{T} h$ because $C h = 0$ (with C defined by (2.24)) and Section 2.1 proved that r, m and e solve (2.1). Proposition 3.1 means that $r = 0$, $m = 0$, and $e = 0$. We split the proof into several steps. To start with, since 1, v and $\mathfrak{E}(v)$ are orthonormal Hermite polynomials, we have

$$\|h\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2\,. \tag{3.1}$$

Lemma 3.2. *With the above notations, the function h as in Proposition 3.1 satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 = 0\,.$$

Proof of Lemma 3.2. It follows from $C h = 0$, $\mathscr{T}^* = -\mathscr{T}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 = 2\,(h, \mathscr{T} h) = 0$. ∎

Collecting the results of Section 2.1 and using $r_0 = \langle r\rangle = 0$, we get:

Lemma 3.3. *Consider the function h as in Proposition 3.1. With the above notations, let $A := \langle \nabla_x^{skew} m\rangle$, $b(t) := \langle m\rangle$ and $c(t) := \langle e\rangle$. Then we have*

$$r(t,x) = -\,x\cdot b'(t) + \tfrac{\xi_2(x)}{2\sqrt{2\,d}}\,c''(t) + \sqrt{\tfrac{2}{d}}\,\xi_\phi\,c(t)\,, \tag{3.2a}$$

$$m(t,x) = A\,x + b(t) - \tfrac{x}{\sqrt{2\,d}}\,c'(t)\,, \tag{3.2b}$$

$$e(t,x) = c(t)\,, \tag{3.2c}$$

where A is a constant skew-symmetric matrix, while b and c are respectively vector valued and scalar functions of t related by (2.12).

We recall that the functions ξ_2 and ξ_ϕ are defined respectively by (2.10) and (2.11). Equation (2.12) in Proposition (2.1) provides us with various estimates on A, b and c which are collected in Sections 3.2, 3.3 and 3.4 in order to prove Proposition 3.1 in Section 3.5.

3.2. *Control of A*

For any $A \in \mathfrak{M}^{\mathrm{skew}}_{d\times d}(\mathbb{R})$, let us define

$$|A|^2 := \int_{\mathbb{R}^d} |A\,x|^2\,\rho(x)\,\mathrm{d}x\,.$$

The vector space $\mathfrak{M}^{\mathrm{skew}}_{d\times d}(\mathbb{R})$ is of finite dimension: all norms are equivalent to $|\cdot|$. This is also why the rigidity constant c_{K} given by (1.14) is positive, which implies the Korn-type inequality

$$\forall\,A \in \mathcal{R}_\phi^\perp\,,\quad \|\nabla_x\phi\cdot A\,x\|^2 \ge c_{\mathrm{K}}\,|A|^2\,. \tag{3.3}$$

By multiplying (2.12) by x_k for $k = 1,\ldots,d$, then integrating against $\rho(x)$ and performing some integrations by part, using that $\rho(x)\,\mathrm{d}x$ is centred and that the terms involving $\nabla\phi$ vanish, we obtain

$$\tfrac{1}{\sqrt{2\,d}}\,\langle 2\,\phi\,x\rangle\,c' + \tfrac{1}{2\sqrt{2\,d}}\,\langle |x|^2\,x\rangle\,c''' - \langle x\otimes x\rangle\,b'' = b\,. \tag{3.4}$$

With the notation of (3.2), let us define X and Y by

$$X := \tfrac{1}{\sqrt{2\,d}}\,\big(2\,\xi_\phi + \nabla_x\phi\cdot x - d\big)\,c + \tfrac{\xi_2}{2\sqrt{2\,d}}\,c'' - x\cdot b' \tag{3.5}$$

and

$$Y := \sqrt{\tfrac{2}{d}}\,\langle\phi\,x\rangle\,c + \tfrac{1}{2\sqrt{2\,d}}\,\langle |x|^2\,x\rangle\,c'' - \langle x\otimes x\rangle\,b'\,. \tag{3.6}$$

Identities (2.12) and (3.4) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(X - Y\cdot\nabla_x\phi) = \nabla_x\phi\cdot A\,x$$

where, according to Lemma 3.3, the r.h.s. is independent of t. As a consequence, we have the following estimate.

Lemma 3.4. *Consider the function h as in Proposition 3.1. The infinitesimal rotation matrix A of Lemma 3.3 satisfies*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\,(X - Y\cdot\nabla_x\phi)\,,\nabla_x\phi\cdot A\,x\Big\rangle = -\,\|\nabla_x\phi\cdot A\,x\|^2 \le -\,c_{\mathrm{K}}\,|A|^2\,.$$

Proof. By the conservation law (2.27), we know that $A\,x = \mathbb{P}(m) \in \mathcal{R}_\phi^\perp$, so that (3.3) applies. ∎

3.3. *Control of b, b'', c' and c'''*

Lemma 3.5. *Consider the function h as in Proposition 3.1. The functions b and c as defined in Lemma 3.3 are such that*

$$|b| + |b''| + |c'| + |c'''| \lesssim |A|\,. \tag{3.7}$$

Proof of Lemma 3.5. Multiplying (2.12) by $\nabla_x\phi$ and integrating against $\rho(x)$, after integration by parts, using that ρ is centred and observing that the terms involving $2\,\xi_\phi - d$ and c''' vanish, it follows that

$$b'' = -\,\langle\nabla_x^2\phi\rangle\,b + \tfrac{1}{\sqrt{2\,d}}\,\langle\nabla_x^2\phi\,x\rangle\,c' + R_1 = -\,b + \tfrac{1}{\sqrt{2\,d}}\,\langle\nabla_x^2\phi\,x\rangle\,c' + R_1 \tag{3.8}$$

with $R_0 := -\,\nabla_x\phi\cdot A\,x$ and $R_1 := \langle R_0\,\nabla_x\phi\rangle = O(A)$. By inserting (3.8) into (2.12), one gets

$$\Psi_1(x)\,c' + \Psi_2(x)\,c''' - \Phi(x)\cdot b = R_2 \tag{3.9}$$

with

$$\Phi(x) := \nabla_x\phi - \langle\nabla_x^2\phi\rangle\,x = \nabla_x\phi - x\,,$$

$$\Psi_1(x) := \tfrac{2\,\xi_\phi(x)+\nabla_x\phi\cdot x-d}{\sqrt{2\,d}} - x\cdot\tfrac{\langle\nabla_x^2\phi\,x\rangle}{\sqrt{2\,d}}\,,\quad \Psi_2(x) := \tfrac{\xi_2(x)}{2\,\sqrt{2\,d}}\,,$$

and $R_2 := R_1\cdot x - R_0$. Let

$$M_\phi := \langle\Phi\otimes\Phi\rangle \in \mathfrak{M}^{\mathrm{sym}}_{d\times d}(\mathbb{R})\,,\quad \alpha_i := \langle\Psi_i\,\Phi\rangle \in \mathbb{R}^d \tag{3.10}$$

with $i = 1, 2$. A multiplication of (3.9) by Φ and an integration against ρ yields

$$M_\phi\,b = \alpha_1\,c' + \alpha_2\,c''' + R_3 \tag{3.11}$$

where $R_3 := -\,\langle R_2\,\Phi\rangle = O(A)$ thanks to the moment bounds on ϕ deduced from (H6). Inverting the matrix M_ϕ allows to control b by c and c'', and rewrite (3.9) as an ordinary differential equation on c, up to an error term of the order of $O(A)$. If M_ϕ is not invertible, a similar estimate can still be done after taking into account the global conservation laws of Corollary 2.4.

We recall that E_ϕ is defined by (1.9). We distinguish three cases.

▷ *Fully non-harmonic case* ($d_\phi = d$). The matrix M_ϕ is invertible (see Lemma A.2 in Appendix A) and (3.11) yields

$$b = M_\phi^{-1} \left(\alpha_1 \, c' + \alpha_2 \, c''' + R_3\right) \tag{3.12}$$

and hence, together with (3.9), it follows that

$$\widetilde{\Psi}_1(x) \, c' + \widetilde{\Psi}_2(x) \, c''' = R_4 \,, \tag{3.13}$$

with $R_4 := R_2 + \Phi(x) \cdot M_\phi^{-1} R_3$ and

$$\widetilde{\Psi}_1(x) := \Psi_1(x) - \Phi(x) \cdot M_\phi^{-1} \, \alpha_1 \,, \quad \widetilde{\Psi}_2(x) := \Psi_2(x) - \Phi(x) \cdot M_\phi^{-1} \, \alpha_2 \,. \tag{3.14}$$

From Lemma A.3 we know that $\mathrm{Rank}(\widetilde{\Psi}_1, \widetilde{\Psi}_2) = 2$ and deduce from (3.13) that $c' = O(A)$ and $c''' = O(A)$. Using then (3.12) and (3.8), we also deduce $b = O(A)$ and $b'' = O(A)$, and the proof is complete in this case.

▷ *Partially harmonic case* ($1 \le d_\phi \le d-1$). Let $\{e_1, \ldots, e_{d_\phi}, e_{d_\phi+1}, \ldots, e_d\}$ be a basis of $\mathbb{R}^d$ such that $\{e_1, \ldots, e_{d_\phi}\}$ generates E_ϕ. For any vector $x \in \mathbb{R}^d$, we shall write $x = (\hat{x}, \check{x})$ with $\hat{x} \in \mathbb{R}^{d_\phi}$ and $\check{x} \in \mathbb{R}^{d-d_\phi}$. Similarly, we use the notation $\xi(x) = (\hat{\xi}(x), \check{\xi}(x))$ for a vector-field $\xi : \mathbb{R}^d \to \mathbb{R}^d$. In particular one has $\Phi = (\widehat{\Phi}, 0)$ and also $\check{b} = 0$ so that $b = (\hat{b}, 0)$ as a consequence of (2.28). Hence (3.9) becomes

$$\Psi_1 \, c' + \Psi_2 \, c''' - \widehat{\Phi} \cdot \hat{b} = R_2 \,. \tag{3.15}$$

The matrix M_ϕ defined in (3.10) is given by

$$M_\phi = \begin{pmatrix} \widehat{M}_\phi & 0 \\ 0 & 0 \end{pmatrix}$$

where

$$\widehat{M}_\phi := \langle \widehat{\Phi} \otimes \widehat{\Phi} \rangle \in \mathfrak{M}^{\mathrm{sym}}_{d_\phi \times d_\phi}(\mathbb{R}) \,. \tag{3.16}$$

Following the same procedure as in the fully non-harmonic case, we obtain after multiplication by $\widehat{\Phi}$ and integration in $\mathrm{L}^2(\rho)$ that

$$\widehat{M}_\phi \, \hat{b} = \hat{\alpha}_1 \, c' + \hat{\alpha}_2 \, c''' + \hat{R}_3 \,, \tag{3.17}$$

with $\hat{R}_3 := -\langle R_2 \, \widehat{\Phi} \rangle = O(A)$, $\hat{\alpha}_1 := \langle \Psi_1 \, \widehat{\Phi} \rangle$ and $\hat{\alpha}_2 := \langle \Psi_2 \, \widehat{\Phi} \rangle$. The matrix $\widehat{M}_\phi$ is invertible (see Lemma A.2 in Appendix A) and (3.17) yields

$$\hat{b} = \widehat{M}_\phi^{-1} \left(\hat{\alpha}_1 \, c' + \hat{\alpha}_2 \, c''' + \hat{R}_3\right) . \tag{3.18}$$

Hence, together with (3.15), it follows that

$$\widehat{\Psi}_1(x) \, c' + \widehat{\Psi}_2(x) \, c''' = \hat{R}_4$$

with $\hat{R}_4 := R_2 + \widehat{\Phi}(x) \cdot \widehat{M}_\phi^{-1} \hat{R}_3 = O(A)$ and

$$\widehat{\Psi}_1(x) := \Psi_1(x) - \widehat{\Phi}(x) \cdot \widehat{M}_\phi^{-1} \hat{\alpha}_1 , \quad \widehat{\Psi}_2(x) := \Psi_2(x) - \widehat{\Phi}(x) \cdot \widehat{M}_\phi^{-1} \hat{\alpha}_2 . \tag{3.19}$$

As in the full rank case, $\mathrm{Rank}(\widehat{\Psi}_1, \widehat{\Psi}_2) = 2$ according to Lemma A.3 and we deduce that $c' = O(A)$ and $c''' = O(A)$. From (3.18) and (3.8), we also get $\hat{b} = O(A)$ and $\hat{b}'' = O(A)$, and since $\check{b} = 0$ we eventually get $b = O(A)$ and $b'' = O(A)$, which completes the proof of the case partially harmonic case.

▷ *Fully harmonic case* ($d_\phi = 0$). We read from (3.2c) and (2.29) that $\langle e \rangle = c = 0 = c' = c''$ and from (3.2b), (2.28) and (H3) that $\langle m \rangle = b = 0 = b' = b''$, which completes the proof of the case fully harmonic case. ■

3.4. *Control of b', c'' and c*

Lemma 3.6. *Consider the function h as in Proposition 3.1. The functions b and c as defined in Lemma 3.3 obey the two differential inequalities*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b' \rangle \le -|b'|^2 + O(|A|^2) \quad \textit{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle -c', c'' \rangle \le -|c''|^2 + O(|A|^2) .$$

Proof of Lemma 3.6. We write

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b' \rangle = \langle -b', b' \rangle + \langle -b, b'' \rangle \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle -c', c'' \rangle = \langle -c'', c'' \rangle + \langle -c', c''' \rangle$$

and notice that $\langle -b, b'' \rangle = O(|A|^2)$ and $\langle -c', c''' \rangle = O(|A|^2)$ by (3.7). ■

Lemma 3.7. *The function c as defined in Lemma 3.3 is such that*

$$|c| \lesssim |b'| + |c''| \quad \textit{and} \quad |c''| \lesssim |b'| + |c| . \tag{3.20}$$

Proof of Lemma 3.7. Multiplying (3.2a) by ξ_ϕ and integrating against ρ, we obtain

$$c\left(\sqrt{\tfrac{2}{d}}\left\langle \xi_\phi^2 \right\rangle + \sqrt{\tfrac{d}{2}}\right) = \langle x\, \xi_\phi \rangle \cdot b' - \frac{\langle \xi_2\, \xi_\phi \rangle}{2\sqrt{2}\, d}\, c''$$

using $\langle r\, \phi \rangle = -\sqrt{\frac{d}{2}}\, c$ and $\langle r \rangle = 0$ by (2.26), which completes the proof. ■

3.5. *A Lyapunov function method*

We define the Lyapunov function

$$\mathcal{F}[h] := \|h\|^2 - \varepsilon_A \left\langle (X - Y \cdot \nabla_x \phi), \nabla_x \phi \cdot A\, x \right\rangle - \varepsilon_b \langle b, b' \rangle - \varepsilon_c \langle c', c'' \rangle ,$$

for some positive constants ε_A, ε_b and ε_c to be chosen later.

Lemma 3.8. *With the above notations, if h is defined as in Proposition 3.1, then*

$$\mathcal{F}[h] \simeq \|h\|^2 \tag{3.21}$$

for ε_A, ε_b and ε_c small enough.

Proof of Lemma 3.8. From (3.1), we know that $\|h\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2$ and it follows from (3.2a), (3.2b) and (3.2c) that

$$\|r\|^2 \lesssim |b'|^2 + |c|^2 + |c''|^2\,, \quad \|m\|^2 \simeq |b|^2 + |A|^2 + |c'|^2 \quad \text{and} \quad \|e\|^2 = |c|^2\,. \tag{3.22}$$

By Lemma 3.5, we obtain

$$\|m\|^2 \lesssim |A|^2\,, \quad \big|\langle b, b'\rangle\big| \lesssim |A|^2 + |b'|^2\,, \quad \big|\langle c', c''\rangle\big| \lesssim |A|^2 + |c''|^2\,.$$

By Lemma 3.7, we know that $|c|^2 \lesssim |b'|^2 + |c''|^2$ and obtain

$$\|r\|^2 \lesssim |b'|^2 + |c''|^2\,, \quad \|c\|^2 \lesssim |b'|^2 + |c''|^2\,, \quad \|e\|^2 \lesssim |b'|^2 + |c''|^2\,,$$

$$\big|\big\langle (X - Y \cdot \nabla_x \phi), \nabla_x \phi \cdot A\,x \big\rangle\big| \lesssim |A|^2 + |b'|^2 + |c''|^2\,.$$

Altogether we have the upper estimate

$$\mathcal{F}[h] \lesssim |A|^2 + |b'|^2 + |c''|^2 \tag{3.23}$$

and, using (3.20),

$$\mathcal{F}[h] \lesssim |A|^2 + |b'|^2 + |c|^2\,.$$

Using (2.1b), we notice that

$$b' = \langle \partial_t m \rangle = -\,\langle \nabla_x r \rangle = -\,\langle r\,\nabla_x \phi \rangle \le \|r\|\,\|\nabla_x \phi\| \lesssim \|r\| \tag{3.24}$$

performing one integration by parts and using Cauchy-Schwarz inequality. It is then clear that $|A|^2 \lesssim \|m\|^2$ and $|c|^2 = \|e\|^2$, so that by (3.1),

$$\mathcal{F}[h] \lesssim \|r\|^2 + \|m\|^2 + \|e\|^2 = \|h\|^2\,.$$

Then, using (3.22) again, we have the lower bound estimate

$$\begin{aligned} 2\,\mathcal{F}[h] - \|h\|^2 \gtrsim\ & \|r\|^2 + \big(|b|^2 + |A|^2 + |c'|^2\big) + |c|^2 \\ & - 2\,\varepsilon_A \left(|A|^2 + |b'|^2 + |c''|^2\right) - 2\,\varepsilon_b \left(|A|^2 + |b'|^2\right) - 2\,\varepsilon_c \left(|A|^2 + |c''|^2\right). \end{aligned}$$

We know from (3.24) that $|b'|^2 \lesssim \|r\|^2$ and, using (3.20), we also have that $|c''|^2 \lesssim |b'|^2 + |c|^2 \lesssim \|r\|^2 + |A|^2$. As a consequence, we have that

$$2\,\mathcal{F}[h] - \|h\|^2 \ge 0$$

if ε_A, ε_b and ε_c are chosen small enough, which completes the proof. ∎

Lemma 3.9. *With the above notations, if h is defined as in Proposition 3.1, then for some ε_A, ε_b and ε_c small enough, there is a positive constant λ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[h] \le -\lambda\,\mathcal{F}[h]\,.$$

Proof of Lemma 3.9. Using Lemma 3.4, Lemma 3.6 and (3.3), we have

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}[h] = -\,\varepsilon_A\,\|\nabla_x\phi\cdot A\,x\|^2 + \varepsilon_b\,|b'|^2 + \varepsilon_c\,|c''|^2 - (\varepsilon_b+\varepsilon_c)\,O(|A|^2)\\ \ge \varepsilon_A'\,|A|^2 + \varepsilon_b\,|b'|^2 + \varepsilon_c\,|c''|^2 \gtrsim \mathcal{F}[h]\,,\end{aligned}$$

by choosing ε_b and ε_c small enough compared to ε_A and using (3.23) in the last inequality. ■

Proof of Proposition 3.1. Let $h_0 = h(t=0)$. Thanks to Grönwall's lemma and the equivalence (3.21), we deduce

$$\forall\,t\ge 0\,,\quad \|h(t)\|^2 \lesssim \mathcal{F}[h(t)] \le e^{-\lambda\,t}\,\mathcal{F}[h_0] \lesssim e^{-\lambda\,t}\,\|h_0\|^2\,.$$

By Lemma 3.3, we know that A is constant in time. Using for instance (3.22), we deduce from

$$|A|^2 \lesssim \lim_{t\to+\infty}\|h(t)\|^2 = 0$$

that $A=0$. By Lemma 3.5, we get that $b=0$ and $c'=0$ for any $t\ge 0$ so that c is independent of t. Taking for instance (3.23) into account, we conclude that $h=0$. ■

Completing the proof of Proposition 3.1 means that Part (1) of Theorem 1.1 is established.

4. Proof of hypocoercivity by the micro-macro method

In this section we prove Part (2) of Theorem 1.1 on hypocoercivity using the *micro-macro decomposition* of the solution as in Section 2.3. The proof of Proposition 3.1 is our a guideline for a new cascade of estimates, but the analysis is however more complex due to the presence of microscopic terms.

4.1. Statement

Theorem 1.1, Part (2) can be rewritten as follows.

Proposition 4.1. *Consider a solution h to (2.22)–(2.23) in $\mathrm{L}^2(\mathcal{M})$ with initial datum h_0. There exist two positive constants C and κ such that*

$$\forall\,t\ge 0\,,\quad \|h(t)\| \le C\,e^{-\kappa\,t}\,\|h_0\|\,.$$

Here C and κ depend only on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant defined in (1.14). We split h into a microscopic part $h^\perp$ and a macroscopic part $h^\parallel$ such that

$$h = h^\parallel + h^\perp = r + m \cdot v + e\,\mathfrak{E}(v) + h^\perp$$

where r, m and e defined by (2.25) evolve according to the macroscopic equations (2.30) involving the matrix valued function $E[h]$ and the vector valued function $\Theta[h]$ defined by (2.31a) and (2.31b). By construction, we have

$$\|h\|^2 = \|r\|^2 + \|m\|^2 + \|e\|^2 + \|h^\perp\|^2 \,. \tag{4.1}$$

Let deviations from averages, or *space* inhomogeneous, terms be defined by

$$r_s := r - \langle \nabla_x r \rangle \cdot x - \tfrac{1}{2\,d} \langle \Delta_x r \rangle\, \xi_2 \,, \tag{4.2a}$$

$$m_s := m - \langle \nabla_x^{\rm skew} m \rangle\, x - \tfrac{1}{d} \langle \nabla_x \cdot m \rangle\, x - \langle m \rangle \,, \tag{4.2b}$$

$$e_s := e - \langle e \rangle \,, \tag{4.2c}$$

$$w := r - \sqrt{\tfrac{2}{d}}\, \langle e \rangle\, \phi \,, \tag{4.2d}$$

$$w_s := r_s - \sqrt{\tfrac{2}{d}}\, \langle e \rangle\, \phi_s \quad \text{with} \quad \phi_s := \xi_\phi - \tfrac{1}{2\,d} \langle \Delta_x \phi \rangle\, \xi_2 \,. \tag{4.2e}$$

We recall that $\xi_2(x) := |x|^2 - \langle |x|^2 \rangle$ and $\xi_\phi(x) := \phi - \langle \phi \rangle$ were already defined in (2.10) and (2.11) while $\nabla_x^{\rm skew} m$ refers to (2.6). In particular

$$w_s = w - \langle \nabla_x w \rangle\, x - \tfrac{1}{2\,d} \langle \Delta w \rangle\, \xi_2 + \sqrt{\tfrac{2}{d}}\, \langle e \rangle\, \langle \phi \rangle \,. \tag{4.3}$$

After introducing some geometric tools in Section 4.2, we split the proof of Proposition 4.1 by considering infinite-dimensional quantities in Section 4.3 and finite-dimensional quantities in Section 4.4; in the latter the analysis closely follows the strategy of Section 3. From now on, we assume that h is as in Proposition 4.1.

4.2. *Witten-Hodge operator and Korn inequality: a toolbox*

Here we collect several classical and less classical estimates that will be used to control the macroscopic quantities. We refer to [9] for references and details of constructive proofs. Assumptions (H3)–(H4)–(H5) coincide with the hypotheses of [9, Section 1.2]. Let

$$\lfloor \nabla\phi \rceil := \sqrt{1 + |\nabla\phi|^2} \,.$$

▷ The *strong Poincaré inequality*

$$\forall\, \varphi \in {\rm H}^1(\rho) \,, \quad \int_{\mathbb{R}^d} |\varphi - \langle \varphi \rangle|^2 \,\lfloor \nabla_x \phi \rceil^2 \,\rho \,{\rm d}x \lesssim \int_{\mathbb{R}^d} |\nabla_x \varphi|^2 \,\rho \,{\rm d}x \tag{4.4}$$

is proven in [9, Proposition 5].

▷ In order to work in ${\rm L}^2(\rho)$, we shall use the operator Ω introduced in (1.15) and considered as an operator acting either on scalar or vector-valued functions. As a consequence of (4.4), we have the *zeroth-order strong Poincaré inequality* (see [9, Proposition 8]) according to which, for any $\varphi \in \mathrm{L}^2(\rho)$,

$$\left\| \Omega^{-1} \nabla_x^2 \varphi \right\| + \left\| \Omega^{-1} \left(\lfloor \nabla_x \phi \rceil \nabla_x \varphi \right) \right\| + \left\| \Omega^{-1} \left(\lfloor \nabla_x \phi \rceil^2 \varphi \right) \right\| \lesssim \|\varphi\| \,. \tag{4.5}$$

▷ The following zeroth order Poincaré inequality, sometimes called the *Poincaré-Lions inequality*,

$$\forall \, \varphi \in \mathrm{L}^2(\rho) \,, \quad \|\varphi - \langle \varphi \rangle\| \lesssim \left\| \Omega^{-\frac{1}{2}} \nabla_x \varphi \right\| \lesssim \|\varphi - \langle \varphi \rangle\| \,, \tag{4.6}$$

is proven in [9, Proposition 5].
▷ The $(-1)^{th}$ *order Poincaré-Lions inequality*

$$\forall \, \varphi \in \mathrm{H}^{-1}(\rho) \,, \quad \left\| \Omega^{-\frac{1}{2}} \left(\varphi - \langle \varphi \rangle \right) \right\| \lesssim \left\| \Omega^{-1} \nabla_x \varphi \right\| \lesssim \left\| \Omega^{-\frac{1}{2}} \left(\varphi - \langle \varphi \rangle \right) \right\| , \tag{4.7}$$

is proven in [9, Lemma 10] as well as its variant

$$\forall \, \varphi \in \mathrm{L}^2(\rho) \,, \quad \|\varphi - \langle \varphi \rangle\| \lesssim \left\| \nabla_x \, \Omega^{-\frac{1}{2}} \varphi \right\| + \left\| \Omega^{-\frac{1}{2}} \nabla_x \varphi \right\| \lesssim \left\| \varphi - \langle \varphi \rangle \right\| . \tag{4.8}$$

▷ Another key estimate is the *zeroth-order Korn-Poincaré inequality*: for any vector field $u : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ such that $\langle u \rangle = 0$ and $\langle \nabla_x^{\mathrm{skew}} u \rangle = 0$,

$$\|u\| \lesssim \left\| \Omega^{-\frac{1}{2}} \nabla_x^{\mathrm{sym}} u \right\| , \tag{4.9}$$

which is established in [9, Theorem 4] using (2.5).

4.3. *Control of infinite-dimensional quantities*

We build an entropy function by assembling dissipative functionals for $h^\perp$ and the space inhomogeneous terms defined in (4.2).

4.3.1. *Control of $h^\perp$.*

We first control the dissipation of the microscopic part.

Lemma 4.2. *If h is a solution to* (2.23) *in* $\mathrm{L}^2(\mathcal{M})$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \|h\|^2 \leq -2 \, \mathrm{c}_{\mathscr{C}} \, \|h^\perp\|^2 \,. \tag{4.10}$$

Proof of Lemma 4.2. Since $C^* = C$ and $\mathscr{T}^* = -\mathscr{T}$, there holds

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|h\|^2 = \langle C h, h \rangle \,.$$

We conclude that (4.10) holds by the spectral gap assumption (H1) on $\mathscr{C}$. ∎

4.3.2. Control of e_s. Let us consider e_s as defined in (4.2c).

Lemma 4.3. *There are some positive constants κ_1 and C such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega^{-1}\,\nabla_x e, \Theta[h]\right\rangle \le -\,\kappa_1\,\|e_s\|^2 + C\,\|h^\perp\|\,\|h\|\,. \tag{4.11}$$

Proof of Lemma 4.3. Recall that $\Theta[h] = \Theta[h^\perp]$ from (2.31b). We compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega^{-1}\,\nabla_x e, \Theta[h]\right\rangle\\
&= \left\langle \Omega^{-1}\,\nabla_x e\,, -\left(1+\tfrac{2}{d}\right)\nabla_x e + \Theta[\mathcal{L}\,h^\perp]\right\rangle + \left\langle \Omega^{-1}\,\nabla_x(\partial_t e)\,, \Theta[h]\right\rangle\\
&\le -\tfrac{1}{2}\left(1+\tfrac{2}{d}\right)\left\|\Omega^{-\frac12}\,\nabla_x e\right\|^2 + C\left\|\Omega^{-\frac12}\,\Theta[\mathcal{L}\,h^\perp]\right\|^2 + C\left\|\Omega^{-1}\,\nabla_x(\partial_t e)\right\|\,\|h^\perp\|\,,
\end{aligned}$$

by using Cauchy-Schwarz and Young inequalities. We read from (4.6) that

$$\|\,\Omega^{-1/2}\,\nabla_x e\|^2 \gtrsim \|e_s\|^2\,.$$

According to (4.6), (H2) and (H6), we have

$$\left\|\Omega^{-\frac12}\,\Theta[\mathcal{L}\,h^\perp]\right. \lesssim \|h^\perp\|\,.$$

It follows from (2.30c) that

$$\Omega^{-1}\,\nabla_x(\partial_t e) = -\sqrt{\tfrac{2}{d}}\,\Omega^{-1}\,\nabla_x(\nabla_x\cdot m) + \Omega^{-1}\,\nabla_x\big(\nabla_x^*\cdot\Theta[h^\perp]\big)\,,$$

so that $\left\|\Omega^{-1}\,\nabla_x(\partial_t e)\right\| \lesssim \|h\|$ by (4.5). This completes the proof of (4.11). ■

4.3.3. Control of m_s. Let us consider $\nabla_x^{\mathrm{sym}} m_s$ as defined by (2.6) and (4.2b).

Lemma 4.4. *There are some positive constants κ_2 and C such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega^{-1}\,\nabla_x^{\mathrm{sym}} m_s, E[h] - \sqrt{\tfrac{2}{d}}\,\langle e\rangle\,\mathrm{Id}_{d\times d}\right\rangle \le -\,\kappa_2\,\|m_s\|^2 + C\,\big(\|e_s\| + \|h^\perp\|\big)\,\|h\|\,. \tag{4.12}$$

Proof of Lemma 4.4. Let us remark that from (4.2b) one has

$$\nabla_x^{\mathrm{sym}}\, m = \nabla_x^{\mathrm{sym}}\, m_s + \tfrac{1}{d}\,\langle\nabla_x\cdot m\rangle\,\mathrm{Id}_{d\times d}\,,$$

and from (2.31a),

$$E[h] - \sqrt{\tfrac{2}{d}}\,\langle e\rangle\,\mathrm{Id}_{d\times d} = \sqrt{\tfrac{2}{d}}\,e_s\,\mathrm{Id}_{d\times d} + E[h^\perp]\,.$$

Moreover, from (2.30c), one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle e\rangle = -\sqrt{\tfrac{2}{d}}\,\langle\nabla_x\cdot m\rangle\,.$$

As a consequence, from (2.30d), one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \Omega^{-1}\,\nabla_x^{\mathrm{sym}}\, m_s, E[h] - \sqrt{\tfrac{2}{d}}\,\langle e\rangle\,\mathrm{Id}_{d\times d}\right\rangle$$

$$
\begin{aligned}
&= \left\langle \Omega^{-1}\, \nabla_x^{\mathrm{sym}} m_s, -2\, \nabla_x^{\mathrm{sym}}\, m + E[\mathcal{L}\, h^\perp] + \tfrac{2}{d} \langle \nabla_x \cdot m\rangle\, \mathrm{Id}_{d\times d} \right\rangle \\
&\qquad + \left\langle \Omega^{-1}\, \nabla_x^{\mathrm{sym}} (\partial_t m_s), E[h] - \sqrt{\tfrac{2}{d}}\, \langle e\rangle\, \mathrm{Id}_{d\times d} \right\rangle \\
&= -2 \left\| \Omega^{-\frac12}\, \nabla_x^{\mathrm{sym}}\, m_s \right\|^2 + \left\langle \Omega^{-\frac12}\, \nabla_x^{\mathrm{sym}}\, m_s\, , \Omega^{-\frac12}\, E[\mathcal{L}\, h^\perp] \right\rangle \\
&\qquad + \left\langle \Omega^{-1}\, \nabla_x^{\mathrm{sym}} (\partial_t m_s), \sqrt{\tfrac{2}{d}}\, e_s\, \mathrm{Id}_{d\times d} + E[h^\perp] \right\rangle .
\end{aligned}
$$

Using the Cauchy-Schwarz inequality, we deduce

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Omega^{-1}\, \nabla_x^{\mathrm{sym}}\, m_s, E[h] - \sqrt{\tfrac{2}{d}}\, \langle e\rangle \mathrm{Id}_{d\times d} \right\rangle \\
&\qquad \le - \left\| \Omega^{-\frac12}\, \nabla_x^{\mathrm{sym}}\, m_s \right\|^2 + C \left\| \Omega^{-\frac12}\, E[\mathcal{L}\, h^\perp] \right\|^2 \\
&\qquad\qquad + C \left\| \Omega^{-1}\, \nabla_x^{\mathrm{sym}} (\partial_t m_s) \right\| \left\| \sqrt{\tfrac{2}{d}}\, e_s \mathrm{Id}_{d\times d} + E[h^\perp] \right\| .
\end{aligned}
$$

Using Korn's inequality (4.9) and observing by (2.30b) that

$$
\left\| \Omega^{-1}\, \nabla_x^{\mathrm{sym}} (\partial_t m_s) \right\| = O(\|h\|) \quad \text{and} \quad \left\| \Omega^{-\frac12}\, E[\mathcal{L}\, h^\perp] \right\| = O(\|h^\perp\|)
$$

from (4.5) and (4.8) as in the proof of Lemma 4.3, we prove (4.12). ∎

4.3.4. Control of w_s. Let us consider w_s as defined in (4.2e).

Lemma 4.5. *There are some positive constants κ_3 and C such that*

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Omega^{-1}\, \nabla_x w_s, m_s \right\rangle \le -\kappa_3\, \|w_s\|^2 + C\, \|e_s\|^2 + C\, \|h^\perp\|^2 + C\, \|m_s\|\, \|h\| \tag{4.13}
$$

and

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left\langle -\Omega^{-1}\, \partial_t w_s, w_s \right\rangle \le - \left\| \Omega^{-\frac12}\, \partial_t w_s \right\|^2 + C\, \|w_s\|\, \|h\| . \tag{4.14}
$$

Proof of Lemma 4.5. Observe that (2.30b), (4.2c) and (4.3) imply

$$
\begin{aligned}
\partial_t m &= -\nabla_x w + \sqrt{\tfrac{2}{d}}\, \nabla_x^* e_s + \nabla_x^* \cdot E[h^\perp] \\
&= -\nabla_x w_s - \langle \nabla_x w\rangle - \tfrac{1}{d}\, \langle \Delta_x w\rangle\, x + \sqrt{\tfrac{2}{d}}\, \nabla_x^* e_s + \nabla_x^* \cdot E[h^\perp] .
\end{aligned}
$$

Integrating (2.30b) and using (4.2d), one gets

$$
\frac{\mathrm{d}}{\mathrm{d}t} \langle m\rangle = -\langle \nabla_x r\rangle = -\langle \nabla_x w\rangle
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla_x \cdot m\rangle &= -\langle \Delta_x r\rangle + \sqrt{\tfrac{2}{d}}\, \langle e\, \Delta_x \phi\rangle + \left\langle \mathrm{Tr}(E[h^\perp]\, \nabla_x^2 \phi) \right\rangle \\
&= -\langle \Delta_x w\rangle + \sqrt{\tfrac{2}{d}}\, \langle e_s\, \Delta_x \phi\rangle + \left\langle \mathrm{Tr}(E[h^\perp]\, \nabla_x^2 \phi) \right\rangle .
\end{aligned}
$$

Finally, by differentiating (2.30b), one has

$$\partial_t \nabla_x m = -\nabla_x^2 r + \sqrt{\tfrac{2}{d}}\, \nabla_x^* \nabla_x e + \sqrt{\tfrac{2}{d}}\, \nabla_x^2 \phi\, e + \nabla_x^* \cdot (\nabla_x \otimes E[h^\perp]) + E[h^\perp]\, \nabla_x^2 \phi$$

and the integration of the skew-symmetric part yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \nabla_x^{\text{skew}} m \rangle = \left\langle \left(E[h^\perp]\, \nabla_x^2 \phi \right)^{\text{skew}} \right\rangle . \tag{4.15}$$

As a consequence of these identities and (4.2b), one gets

$$\partial_t m_s = -\nabla_x w_s + \sqrt{\tfrac{2}{d}} \left(\nabla_x^* e_s - \tfrac{1}{d} \langle e_s\, \Delta_x \phi \rangle\, x \right) + m_E$$

where $m_E := \nabla_x^* \cdot E[h^\perp] - \left\langle \left(E[h^\perp]\, \nabla_x^2 \phi \right)^{\text{skew}} \right\rangle x - \tfrac{1}{d} \left\langle \mathrm{Tr}\left(E[h^\perp]\, \nabla_x^2 \phi \right) \right\rangle x$. Hence

$$\begin{aligned} \left\langle \Omega^{-1}\, \nabla_x w_s, \partial_t m_s \right\rangle = &- \left\| \Omega^{-\frac{1}{2}}\, \nabla_x w_s \right\|^2 + \left\| \Omega^{-\frac{1}{2}}\, \nabla_x w_s \right\| \left\| \Omega^{-\frac{1}{2}}\, m_E \right\| \\ &+ \sqrt{\tfrac{2}{d}} \left\| \Omega^{-\frac{1}{2}}\, \nabla_x w_s \right\| \left\| \Omega^{-\frac{1}{2}} \left(\nabla_x^* e_s - \tfrac{1}{d} \langle e_s\, \Delta_x \phi \rangle\, x \right) \right\| \end{aligned}$$

Using the zeroth order Poincaré inequality (4.6) and (4.2c), we can estimate $\|\Omega^{-1/2}\, \nabla_x^* e_s\|$ by $\|e_s\|$. Up to a few integrations by parts, using (H4), (H6) and (H7), we end up for some constant $C > 0$ with

$$\left\langle \Omega^{-1}\, \nabla_x w_s, \partial_t m_s \right\rangle \le -\frac{1}{2} \left\| \Omega^{-\frac{1}{2}}\, \nabla_x w_s \right\|^2 + C \left(\|e_s\|^2 + \|h^\perp\|^2 \right) . \tag{4.16}$$

From the definitions (4.2a) and (4.2e), we have

$$\partial_t w_s = \partial_t r - \langle \nabla_x \partial_t r \rangle \cdot x - \tfrac{1}{2d} \langle \Delta_x \partial_t r \rangle\, \xi_2, -\sqrt{\tfrac{2}{d}}\, \langle \partial_t e \rangle\, \phi_s ,$$

so that, by (2.30a) and (2.30c),

$$\partial_t w_s = \nabla_x^* m - \langle \nabla_x \nabla_x^* \cdot m \rangle \cdot x - \tfrac{1}{2d} \langle \Delta_x \nabla_x^* \cdot m \rangle\, \xi_2 - \tfrac{2}{d} \langle \nabla_x \cdot m \rangle\, \phi_s . \tag{4.17}$$

Using (4.5) in order to estimate the first term, and performing several integration by parts and using the boundedness assumption (H6) on ϕ in order to estimate the three last terms, we obtain

$$\|\Omega^{-1}\, \nabla_x\, \partial_t w_s\| \lesssim \|m\| \lesssim \|h\| . \tag{4.18}$$

Inserting (4.16) and (4.18) in

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle \Omega^{-1}\, \nabla_x w_s, m_s \right\rangle = \left\langle \Omega^{-1}\, \nabla_x\, \partial_t w_s, m_s \right\rangle + \left\langle \Omega^{-1}\, \nabla_x w_s, \partial_t m_s \right\rangle$$

completes the proof of (4.13).

In order to control the time-derivative of w_s, we write

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle -\Omega^{-1}\, \partial_t w_s, w_s \right\rangle = -\left\| \Omega^{-\frac{1}{2}}\, \partial_t w_s \right\|^2 - \left\langle \Omega^{-1}\, \partial_{tt}^2 w_s\, , w_s \right\rangle . \tag{4.19}$$

Differentiating (4.17) with respect to t, we have

$$\partial_{tt}^2 w_s = \nabla_x^* \cdot (\partial_t m) - \langle \nabla_x \nabla_x^* \cdot (\partial_t m)\rangle \cdot x - \tfrac{1}{2d}\langle \Delta_x \nabla_x^* \cdot (\partial_t m)\rangle\, \xi_2 - \tfrac{2}{d}\langle \nabla_x \cdot (\partial_t m)\rangle\, \phi_s\,,$$

where the first term is obtained by differentiating (2.30b) and amounts to

$$\nabla_x^* (\partial_t m) = -\nabla_x^* \cdot \nabla_x w + \sqrt{\tfrac{2}{d}}\, \nabla_x^* \cdot \nabla_x^* e_s + \nabla_x^* \cdot \nabla_x^* \cdot E[h^\perp]\,,$$

using (4.2c) and (4.2d). Similar expressions hold for the three next terms. Arguing similarly as for (4.18), we have

$$\Omega^{-1}\, \partial_{tt}^2 w_s = \mathcal{O}(\|h\|)\,.$$

Together with (4.19) this proves (4.14). ∎

4.3.5. First Lyapunov functional. We end this section by introducing a first, partial Lyapunov functional

$$\mathcal{F}_1[h] := \|h\|^2 + \varepsilon_1 \left\langle \Omega^{-1}\, \nabla_x e, \Theta[h]\right\rangle + \varepsilon_2 \left\langle \Omega^{-1}\, \nabla_x^{\rm sym}\, m_s, E[h] - \sqrt{\tfrac{2}{d}}\, \langle e\rangle\, \mathrm{Id}_{d\times d}\right\rangle$$
$$+ \varepsilon_3 \left\langle \Omega^{-1}\, \nabla_x w_s, m_s\right\rangle + \varepsilon_4 \left\langle -\Omega^{-1}\, \partial_t w_s, w_s\right\rangle \quad (4.20)$$

where

$$\varepsilon_1 = \varepsilon\,, \quad \varepsilon_2 = \varepsilon^{3/2}\,, \quad \varepsilon_3 = \varepsilon^{7/4}\,, \quad \varepsilon_4 = \varepsilon^{15/8}\,. \quad (4.21)$$

Let us define the dissipation functional

$$\mathcal{D}_1[h] := \|h^\perp\|^2 + \|e_s\|^2 + \|m_s\|^2 + \|w_s\|^2 + \left\|\, \Omega^{-\frac{1}{2}}\, \partial_t w_s\right\|^2. \quad (4.22)$$

Lemma 4.6. *There are some positive constants κ_0, C_0 and κ such that for any $\varepsilon > 0$ small enough, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1[h] \le -\kappa_0\, \|h^\perp\|^2 - \varepsilon^{\frac{15}{8}}\, \kappa\, \mathcal{D}_1[h] + \varepsilon^2\, C_0\, \|h\|^2\,.$$

Proof of Lemma 4.6. By collecting the results of Lemmata 4.2, 4.3, 4.4 and 4.5, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_1[h] \le -2\,\mathrm{c}_{\mathscr{C}}\, \|h^\perp\|^2 - \varepsilon_1\, \kappa_1\, \|e_s\|^2 + \varepsilon_1\, C\, \|h^\perp\|\, \|h\| \\ - \varepsilon_2\, \kappa_2\, \|m_s\|^2 + \varepsilon_2\, C\left(\|h^\perp\| + \|e_s\|\right) \|h\| \\ - \varepsilon_3\, \kappa_3\, \|w_s\|^2 + \varepsilon_3\, C\left(\|e_s\|^2 + \|h^\perp\|^2 + \|m_s\|\, \|h\|\right) \\ - \varepsilon_4\, \|\, \Omega^{-\frac{1}{2}}\, \partial_t w_s\|^2 + \varepsilon_4\, C\, \|w_s\|\, \|h\|\end{aligned}$$

for any $\varepsilon_i \in (0,1)$, $i = 1,2,3,4$, up to a renaming of the generic constant $C > 0$. Using repeatedly Young's inequality, we have

$$\varepsilon_1\, C\, \|h^\perp\|\, \|h\| \le \tfrac{1}{2}\, \mathrm{c}_{\mathscr{C}}\, \|h^\perp\|^2 + \varepsilon_1^2\, \tfrac{C^2}{2\,\mathrm{c}_{\mathscr{C}}}\, \|h\|^2\,,$$
$$\varepsilon_2\, C\, \|h^\perp\|\, \|h\| \le \tfrac{1}{2}\, \mathrm{c}_{\mathscr{C}}\, \|h^\perp\|^2 + \varepsilon_2^2\, \tfrac{C^2}{2\,\mathrm{c}_{\mathscr{C}}}\, \|h\|^2\,,$$

$$\varepsilon_2\, C\, \|e_s\|\, \|h\| \le \tfrac12\, \varepsilon_1\, \kappa_1\, \|e_s\|^2 + \tfrac{\varepsilon_2^2}{\varepsilon_1}\, \tfrac{C^2}{2\,\kappa_1}\, \|h\|^2\,,$$
$$\varepsilon_3\, C\, \|m_s\|\, \|h\| \le \tfrac12\, \varepsilon_2\, \kappa_2\, \|m_s\|^2 + \tfrac{\varepsilon_3^2}{\varepsilon_2}\, \tfrac{C^2}{2\,\kappa_2}\, \|h\|^2\,,$$
$$\varepsilon_4\, C\, \|w_s\|\, \|h\| \le \tfrac12\, \varepsilon_3\, \kappa_3\, \|w_s\|^2 + \tfrac{\varepsilon_4^2}{\varepsilon_3}\, \tfrac{C^2}{2\,\kappa_3}\, \|h\|^2\,,$$

and therefore

$$\begin{aligned}\frac{\mathrm d}{\mathrm dt}\mathcal F_1[h] \le -\,(\mathrm c_{\mathscr C} - \varepsilon_3\, C)\, \|h^\perp\|^2 - \varepsilon_1\, \kappa_1\, \big(\tfrac12 - \tfrac{\varepsilon_3}{\kappa_1\,\varepsilon_1}\, C\big)\|e_s\|^2 - \tfrac12\, \varepsilon_2\, \kappa_2\, \|m_s\|^2 \\ -\,\tfrac12\, \varepsilon_3\, \kappa_3\, \|w_s\|^2 - \varepsilon_4\, \Big\|\Omega^{-\frac12}\, \partial_t w_s\Big\|^2 \\ +\,\tfrac12\, C^2\left(\tfrac{\varepsilon_1^2}{\mathrm c_{\mathscr C}} + \tfrac{\varepsilon_2^2}{\mathrm c_{\mathscr C}} + \tfrac{\varepsilon_2^2}{\kappa_1\,\varepsilon_1} + \tfrac{\varepsilon_3^2}{\kappa_2\,\varepsilon_2} + \tfrac{\varepsilon_4^2}{\kappa_3\,\varepsilon_3}\right)\|h\|^2\,.\end{aligned}$$

The choice $\kappa_0 = \frac14\, \mathrm c_{\mathscr C}$, $\kappa = \min\big\{\frac{\mathrm c_{\mathscr C}}{4}, \frac{\kappa_1}{4}, \frac{\kappa_2}{2}, \frac{\kappa_3}{2}, 1\big\}$ and $C_0 = \frac12\, C^2\, \big(\frac{2}{\mathrm c_{\mathscr C}} + \frac{1}{\kappa_1} + \frac{1}{\kappa_2} + \frac{1}{\kappa_3}\big)$ with ε_i, $i = 1, \ldots, 4$, given by (4.21) and

$$0 < \varepsilon < \min\left\{1, \big(\tfrac{4\,C}{\kappa_1}\big)^{-4/3}, \big(\tfrac{2\,C}{\mathrm c_{\mathscr C}}\big)^{-4/7}\right\}$$

completes the proof. ■

4.4. Control of finite-dimensional quantities

After estimating the decay of the *deviations from averages terms* defined by (4.2), let us consider the time-dependent *global scalar* quantities $\langle e\rangle$, $\langle \nabla_x^{\rm skew} m\rangle$, $\langle \nabla_x \cdot m\rangle$, $\langle m\rangle$, $\langle \nabla_x r\rangle$, $\langle \Delta_x r\rangle$. We proceed as in the proof of Proposition 3.1. Let

$$A(t) := \langle \nabla_x^{\rm skew} m\rangle\,, \quad b(t) := \langle m\rangle\,, \quad c(t) := \langle e\rangle\,, \tag{4.23a}$$
$$z(t,x) := r(t,x) + b'(t)\cdot x - c''(t)\,\frac{\xi_2(x)}{2\sqrt{2\,d}} - c(t)\,\sqrt{\tfrac2d}\,\xi_\phi(x)\,. \tag{4.23b}$$

By comparison with (3.2a), we know that $z = 0$ if h corresponds to a special macroscopic mode. We observe here that there is no reason for $A(t)$ to be neither the orthogonal projection of m onto infinitesimal rotation matrices nor independent of t. We shall have to take this fact into account later and remember that, according to (4.15),

$$A'(t) = \Big\langle \big(E[h^\perp]\, \nabla_x^2\phi\big)^{\rm skew}\Big\rangle\,. \tag{4.24}$$

4.4.1. The macroscopic equations. As defined by (2.25), the functions r, m and e can be rewritten in the new variables as follows.

Lemma 4.7. *With previous notations, if h solves (2.22)–(2.23) in $\mathrm L^2(\mathcal M)$, then*

$$r(t,x) = -\,b'(t)\cdot x + c''(t)\,\frac{\xi_2(x)}{2\sqrt{2\,d}} + c(t)\,\sqrt{\tfrac2d}\,\xi_\phi(x) + z(t,x)\,, \tag{4.25a}$$
$$m(t,x) = A(t)\,x + b(t) - c'(t)\,\frac{1}{\sqrt{2\,d}}\,x + m_s(t,x)\,, \tag{4.25b}$$
$$e(t,x) = c(t) + e_s(t,x)\,, \tag{4.25c}$$

where z obeys the bounds

$$\|z\|^2 \lesssim \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2, \tag{4.26a}$$

$$\left\| \Omega^{-\frac{1}{2}} \, \partial_t z \right\|^2 \lesssim \left\| \Omega^{-\frac{1}{2}} \, \partial_t w_s \right\|^2 + \|m_s\|^2 + \|h^\perp\|^2 . \tag{4.26b}$$

Proof of Lemma 4.7. The expression (4.25a) follows from the definition of z in (4.23b), while (4.25c) is no more than a rewriting of (4.2c). From (2.30c) one observes that

$$c' = \frac{\mathrm{d}}{\mathrm{d}t} \langle e \rangle = -\sqrt{\tfrac{2}{d}} \, \langle \nabla_x \cdot m \rangle \,, \tag{4.27}$$

so that (4.25b) follows from the definition (4.2b) of m_s.

From (2.30b) we have

$$b' = \frac{\mathrm{d}}{\mathrm{d}t} \langle m \rangle = -\langle \nabla_x r \rangle \,. \tag{4.28}$$

Using (4.2a) and (4.2e), we write

$$\begin{aligned} r &= w_s + \sqrt{\tfrac{2}{d}} \, \langle e \rangle \, \phi_s + \langle \nabla_x r \rangle \cdot x + \tfrac{1}{2\,d} \, \langle \Delta_x r \rangle \, \xi_2 \\ &= w_s + \sqrt{\tfrac{2}{d}} \, c \left(\xi_\phi - \tfrac{1}{2\,d} \, \langle \Delta_x \phi \rangle \, \xi_2 \right) - b' \cdot x + \tfrac{1}{2\,d} \, \langle \Delta_x r \rangle \, \xi_2 \,. \end{aligned}$$

From (4.25a), we deduce

$$z = w_s + \tfrac{1}{\sqrt{2}\,d} \left(\tfrac{1}{\sqrt{2}\,d} \, \langle \Delta_x r \rangle - \tfrac{1}{2} \, c'' - \tfrac{1}{d} \, \langle \Delta_x \phi \rangle \, c \right) \xi_2 \,. \tag{4.29}$$

Finally, thanks to (4.27) and (2.30b), we compute

$$c'' = -\sqrt{\tfrac{2}{d}} \, \langle \nabla_x \cdot \partial_t m \rangle = \sqrt{\tfrac{2}{d}} \, \langle \Delta_x r \rangle - \tfrac{2}{d} \left\langle \nabla_x \cdot \nabla_x^* e \right\rangle - \sqrt{\tfrac{2}{d}} \left\langle \nabla_x \cdot \left(\nabla_x^* \cdot E[h^\perp] \right) \right\rangle ,$$

and thus obtain

$$c'' = \sqrt{\tfrac{2}{d}} \, \langle \Delta_x r \rangle - \tfrac{2}{d} \, \langle e \, \Delta \, \phi \rangle - \sqrt{\tfrac{2}{d}} \left\langle \mathrm{Tr}\!\left(E[h^\perp] \, \nabla_x^2 \phi \right) \right\rangle . \tag{4.30}$$

By inserting (4.30) in (4.29), we obtain

$$z = w_s + \tfrac{1}{2\,d} \left\langle E[h^\perp] : \nabla_x^2 \phi \right\rangle \xi_2 + \tfrac{1}{d\,\sqrt{2}\,d} \, \langle e_s \, \Delta_x \phi \rangle \, \xi_2 \,,$$

from which (4.26a) follows. By differentiating z with respect to t, we get

$$\partial_t z = \partial_t w_s + \tfrac{1}{2\,d} \left\langle \partial_t E[h^\perp] : \nabla_x^2 \phi \right\rangle \xi_2 + \tfrac{1}{d\,\sqrt{2}\,d} \, \langle \partial_t e_s \, \Delta_x \phi \rangle \, \xi_2 \,.$$

By (2.30d) and (2.31a), we know that

$$\partial_t E[h] = -2 \, \nabla_x^{\mathrm{sym}} m + E[\mathcal{L} \, h^\perp] = \sqrt{\tfrac{2}{d}} \, \partial_t e \, \mathrm{Id}_{d\times d} + \partial_t E[h^\perp] \,.$$

Besides, we learn from (4.2b) and (4.27) that

$$\nabla_x^{\mathrm{sym}} m = \nabla_x^{\mathrm{sym}} m_s + \tfrac{1}{d} \, \langle \nabla_x \cdot m \rangle \, \mathrm{Id}_{d\times d} = \nabla_x^{\mathrm{sym}} m_s - \tfrac{1}{\sqrt{2}\,d} \, \langle \partial_t e \rangle \, \mathrm{Id}_{d\times d}$$

and, as a consequence,

$$\partial_t E[h^\perp] = -2\,\nabla_x^{\rm sym}\, m_s + E[\mathcal{L}\, h^\perp] - \sqrt{\tfrac{2}{d}}\,\partial_t e_s\, \mathrm{Id}_{d\times d}\,.$$

Hence

$$\partial_t z = \partial_t w_s - \tfrac{1}{d}\left\langle \nabla_x^{\rm sym}\, m_s : \nabla_x^2\phi\right\rangle \xi_2 + \tfrac{1}{2\,d}\left\langle E[\mathcal{L}\, h^\perp] : \nabla_x^2\phi\right\rangle \xi_2$$

and Estimate (4.26b) follows using an integration by parts and (H6). ■

Using (4.25a) on the one hand, and (2.30a) combined with (4.25b) on the other hand, we write

$$\begin{aligned}\partial_t r &= \sqrt{\tfrac{2}{d}}\,\xi_\phi\, c' - x\cdot b'' + \tfrac{\xi_2}{2\sqrt{2\,d}}\, c''' + \partial_t z\\ &= \nabla_x^*\cdot m_s + \nabla_x\phi\cdot A\,x - \tfrac{1}{\sqrt{2\,d}}\,(\nabla_x\phi\cdot x - d)\, c' + \nabla_x\phi\cdot b\,.\end{aligned}$$

We deduce a differential equation which is very similar to (2.12) up to additional terms involving m_s and z, namely

Proposition 4.8. *The functions A, b, c, z and m_s defined by* (4.23) *and* (4.2b) *solve*

$$\frac{2\,\xi_\phi(x)+\nabla_x\phi\cdot x-d}{\sqrt{2\,d}}\, c' + \frac{\xi_2}{2\sqrt{2\,d}}\, c''' - \nabla_x\phi\cdot b - x\cdot b'' - \nabla_x\phi\cdot A\,x = \nabla_x^*\cdot m_s - \partial_t z\,. \tag{4.31}$$

4.4.2. Control of A. The counterpart of Section 3.2 goes as follows.

Lemma 4.9. *There are some positive constants κ_4 and C such that*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle\,(X - Y\cdot\nabla_x\phi)\,,\nabla_x\phi\cdot A\,x\Big\rangle \le -\,\kappa_4\,|A|^2 + C\left(\mathcal{D}_1[h] + C\,\|h^\perp\|\,\|h\|\right), \tag{4.32}$$

where $X = X(b', c, c'')$, $Y = Y(b', c, c'')$ and $\mathcal{D}_1[h]$ are defined respectively in (3.5), (3.6) *and* (4.22).

Proof of Lemma 4.9. We argue as for Lemma 3.4. We multiply (4.31) by x_k for $k = 1,\ldots,d$ and after integration, we get

$$\sqrt{\tfrac{2}{d}}\,\big\langle \xi_\phi\, x\big\rangle\, c' + \tfrac{1}{2\sqrt{2\,d}}\,\langle \xi_2\, x\,\rangle\, c''' - b - \langle x\otimes x\rangle\, b'' = \langle m_s\rangle - \langle x\,\partial_t z\rangle\,. \tag{4.33}$$

Using the definitions of X and Y, (4.31) and (4.33) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}(X - Y\cdot\nabla_x\phi) = \nabla_x\phi\cdot A\,x + \nabla_x^*\cdot m_s - \langle m_s\rangle\cdot\nabla_x\phi - \partial_t z + \langle x\,\partial_t z\rangle\cdot\nabla_x\phi\,.$$

Using (4.23a) and (4.24), we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big\langle -(X - Y\cdot\nabla_x\phi), \nabla_x\phi\cdot A\,x\Big\rangle\\ &= -\,\|\nabla_x\phi\cdot A\,x\|^2 - \langle\nabla_x^*\cdot m_s, \nabla_x\phi\cdot A\,x\rangle + \Big\langle\langle m_s\rangle\cdot\nabla_x\phi\,,\nabla_x\phi\cdot A\,x\Big\rangle\end{aligned}$$

$$+\ \langle \partial_t z, \nabla_x \phi \cdot A\,x\rangle - \big\langle \langle x\, \partial_t z\rangle \cdot \nabla_x \phi\,, \nabla_x \phi \cdot A\,x\big\rangle$$
$$-\Big\langle \big(X - Y \cdot \nabla_x \phi\big), \nabla_x \phi \cdot \big\langle \big(E[h^\perp]\, \nabla_x^2 \phi\big)^{\text{skew}} \big\rangle\, x\Big\rangle\,.$$

For the first term and thanks to the conservation law (2.27), we note that

$$\mathcal{R}_\phi^\perp \ni \mathbb{P}(m) = A\,x + \mathbb{P}(m_s)\,,$$

so that we can apply inequality (3.3) to $x \mapsto A\,x + \mathbb{P}(m_s)$ to get

$$c_{\mathrm{K}}\,\|A\,x + \mathbb{P}(m_s)\|^2 \le \|\nabla_x \phi \cdot A\,x + \nabla_x \phi \cdot \mathbb{P}(m_s)\|^2\,,$$

which yields

$$c_{\mathrm{K}}\,|A|^2 = c_{\mathrm{K}}\,\|A\,x\|^2 \le 4\,\|\nabla_x \phi \cdot A\,x\|^2 + C\,\|m_s\|^2 \tag{4.34}$$

for any $C > 4 + c_{\mathrm{K}}$. In order to estimate the other terms, we use

$$\Big\langle \big(X - Y \cdot \nabla_x \phi\big), \nabla_x \phi \cdot \big\langle \big(E[h^\perp]\, \nabla_x^2 \phi\big)^{\text{skew}} \big\rangle\, x\Big\rangle$$
$$\lesssim \big\|\Omega^{-1}\,(X - Y \cdot \nabla_x \phi)\big\|\,\big\|\Omega\,(\nabla_x \phi\, x)\big\|\,\big|\big\langle \big(E[h^\perp]\, \nabla_x^2 \phi\big)^{\text{skew}} \big\rangle\big|$$
$$\lesssim \big(|b'| + |c| + |c''|\big)\,\|h^\perp\|\,,$$

and

$$\big\langle \langle m_s\rangle \cdot \nabla_x \phi\,, \nabla_x \phi \cdot A\,x\big\rangle \lesssim \|m_s\|\,\|\nabla_x \phi \cdot A\,x\| \lesssim \|m_s\|\,|A|\,.$$

Thanks to the zeroth order Poincaré inequality (4.6), we also have

$$\big\langle \nabla_x^* \cdot m_s, \nabla_x \phi \cdot A\,x\big\rangle = \big\langle \Omega^{-\frac12}\,\big(\nabla_x^* \cdot m_s\big)\,,\ \Omega^{\frac12}\,\big(\nabla_x \phi \cdot A\,x\big)\big\rangle \lesssim \|m_s\|\,|A|$$

as well as similar estimates for the terms in $\partial_t z$ and $\langle x\,\partial_t z\rangle \cdot \nabla_x \phi$. Collecting these estimates with (4.34), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle -\big(X - Y \cdot \nabla_x \phi\big), \nabla_x \phi \cdot A\,x\big\rangle$$
$$\le -\tfrac14\,c_{\mathrm{K}}\,|A|^2 + C\,\big(\,\|m_s\| + \big\|\Omega^{-\frac12}\,\partial_t z\big\|\,\big)\,|A| + \big(|b'| + |c| + |c''|\big)\,\|h^\perp\|\,,$$

for some $\kappa_4 < \frac14\,c_{\mathrm{K}}$ and $C > 0$ large enough. Young's inequality and (4.26b) conclude the proof of (4.32). ∎

4.4.3. Control of b, b', b'' and c, c', c'' and c'''. It follows from three lemmata.

Lemma 4.10. *The following estimate holds*

$$|b| + |b''| + |c'| + |c'''| \lesssim |A| + \big\|\Omega^{-\frac12}\,\partial_t w_s\big\| + \|m_s\| + \|h^\perp\|\,. \tag{4.35}$$

Proof. Using (4.31), we can write

$$\tfrac{2\,\xi_\phi + \nabla_x \phi \cdot x - d}{\sqrt{2\,d}}\,c' + \tfrac{\xi_2}{2\,\sqrt{2\,d}}\,c''' - \nabla_x \phi \cdot b - x \cdot b'' = R_0$$

with $R_0 := \nabla_x\phi \cdot A\,x + \nabla_x^* \cdot m_s - \partial_t z$. Arguing as in the proof of Lemma 3.5 with this new definition of R_0, we obtain that (3.11) holds with

$$R_3 := -\,\langle R_0\,\widetilde{\Phi}\rangle\,,\quad \widetilde{\Phi} := \nabla\phi - x - \langle(\nabla\phi - x)\otimes x\rangle\,\nabla\phi\,.$$

Observing that

$$R_3 = -\left\langle \nabla_x\phi \cdot A\,x\,\widetilde{\Phi}\right\rangle - \left\langle {}^T\!\big(D\,\widetilde{\Phi}\big)\,m_s\right\rangle + \left\langle \Omega^{\frac12}\,\widetilde{\Phi}, \Omega^{-\frac12}\,\partial_t z\right\rangle \\ = O\big(\,|A| + \|m_s\| + \big\|\,\Omega^{-\frac12}\,\partial_t z\big\|\,\big)$$

and using (4.26b), the proof of (4.35) follows for the same reasons as in the proof of Lemma 3.5. ∎

With $\mathcal{D}_1[h]$ defined in (4.22), we obtain the counterpart of Lemma 3.6.

Lemma 4.11. *There exists a constant $C > 0$ so that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -b, b'\rangle \le -\,|b'|^2 + C\,|A|^2 + C\,\mathcal{D}_1[h]\,,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle -c', c''\rangle \le -\,|c''|^2 + C\,|A|^2 + C\,\mathcal{D}_1[h]\,.$$

Lemma 4.12. *The following estimates hold*

$$|c| \lesssim |b'| + |c''| + \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2, \tag{4.36}$$
$$\|r\| \lesssim |b'| + |c''| + \|w_s\|^2 + \|e_s\|^2 + \|h^\perp\|^2\,. \tag{4.37}$$

Proof. According to (4.25a), we can write r as

$$r = \sqrt{\tfrac{2}{d}}\,\xi_\phi\,c + R_5 \tag{4.38}$$

where $R_5 := z - x\cdot b' + \frac{\xi_2}{2\sqrt{2\,d}}\,c'' = O(|b'| + |c''| + \mathcal{D}_1[h])$ because of (4.26a). Using this expression in (2.26) and recalling that $\langle e\rangle = c$ yields

$$\sqrt{\tfrac{d}{2}}\,c\left(1 + \tfrac{2}{d}\,\langle\xi_\phi^2\rangle\right) = -\left\langle \xi_\phi\,R_5\right\rangle\,,$$

from which (4.36) follows. Coming back to (4.38), we establish (4.37). ∎

4.4.4. Second Lyapunov functional. Let us introduce the Lyapunov function

$$\mathcal{F}_2[h] := \mathcal{F}_1[h] - \varepsilon_5\,\langle(X - Y\cdot\nabla_x\phi), \nabla_x\phi\cdot A\,x\rangle - \varepsilon_6\,\langle b, b'\rangle - \varepsilon_6\,\langle c', c''\rangle \tag{4.39}$$

for some additional small parameters ε_5 and ε_6, and the associated dissipation functional

$$\mathcal{D}_2[h] := \mathcal{D}_1[h] + |A|^2 + |b'|^2 + |c''|^2\,. \tag{4.40}$$

Lemma 4.13. *For any $0 < \varepsilon_6 < \varepsilon_5 < \varepsilon_4 < \varepsilon_3 < \varepsilon_2 < \varepsilon_1$ with ε_1 small enough, there holds*

$$\|h\|^2 \lesssim \mathcal{F}_2[h] \lesssim \mathcal{D}_2[h] \lesssim \|h\|^2\,. \tag{4.41}$$

Proof of Lemma 4.13. We can control all quantities involved in the definitions of $\mathcal{F}_2$ and $\mathcal{D}_2$ by $\|h\|^2$. Indeed, from (4.1) and (4.23a), we have

$$\|r\| + \|m\| + \|e\| + \|h^\perp\| + |b| + |c| \lesssim \|h\| \tag{4.42}$$

and thus also $\|e_s\| \lesssim \|e\| + |c| \lesssim \|h\|$ from (4.25c). Next, we observe from (4.27) that

$$|c'| = \sqrt{\tfrac{2}{d}}\, |\langle m \cdot \nabla\phi\rangle| \lesssim \|m\| \le \|h\|\,,$$

from (4.23a) that

$$|A| = |\langle m \nabla_x^{\mathrm{skew}} \phi\rangle| \lesssim \|m\| \le \|h\|\,,$$

and thus also $\|m_s\| \lesssim \|m\| + |A| + |b| + |c'| \lesssim \|h\|$ from (4.25b). Similarly, we observe from (4.28) that

$$|b'| = |\langle r \nabla_x \phi\rangle| \lesssim \|r\| \le \|h\|\,.$$

Coming back to the definition of w_s and using (4.28), we get

$$w_s = r - \sqrt{\tfrac{2}{d}}\, c \left(\xi_\phi - \tfrac{1}{2\,d} \langle \Delta_x \phi\rangle\, \xi_2\right) + b' \cdot x - \tfrac{1}{2\,d} \left\langle r \left(|\nabla\phi|^2 - \Delta\phi\right)\right\rangle \xi_2\,,$$

and deduce $\|w_s\| \lesssim \|r\| + |c| + |b'| \lesssim \|h\|$. Similarly, from (4.30), we also have $|c''| \lesssim \|r\| + \|e\| + \|h^\perp\| \le \|h\|$. Summing up, we have proved

$$\|e_s\| + \|m_s\| + \|w_s\| + |A| + |c'| + |b'| + |c''| \lesssim \|h\|\,. \tag{4.43}$$

We finally have to control the terms $\|\,\Omega^{-1/2}\,\partial_t w_s\|$. From (4.2e), (4.2a) and (2.30), we have

$$\partial_t w_s = \nabla_x^* \cdot m - \langle \nabla_x \nabla_x^* \cdot m\rangle \cdot x - \tfrac{1}{2\,d} \left\langle \Delta_x \nabla_x^* \cdot m\right\rangle \xi_2 - \sqrt{\tfrac{2}{d}}\, c' \phi_s$$

and, after performing several integration by parts,

$$\left\|\Omega^{-\frac{1}{2}}\, \partial_t w_s\right\| \lesssim \|m\| + |c'| \lesssim \|h\|\,. \tag{4.44}$$

As a consequence of the estimates (4.42), (4.43), (4.44) and of the definition (4.39) of $\mathcal{F}_2$ (also see (4.20)), we have

$$\left|\|h\|^2 - \mathcal{F}_2[h]\right| \le C\,\varepsilon_1\, \|h\|^2\,.$$

This completes the proof of the first equivalence in (4.41). For the same reason, we have $\mathcal{D}_2[h] \lesssim \|h\|^2$. On the other way round, from (4.25a) and (4.36), we have

$$\|r\| \lesssim |b'| + |c''| + |c| + \|w_s\| + \|e_s\| + \|h^\perp\| \lesssim |b'| + |c''| + \|w_s\| + \|e_s\| + \|h^\perp\|$$

and similarly, from (4.25b) and (4.35), we have

$$\|m\| \lesssim |A| + |b| + |c'| + \|m_s\| \lesssim |A| + |c'| + \|m_s\| + \|\,\Omega^{-\frac{1}{2}}\,\partial_t w_s\| + \|h^\perp\|\,.$$

Combining the last two estimates, (4.1), (4.25c) and the definition (4.22) of $\mathcal{D}_2$, we deduce the reverse inequality $\|h\|^2 \lesssim \mathcal{D}_2[h]$, which completes the proof of the second equivalence in (4.41). ■

4.5. *Proof of Proposition* 4.1

Proof of Proposition 4.1. We differentiate with respect to t the Lyapunov function $\mathcal{F}_2[h]$ and use Lemmata 4.6, 4.9 and 4.11 to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le & -\kappa_0\,\|h^\perp\|^2 - \varepsilon^{\frac{15}{8}}\,\kappa\,\mathcal{D}_1[h] + \varepsilon^2\,C_0\,\|h\|^2\\ & -\kappa_4\,\varepsilon_5\,|A|^2 + C\,\varepsilon_5\left(\mathcal{D}_1[h] + C\,\|h^\perp\|\,\|h\|\right)\\ & +\varepsilon_6\left(-|b'|^2 - |c''|^2 + 2\,C\,|A|^2 + 2\,C\,\mathcal{D}_1[h]\right).\end{aligned}$$

Using Young's inequality we have

$$\varepsilon_5\,C^2\,\|h^\perp\|\,\|h\| \le \kappa_0\,\|h^\perp\|^2 + \varepsilon_5^2\,\kappa_0^{-1}\,C^4\,\|h\|^2$$

and we deduce for some new constants $C_1, C_2, C_3 > 0$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le & -\varepsilon^{\frac{15}{8}}\,\kappa\,\mathcal{D}_1[h] + \varepsilon^2\,C_0\,\|h\|^2\\ & -\kappa_4\,\varepsilon_5\,|A|^2 + C_1\,(\varepsilon_5+\varepsilon_6)\,\mathcal{D}_1[h] + C_2\,\varepsilon_5^2\,\|h\|^2\\ & -\varepsilon_6\left(|b'|^2 + |c''|^2\right) + C_3\,\varepsilon_6\,|A|^2.\end{aligned}$$

As in the proof of Lemma 4.6, we choose appropriately the small parameters ε_i such that the quantities ε_5, ε_6, $\varepsilon_5/\varepsilon_4$ and $\varepsilon_6/\varepsilon_5$ are small enough in terms of ε. With $\varepsilon_5 := \varepsilon^{61/32}$ and $\varepsilon_6 := \varepsilon^{62/32}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le & -\varepsilon^{15/8}\left(\kappa - 2\,\varepsilon^{1/32}\,C_1\right)\mathcal{D}_1[h] - \varepsilon^{61/32}\left(\kappa_4 - \varepsilon^{1/32}\,C_3\right)|A|^2\\ & -\varepsilon^{62/32}\left(|b'|^2 + |c''|^2\right) + \varepsilon^2\left(C_0 + C_2\,\varepsilon^{29/16}\right)\|h\|^2.\end{aligned}$$

Choosing $\varepsilon > 0$ small enough, the differential inequality simplifies into

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le -\varepsilon^{62/32}\,\mathcal{D}_2[h] + 5\,C\,\varepsilon^2\,\|h\|^2.$$

Because of the equivalences established in Lemma 4.13, there are two constants $K_i > 0$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le -\varepsilon^{62/32}\left(K_1 - K_2\,\varepsilon^{2/32}\right)\mathcal{F}_2[h].$$

Choosing $\varepsilon > 0$ smaller if necessary, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \le -\kappa\,\mathcal{F}_2[h],$$

for some $\kappa > 0$, which implies $\mathcal{F}_2[h(t)] \le e^{-\kappa\,t}\,\mathcal{F}_2[h_0]$. This completes the proof of Proposition 4.1, that is, of Part (2) of Theorem 1.1, by using once again the equivalences of Lemma 4.13. ∎

5. Proof of hypocoercivity by the commutator method

In this section we give an alternative proof of our main result in Theorem 1.1 using a commutator method, under the additional hypotheses that

$$\textit{The linear collision operator } \mathscr{C} \textit{ is bounded in } \mathrm{L}^2(\mu^{-1})\,, \tag{H9}$$

$$\|\nabla_x^2\,\phi\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < \infty \textit{ and } \int_{\mathbb{R}^d} x\,\phi(x)\,e^{-\phi(x)}\,\mathrm{d}x = 0\,. \tag{H10}$$

Assumption (H9) means that the operator C as defined in (2.24) is bounded on $\mathrm{L}^2(\mu)$ while (H10) means that the potential ϕ has bounded second derivatives and is superlinear at infinity. These assumptions are added merely in order to simplify the computations but the bound (H9) on the collision operator can be relaxed into just (H2) by using $\tilde{A}_i := \Pi^* A_i \Pi$ instead of the A_i's defined in (5.2) in Proposition 5.2 (where $\Pi h = h^{\parallel}$ is the orthogonal projection on the macroscopic part), and the bound in (H10) can be relaxed into simply $|\nabla_x^2\phi| \lesssim 1 + |\nabla_x\phi|$ thanks to the additional weight $\lfloor\nabla_x\phi\rceil$ in the Poincaré inequality (H5). We include the commutator method because of its interesting algebraic properties and potential applications to a larger class of equations. Then part (2) of Theorem 1.1 writes:

Proposition 5.1. *Assume that* (H0)–(H10) *hold and consider a solution h to* (2.22)–(2.23) *in* $\mathrm{L}^2(\mathcal{M})$. *Then there are explicit constants $C > 0$ and $\kappa > 0$ such that*

$$\|h(t)\| \le C\,e^{-\kappa\,t}\,\|h_0\|$$

where C and κ depend only on bounded moments constants, spectral gap constants or explicitly computable quantities associated to ϕ such as the rigidity constant defined in (1.14).

While ∇_x and ∇_v map scalar functions to vector-valued functions, their adjoints in $\mathrm{L}^2(\mathcal{M})$ are $\nabla_x^* = -\nabla_x\cdot + \nabla_x\phi$ and $\nabla_v^* = -\nabla_v\cdot + v\,\cdot$ and map vector-valued functions back to scalar functions. The operators ∇_x and ∇_v commute but each does not commute with its adjoint. We have

$$\begin{aligned} &[\nabla_v, \mathscr{T}] = -\nabla_x\,, && [\nabla_x, \mathscr{T}] = H_\phi\,\nabla_v\,,\\ &[\nabla_v^*, \mathscr{T}] = -\nabla_x^*\,, && [\nabla_x^*, \mathscr{T}] = \left(\nabla_x^2\,\nabla_v\right)^*, \end{aligned} \tag{5.1}$$

where $[A, B] = A\,B - B\,A$ is the commutator and $H_\phi := (\partial^2_{x_i x_j}\phi)_{i,j}$.

In addition to $\Omega = \nabla_x^*\cdot\nabla_x + 1$ defined in (1.15), we also introduce

$$\Gamma = \nabla_v^*\cdot\nabla_v + 1\,,\quad \Lambda = \nabla_v^*\cdot\nabla_v + \nabla_x^*\cdot\nabla_x + 1\,.$$

These scalar operators also act, coordinate by coordinate, on tensors.

From, *e.g.*, [9] (see Sections 4.4-4.5) or [25, 27], these operators are self-adjoint in $\mathrm{L}^2(\mathcal{M})$. As in Section 4, we construct a cascade of estimates.

5.1. Cascade of infinite-dimensional correctors

The three following operators play the role of *correctors*:

$$\begin{cases} A_0 := \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^* \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}} \nabla_v \otimes \nabla_x \otimes \nabla_x \,, \\ A_1 := \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac{1}{2}} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac{1}{2}} \nabla_v \otimes \nabla_v \otimes \nabla_x \,, \\ A_2 := \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_v \otimes \nabla_v \,. \end{cases} \tag{5.2}$$

From [25, 27] or by standard pseudo-differential calculus arguments (see Lemma A.4), A_0, A_1, A_2 are bounded operators in $\mathrm{L}^2(\mathcal{M})$. Let $\mathbb{R}_n[V]$ be the space of real polynomials of v with degree less or equal than $n \in \mathbb{N}$, then

$$\begin{cases} \nabla_v^{\otimes 3} (\mathbb{R}_2[V]) = \{0\} \,, \quad \nabla_v^{\otimes 2} (\mathbb{R}_1[V]) = \{0\} \,, \quad \nabla_v (\mathbb{R}_0[V]) = \{0\} \,, \\ A_2 (\mathbb{R}_2[V]) = \{0\} \,, \quad A_1 (\mathbb{R}_1[V]) = \{0\} \,, \quad A_0 (\mathbb{R}_0[V]) = \{0\} \,. \end{cases}$$

This means for instance that A_2 can access the local energy without seeing the local density and local momentum, in a descending cascade. Note that the simplest "order 1" corrector A_0 is inspired by the corrector $\nabla_x^* \Lambda^{-1/2} : \Lambda^{-1/2} \nabla_v$ introduced in [27] for Fokker-Planck type equations. We define macroscopic deviations from averages quantities that are slighlty different from (4.2a)–(4.2c):

$$\begin{aligned} \tilde{e} &:= e - \langle e \rangle \,, \\ \tilde{m} &:= m - \langle \nabla_x m \rangle \, x - \langle m \rangle \,, \\ \tilde{r} &:= r - \frac{1}{2} \langle \nabla_x^{\otimes 2} r \rangle : \big(x \otimes x - \langle x \otimes x \rangle \big) - \langle \nabla_x r \rangle \cdot x - \langle r \rangle \,. \end{aligned}$$

The core commutator estimates are:

Proposition 5.2. *For all $i \in \{0, 1, 2\}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle A_i \, h, h \rangle = - \langle \Lambda_i \, h, h \rangle + \langle B_i h, h \rangle \tag{5.3}$$

where the B_i's are bounded operators that satisfy

$$\begin{aligned} \langle B_0 h, h \rangle &\lesssim \|h\| \left(\|h^\perp\| + \|\tilde{e}\| + \|\tilde{m}\| \right) , \\ \langle B_1 h, h \rangle &\lesssim \|h\| \left(\|h^\perp\| + \|\tilde{e}\| \right) , \\ \langle B_2 h, h \rangle &\lesssim \|h\| \, \|h^\perp\| \,, \end{aligned}$$

and where the Λ_i's are nonnegative self-adjoint operators that satisfy, for some C_0, C_1, $C_2 > 0$ and $\bar{\lambda}_0$, $\bar{\lambda}_1$, $\bar{\lambda}_2 > 0$

$$\begin{aligned} - \langle \Lambda_0 \, h, h \rangle &\le -\bar{\lambda}_0 \, \|\tilde{r}\|^2 + C_0 \left(\|\tilde{m}\| + \|\tilde{e}\| + \|h^\perp\| \right) \|h\| \,, \\ - \langle \Lambda_1 \, h, h \rangle &\le -\bar{\lambda}_1 \, \|\tilde{m}\|^2 + C_1 \left(\|\tilde{e}\| + \|h^\perp\| \right) \|h\| \,, \\ - \langle \Lambda_2 \, h, h \rangle &\le -\bar{\lambda}_2 \, \|\tilde{e}\|^2 + C_2 \, \|h^\perp\| \, \|h\| \,. \end{aligned}$$

Proof of Proposition 5.2. Since C is self-adjoint and $\mathscr{T}$ is skew-adjoint,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A_i\, h, h\rangle = \Big\langle \left([A_i, \mathscr{T}] + A_i\, C + C\, A_i\right) h, h\Big\rangle \quad \text{for} \quad i \in \{0,1,2\}.$$

We can therefore write $[A_i, \mathscr{T}] + A_i\, C + C\, A_i =: B_i - \Lambda_i$ where, by using (5.1), we have the following explicit formulas

$$\begin{aligned}
B_0 :=\; & A_0\, C + C\, A_0 \\
& + \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^* \big[\Lambda^{-\frac{3}{2}}, \mathscr{T}\big] : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Lambda^{-\frac{3}{2}} : \big[\Lambda^{-\frac{3}{2}}, \mathscr{T}\big] \nabla_v \otimes \nabla_x \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes (H_\phi \nabla_v) \\
& + \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^* \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes (H_\phi \nabla_v) \otimes \nabla_x \\
& + (H_\phi \nabla_v)^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_x^* \otimes (H_\phi \nabla_v)^*\, \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x \\
& + \nabla_x^* \otimes (H_\phi \nabla_v)^* \otimes \nabla_x^* \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x\,, \\
B_1 :=\; & A_1\, C + C\, A_1 \\
& + \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \big[\Lambda^{-1}\, \Gamma^{-\frac{1}{2}}, \mathscr{T}\big] : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_v \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Lambda^{-\frac{3}{2}} : \big[\Lambda^{-1}\, \Gamma^{-\frac{1}{2}}, \mathscr{T}\big] \nabla_v \otimes \nabla_v \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_v \otimes (H_\phi \nabla_v) \\
& + \nabla_x^* \otimes \nabla_v^* \otimes (H_\phi \nabla_v)^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_v \otimes \nabla_x \\
& - \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_v \otimes \nabla_x \\
& + (H_\phi \nabla_v)^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_v \otimes \nabla_x\,, \\
B_2 :=\; & A_2\, C + C A_2 \\
& + \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \big[\Lambda^{-\frac{1}{2}}\, \Gamma^{-1}, \mathscr{T}\big] : \Gamma^{-1}\, \Lambda^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_v \otimes \nabla_v \\
& + \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Lambda^{-\frac{1}{2}}\, \Gamma^{-1} : \big[\Gamma^{-1}\, \Lambda^{-\frac{1}{2}}, \mathscr{T}\big] \nabla_v \otimes \nabla_v \otimes \nabla_v \\
& + \nabla_v^* \otimes \nabla_v^* \otimes (H_\phi \nabla_v)^*\, \Gamma^{-1}\, \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}}\, \Gamma^{-1}\, \nabla_v \otimes \nabla_v \otimes \nabla_v \\
& - \nabla_v^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Gamma^{-1}\, \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}}\, \Gamma^{-1} \nabla_v \otimes \nabla_v \otimes \nabla_v\,, \\
& - \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-1}\, \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}}\, \Gamma^{-1} : \nabla_v \otimes \nabla_v \otimes \nabla_v\,,
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_0 :=\; & \nabla_x^* \otimes \nabla_x^* \otimes \nabla_x^*\, \Lambda^{-\frac{3}{2}} : \Lambda^{-\frac{3}{2}}\, \nabla_x \otimes \nabla_x \otimes \nabla_x\,, \\
\Lambda_1 :=\; & \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_x \otimes \nabla_v \otimes \nabla_x \\
& + \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-\frac{1}{2}}\, \Lambda^{-1} : \Lambda^{-1}\, \Gamma^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x\,, \\
\Lambda_2 :=\; & \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^*\, \Gamma^{-1}\, \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}}\, \Gamma^{-1}\, \nabla_x \otimes \nabla_v \otimes \nabla_v
\end{aligned}$$

$$+ \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_x \otimes \nabla_v$$
$$+ \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-1} \Lambda^{-\frac{1}{2}} : \Lambda^{-\frac{1}{2}} \Gamma^{-1} \nabla_v \otimes \nabla_v \otimes \nabla_x .$$

The Λ_i's operators are nonnegative, self-adjoint (see Appendix A.3) and bounded (see Lemma A.4 again).

Note that for all $i \in \{0, 1, 2\}$:

$$|\langle A_i \, C \, h, h\rangle| = |\langle A_i \, C \, h^\perp, h\rangle| \lesssim \|h^\perp\| \, \|h\| ,$$
$$|\langle C \, A_i \, h, h\rangle| = |\langle A_i \, h, C h\rangle| = |\langle A_i \, h, C h^\perp\rangle| \lesssim \|h\| \, \|h^\perp\| ,$$

because C is self-adjoint and bounded by (H9).

Now we deal with B_2. Let us denote by $B_{2,\ell}$ the ℓ-th line. Since $\Lambda^{1/2} \, \Gamma \, [\Lambda^{-1/2} \, \Gamma^{-1}, \mathscr{T}]$ is bounded by standard pseudo-differential calculus (see [25]),

$$b_{2,2} := \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \, [\Lambda^{-\frac{1}{2}} \, \Gamma^{-1}, \mathscr{T}]$$

is bounded and

$$\left|\left\langle B_{2,2} \, h, h\right\rangle\right| = \left|\left\langle b_{2,2} : \Gamma^{-1} \, \Lambda^{-\frac{1}{2}} \, \nabla_v \otimes \nabla_v \otimes \nabla_v h, h\right\rangle\right|$$
$$= \left|\left\langle b_{2,2} : \Gamma^{-1} \, \Lambda^{-\frac{1}{2}} \, \nabla_v \otimes \nabla_v \otimes \nabla_v h^\perp, h\right\rangle\right| \lesssim \|h\| \, \|h^\perp\|$$

since the three derivatives in velocity on the right hand side cancel all macroscopic quantities. The other lines are dealt with similarly, using the boundedness of H_ϕ, and we deduce $|\langle B_2 h, h\rangle| \lesssim \|h\| \, \|h^\perp\|$.

Now we deal with B_1. We consider the second line $B_{1,2}$. Then $\Lambda \, \Gamma^{1/2} \, [\Lambda^{-1} \, \Gamma^{-1/2}, \mathscr{T}]$ is bounded by standard pseudo-differential calculus and so

$$b_{1,2} := \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \left[\Lambda^{-1} \, \Gamma^{-\frac{1}{2}}, \mathscr{T}\right]$$

is bounded. Thus

$$\left|\left\langle B_{1,2} h, h\right\rangle\right| = \left|\left\langle b_{1,2} : \Gamma^{-\frac{1}{2}} \, \Lambda^{-1} \, \nabla_v \otimes \nabla_v \otimes \nabla_x h, h\right\rangle\right|$$
$$\lesssim \left|\left\langle B_{1,2} : \Gamma^{-\frac{1}{2}} \, \Lambda^{-1} \, \nabla_v \otimes \nabla_v \otimes \nabla_x (e \, \mathfrak{E} + h^\perp), h\right\rangle\right| .$$

The macroscopic quantities r and m are canceled since two derivatives in velocity are involved. We then use $\nabla_v \otimes \nabla_v \otimes \nabla_x e \, \mathfrak{E} = \nabla_x e \otimes \mathrm{Id}_{d\times d}$ and Appendix A.4 shows (using $\Omega \geq 1$ for the first inequality)

$$\left\|\Omega^{-1} \, \nabla_x e\right\| \leq \left\|\Omega^{-\frac{1}{2}} \, \nabla_x e\right\| \lesssim \|\tilde{e}\| \quad \text{so} \quad \left|\left\langle B_{1,2} h, h\right\rangle\right| \lesssim \|h\| \left(\|\tilde{e}\| + \|h^\perp\|\right) .$$

The other lines are similar and yield the same estimate.

We then deal with B_0, and focus on the second line $B_{0,2}$ again. The operator

$$b_{0,2} := \nabla_v^* \otimes \nabla_v^* \otimes \nabla_x^* \left[\Lambda^{-\frac{3}{2}}, \mathscr{T}\right]$$

is bounded arguing as before, and

$$\left|\left\langle B_{0,2}h, h\right\rangle\right| = \left|\left\langle b_{0,2} : \Lambda^{-\frac{3}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x h, h\right\rangle\right| \lesssim \|h\| \left(\|\tilde{e}\| + \|\tilde{m}\| + \|h^{\perp}\|\right). \tag{5.4}$$

Indeed a direct computation gives

$$\Gamma^{-1}\, \Lambda^{-\frac{1}{2}}\, \nabla_v \otimes \nabla_x \otimes \nabla_x h = \Omega^{-\frac{3}{2}}\, \nabla_x \otimes \nabla_x m + \tilde{\Omega}^{-\frac{3}{2}}\, \nabla_x \otimes \nabla_x \otimes (v\, e)$$

with $\tilde{\Omega} := (\nabla_x^* \nabla_x + 2)$. The factor 2 comes from the fact that for all $\alpha \in \mathbb{R}$,

$$\Lambda^{\alpha}(v\, e) = (\nabla_x^* \nabla_x + \nabla_v^* \nabla_v + 1)^{\alpha}\, (v\, e) = (\nabla_x^* \nabla_x + 1 + 1)^{\alpha}\, (v\, e)$$

since v is an eigenfunction of $\nabla_v^* \nabla_v$ with eigenvalue 1. To complete the proof of (5.4), it is sufficient to notice that $\|\Omega^{-\frac{3}{2}}\, \nabla_x^2 e\| \le \|\Omega^{-1}\, \nabla_x^2 e\| \lesssim \|\tilde{e}\|$ and also that, by Appendix A.4, $\|\Omega^{-\frac{3}{2}}\, \nabla_x^2 m\| \le \|\Omega^{-1}\, \nabla_x^2 m\| \lesssim \|\tilde{m}\|$. The other lines in B_0 are treated similarly again.

Now we deal with the main nonnegative terms $\langle \Lambda_i\, h, h\rangle$ in (5.3). We first compute the contribution of Λ_0, Λ_1 and Λ_2 acting respectively on r, m and e. Due to the number of derivatives in velocity in the right hand side of the expressions giving the Λ_i's, we have $\Lambda_i(\mathbb{R}_{i-1}[V]) = \{0\}$ for $i \in \{1, 2\}$ and $\Lambda_i(\mathbb{R}_i[V]) \subset \mathbb{R}_i[V]$ for $i \in \{0, 1, 2\}$ and

$$\langle \Lambda_0\, r, r\rangle = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left|\Omega^{-\frac{3}{2}}\, \nabla_x^3 r\right|^2 \mathcal{M}\, \mathrm{d}x\, \mathrm{d}v = \int_{\mathbb{R}^d} \left|\Omega^{-\frac{3}{2}}\, \nabla_x^3 r\right|^2 \rho\, \mathrm{d}x\,, \tag{5.5}$$

$$\begin{aligned}\left\langle \Lambda_1\, (m(x)\cdot v)\,, (m(x)\cdot v)\right\rangle &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Big(\left(\Omega^{-1}\partial^2_{x_i x_k} m_j\right) \left(\Omega^{-1}\, \partial^2_{x_i x_k} m_j\right) \\ &\qquad + \left(\Omega^{-1}\partial^2_{x_i x_k} m_j\right) \left(\Omega^{-1}\, \partial^2_{x_j x_k} m_i\right) \Big)\, \mathcal{M}\, \mathrm{d}x\, \mathrm{d}v \\ &= 2 \int_{\mathbb{R}^d} \left|\Omega^{-1} \nabla_x \nabla_x^{\mathrm{sym}}\, m\right|^2 \rho\, \mathrm{d}x\,,\end{aligned} \tag{5.6}$$

$$\begin{aligned}&\left\langle \Lambda_2\, (e(x)\, \mathfrak{E}(v))\,, (e(x)\, \mathfrak{E}(v))\, \right\rangle \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \Big(\left(\Omega^{-\frac{1}{2}}\, \partial_{x_k} e\right) \left(\partial^2_{v_i\, v_j}\, \mathfrak{E}\right) \left(\Omega^{-\frac{1}{2}}\, \partial_{x_k} e\right) \left(\partial^2_{v_i\, v_j}\, \mathfrak{E}\right) \\ &\quad + \left(\Omega^{-\frac{1}{2}}\, \partial_{x_j} e\right) \left(\partial^2_{v_i v_k}\, \mathfrak{E}\right) \left(\Omega^{-\frac{1}{2}}\, \partial_{x_k} e\right) \left(\partial^2_{v_i\, v_j}\, \mathfrak{E}\right) \\ &\quad + \left(\Omega^{-\frac{1}{2}}\, \partial_{x_i} e\right) \left(\partial^2_{v_j v_k}\, \mathfrak{E}\right) \left(\Omega^{-\frac{1}{2}}\, \partial_{x_k} e\right) \left(\partial^2_{v_i\, v_j}\, \mathfrak{E}\right) \Big)\, \mathcal{M}\, \mathrm{d}x\, \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \left(2 \left(\Omega^{-\frac{1}{2}}\, \partial_{x_k} e\right)^2 + \frac{2}{d} \left(\Omega^{-\frac{1}{2}}\, \partial_{x_i} e\right)^2 + \frac{2}{d} \left(\Omega^{-\frac{1}{2}}\, \partial_{x_j} e\right)^2 \right) \rho\, \mathrm{d}x \\ &= \left(\frac{4}{d} + 2\right) \int_{\mathbb{R}^d} \left|\Omega^{-\frac{1}{2}}\, \nabla_x e\right|^2 \rho\, \mathrm{d}x\,.\end{aligned} \tag{5.7}$$

Next we use the cascade of Poincaré inequalities of Lemma A.6. For the density r, this implies that there isa constant $\overline{\lambda}_0 > 0$ such that

$$\begin{aligned}\langle \Lambda_0\, r, r\rangle &= \int_{\mathbb{R}^d} \left|\Omega^{-\frac{3}{2}}\, \nabla_x^3 r\right|^2 \rho\, \mathrm{d}x \\ &\ge 2\, \overline{\lambda}_0 \left\| r - \langle r\rangle - \langle \nabla_x r\rangle \cdot x - \tfrac{1}{2} \langle \nabla_x^2 r\rangle : (x \otimes x - \langle x \otimes x\rangle)\right\|^2 = 2\, \overline{\lambda}_0\, \|\tilde{r}\|^2\,.\end{aligned} \tag{5.8}$$

Regarding the momentum m, one first observes that

$$|\Omega^{-1}\nabla_x\nabla_x^{\rm sym}\, m|^2 \ge \frac{1}{9}\,|\Omega^{-1}\,\nabla_x^2 m|^2$$

thanks to the Schwarz lemma written as

$$\forall\, i,j,k\in\{1,\ldots,d\}\,,\ \partial^2_{ij} m_k = \partial_i\,(\nabla^{\rm sym} m)_{jk} + \partial_j\left(\nabla_x^{\rm sym}\, m\right)_{ik} - \partial_k\left(\nabla_x^{\rm sym}\, m\right)_{ij}\,.$$

Applied to (5.5)–(5.7), the cascade of Poincaré inequalities at order 2 stated in Lemma A.6 implies that there exists a constant $\overline{\lambda}_1$ such that

$$\begin{aligned}\langle \Lambda_1\,(m\cdot v),(m\cdot v)\rangle &\ge \frac{2}{9}\,\|\,\Omega^{-1}\,\nabla_x^2 m\|^2\\ &\ge 2\,\overline{\lambda}_1\,\|m-\langle m\rangle-\langle\nabla_x m\rangle\,x\|^2 = 2\,\overline{\lambda}_1\,\|\tilde m\|^2.\end{aligned}\tag{5.9}$$

Regarding the energy e, we use the standard Poincaré inequality in $\mathrm{L}^2(\rho)$ (the order 1 inequality of Lemma A.6) to get $\overline{\lambda}_2>0$ such that

$$\langle \Lambda_2\,(e\,\mathfrak{E}(v)),(e\,\mathfrak{E}(v))\rangle = \left(\tfrac{4}{d}+2\right)\|\,\Omega^{-\frac12}\,\nabla_x e\|^2 \ge 2\,\overline{\lambda}_2\,\|e-\langle e\rangle\|^2 = 2\,\overline{\lambda}_2\,\|\tilde e\|^2\,.\tag{5.10}$$

With the above estimates in hand, we can investigate all terms appearing in $\langle\Lambda_i\,h,h\rangle$. According to the number of velocity and space gradients in Λ_2, we get that

$$\langle \Lambda_2\,h,h\rangle = \Big\langle \Lambda_2\,(e\,\mathfrak{E}(v)),e\,\mathfrak{E}(v)\Big\rangle + \left\langle \Lambda_2\,(e\,\mathfrak{E}(v)),h^\perp\right\rangle + \left\langle \Lambda_2\,h^\perp,h\right\rangle\,,$$

from which one obtains with (5.10) that

$$-\langle\Lambda_2\,h,h\rangle \le -\overline{\lambda}_2\,\|\tilde e\|^2 + O\left(\|h^\perp\|\,\|h\|\right).$$

Similarly for m, using in addition that Λ_1 is self-adjoint and $\Lambda_1\,(e\,\mathfrak{E}(v)) = \Lambda_1\,(\tilde e\,\mathfrak{E}(v))$, one has

$$\begin{aligned}\langle \Lambda_1\,h,h\rangle = \langle \Lambda_1\,(m\cdot v),m\cdot v\rangle &+ \left\langle \Lambda_1\,(m\cdot v),(e-\langle e\rangle)\,\mathfrak{E}(v)+h^\perp\right\rangle\\ &+\left\langle \Lambda_1\left((e-\langle e\rangle)\,\mathfrak{E}(v)+h^\perp\right),h\right\rangle\,,\end{aligned}$$

which implies using (5.9) that

$$-\langle\Lambda_1\,h,h\rangle \le -\overline{\lambda}_1\,\|\tilde m\|^2 + O\big(\|\tilde e\|\,\|h\|\big) + O\big(\|h^\perp\|\,\|h\|\big)\,.$$

Finally, regarding the local density r, we get similarly

$$\begin{aligned}\langle\Lambda_0\,h,h\rangle &= \langle\Lambda_0\,r,r\rangle\\ &\quad+\left\langle\Lambda_0\,r,\tilde m\cdot v+\tilde e\mathfrak{E}(v)+h^\perp\right\rangle + \left\langle\Lambda_0\,(\tilde m\cdot v+\tilde e\mathfrak{E}(v)+h^\perp),h\right\rangle\end{aligned}$$

and it follows from (5.8) that

$$-\langle\Lambda_0\,h,h\rangle \le -\overline{\lambda}_0\,\|\tilde r\|^2 + O\big(\|\tilde m\|\,\|h\|\big) + O\big(\|\tilde e\|\,\|h\|\big) + O\big(\|h^\perp\|\,\|h\|\big)\,.$$

The proof of the proposition is complete. ■

We collect the previous estimates into a first partial Lyapunov inequality:

Lemma 5.3. *Define the following norm*

$$\|h\|^2_{\mathcal{H}_1} := \|h\|^2 + \varepsilon_0 \langle A_0\, h\,, h\rangle + \varepsilon_1 \langle A_1\, h, h\rangle + \varepsilon_2 \langle A_2\, h, h\rangle$$

for $\varepsilon_0, \varepsilon_1, \varepsilon_2 > 0$*, then for* $c_{\mathscr{C}} \gg \varepsilon_2 \gg \varepsilon_1 \gg \varepsilon_0$*, we have*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_1} \le -\frac{c_{\mathscr{C}}}{2}\|h^\perp\|^2 - \frac{\varepsilon_2}{2}\bar\lambda_2\|\widetilde{e}\|^2 - \frac{\varepsilon_1}{2}\bar\lambda_1\|\widetilde{m}\|^2 - \varepsilon_0\bar\lambda_0\|\widetilde{r}\|^2 + \eta_1\|h\|^2,$$

for some $0 < \eta_1 \ll \varepsilon_0$.

Proof of Lemma 5.3. Propositions 5.2 combined with Lemma 4.2 imply

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_1} \le &- c_{\mathscr{C}}\|h^\perp\|^2 - \varepsilon_2\bar\lambda_2\|\widetilde{e}\|^2 - \varepsilon_1\bar\lambda_1\|\widetilde{m}\|^2 - \varepsilon_0\bar\lambda_0\|\widetilde{r}\|^2\\ &+ C\,\varepsilon_2\|h\|\,\|h^\perp\| + C\,\varepsilon_1\|h\|\left(\|h^\perp\| + \|\widetilde{e}\|\right)\\ &+ C\,\varepsilon_0\|h\|\left(\|h^\perp\| + \|\widetilde{m}\| + \|\widetilde{e}\|\right)\end{aligned}$$

for some constant $C > 0$. The statement then follows from repeated uses of Young's inequality for products. ■

In fact the time derivatives of the local density, momentum and energy can also be controlled as follows:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_t\widetilde{r}\,,\Omega^{-1}\widetilde{r}\right\rangle \ge \left\|\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{r}\right\|^2 - O\big(\|h\|\;\|\widetilde{r}\|\,\big)\,,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_t\widetilde{m}\,,\Omega^{-1}\,\widetilde{m}\,\right\rangle \ge \|\,\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{m}\,\|^2 - O\big(\|h\|\,\|\widetilde{m}\|\big)\,,\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left\langle \partial_t\widetilde{e}\,,\Omega^{-1}\,\widetilde{e}\,\right\rangle \ge \|\,\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{e}\,\|^2 - O\big(\|h\|\,\|\widetilde{e}\|\big)\,.\end{cases} \tag{5.11}$$

This leads to second improved partial Lyapunov inequality:

Lemma 5.4. *Given* $1 \gg \varepsilon_2 \gg \varepsilon_2' \gg \varepsilon_1 \gg \varepsilon_1' \gg \varepsilon_0 \gg \varepsilon_0' \gg \eta_1$*, the norm*

$$\|h\|^2_{\mathcal{H}_2} := \|h\|^2_{\mathcal{H}_1} - \varepsilon_2'\left\langle \partial_t\widetilde{e}\,,\Omega^{-1}\,\widetilde{e}\,\right\rangle - \varepsilon_1'\left\langle \partial_t\widetilde{m}\,,\Omega^{-1}\,\widetilde{m}\,\right\rangle - \varepsilon_0'\left\langle \partial_t\widetilde{r}\,,\Omega^{-1}\,\widetilde{r}\right\rangle\,.$$

satisfies

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_2} \le &-\frac{c_{\mathscr{C}}}{4}\|h^\perp\|^2 - \frac{\varepsilon_2}{4}\bar\lambda_2\|\widetilde{e}\|^2 - \frac{\varepsilon_1}{4}\bar\lambda_1\|\widetilde{m}\|^2 - \frac{\varepsilon_0}{2}\bar\lambda_0\|\widetilde{r}\|^2\\ &- \varepsilon_2'\bar\lambda_2\left\|\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{e}\right\|^2 - \varepsilon_1'\bar\lambda_1\left\|\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{m}\right\|^2 - \varepsilon_0'\bar\lambda_0\left\|\Omega^{-\frac{1}{2}}\,\partial_t\widetilde{r}\right\|^2\\ &+ \eta_2\|h\|^2\end{aligned}$$

for some $0 < \eta_2 \ll \varepsilon_0'$.

Proof of Lemma 5.4. This follows from (5.11) and

$$\left|\left\langle \partial_t\widetilde{e}\,,\Omega^{-1}\,\widetilde{e}\,\right\rangle\right| + \left|\left\langle \partial_t\widetilde{m}\,,\Omega^{-1}\,\widetilde{m}\,\right\rangle\right| + \left|\left\langle \partial_t\widetilde{r}\,,\Omega^{-1}\,\widetilde{r}\right\rangle\right| \lesssim \|h\|^2$$

and the fact that second order time derivatives of the macroscopic quantities can be controlled by Ω (as in Section 4). ■

5.2. *Cascade of finite-dimensional correctors*

In view of Lemma 5.3, what remains to be controlled are the finite dimensional terms

$$\langle r \rangle\,, \quad \langle \nabla_x m \rangle\, x + \langle m \rangle \quad \text{and} \quad \frac{1}{2}\,\langle \nabla_x^{\otimes 2} r \rangle : \big(x \otimes x - \langle x \otimes x \rangle\big) - \langle \nabla_x r \rangle \cdot x - \langle r \rangle\,.$$

5.2.1. Control of moments of the local momentum. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle v_i\, v_j\, h \rangle = 2 \left\langle \big(\nabla_x^{\mathrm{sym}}\, m\big)_{ij} \right\rangle + \big\langle \big(\mathscr{T}(v_i\, v_j) + C(v_i\, v_j)\big)\, h^\perp \big\rangle\,,$$

which yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \left(\langle v_i\, v_j\, h \rangle\, \frac{\mathrm{d}}{\mathrm{d}t} \langle v_i\, v_j\, h \rangle \right)\\ &\geq 2 \left\langle \big(\nabla_x^{\mathrm{sym}}\, m\big)_{i,j} \right\rangle^2 - \left| \big\langle \big(\mathscr{T}(v_i\, v_j) + C(v_i\, v_j)\big)\, h^\perp \big\rangle^2 \right| + \langle v_i\, v_j\, h \rangle\, \frac{\mathrm{d}^2}{\mathrm{d}^2 t} \langle v_i\, v_j\, h \rangle\end{aligned}$$

and

$$\begin{cases} \langle v_i\, v_j\, h \rangle\, \dfrac{\mathrm{d}}{\mathrm{d}t} \langle v_i\, v_j\, h \rangle \lesssim \|h\|\, \|h^\perp\|\,, \\[2mm] \left| \big\langle \big(\mathscr{T}(v_i\, v_j) + C(v_i\, v_j)\big)\, h^\perp \big\rangle^2 \right| + \langle v_i\, v_j\, h \rangle\, \dfrac{\mathrm{d}^2}{\mathrm{d}^2 t} \langle v_i\, v_j\, h \rangle \lesssim \|h\|\, \|h^\perp\|\,. \end{cases}$$

Define for all $i \in \{1, \ldots, d\}$

$$\psi_i(v) := 1 + \sqrt{\tfrac{d}{2}} \left(1 + \tfrac{4}{d}\right) \mathfrak{E}(v) - \sqrt{\tfrac{d}{2}}\, |v_i|^2\, \mathfrak{E}(v)$$

which is orthogonal to 1, v, $|v|^2$. We then compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_i\, h \rangle = 4 \left\langle \tfrac{1}{d}\, \nabla_x \cdot m - \partial_{x_i} m_i \right\rangle + \left\langle [\mathscr{T}(\psi_i) + C(\psi_i)]\, h^\perp \right\rangle\,,$$

which yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \left(\langle \psi_i\, h \rangle\, \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_i\, h \rangle \right)\\ &\geq 8 \left\langle \tfrac{1}{d}\, \nabla_x \cdot m - \partial_{x_i} m_i \right\rangle^2 - \left\langle [\mathscr{T}(\psi_i(v)) + C(\psi_i(v))] h^\perp \right\rangle^2 + \langle \psi_i\, h \rangle\, \frac{\mathrm{d}^2}{\mathrm{d}^2 t} \langle \psi_i\, h \rangle\end{aligned}$$

and

$$\begin{cases} \langle \psi_i\, h \rangle\, \dfrac{\mathrm{d}}{\mathrm{d}t} \langle \psi_i\, h \rangle \lesssim \|h\|^2\,, \\[2mm] \left| \left\langle [\mathscr{T}(\psi_i(v)) + C(\psi_i(v))] h^\perp \right\rangle \right|^2 + \langle \psi_i\, h \rangle\, \dfrac{\mathrm{d}^2}{\mathrm{d}^2 t} \langle \psi_i\, h \rangle \lesssim \|h\|\, \|h^\perp\|\,. \end{cases}$$

We finally introduce the third norm

$$\|h\|_{\mathcal{H}_3}^2 := \|h\|_{\mathcal{H}_2}^2 - \varepsilon_3 \sum_{i \neq j} \langle v_i\, v_j\, h \rangle\, \frac{\mathrm{d}}{\mathrm{d}t} \langle v_i\, v_j\, h \rangle - \varepsilon_3\, \langle \psi_i\, h \rangle\, \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi_i\, h \rangle$$

$$- \varepsilon_3' \sum_{i \neq j} \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \big(\nabla_x^{\mathrm{sym}} m\big)_{i,j} \Big\rangle \Big\langle \big(\nabla_x^{\mathrm{sym}} m\big)_{i,j} \Big\rangle$$

$$- \varepsilon_3' \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \Big(\tfrac{1}{d} \nabla_x \cdot m - \partial_{x_i} m_i \Big) \Big\rangle \Big\langle \tfrac{1}{d} \nabla_x \cdot m - \partial_{x_i} m_i \Big\rangle$$

for $1 \gg \varepsilon_3 \gg \varepsilon_3' \gg \varepsilon_2 \gg \varepsilon_2' \gg \varepsilon_1 \gg \varepsilon_1' \gg \varepsilon_0 \gg \varepsilon_0'$. Therefore, defining as in (4.2b) and (4.2c) the *space* inhomogeneous terms

$$m_s := m - \langle \nabla_x^{\mathrm{skew}} m \rangle \, x - \tfrac{1}{d} \langle \nabla_x \cdot m \rangle \, x - \langle m \rangle \quad \text{and} \quad e_s := \tilde{e} = e - \langle e \rangle \,,$$

we obtain

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|h\|_{\mathcal{H}_3}^2 \leq & - \frac{c_{\mathscr{C}}}{8} \|h^\perp\|^2 - \frac{\varepsilon_2}{4} \bar{\lambda}_2 \|e_s\|^2 \\ & - \frac{\varepsilon_1}{8} \bar{\lambda}_1 \|\widetilde{m}\|^2 - \frac{\varepsilon_1}{8} \bar{\lambda}_1 \|m_s\|^2 - \frac{\varepsilon_0}{2} \bar{\lambda}_0 \|\widetilde{r}\|^2 \\ & - \frac{\varepsilon_2'}{2} \bar{\lambda}_2 \left\| \Omega^{-\frac{1}{2}} \partial_t \widetilde{e} \right\|^2 \\ & - \frac{\varepsilon_1'}{2} \bar{\lambda}_1 \left\| \Omega^{-\frac{1}{2}} \partial_t \widetilde{m} \right\|^2 - \frac{\varepsilon_1'}{2} \bar{\lambda}_1 \left\| \Omega^{-\frac{1}{2}} \partial_t m_s \right\|^2 \\ & - \varepsilon_0' \bar{\lambda}_0 \left\| \Omega^{-\frac{1}{2}} \partial_t \widetilde{r} \right\|^2 + \eta_3 \|h\|^2 \end{aligned}$$

for another $0 < \eta_3 \ll \varepsilon_0'$.

5.2.2. Control of moments of the local density and energy. We now control the difference between the finite dimensional quantities

$$r - \frac{1}{2} \langle \nabla_x^{\otimes 2} r \rangle : \big(x \otimes x - \langle x \otimes x \rangle \big) - \langle \nabla_x r \rangle \cdot x - \langle r \rangle$$

and

$$\begin{aligned} w_s := r & - \frac{1}{2\,d} \langle \Delta_x r \rangle \left(|x|^2 - \langle |x|^2 \rangle \right) - \langle \nabla_x r \rangle \cdot x \\ & - \sqrt{\frac{2}{d}} \langle e \rangle \left[\phi - \langle \phi \rangle - \frac{1}{2\,d} \langle \Delta_x \phi \rangle \left(|x|^2 - \langle |x|^2 \rangle \right) \right] \end{aligned}$$

defined in (4.2e) in Section 4, which is made of the two terms (using that $\langle r \rangle = 0$)

$$\begin{cases} I_1 := -\dfrac{1}{2} \displaystyle\sum_{1 \leq i,j \leq d} \left(\langle \partial^2_{x_i x_j} r \rangle - \dfrac{1}{d} \langle \Delta_x r \rangle \delta_{ij} \right) \left(x_i \, x_j - \langle x_i x_j \rangle \right) \,, \\ I_2 := \sqrt{\dfrac{2}{d}} \langle e \rangle \left[\phi - \langle \phi \rangle - \dfrac{1}{2\,d} \langle \Delta_x \phi \rangle \left(|x|^2 - \langle |x|^2 \rangle \right) \right] \,. \end{cases}$$

The first term is controlled by using

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\Big\langle (\nabla_x^{\mathrm{sym}} m)_{ij} \Big\rangle - \frac{1}{d} \langle \nabla_x \cdot m \rangle \, \delta_{ij} \right)$$

$$= -\left(\langle \partial^2_{x_i x_j} r\rangle - \frac{1}{d}\,\langle \Delta_x r\rangle \delta_{ij}\right) + \text{ controlled terms}$$

since the left hand side is already under control, and the second term is controlled by observing that $\langle e\rangle = \langle r\,\phi\rangle$ due to the energy conservation, and

$$\langle r\,\phi\rangle = \langle \tilde r\,\phi\rangle + \frac{1}{2}\,\langle \nabla^2_x r\rangle : \langle (x\otimes x - \langle x\otimes x\rangle)\,\phi\rangle$$

provided that $\int_{\mathbb{R}^d} x\,\phi\,e^{-\phi}\,\mathrm{d}x = 0$, and $\nabla^2_x r = \nabla^2_x \tilde r$, so finally

$$|\langle e\rangle| = |\langle r\,\phi\rangle| \lesssim \|\tilde r\|\,.$$

This allows to define the final and fourth norm

$$\begin{aligned}\|h\|^2_{\mathcal{H}_4} := \|h\|^2_{\mathcal{H}_3}&\\ &+ \varepsilon_w\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\langle (\nabla^{\rm sym}_x m)_{ij}\right\rangle - \frac{1}{d}\,\langle \nabla_x\cdot m\rangle\,\delta_{ij}\right)\left(\langle \partial^2_{x_i x_j} r\rangle - \frac{1}{d}\,\langle \Delta_x r\rangle\,\delta_{ij}\right)\\ &- \varepsilon'_w\left(\langle \partial^2_{x_i x_j} r\rangle - \frac{1}{d}\,\langle \Delta_x r\rangle\,\delta_{ij}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\langle \partial^2_{x_i x_j} r\rangle - \frac{1}{d}\,\langle \Delta_x r\rangle\,\delta_{ij}\right)\end{aligned}$$

for $1 \gg \varepsilon_3 \gg \varepsilon'_3 \gg \varepsilon_2 \gg \varepsilon'_2 \gg \varepsilon_1 \gg \varepsilon'_1 \gg \varepsilon_0 \gg \varepsilon'_0 \gg \varepsilon_w \gg \varepsilon'_w$, with

$$\begin{aligned}\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_4} \le &- \frac{\mathrm{c}_{\mathscr{C}}}{16}\,\|h^\perp\|^2 - \frac{\varepsilon_2}{8}\,\bar\lambda_2\,\|e_s\|^2 - \frac{\varepsilon_w}{2}\,\|w_s\|^2\\ &- \frac{\varepsilon_1}{8}\,\bar\lambda_1\,\|\widetilde m\|^2 - \frac{\varepsilon_1}{16}\,\bar\lambda_1\,\|m_s\|^2 - \frac{\varepsilon_0}{2}\,\bar\lambda_0\,\|\widetilde r\|^2\\ &- \frac{\varepsilon'_2}{2}\,\bar\lambda_2\left\|\Omega^{-\frac12}\,\partial_t\widetilde e\right\|^2 - \frac{\varepsilon'_1}{2}\,\bar\lambda_1\left\|\Omega^{-\frac12}\,\partial_t\widetilde m\right\|^2\\ &- \frac{\varepsilon'_1}{2}\,\bar\lambda_1\left\|\Omega^{-\frac12}\,\partial_t m_s\right\|^2 - \varepsilon'_0\,\bar\lambda_0\left\|\Omega^{-\frac12}\,\partial_t\widetilde r\right\|^2\\ &- \varepsilon'_w\left\|\Omega^{-\frac12}\,\partial_t w_s\right\|^2 + \eta_4\,\|h\|^2\end{aligned}$$

for $0 < \eta_4 \ll \varepsilon'_w$. Denote by $C_4[h]$ the semi-norm of the controlled quantities

$$\begin{aligned}C_4[h] := \Big(\|h^\perp\|^2 + \|\widetilde e\|^2 + \|\widetilde m\|^2 + \|m_s\|^2 + \|\widetilde r\|^2 + \|w_s\|^2 + \left\|\Omega^{-\frac12}\,\partial_t e_s\right\|^2&\\ + \left\|\Omega^{-\frac12}\,\partial_t \widetilde m\right\|^2 + \left\|\Omega^{-\frac12}\,\partial_t m_s\right\|^2 + \left\|\Omega^{-\frac12}\,\partial_t \widetilde r\right\|^2 + \left\|\Omega^{-\frac12}\,\partial_t w_s\right\|^2\Big)^{1/2}&\end{aligned}$$

and adjust the constants to get, for some $0 < \eta \ll \varepsilon \ll 1$,

$$\frac{1}{2}\,\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2_{\mathcal{H}_4} \le -\,\varepsilon\,C_4[h]^2 + \eta\,\|h\|^2\,. \tag{5.12}$$

5.3. *Control of the remaining finite-dimensional quantities related to the special macroscopic modes*

Estimate (5.12) controls the same microscopic and macroscopic parts of the solution as in Lemma 4.6 in the micro-macro method. The remaining finite-dimensional quantities related to the special macroscopic modes can then be treated exactly as in Sections 4.4 and 4.5. This completes the proof of Proposition 5.1.

Appendix A. Some technical computations

A.1. *Momentum conservation* versus *infinitesimal rotations*

Here we prove (2.19) for a solution $f \in C\big(\mathbb{R}^+; \mathrm{L}^2(\mathcal{M}^{-1})\big)$ to (1.1) with initial datum $f_0 \in \mathrm{L}^2(\mathcal{M}^{-1})$. With $x \mapsto A\,x := \mathbb{P}_\phi m_0(x)$,

$$m_0(x) := e^{\phi(x)} \int_{\mathbb{R}^d} v\, f_0(x,v)\,\mathrm{d}v\,, \qquad m_f(t,x) := e^{\phi(x)} \int_{\mathbb{R}^d} v\, f(t,x,v)\,\mathrm{d}v\,,$$
$$r_f(t,x) := e^{\phi(x)} \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\,, \qquad e_f(t,x) := e^{\phi(x)} \int_{\mathbb{R}^d} \mathfrak{E}(v)\, f(t,x,v)\,\mathrm{d}v\,,$$

let us define $h^\perp$ such that $f = r_f\,\mathcal{M} + m_f \cdot v\,\mathcal{M} + e_f\,\mathfrak{E}\,\mathcal{M} + h^\perp \mathcal{M}$.

Lemma A.1. *With the above notations, we have have $\mathbb{P}(m_f - m_0) \in \mathcal{R}_\phi^\perp$.*

Proof of Lemma A.1. At $t = 0$, we have $\mathbb{P}\,m_f(0) - \mathbb{P}\,m_0 = 0$. Let $B \in \mathcal{R}_\phi$. To prove that $\mathbb{P}(m_f - m_0)$ is orthogonal to $x \mapsto B\,x$, it is sufficient to prove that

$$\forall\, t \geq 0\,, \quad \big\langle m_f(t) - m_0, B\,x \big\rangle = 0\,.$$

By direct computation, we have

$$\partial_t m_f = -\nabla_x r_f + \sqrt{\tfrac{2}{d}}\,\nabla_x^* e_f + \nabla_x^* \cdot E[h^\perp]$$

where $E[h^\perp] = \int_{\mathbb{R}^d} (v \otimes v - \mathrm{Id}_{d\times d})\, h^\perp\, \mu\,\mathrm{d}v$, and use it to compute

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle m_f - m_0, B\,x\rangle \\
&= \langle \partial_t m_f, B\,x\rangle \\
&= \Big\langle -\nabla_x r_f + \sqrt{\tfrac{2}{d}}\,\nabla_x^* e_f + \nabla_x^* \cdot E\,(h^\perp), B\,x \Big\rangle \\
&= -\big\langle r_f, \nabla_x^* \cdot B\,x \big\rangle + \sqrt{\tfrac{2}{d}}\,\Big\langle e_f, \nabla_x \cdot B\,x \Big\rangle + \big\langle E[h^\perp] : \nabla \otimes B\,x \big\rangle
\end{aligned}$$

where the last line follows from an integration by parts. The first term in the right hand side vanishes because $\nabla_x^* \cdot B\,x = -\nabla_x \cdot B\,x + \nabla\phi \cdot B\,x = 0$ since B is skew-symmetric and $(x \mapsto B\,x) \in \mathcal{R}_\phi$. The second term vanishes as well because $\nabla_x \cdot B\,x = 0$. Since $E[h^\perp] : \nabla \otimes B\,x = -E[h^\perp] : B = 0$ because $E[h^\perp]$ is symmetric and B is skew-symmetric, the third term also vanishes. This proves that $\frac{\mathrm{d}}{\mathrm{d}t}\langle m(t) - m_0, B\,x\rangle = 0$ and completes the proof. ∎

A.2. *Special macroscopic modes: the invertibility and rank*

We state and prove two results used in Section 3.3 and implicitely in Section 4.5. The first result deals with the invertibility of the matrices M_ϕ and $\widehat{M}_\phi$ defined respectively in (3.10) and (3.16).

Lemma A.2. *If $d_\phi = d$, the matrix M_ϕ is invertible. If $1 \le d_\phi \le d-1$, the matrix $\widehat{M}_\phi$ is invertible.*

Proof of Lemma A.2. Assume that $d_\phi = d$ in (1.9). Let $u \in \mathbb{R}^d$ be such that $M_\phi u = 0$. Then $M_\phi u \cdot u = \langle |\Phi \cdot u|^2 \rangle = 0$, which implies that $\Phi(x) \cdot u = 0$ for any $x \in \mathbb{R}^d$, hence $u = 0$. This means that $\operatorname{Ker} M_\phi = \{0\}$. The proof in the case $d_\phi \le d-1$ follows exactly the same scheme. ■

The second result deals with the linear independence of the two functions $\widetilde{\Psi}_1$ and $\widetilde{\Psi}_2$ defined in (3.14), and similarly for $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$ defined in (3.19).

Lemma A.3. *If $d_\phi = d$, we have* $\operatorname{Rank}(\widetilde{\Psi}_1, \widetilde{\Psi}_2) = 2$.
If $1 \le d_\phi \le d-1$, we have $\operatorname{Rank}(\widehat{\Psi}_1, \widehat{\Psi}_2) = 2$.

Proof of Lemma A.3. Let us assume that $d_\phi = d$ and argue by contradiction. Assume that $\widetilde{\Psi}_1 = \lambda \widetilde{\Psi}_2$ for some $\lambda \in \mathbb{R}^*$, that is, there are constants $\alpha, \beta \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$ such that

$$\phi + \tfrac{1}{2} \nabla_x \phi \cdot x + \alpha \cdot \nabla_x \phi = \tfrac{\lambda}{4} |x|^2 + \beta \cdot x + \gamma \,. \tag{A.1}$$

We first look for quadratic solutions to (A.1) of the form $\phi_0 = x \cdot M_0 \, x + b_0 \cdot x + c_0$ with $M_0 \in \mathfrak{M}_{d\times d}(\mathbb{R})$, $b_0 \in \mathbb{R}^d$ and $c_0 \in \mathbb{R}$. Plugging ϕ_0 into (A.1), one obtains $M_0 = \frac{\lambda}{8} \mathrm{Id}_{d\times d}$, $b_0 = \frac{2}{3} \left(\beta - \frac{\lambda}{4} \alpha\right)$ and $c_0 = \gamma - \frac{2}{3} \alpha \cdot \left(\beta - \frac{\lambda}{4} \alpha\right)$. Now let ϕ be a solution to (A.1). Define $\psi_0(x) = \phi(x) - \phi_0(x)$ and then $\psi(y) = \psi_0(y - 2\,\alpha)$, which hence verifies

$$\psi(y) - \tfrac{1}{2} \nabla_y \psi(y) \cdot y = 0 \,.$$

Let $\zeta(y) = |y|^2 \, \psi(y)$ so that $\nabla_y \zeta(y) \cdot y = 2\,|y|^2 \left(\psi(y) - \frac{1}{2} \nabla_y \psi(y) \cdot y\right) = 0$ for any $y \in \mathbb{R}^d$. In polar coordinates (r, θ), this implies that $\zeta(y) = \zeta(\theta)$ and hence

$$\forall\, r > 0 \,, \quad \psi(r, \theta) = \frac{\zeta(\theta)}{r^2} \,.$$

But ψ is by assumption continuous at the origin, therefore $\lim_{r\to 0} \psi(r,\theta)$ is finite, which in turn implies that $\psi(r,\theta) = 0$. Finally one gets $\phi = \phi_0$. Thanks to the normalizations (H3) and (H7), one gets $\phi(x) = \frac{1}{2} |x|^2 + \frac{d}{2} \log(2\,\pi)$ and, by definition (1.9), $E_\phi = \{0\}$, which contradicts the hypothesis $d_\phi = d$. This completes the proof when $d_\phi = d$. When $d_\phi \le d-1$, we argue similarly on $\widehat{\Psi}_1$ and $\widehat{\Psi}_2$. ■

A.3. Some computations for the commutator method

Here we prove technical claims used in Section 5. The first result is concerned with boundedness of the operators defined in Section 5.1 under Assumptions (H9) and (H10).

Lemma A.4. *Assume that* (H0)–(H1)–(H2)–(H3)–(H4)–(H5)–(H6)–(H7)–(H8)– (H9)–(H10) *hold. Then the operators Λ_i, A_i and B_i, $i \in \{1, \ldots, 3\}$, are bounded.*

Proof of Lemma A.4. As a typical example, we focus on Λ_1 for which it is sufficient to show that $\Lambda^{-1}\,\Gamma^{-1/2}\,\partial_{x_i}\partial_{v_j}\partial_{v_k}$ is bounded in $\mathrm{L}^2(\mathcal{M})$. Adopting the point of view of [25, Proposition A.7], we first conjugate with $\mathcal{M}^{1/2}$ and only have to check that $\tilde{\Gamma}^{-1/2}\,(\partial_{v_k}+v_k/2)$ and $\tilde{\Lambda}^{-1}\,(\partial_{x_i}+x_i/2)\,(\partial_{v_j}+v_j/2)$ are bounded in $\mathrm{L}^2(\mathbb{R}^d,\mathrm{d}x\mathrm{d}v)$, where

$$\tilde{\Lambda}=\sum_i\left(-\partial_{x_i}+\frac{x_i}{2}\right)\left(\partial_{x_i}+\frac{x_i}{2}\right)+\left(-\partial_{v_i}+\frac{v_i}{2}\right)\left(\partial_{v_i}+\frac{v_i}{2}\right),$$

$$\tilde{\Gamma}=\sum_i\left(-\partial_{v_i}+\frac{v_i}{2}\right)\left(\partial_{v_i}+\frac{v_i}{2}\right).$$

For $\tilde{\Gamma}^{-1/2}(\partial_{v_k}+v_k/2)$ this is due to the fact that $\tilde{\Gamma}^{-1/2}$ is of order -1 and $\partial_{v_k}+v_k/2$ of order 1 in the pseudo-differential calculus associated to the metric $(\mathrm{d}v^2+\mathrm{d}\eta^2)/(1+|v|^2+|\eta|^2)$, η being the dual variable of v. The composition is then of order 0 and the Calderón-Vaillancourt Theorem (see [8]) implies the boundedness. For $\tilde{\Lambda}^{-1}(\partial_{x_i}+x_i/2)(\partial_{v_j}+v_j/2)$, the result is also true because Λ^{-1} is of order -2 and $(\partial_{x_i}+x_i/2)(\partial_{v_j}+v_j/2)$ is of order 2 in the pseudo-differential calculus associated to the metric $(\mathrm{d}x^2+\mathrm{d}v^2+\mathrm{d}\xi^2+\mathrm{d}\eta^2)/(1+|\eta|^2+|v|^2+|\nabla\phi|^2+|\xi|^2)$, ξ being the dual variable of v. This implies the desired boundedness. Such calculus with two levels (involving Λ in all variable and Γ only in velocity variables) is also at the core of the boundedness of terms like $\Lambda\,\Gamma^{1/2}\left[\Lambda^{-1}\,\Gamma^{-1/2},\,\mathscr{T}\right]$ where we use that Λ and Γ commute and that the commutation decrease the order by 1, so that this operator is of order 0 and therefore bounded by the Calderón-Vaillancourt Theorem. Note in addition that H_ϕ (which appears, *e.g.*, in the B_i's) is of order 0 which greatly simplifies the proofs. For all other terms Λ_i, A_i and B_i, similar computations give the result. ■

The second result deals with the symmetry and nonnegativity of Λ_1.

Lemma A.5. *The operator Λ_1 is symmetric and nonnegative.*

Proof of Lemma A.5. First we check that

$$\nabla_x^*\,\nabla_v^*\,\nabla_x^*\,\Gamma^{-\frac12}\,\Lambda^{-1}:\Lambda^{-1}\,\Gamma^{-\frac12}\,\nabla_v\,\nabla_x\,\nabla_x$$

is symmetric, since for the other part of Λ_1 this is obvious:

$$\begin{aligned}
&\left\langle\nabla_x^*\,\nabla_v^*\,\nabla_x^*\,\Gamma^{-\frac12}\,\Lambda^{-1}:\Lambda^{-1}\,\Gamma^{-\frac12}\,\nabla_v\,\nabla_x\nabla_x f,g\right\rangle\\
&=\sum_{i,j,k}\left\langle\partial_{x_i}^*\,\partial_{v_j}^*\,\partial_{x_k}^*\,\Gamma^{-\frac12}\,\Lambda^{-1}:\Lambda^{-1}\,\Gamma^{-\frac12}\,\partial_{v_k}\,\partial_{x_j}\,\partial_{x_i}\,f,\,g\right\rangle\\
&=\sum_{i,j,k}\left\langle\Lambda^{-1}\,\Gamma^{-\frac12}\,\partial_{v_k}\,\partial_{x_j}\,\partial_{x_i}\,f,\,\Lambda^{-1}\,\Gamma^{-\frac12}\,\partial_{x_k}\,\partial_{v_j}\,\partial_{x_i}\,g\right\rangle\\
&=\sum_{i,j,k}\left\langle\Lambda^{-1}\,\Gamma^{-\frac12}\,\partial_{x_j}\,\partial_{v_k}\,\partial_{x_i}\,f,\,\Lambda^{-1}\,\Gamma^{-\frac12}\,\partial_{v_j}\,\partial_{x_k}\,\partial_{x_i}\,g\right\rangle\\
&=\sum_{i,j,k}\langle f,\,\partial_{x_i}^*\,\partial_{v_k}^*\,\partial_{x_j}^*\,\Gamma^{-\frac12}\,\Lambda^{-2}\,\Gamma^{-\frac12}\,\partial_{v_j}\,\partial_{x_k}\,\partial_{x_i}\,g\rangle
\end{aligned}$$

$$= \left\langle f,\, \nabla_x^* \nabla_v^* \nabla_x^* \Gamma^{-\frac12} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac12} \nabla_v \nabla_x \nabla_x g \right\rangle .$$

Next check that Λ_1 is indeed a nonnegative operator:

$$\begin{aligned}\left\langle \Lambda_1 f, f \right\rangle &= \left\langle \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac12} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac12} \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* f,\, f \right\rangle \\ &\quad + \left\langle \nabla_x^* \otimes \nabla_v^* \otimes \nabla_x^* \Gamma^{-\frac12} \Lambda^{-1} : \Lambda^{-1} \Gamma^{-\frac12} \nabla_v^* \otimes \nabla_x^* \otimes \nabla_x^* f, f \right\rangle \\ &= \sum_{i,j,k} \left\langle \Lambda^{-1} \Gamma^{-\frac12} \partial_{x_k} \partial_{v_j} \partial_{x_i} f,\, \Lambda^{-1} \Gamma^{-\frac12} \partial_{x_k} \partial_{v_j} \partial_{x_i} f \right\rangle \\ &\quad + \left\langle \Lambda^{-1} \Gamma^{-\frac12} \partial_{v_k} \partial_{x_j} \partial_{x_i} f,\, \Lambda^{-1} \Gamma^{-\frac12} \partial_{x_k} \partial_{v_j} \partial_{x_i} f \right\rangle \\ &= \sum_{i,j,k} \frac12 \left\| \Lambda^{-1} \Gamma^{-\frac12} \left(\partial_{x_k} \partial_{v_j} + \partial_{v_k} \partial_{x_j} \right) \partial_{x_i} f \right\|^2 .\end{aligned}$$

This completes the proof. ■

A.4. A cascade of Poincaré-Lions inequalities

Under Assumptions (H9) and (H10), we prove several inequalities used in Section 5.1. Let φ a smooth function in $\mathrm{L}^2(\rho)$ with compact support and

$$\begin{aligned} P_0(\varphi) &:= \langle \varphi \rangle , \\ P_1(\varphi) &:= \langle \varphi \rangle + \langle \nabla_x \varphi \rangle \cdot x , \\ P_2(\varphi) &:= \langle \varphi \rangle + \langle \nabla_x \varphi \rangle \cdot x + \frac12 \left\langle (\nabla_x \otimes \nabla_x) \varphi \right\rangle : \left(x \otimes x - \langle x \otimes x \rangle \right) . \end{aligned}$$

Lemma A.6. *Let $n \in \{1, 2, 3\}$. Then there exists a constant $c_{P,n} > 0$ such that for all smooth φ with compact support we have*

$$c_{P,n} \, \| \varphi - P_{n-1}(\varphi) \|^2 \le \left\| \Omega^{-n/2} \nabla_x^{\otimes n} \varphi \right\|^2 .$$

Proof of Lemma A.6. For $n = 1$ this is exactly the Poincaré-Lions Theorem as stated in [9, Proposition 5] and recalled in (4.6). Let us prove the result for $n = 2$. Let φ be smooth and with compact support. We have

$$\langle \varphi - P_1(\varphi) \rangle = \langle \varphi \rangle - \langle \varphi \rangle - \langle \nabla_x \varphi \rangle \cdot \langle x \rangle = 0$$

because $\langle x \rangle = 0$. We therefore apply the Poincaré-Lions inequality (*i.e.*, the case $n = 1$) to $\varphi - P_1(\varphi)$, which gives

$$\| \varphi - P_1(\varphi) \|^2 \le c_{\mathrm{P},1}^{-1} \left\| \Omega^{-\frac12} \nabla_x \left(\varphi - P_1(\varphi) \right) \right\|^2 = c_{\mathrm{P},1}^{-1} \left\| \Omega^{-\frac12} \left(\nabla_x \varphi - \langle \nabla_x \varphi \rangle \right) \right\|^2 .$$

We then apply the "-1-order" Poincaré-Lions inequality in [9, Lemma 10] recalled in (4.7) to $\nabla_x \varphi$ to get, for some $C_{\mathrm{LPL}} > 0$ depending only on ϕ,

$$\left\| \Omega^{-\frac12} \left(\nabla_x \varphi - \langle \nabla_x \varphi \rangle \right) \right\|^2 \le C_{\mathrm{LPL}} \left\| \Omega^{-1} \nabla_x^2 \varphi \right\|^2 .$$

This proves the case $n = 2$ with $c_{\mathrm{P},2} = c_{\mathrm{P},1}/C_{\mathrm{LPL}}$.

In the case $n = 3$, we define $\psi := \varphi - \frac{1}{2}\langle\nabla^2\phi\rangle : x \otimes x$ and we compute

$$\begin{aligned}\psi - P_1(\psi) &= \psi - \langle\psi\rangle - \langle\nabla_x\psi\rangle \cdot x \\ &= \varphi - \tfrac{1}{2}\langle\nabla^2\varphi\rangle : x \otimes x - \langle\varphi\rangle \\ &\quad + \tfrac{1}{2}\langle\nabla^2\varphi\rangle : \langle x \otimes x\rangle - \langle\nabla_x\varphi\rangle \cdot x + \langle\nabla_x^2\varphi\rangle : \langle x\rangle \otimes x \\ &= \varphi - P_2(\varphi)\end{aligned}$$

since $\langle x\rangle = 0$. We apply the inequality for $n = 2$ and obtain

$$\begin{aligned}\|\varphi - P_2(\varphi)\|^2 &= \|\psi - P_1(\psi)\|^2 \\ &\le c_{\mathrm{P},2}^{-1}\left\|\Omega^{-1}\nabla_x^2\psi\right\|^2 = c_{\mathrm{P},2}^{-1}\left\|\Omega^{-1}(\nabla_x^2\varphi - \langle\nabla_x^2\varphi\rangle)\right\|^2. \end{aligned} \tag{A.2}$$

Arguing as for the proof of the "−1 order" Poincaré-Lions inequality in [9, Lemma 10], we prove the "−2 order" Poincaré-Lions inequality with constant $C_{\mathrm{LPL}} > 0$:

$$\left\|\Omega^{-1}(f - \langle f\rangle)\right\|^2 \le C_{\mathrm{LPL}}\left\|\Omega^{-\frac{3}{2}}\nabla_x f\right\|^2$$

for any $f \in C_c^\infty$. Applying this estimate in (A.2) to $\nabla_x^2\varphi$ gives

$$\|\varphi - P_2(\varphi)\|^2 \le c_{\mathrm{P},2}^{-1} C_{\mathrm{LPL}}\left\|\Omega^{-\frac{3}{2}}\nabla_x^3\varphi\right\|^2$$

and concludes for $n = 3$ with $c_{\mathrm{P},3} = c_{\mathrm{P},2}/C_{\mathrm{LPL}}$, and completes the proof. ∎

Appendix B. Extension to weakly coercive collision operators

Our method covers the case of collision operators $\mathscr{C}$ that do not possess a spectral gap (assumption (H1) in Section 1) but only satisfy a weaker coercivity property (see (H1')) below). In this appendix, we state some results and changes to be done in the proofs.

B.1. Results on decay rates

We assume that $\mathscr{C}$ satisfies, for some $\alpha > 0$, the *weak coercivity property*

$$-\int_{\mathbb{R}^d}\big(\mathscr{C}f(v)\big)\, f(v)\,\mu(v)^{-1}\,\mathrm{d}v \ge \mathrm{c}_{\mathscr{C}}\,\|f - \Pi f\|^2_{\mathrm{L}^2(\lfloor v\rceil^{-\alpha}\mu^{-1})} \tag{H1'}$$

for some constant $\mathrm{c}_{\mathscr{C}} > 0$, and for all f in the domain of $\mathscr{C}$, where Π is the $\mathrm{L}^2(\mu^{-1})$-orthogonal projection onto $\operatorname{Ker}\mathscr{C}$. Here $\lfloor v\rceil$ denotes the weight $\sqrt{1+|v|^2}$. Moreover, we suppose that for any polynomial function $p(v) : \mathbb{R}^d \to \mathbb{R}$ of degree at most 4, the function $p\,\mu$ is in the domain of $\mathscr{C}$ and

$$C(p) := \|\mathscr{C}(p\,\mu)\|_{\mathrm{L}^2(\lfloor v\rceil^{\alpha}\mu^{-1})} < \infty\,. \tag{H2'}$$

The analog of our main result in Theorem 1.1 then becomes:

Theorem B.1. *Assume that the potential ϕ and the collision operator $\mathscr{C}$ satisfy assumptions* (H0)–(H1')–(H2')–(H3)–(H4)–(H5)–(H6)–(H7)–(H8). *Then*

(1) *All special macroscopic modes of* (1.4) *are given by* (1.12), i.e., *are linear combinations of the Maxwellian, the energy mode, rotation modes compatible with ϕ, and harmonic directional or pulsating modes if allowed by ϕ.*

(2) *There exists a norm $|||\cdot|||_{L^2(\mathcal{M}^{-1})}$ on $L^2(\mathcal{M}^{-1})$, which is equivalent to $\|\cdot\|_{L^2(\mathcal{M}^{-1})}$ (with quantitative comparison constants), and some explicit $\lambda > 0$ such that, for any solution $f \in C\big(\mathbb{R}^+; L^2(\mathcal{M}^{-1})\big)$ to* (1.1) *with initial datum $f_0 \in L^2(\mathcal{M}^{-1})$, there exists a unique special macroscopic mode F (determined by f_0) such that*

$$\forall\, t \ge 0\,, \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|||f(t) - F(t)|||^2_{L^2(\mathcal{M}^{-1})} \le -\lambda\, \|f(t) - F(t)\|^2_{L^2(\lfloor v \rceil^{-\alpha}\mathcal{M}^{-1})}\,. \tag{B.1}$$

The differential inequality (B.1) alone is not sufficient to prove a decay estimate when f_0 is merely in $L^2(\mathcal{M}^{-1})$. In order to get such an estimate, one needs to assume more decay at infinity for f_0 and the differential inequality (B.1) has to be replaced by an inequality in spaces with stronger weights. For instance, assume that, for some $\beta > 0$,

$$t \mapsto e^{t\mathscr{L}} \text{ is a strongly continuous uniformly bounded semigroup on } L^2(\mathcal{M}^{-1-\beta})\,, \tag{H9'}$$

with $\mathscr{L}$ as in (1.1). In the spirit of [43], we obtain the following decay rate.

Corollary B.2. *Assume that the potential ϕ and the collision operator $\mathscr{C}$ satisfy assumptions* (H0)–(H1')–(H2')–(H3)–(H4)–(H5)–(H6)–(H7)–(H8)–(H9'). *Then there are explicit constants $C_0 > 0$ and $\Lambda > 0$ such that, for any solution $f \in C\big(\mathbb{R}^+; L^2(\mathcal{M}^{-1})\big)$ to* (1.1) *with initial datum $f_0 \in L^2(\mathcal{M}^{-1-\beta})$, there exists a unique special macroscopic mode F (determined by f_0) such that*

$$\forall\, t \ge 0\,, \quad \|f(t) - F(t)\|_{L^2(\mathcal{M}^{-1})} \le C_0 \exp\Big(-\Lambda\, t^{\frac{2}{2+\alpha}}\Big)\, \|f_0 - F(0)\|_{L^2(\mathcal{M}^{-1-\beta})}\,.$$

As in the proof of Theorem 1.1, it is convenient to work with the function h defined by (2.22). We use various norms: $\|\cdot\|$ defined by (1.13) and also

$$\|h\|^2_\star := \iint_{\mathbb{R}^d\times\mathbb{R}^d} |h|^2\, \lfloor v \rceil^{-\alpha}\, \mathcal{M}\, \mathrm{d}x\, \mathrm{d}v \quad \text{and} \quad \|h\|^2_{1-\beta} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} |h|^2\, \mathcal{M}^{1-\beta}\, \mathrm{d}x\, \mathrm{d}v\,.$$

Observe that $\|\cdot\| = \|\cdot\|_1 \le \|\cdot\|_{1-\beta}$, $\|\cdot\|_\star = \|\cdot\|^2_{L^2(\lfloor v \rceil^{-\alpha}\mathcal{M}^{-1})} \le \|\cdot\|$ and

$$\|h\|^2_\star \approx \|h^\perp\|^2_\star + \|r\|^2 + \|m\|^2 + \|e\|^2.$$

B.2. Proof of Theorem B.1

Proposition 3.1 applies: the proof of Part (1) is the same as Part (1) of Theorem 1.1. To prove Part (2), we argue as in the proof of Proposition 4.1, using the new assumptions. Thanks to (H1') and using that $\langle \mathscr{T} h, h\rangle = 0$, the function h defined by (2.22) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|h\|^2 = \langle C h, h\rangle + \langle \mathscr{T} h, h\rangle \le -\mathrm{c}_{\mathscr{C}}\, \|h^\perp\|^2_\star\,.$$

This replaces the estimate of Lemma 4.2. We can then use (H2') and the above estimate to prove counterparts of estimates between Lemma 4.3 and Lemma 4.12 with $\|h^\perp\|$ and $\|h\|$ respectively replaced by $\|h^\perp\|_\star$ and $\|h\|_\star$. Then $\mathcal{F}_2$ defined in (4.39) using (4.20), with an appropriate choice of parameters $0 \ll \varepsilon_6 \ll \varepsilon_5 \ll \varepsilon_4 \ll \varepsilon_3 \ll \varepsilon_2 \ll \varepsilon_1 \ll 1$, is equivalent to $\|\cdot\|^2$ and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h] \leq -\kappa\,\overline{\mathcal{D}}_2[h]$$

for some constant $\kappa > 0$, where $\overline{\mathcal{D}}_2[h]$ is defined as $\mathcal{D}_2[h]$ in (4.40) with $\|h^\perp\|^2$ replaced by $\|h^\perp\|_\star^2$ in (4.22). This concludes the proof with

$$|||f - F|||^2_{\mathrm{L}^2(\mathcal{M}^{-1})} := \mathcal{F}_2[h]$$

since $\overline{\mathcal{D}}_2$ is equivalent to $\|\cdot\|_\star^2$. ■

B.3. *Proof of Corollary* B.2

Let $h_0 \in \mathrm{L}^2(\mathcal{M}^{1-\beta})$ and consider the solution $t \mapsto h(t)$ to the equation $\partial_t h = \mathcal{L}h$ with initial datum $h(0) = h_0$. Thanks to (H9'), there is some $C > 0$ such that

$$\forall\, t \geq 0\,, \quad \|h(t)\|_{1-\beta} \leq C\,\|h_0\|_{1-\beta}\,.$$

We now observe that, for any $R > 0$, the following interpolation inequality holds

$$\|g\|^2 \leq \left(1 + R^2\right)^{\alpha/2}\|g\|_\star^2 + \bar{\mu}^\beta(R)\,\|g\|_{1-\beta}^2$$

with $\bar{\mu}(R) := (2\pi)^{-d/2}\left\|e^{-\phi}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} e^{-R^2/2}$. Therefore, thanks to (B.1) and using fact that $h \mapsto |||f - F|||^2_{\mathrm{L}^2(\mathcal{M}^{-1})} = \mathcal{F}_2[h]$ is equivalent to $h \mapsto \|h\|^2$, one deduces that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_2[h(t)] &\leq -\Lambda\left(1 + R^2\right)^{-\alpha/2}\mathcal{F}_2[h(t)] + \lambda\left(1 + R^2\right)^{-\alpha/2}\bar{\mu}^\beta(R)\,\|h(t)\|_{1-\beta}^2\\ &\leq -\Lambda\left(1 + R^2\right)^{-\alpha/2}\mathcal{F}_2[h(t)] + \lambda\,C\left(1 + R^2\right)^{-\alpha/2}\bar{\mu}^\beta(R)\,\|h_0\|_{1-\beta}^2\end{aligned}$$

for any $R > 0$ and all $t \geq 0$, and for some positive constant Λ. This yields, for all $t \geq 0$,

$$\mathcal{F}_2[h(t)] \leq \exp\left(-\Lambda\left(1 + R^2\right)^{-\alpha/2} t\right)\mathcal{F}_2[h_0] + \frac{\lambda}{\Lambda}\,C\,\bar{\mu}^\beta(R)\,\|h_0\|_{1-\beta}^2\,.$$

Taking $R > 0$ such that $1 + R^2 = t^{2/(2+\alpha)}$, we obtain

$$\forall\, t \geq 0\,, \quad \mathcal{F}_2[h(t)] \lesssim \exp\left(-\Lambda\,t^{\frac{2}{2+\alpha}}\right)\|h_0\|_{1-\beta}^2\,,$$

which completes the proof. ■

B.4. *Comments and open questions*

In order to apply Theorem B.1 to the linearized Boltzmann and Landau operators with very soft potentials, one has to establish (H9'), which is so far an open question. Instead of proving *stretched exponential* decay rates as in Corollary B.2, *polynomial* decay rates could also be achieved with (H9') replaced, for some $k > 0$ large enough, by

$$t \mapsto e^{t\mathscr{L}} \textit{ is a strongly continuous uniformly bounded semigroup on } \mathrm{L}^2(\mathscr{H}^k \mathcal{M}^{-1}) \tag{H9''}$$

where $\mathscr{H}(x,v) = \phi(x) + \frac{1}{2}|v|^2 - \min_{\mathbb{R}^d}\phi$. Such a condition is also open in the case of the linearized Boltzmann and Landau operators with very soft potentials, but might be easier to prove in the spirit of [34, Appendix A].

Appendix C. Examples and remarks

C.1. *Examples of collision operators*

We list some examples of linear collision operators $\mathscr{C}$ satisfying the hypotheses of Theorem 1.1, in particular the spectral gap property (H1) and the bounded moment property (H2).

Example C.1 (The full linear Boltzmann operator). *Consider*

$$\mathscr{C}f := -\left(f - r_f\,\mathcal{M} - m_f\cdot v\,\mathcal{M} - e_f\,\mathfrak{E}\,\mathcal{M}\right)$$

where r_f, m_f and e_f are defined by

$$r_f(t,x) := \left(\int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v\right) e^{\phi(x)}\,, \qquad \textit{(local) density}\,,$$
$$m_f(t,x) := \left(\int_{\mathbb{R}^d} v\,f(t,x,v)\,\mathrm{d}v\right) e^{\phi(x)}\,, \qquad \textit{(local) momentum}\,,$$
$$e_f(t,x) := \left(\int_{\mathbb{R}^d} \mathfrak{E}(v)\,f(t,x,v)\,\mathrm{d}v\right) e^{\phi(x)}\,, \qquad \textit{(local) thermal energy}\,.$$

By construction, $\mathscr{C}$ satisfies the spectral gap condition (H1) *and since it is bounded, it satisfies also the bounded moment property* (H2)*.*

Example C.2 (The linearized Boltzmann collision operator). *Consider*

$$\mathscr{C}f := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left(f'\,\mu'_* + f'_*\,\mu' - f\,\mu_* - f_*\,\mu\right)|v - v_*|^\gamma\, b(\theta)\,\mathrm{d}\sigma\,\mathrm{d}v_*$$

with the notation $f' = f(v')$, $f_* = f(v_*)$ and $f'_* = f(v'_*)$ and

$$v' := \frac{v+v_*}{2} + \sigma\,\frac{|v-v_*|}{2}\,, \quad v'_* := \frac{v+v_*}{2} - \sigma\,\frac{|v-v_*|}{2}\,, \tag{C.1}$$

and θ is the deviation angle *defined by* $\cos\theta := \frac{(v-v_*)}{|v-v_*|} \cdot \sigma$*, and where* $\gamma \in (-d, +\infty)$*. We assume that b is positive, smooth away from* $\theta = 0$ *and bounded by* $b(\theta) \lesssim \theta^{-(d-1)-s}$ *with* $s \in [0, 2)$*. This framework includes the short-range so-called* hard spheres *interactions, as well as the long-range so-called* hard potentials *and* moderately soft potentials *interactions. This operator satisfies the spectral gap property* (H1) *when* $\gamma + s \geq 0$ *but only satisfies the weaker coercivity property* (H1') *with* $\alpha = \gamma + s$ *when* $\gamma + s < 0$ *(see* [2,35,37] *for quantitative estimates). It is in general not bounded on* $L^2(\mu^{-1})$*. Polynomials multiplied by* μ *are however in the domain of* $\mathscr{C}$ *and it satisfies the boundedness property* (H2)*.*

Example C.3 (The linearized Landau collision operator). *With same convention as in Example* C.2*, consider* $\mathscr{C} f = \mu C h$ *with* $h = f/\mu$ *and*

$$Ch := \nabla_v \cdot \left(\int_{\mathbb{R}^d} \mathsf{B}_\gamma(v, v_*) \left(\nabla h - \nabla h_*\right) \mu\, \mu_*\, \mathrm{d}v_* \right)$$

where the cross-section is defined by

$$\mathsf{B}_\gamma(v, v_*) := |v - v_*|^{\gamma+2} \left(\mathrm{Id} - \frac{v - v_*}{|v - v_*|} \otimes \frac{v - v_*}{|v - v_*|} \right)$$

with parameter $\gamma \in [-d, 1]$*. This operator is non-local, of order* 2 *in velocity (of* diffusive *type) and therefore not bounded. It satisfies the spectral gap condition* (H1) *when* $\gamma \in [-2, 1]$ *but only the weaker coercivity property* (H1') *with* $\alpha = \gamma + 2$ *when* $\gamma \in [-d, -2)$ *(see,* e.g., [2, 37] *for constructive estimates). Again all polynomials in velocity multiplied by* μ *are in its domain and it satisfies the boundedness property* (H2)*. Note that the main physical case, the linearisation of the so-called* Landau-Coulomb collision operator *(describing statistically collisions for a gas of electrons with Coulomb interactions) corresponds to* $\gamma = -3$ *in dimension* $d = 3$ *and is covered by our (extended) assumption* (H1')*.*

Remark C.4. *Examples* C.2 *and* C.3 *are obtained after a linearization of the bilinear form associated with the original nonlinear collision kernel around the Gaussian* μ *and not around the Maxwellian* $\mathcal{M}$*: when linearizing the full nonlinear inhomogeneous kinetic models around a Maxwellian* $\mathcal{M}$*, one gets an additional term* $\rho(x)$ *in front of the collision operator that goes to zero at infinity. We have not considered this degeneracy in the present paper: it is likely to create significant difficulties since there is then no uniform-in-x spectral gap for* $\rho\,\mathscr{C}$*.*

C.2. Examples of potentials

Let us discuss and illustrate the hypotheses (H5) and (H6) on the potential ϕ. The bounded moment hypothesis (H6) is not restrictive. Functions like $\phi(x) = \frac{d+5}{2} \ln(1 + |x|^2) - Z_\phi$ which are very slowly increasing at infinity satisfy this hypothesis, as well as fast increasing ones like $\phi(x) = e^{|x|^4} - Z_\phi$ (here Z_ϕ is the constant of normalization of $e^{-\phi}$ in L^1). Regarding the Poincaré inequality (H5), many works have been devoted to the study of sufficient conditions in order to guarantee the existence of a spectral gap. Here are some examples.

Example C.5. *The harmonic potential* $\phi(x) = \frac{1}{2}\,|x|^2 + \frac{d}{2}\log(2\,\pi)$ *satisfies the Poincaré inequality with constant* $c_P = 1$. *The inequality is equivalent to the spectral gap inequality for the operator* Ω *defined in* (1.15). *In the flat* L^2 *space, the change of unknown* $u = v\,e^{-\phi/2}$ *shows that the Poincaré inequality is also equivalent to the spectral gap inequality for the quantum harmonic oscillator operator* $-\Delta_x + \frac{1}{4}\,|x|^2 - \frac{d}{2}$.

Example C.6. *For a general* ϕ, *the change of unknown* $u = v\,e^{-\phi/2}$ *yields the following Schrödinger-type operator*

$$P_\phi = -\Delta_x + \frac{1}{4}\,|\nabla_x \phi|^2 - \frac{1}{2}\,\Delta_x \phi\,.$$

According to the so-called Bakry-Emery theory *(see for instance to* [1]*), there is a spectral gap as soon as the Hessian* $\nabla_x^2 \phi$ *is uniformly strictly positive at infinity. When it is uniformly strictly positive everywhere the following estimate is available on the spectral gap* c_P*:*

$$c_P \geq \frac{1}{2}\,\inf_{x\in\mathbb{R}^d} \lambda_1(\nabla_x^2\phi)$$

where $\lambda_1(\nabla_x^2\phi) > 0$ *is the lowest eigenvalue of* $\nabla_x^2\phi$.

Example C.7. *All potentials* ϕ *such that* P_ϕ *has compact resolvent satisfy the Poincaré inequality* (H5). *This happens in particular when*

$$\lim_{|x|\to\infty} \left(\tfrac{1}{4}\,|\nabla_x\phi|^2 - \tfrac{1}{2}\,\Delta_x\phi\right) = +\infty\,,$$

which is implied for instance by the stronger assumption

$$\lim_{|x|\to\infty} |\nabla_x\phi| = +\infty\,, \quad \text{and} \quad \lim_{|x|\to\infty} \frac{\Delta_x\phi(x)}{|\nabla_x\phi(x)|^2} = 0\,. \tag{C.2}$$

This is a standard result on Schrödinger operators, see for instance [38, Theorem XIII.67 p. 249], *and* 0 *is then a simple discrete eigenvalue. The argument in the latter reference is not constructive, and for a simpler constructive argument we refer for instance to* [46, Theorem A.1] *or the IMS truncation method in* [41].

Example C.8. *Here is an exotic example of potential that does not satisfy* (C.2) *nor the Bakry-Émery criterion (uniform convexity of* ϕ*) and for which the Poincaré inequality holds. Consider on* $\mathbb{R}^2$

$$\phi(x,y) = x^2\left(1+y^2\right)^2 - Z_\phi$$

where Z_ϕ *is the normalization constant so that* $\rho = e^{-\phi}$ *is a probability density. One can check that* P_ϕ *has a spectral gap, although* ϕ *is constant on the unbounded set* $\{x = 0\}$.

C.3. Change of coordinates

Let us discuss the reduction to the normalization (H7). Note that the formulas for Ker $\mathscr{C}$ are invariant by orthonormal change of coordinates in the velocity variable. By orthonormal change of coordinates in both the velocity and space variables, we can then reduce to the case when ϕ satisfies

$$\langle \nabla_x^2 \phi \rangle = \begin{pmatrix} p_1^2 & 0 & 0 & \cdots & 0 \\ 0 & p_2^2 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & p_d^2 \end{pmatrix}. \tag{C.3}$$

where we suppose without loss of generality that all p_j's are positive. The analysis of the present paper can be adapted to this case, including the main Theorem 1.1, with the following changes. We define the set of adapted centred rotational modes compatible with ϕ as in (1.8):

$$\mathfrak{R}_\phi = \left\{(x, v) \mapsto (A\, x \cdot v)\, \mathcal{M} \;:\; A \in \mathcal{R}_\phi\right\} . \tag{C.4}$$

We then choose orthonormal coordinates $x = (x_1, x_2, \dots, x_d)$ such that $\partial_{x_j}\phi = p_j^2\, x_j$ for some $p_j > 0$ if and only if $j \in I_\phi := \{d_\phi + 1, \dots, d\}$, and $x_j = 0$ for any $j \in I_\phi$ if $x \in E_\phi$ (the linear subspace defined in (1.9)). We define the set of *harmonic directional modes* by

$$\mathfrak{D}_\phi = \operatorname{Span}\left\{f_j^-(t, x, v)\,,\; f_j^+(t, x, v)\right\}_{j \in I_\phi} , \tag{C.5}$$

where

$$\begin{aligned} f_j^-(t, x, v) &:= \big(p_j\, x_j\, \cos(p_j\, t) - v_j\, \sin(p_j\, t)\big)\, \mathcal{M}(x, v)\,,\\ f_j^+(t, x, v) &:= \big(p_j\, x_j\, \sin(p_j\, t) + v_j\, \cos(p_j\, t)\big)\, \mathcal{M}(x, v)\,. \end{aligned}$$

If $d_\phi = 0$ and for some $p > 0$, $p_j = p$ for all $j \in \{1, \dots, d\}$, we define the set of *harmonic pulsating modes* by

$$\mathfrak{P}_\phi = \operatorname{Span}\left\{f^-(t, x, v)\,,\; f^+(t, x, v)\right\}$$

where

$$\begin{aligned} f^-(t, x, v) &:= \left(\tfrac{1}{2}\left(|p\,x|^2 - |v|^2\right)\cos(2\,p\,t) - p\,x \cdot v\, \sin(2\,p\,t)\right) \mathcal{M}(x, v)\,,\\ f^+(t, x, v) &:= \left(\tfrac{1}{2}\left(|p\,x|^2 - |v|^2\right)\sin(2\,p\,t) + p\,x \cdot v\, \cos(2\,p\,t)\right) \mathcal{M}(x, v)\,. \end{aligned}$$

The functions in $\mathfrak{R}_\phi$, $\mathfrak{D}_\phi$ and $\mathfrak{P}_\phi$ are *special macroscopic modes* of (1.1). With these definitions, the proof of Theorem 1.1 can be adapted to prove a hypocoercivity result taking into account all special macroscopic modes.

C.4. Spectral interpretation

We have focused so far on *real* solutions to (1.1), which is natural since physical solutions (densities of probability) are real valued. By considering complex solutions, we can interpret the results in terms of the *complex* spectrum of the nonnegative operator

$$-\mathscr{L} = v \cdot \nabla_x - \nabla_x \phi \cdot \nabla_v - \mathscr{C}$$

in $\mathrm{L}^2_{\mathbb{C}}(\mathcal{M}^{-1})$, the complexification of $\mathrm{L}^2(\mathcal{M}^{-1})$. We consider ϕ as in (C.3). We can then describe precisely the spectrum of $-\mathscr{L}$ and obtain resolvent estimates in a half-plane that includes the imaginary axis. Notice first that 0 is in the spectrum of $-\mathscr{L}$ with associated eigenspace

$$\operatorname*{Span}_{\mathbb{C}}(\mathcal{M}) \oplus \operatorname*{Span}_{\mathbb{C}}(\mathcal{H}\,\mathcal{M}) \oplus \mathfrak{R}_{\phi,\mathbb{C}}$$

where $\mathfrak{R}_{\phi,\mathbb{C}}$ is the set of rotation modes as defined in (C.4) but extended to the corresponding $\mathbb{C}$-vector space. This set is then of (complex) dimension $2+\dim(\mathfrak{R}_\phi)$. Depending on the harmonicity of ϕ we have three cases which are summarized in Figure 1.

(a) Case with no harmonic modes ($d_\phi = d$). In this case ϕ has no harmonic directions and there no non-zero eigenvalue on the imaginary axis.

(b) Case with harmonic directional modes but no pulsating modes ($1 \le d_\phi \le d-1$). In this case, the *real* vector space of functions $\mathfrak{D}_\phi$ in (C.5) yields the complex set

$$\mathfrak{D}_{\phi,\mathbb{C}} = \operatorname*{Span}_{\mathbb{C}} \left\{ \left(p_j\,x_j - i\,v_j\right) e^{-i\,p_j\,t}\mathcal{M}(x,v)\,, \left(p_j\,x_j + i\,v_j\right) e^{i\,p_j\,t}\mathcal{M}(x,v)\right\}_{j\in I_\phi}$$

where $I_\phi := \left\{d_\phi+1,\ldots,d\right\}$, to which we can associate the eigenfunctions of $(-\mathscr{L})$ corresponding to the eigenvalues $\mp\,i\,p_j$ and given by

$$(x,v) \mapsto f_j^{\pm}(x,v) = (p_j\,x_j \pm i\,v_j)\mathcal{M}(x,v)\,.$$

(c) Case with harmonic directional and pulsating modes ($d_\phi = 0$). In this last case necessarily all p_j's are equal to a common value $p>0$ and $\phi(x) = \frac{1}{2}|p\,x|^2 + \frac{d}{2}\log(2\,\pi) - d\log(p)$. All possible harmonic directional modes exist, as well as all possible infinitesimal rotational modes $\mathfrak{R}_{\phi,\mathbb{C}}$ with $\mathcal{R}_\phi = \mathfrak{M}^{\text{skew}}_{d\times d}(\mathbb{C})$. The complexification of the set $\mathfrak{P}_\phi$ defined in (C.5) is

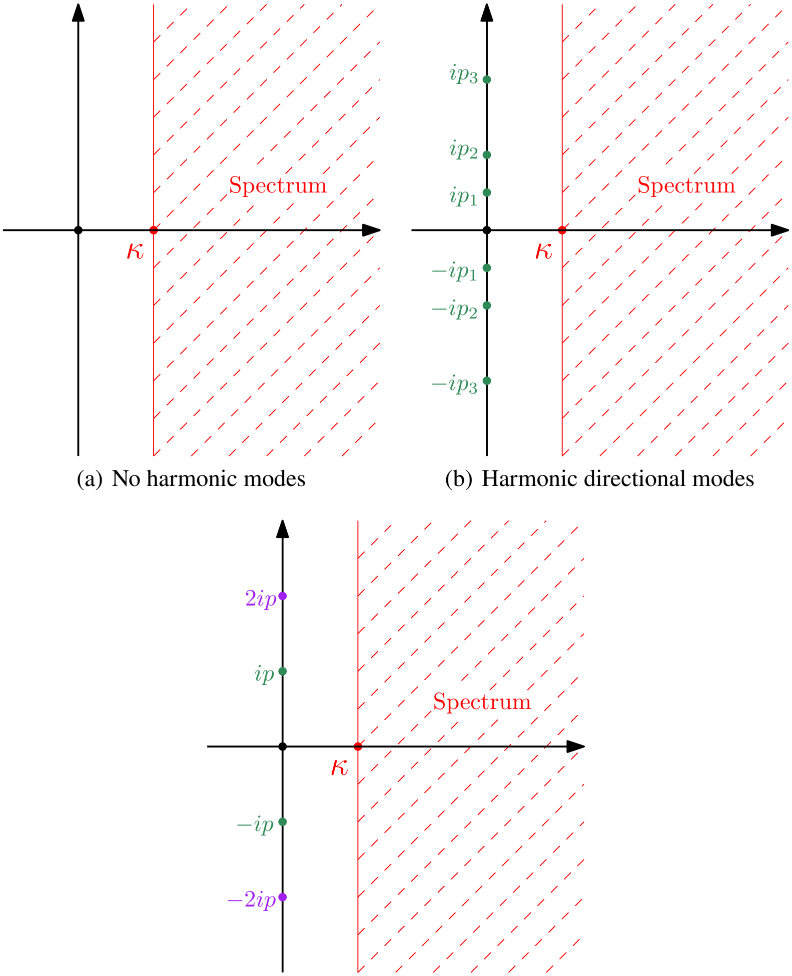
$$\mathfrak{P}_{\phi,\mathbb{C}} = \operatorname{Span}\left\{e^{2\,i\,p\,t}\,f^{+}(x,v)\,, e^{-2\,i\,p\,t}\,f^{-}(x,v)\right\}$$

where $f^{\pm}(x,v) := \left[p\,x\cdot v \pm \frac{i}{2}\left(|p\,x|^2 - |v|^2\right)\right]\mathcal{M}(x,v)$ are eigenfunctions of $(-\mathscr{L})$ with eigenvalues $\pm\,2\,i\,p$.

The analysis of the paper can be extended to the complex Hilbertian space $\mathrm{L}^2_{\mathbb{C}}(\mathcal{M}^{-1})$ with a set of *special macroscopic modes* defined by

$$\mathcal{S} := \operatorname*{Span}_{\mathbb{C}}(\mathcal{M}) \oplus \operatorname*{Span}_{\mathbb{C}}(\mathcal{H}\,\mathcal{M}) \oplus \mathfrak{R}_{\phi,\mathbb{C}} \oplus \operatorname{Span}\left\{f_j^{\pm}\right\}_{j\in I_\phi} \oplus \operatorname{Span}\left\{f^{\pm}\right\}$$

where the f_j's and the $f^{\pm}$'s are defined above (when ϕ has the relevant harmonicity). Let $\mathcal{S}^{\perp}$ be the orthogonal of $\mathcal{S}$ in $\mathrm{L}^2_{\mathbb{C}}(\mathcal{M}^{-1})$. We note that since $\mathscr{L}$ is a real operator, both $\mathcal{S}$



(a) No harmonic modes

(b) Harmonic directional modes

(c) Harmonic directional and pulsating modes

Fig. 1. Complex spectrum of $-\mathscr{L}$.

and $\mathcal{S}^\perp$ are stable by conjugation and therefore stable by $\mathscr{L}$ and $\mathscr{L}^*$. Using the Laplace transform, we obtain from Theorem 1.1 the following resolvent estimate for $-\mathscr{L}|_{\mathcal{S}^\perp}$:

$$\forall\, z \in \mathbb{C} \quad \text{with} \quad \Re(z) < \kappa\,, \quad \left\| (z\mathrm{Id} + \mathscr{L}|_{\mathcal{S}^\perp})^{-1} \right\|_{\mathcal{B}(\mathcal{S}^\perp)} \le \frac{\tilde{C}}{\kappa - \Re(z)}$$

where $\tilde{C}$ is an explicit constant depending on κ and C in Theorem 1.1 and $\|\cdot\|_{\mathcal{B}(\mathcal{S}^\perp)}$ stands for the operator norm on $\mathcal{S}^\perp$. The provides the resolvent estimates in the left half-planes in Figure 1.

C.5. Special macroscopic modes for the full nonlinear Boltzmann equation

The *special macroscopic modes* which minimize the entropy for the full nonlinear Boltzmann equation are the nonlinear counterparts to the linearized special macroscopic modes studied in the present paper. They appear for the first time in the literature in Boltzmann's paper [5] as mentioned in the introduction. The full nonlinear inhomogeneous Boltzmann equation is

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \phi \cdot \nabla_v F = \partial_t F + \mathscr{T} F = \mathscr{Q}(F, F) \tag{C.6}$$

where, with the classical notations $F' = F(v')$, $F_* = F(v_*)$ and $F'_* = F(v'_*)$ associated to elastic collisions $(v, v_*) \mapsto (v', v'_*)$, such that the microscopic conservation of momentum $v' + v'_* = v + v_*$ and energy $|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2$ hold, the Boltzmann collision operator writes

$$\mathscr{Q}(F, F) := \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathsf{B}(v - v_*, \sigma) \left(F' F'_* - F F_* \right) \mathrm{d}\sigma \, \mathrm{d}v_* \, .$$

Here $\mathsf{B} \geq 0$ is the *cross-section*. We refer to [10] for more details. Let us assume the normalization (H7) on ϕ. We consider the functions in the space $\mathcal{S}$ of *special macroscopic modes* generated by

▷ the set $\mathfrak{R}_\phi$ of *rotation modes compatible with* ϕ if ϕ admits any,
▷ the set $\mathfrak{D}_\phi$ of *harmonic directional modes* if ϕ has harmonic directions,
▷ the set $\mathfrak{P}_\phi$ of *harmonic pulsating modes* if ϕ is fully harmonic.

For any $f \in \mathcal{S}$, the function $F(t,x,v) := e^{h(t,x,v)} \mathcal{M}(x,v)$ with $h = f/\mathcal{M}$ is a time-periodic solution to (C.6). Indeed $h(t,x,\cdot)$ is a linear combination of 1, v_i, $i \in \{1, \ldots, d\}$ and $|v|^2$ for each t, x, and therefore the microscopic conservation of momentum and energy imply

$$\forall\, t, x, v, \sigma, \quad h(t,x,v') + h(t,x,v'_*) = h(t,x,v) + h(t,x,v_*)$$

where the four velocities v, v_*, v', v'_* satisfy (C.1). This proves the identity $e^{h'} \mathcal{M}' \, e^{h'_*} \mathcal{M}'_* = e^h \mathcal{M} \, e^{h_*} \mathcal{M}_*$ and thus $\mathscr{Q}(e^h \mathcal{M}, e^h \mathcal{M}) = 0$. Finally we obtain $\mathscr{T}(e^h \mathcal{M}) = \mathscr{T}(e^h) \mathcal{M} + e^h \mathscr{T}(\mathcal{M}) = e^h [\mathscr{T}(h) \mathcal{M} + \mathscr{T}(\mathcal{M})] = 0$, where we have used that $\mathscr{T}$ is a first order operator and $\mathscr{T}(\mathcal{M}) = \mathscr{T}(h) = 0$ as calculated before.

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