Defocusing Nonlinear Schrödinger equation: confinement, stability and asymptotic stability

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Abstract

This paper is devoted to the asymptotic properties of the Nonlinear Schrödinger equation in the defocusing case. We recover dispersion rates by the mean of time-dependent rescalings in the power law case and prove a new result in the case of the logarithmic Nonlinear Schrödinger equation. The rescaled equation is then used to obtain an asymptotic result of nonlinear stability, which is the main result of this paper.

Keywords. Nonlinear Schrödinger equation – Dispersion –Pseudo-conformal law – Time-dependent rescalings – Stability – Large time asymptotics – Minimzers – Relative entropy – Csiszár-Kullback inequality

1 Introduction and main results

In this paper, we prove decay estimates for the defocusing nonlinear Schödinger equation (NLS) by the mean of time-dependent rescalings. Although the results can be extended to general nonlinearities under appropriate assumptions, we shall mainly work with power law nonlinearities. These decay rates are known by the mean of other methods but time-dependent rescalings turn out to be a really simple tool for getting such rates, as we shall see on the example of the logarithmic NLS.

Let us consider the defocusing nonlinear Schrödinger equation (NLS)

$$i\psi_t = -\Delta\psi + |\psi|^{p-1}\psi \tag{1}$$

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in $\mathbb{R}^+ \times \mathbb{R}^d$ where $\psi = \psi(t,x)$ is a complex valued function, $1 < p < 1 + 4/(d-2)$ for $d > 2$ and $1 < p < +\infty$ for $d = 1, 2$. We assume that at $t = 0$

$$\psi(0,\cdot) = \psi_0 \in H^1(\mathbb{R}^d) \ . \tag{2}$$

Our first result is the following.

Theorem 1 *Assume that $1 < p < 1+4/(d-2)$ if $d \geq 3$ and $p \in (1, +\infty)$ if $d = 1$ or 2, and let ψ be a solution of (1) corresponding to an initial data $\psi_0 \in H^1(\mathbb{R}^d)$ such that $(1+|\cdot|^2)^{1/2}\,\psi_0 \in L^2(\mathbb{R}^d)$. Let r be such that $2 \leq r \leq \infty$ if $d = 1$, $2 \leq r < \infty$ if $d = 2$ and $2 \leq r \leq 2 + \dfrac{4}{d-2}$ if $d \geq 3$. Then there exists a constant $C > 0$ such that, if R is the solution of*

$$\ddot{R}R = R^{-c_p - 1} \quad ith\ c_p = \min\left(\frac{d}{2}(p-1), 2\right) , \quad R(0) = 1 \ , \quad \dot{R}(0) = 0 \ ,$$

then

$$\|\psi(t,\cdot)\|_{L^r(\mathbb{R}^d)} \leq C R(t)^{-d\left(\frac{1}{2}-\frac{1}{r}\right)(1-\epsilon)} \quad \forall\, t \geq 0 \ ,$$

where $\epsilon = \epsilon(r,p)$ is such that $\epsilon = 0$ if $p \in [1 + \frac{4}{d}, 1 + 4/(d-2))$ is critical or supercritical, $\epsilon = 0$ if $2 \leq r \leq p+1$, and $\epsilon = \frac{(r-(p+1))(4-d(p-1))}{(r-2)(4-d(p-1)+2(p-1))}$ if $r > p+1$ and $1 < p < 1 + \frac{4}{d}$. Moreover, C depends only on d, p, r and

$$E_0 := \frac{1}{2}\|\nabla\psi_0\|^2_{L^2} + \frac{1}{4}\|\,|x|\,\psi_0\|^2_{L^2} + \frac{1}{p+1}\|\psi_0\|^{p+1}_{L^{p+1}} \ .$$

The method consists in rescaling the equation by the mean of a time-dependent rescaling and to use the energy of the rescaled equation to obtain by interpolation decay estimates in appropriate norms. The asymptotic rate is usually obtained directly by using the pseudo-conformal law [2, 4] and the above result was actually partially proved in [10], under a slightly different point of view: look for a time-dependent change of coordinates which preserves the galilean invariance (an inertial motion remains inertial after the change of variables) and build directly a Lyapunov functional by an appropriate ansatz. This Lyapunov functional is of course the energy of the rescaled equation.

Our purpose here is to study with more details the rescaled wave function and its energy. It turns out that the method provides rates which are apparently entirely new in the limiting case of the logarithmic nonlinear Schrödinger equation.

Because of the reversibility of the Schrödinger equation and standard results of scattering theory, one cannot expect the convergence of the rescaled wave function to some *a priori* given limiting wave function, but it turns out that some convexity properties of the energy can be used to state an asymptotic stability result which is the main issue of this paper.

Theorem 2 *With the same notations as in Theorem 1, in the subcritical case with $p < 3$, there exists a wave function ψ_∞ which takes the form*

$$\psi_\infty(t,x) = \phi_\infty\left(\frac{x}{R}\right)$$

with $R = R(t)$, such that

$$\|\,|\psi(t,\cdot)|^2 - |\psi_\infty(t,\cdot)|^2\,\|_{L^{r/2}(\mathbb{R}^d)} \leq C R(t)^{-d\left(\frac{1}{2}-\frac{1}{r}\right)(1-\epsilon)} \quad \forall\, t \geq 0 \ ,$$

for a constant $C > 0$, which can be made arbitrarily small for an appropriate initial data ψ_0. The expression of ϕ_∞ is given in Proposition 1.

The rest of this paper is organized as follows. In Section 2, we introduce a time-dependent rescaling and prove Theorem 1. Section 3 is devoted to considerations on convexity and nonlinear stability, which are then applied to the rescaled equation. Here we shall use Csiszár-Kullback type inequalities. The last section is devoted estimates of rates of dispersion for the logarithmic nonlinear Schrödinger equation.

Notations. Unless it is explicitely specified integrals and norms are taken over $\mathbb{R}^d$.

2 Decay estimates

¿From the general theory of Schrödinger equations [2, 4], it is well known that the Cauchy problem (1)-(2) is well posed for any initial data in $H^1(\mathbb{R}^d)$ when $1 < p < 1 + 4/(d-2)$ ($1 < p < \infty$ if $d = 1, 2$) and that the solution ψ belongs to $\in C(\mathbb{R}^+, H^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}^+, H^{-1}(\mathbb{R}^d))$. As usual for Schrödinger equations, we shall distinguish three cases:

(i) *subcritical* when $p < 1 + 4/d$,

(ii) *critical* (or pseudo-conformal invariant case) when $p = 1 + 4/d$,

(iii) *supercritical* when $p > 1 + 4/d$.

To investigate the large time behaviour ofψ, we are going to use time-dependent rescalings and time-dependent rescalings are indeed clearly related to the pseudo-conformal invariance law, at least in the critical case. Let ϕ be such that

$$\psi(t,x) = R(t)^{-\frac{d}{2}} e^{i\,S(t)\,\frac{|x|^2}{2}}\,\phi\left(\tau(t), \frac{x}{R(t)}\right)$$

where R, S and τ are positive derivable real functions of the time. It is traightforward to check that with this change of coordinates, ϕ satisfies the following equation

$$i\dot\tau\phi_\tau = -\tfrac{1}{R^2}\Delta\phi + R^{-\frac{d}{2}(p-1)}|\phi|^{p-1}\phi + \tfrac{R^2}{2}(\dot S + 2S^2)|\xi|^2\phi + i\left(\tfrac{\dot R}{R} - 2S\right)\left(\tfrac{d}{2}\,\phi + \xi\cdot\nabla\phi\right),$$

where $\dot{} = \frac{d}{dt}$. With the choice $S = \dot R/2R$, which means $\dot S = \ddot R/2R - 2S^2$, ϕ and ψ are related by

$$\psi(t,x) = R^{-\frac{d}{2}} e^{\frac{i}{4}\frac{\dot R}{R}|x|^2}\,\phi\left(\tau, \frac{x}{R}\right) \Longleftrightarrow \phi(\tau,\xi) = R^{\frac{d}{2}} e^{-\frac{i}{4}\dot R R|\xi|^2}\,\psi(t, R\xi) \quad (3)$$

where $\tau = \tau(t)$ and $\xi = x/R(t)$, and ϕ has to satisfy the following time-dependent defocusing nonlinear Schrödinger equation

$$i\dot\tau\phi_\tau = -\frac{1}{R^2}\,\Delta\phi + R^{-\frac{d}{2}(p-1)}\,|\phi|^{p-1}\phi + \frac{1}{4}\ddot R R|\xi|^2\phi\,.$$

Note that $|\psi(x,t)| = R^{-d/2}\,|\phi\,(\tau, x/R)|$, so that $\|\psi(t,\cdot)\|_{L^2} = \|\phi(\tau(t),\cdot)\|_{L^2} = \|\psi_0\|_{L^2}$ for all $t \geq 0$. Also note that if $\dot R(0) = \tau(0) = 0$ and $R(0) = 1$, then

$$\phi(0,\cdot) = \phi(0,\cdot) = \psi_0 \ . \tag{4}$$

To extract the dominant effects as $t \to +\infty$, we fix τ and R such that

$$\dot\tau = \frac{1}{2}\ddot R R = R^{-c_p} \quad \text{where } c_p = \min\,(\frac{d}{2}\,(p-1), 2)\ . \tag{5}$$

This ansatz is indeed the only one that sets to 1 at least three of the four coefficients in the equation for ϕ, with $\lim_{t\to+\infty} R(t) = +\infty$. Thus $c_p = d(p-1)/2$ if p is subcritical and $c_p = 2$ if p is critical or supercritical, and ϕ solves the equation

$$i\phi_\tau = -R^{c_p-2}\Delta\phi + R^{c_p-\frac{d}{2}(p-1)}|\phi|^{p-1}\phi + \frac{1}{2}|\xi|^2\,\phi\ . \tag{6}$$

Note that $c_p - 2 = 0$ when p is critical or supercritical and $c_p - \frac{d}{2}(p-1) = 0$ when p subcritical or critical.

With the choice $R(0) = 1$ and $\dot R(0) = 0$, one integration of (5) with respect to t gives $\dot R^2 = 4(1 - R(t)^{-c_p})/c_p$ and this is possible if, and only if, $R \geq 1$ for all $t \geq 0$. The function $t \mapsto R(t)$ is therefore globally defined on $\mathbb{R}^+$, increasing, $\lim_{t\to+\infty} R(t) = 2\sqrt{c_p}$ and $R(t) \sim t$ as $t \to +\infty$. Assuming that $\tau(0) = 0$, the rescaled time τ is an increasing positive function such that

$$\lim_{t\to+\infty} \tau(t) = \tau_\infty > 0$$

where $\tau_\infty = +\infty$ if $p \leq 1 + 2/d$ and $\tau_\infty < +\infty$ if $p > 1 + 2/d$.

Consider now the energy functional associated to equation (6)

$$E(\tau) = \frac{R^{c_p-2}}{2}\int_{\mathbb{R}^d}|\nabla\phi|^2 d\xi + \frac{1}{4}\int_{\mathbb{R}^d}|\xi|^2|\phi|^2 d\xi + \frac{R^{c_p-\frac{d}{2}(p-1)}}{p+1}\int_{\mathbb{R}^d}|\phi|^{p+1} d\xi \tag{7}$$

where R has to be understood as a function of τ. Here we shall use the notation $' = \frac{d}{d\tau}$.

Lemma 1 *Assume that $1 < p < 1 + 4/(d-2)$ if $d \geq 3$ and $p \in (1,+\infty)$ if $d = 1$ or 2, and let ψ be a solution of (1) corresponding to an initial data $\psi_0 \in H^1(\mathbb{R}^d)$ such that $(1 + |\cdot|^2)^{1/2}\psi_0 \in L^2(\mathbb{R}^d)$. With the above notations, E is a decreasing positive functional satisfying*

$$E' = -\dot R R^{2c_p-1}\left(\tfrac{2-c_p}{2R^2}\int_{\mathbb{R}^d}|\nabla\phi|^2 d\xi + \tfrac{d\,(p-1)-2\,c_p}{2\,(p+1)\,R^{d\,(p-1)/2}}\int_{\mathbb{R}^d}|\phi|^{p+1} d\xi\right)\ .$$

For any $\tau \in [0,\tau_0)$, $E(\tau)$ is therefore bounded by $E(0) = E_0$ (with the notations of Theorem 1).

Proof. The proof follows by a direct computation. Because of (6), only the coefficients of $\int_{\mathbb{R}^d}|\nabla\phi|^2 d\xi$ and $\int_{\mathbb{R}^d}|\phi|^{p+1} d\xi$ contribute to the decay of the energy. □

Using Lemma 1, we recover the classical dispersion rates (see Theorem 7.2.1 of [4]) for dispersive NLS. Moreover, we obtain constants which can be explicitely expressed in terms of quantities depending only on the initial data.

Proof of Theorem 1. Assume first that p is critical or supercritical. By Lemma 1 and according to the time-dependent rescaling (3),

$$R^2 \int_{\mathbb{R}^d} |\nabla\psi - \frac{i\dot{R}}{2R}\psi x|^2 \, dx = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2 \, d\xi \le E(\tau) \le E(0)$$

Thus

$$\int_{\mathbb{R}^d} |\nabla|\psi|\,|^2 \, dx = \int_{\mathbb{R}^d} |\nabla|e^{-\frac{i\dot{R}}{2R}|x|^2}\psi|\,|^2 \, dx \le \int_{\mathbb{R}^d} |\nabla\psi - \frac{i\dot{R}}{2R}\,x\,\psi|^2 \, dx$$

is bounded by $2E(0)R(t)^{-2}$. The rest of the proof follows the same lines that in Theorem 7.2.1 of [4], using the conservation of the L^2-norm and the Sobolev-Gagliardo-Nirenberg inequality.

If p is subcritical, it follows from Lemma 1 and from the rescaling (3) that the two following estimates hold:

$$\frac{1}{p+1}\,R^{d(p+1)\left(\frac{1}{2}-\frac{1}{p+1}\right)}\int_{\mathbb{R}^d} |\psi|^{p+1} dx = \frac{1}{p+1}\int_{\mathbb{R}^d} |\phi|^{p+1} d\xi \le E(0)\ ,$$

$$\frac{1}{2}R^{\frac{d}{2}(p-1)}\int_{\mathbb{R}^d} |\nabla\psi - \frac{i\dot{R}}{2R}\,x\,\psi|^2 \, dx = \frac{1}{2}\,R^{c_p-2}\int_{\mathbb{R}^d} |\nabla\phi|^2 \, d\xi \le E(0)\ .$$

Up to slightly different interpolations, the method is then the same as in the supercritical case. □

The method of time-dependent rescalings does not improve substantially known rates based on pseudo-conformal identities in the case of power law nonlinearities but it gives precised estimates (explicit constants). It can easily be extended to general convex nonlinearities and it is well suited for the understanding of limiting cases, like the logarithmic nonlinear Schrödinger equation (see Section 4). However, the full interest of time-dependent rescalings will be made clear in the next section.

3 Convexity and nonlinear stability

In this Section, we shall first establish convexity properties of functionals related to the energy when they are written in terms of the density function (the square of the modulus of the wave function). We shall also use Csiszár-Kullback type inequalities to control usual norms in terms of the nonlinearity. These estimates are then applied to solutions of nonlinear Schrödinger equations to get stability results. As a special case, when applied to the equation obtained by the mean of time-dependent rescalings from NLS, they give asymptotic stability results.

3.1 Convexity

In this section, we assume that V is a nonnegative potential such that

$$\lim_{r\to+\infty} \operatorname{infess}_{|x|>r} V(x) = +\infty \qquad (8)$$

(less restrictive conditions on V could be given, but this one is sufficient at least for the study of the asymptotic stability). Consider a critical point of the functional

$$E[\phi] := \frac{A}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2\, dx + \frac{B}{p+1}\int_{\mathbb{R}^d} |\phi|^{p+1}\, dx + \frac{1}{2}\int_{\mathbb{R}^d} V(x)|\phi|^2\, dx$$

with $A,\ B > 0$, under the constraint $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$. As we shall see below, for $p \in [1, 2^* = \frac{d+2}{d-2})$, there is a real critical point ϕ_∞, which has to satisfy:

$$-A\,\Delta\phi_\infty + B\,|\phi_\infty|^{p-1}\phi_\infty + V(x)\phi_\infty - \lambda\phi_\infty = 0 \tag{9}$$

for some Lagrange multiplier λ. Multiplying (9) by ϕ_∞, an integration by parts shows that λ is positive. The existence of such a critical point holds for any $p \in [1, 2^*)$. We shall indeed see below that existence and uniqueness up to a sign change of a real solution follow from elementary considerations on appropriate functionals. Since ϕ_∞ is a minimizer of the quotient $E[\phi]/\|\phi\|_{L^2(\mathbb{R}^d)}$, we may assume that it is nonnegative. By the Maximum Principle applied to the operator $(-\Delta + \lambda)$, the solution ϕ_∞ has to be positive. We may also remark that if V is radial and increasing, using [5], one gets that ϕ_∞ is radial and strictly decreasing.

Next, it is clear that ϕ_∞ realizes the minimum of the functional

$$G[\phi] := F[\phi] - F[\phi_\infty]\ ,$$

where $F[\phi] := E[\phi] - \frac{\lambda}{2}\int_{\mathbb{R}^d} |\phi|^2\, dx$. The functional G can be rewritten as

$$G[\phi] := F[\phi] - F[\phi_\infty] - DF[\phi_\infty]\cdot(\phi - \phi_\infty)$$

using the fact that $DF[\phi_\infty] = 0$. Up to an integration by part, we get

$$\begin{aligned} G[\phi] = \ & \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\phi - \nabla\phi_\infty|^2\, dx \\ & + \tfrac{B}{p+1}\int_{\mathbb{R}^d} \left(|\phi|^{p+1} - |\phi_\infty|^{p+1} - (p+1)|\phi_\infty|^{p-1}\phi_\infty\cdot(\phi - \phi_\infty)\right) dx \\ & + \tfrac{1}{2}\int_{\mathbb{R}^d} V(x)|\phi - \phi_\infty|^2\, dx - \tfrac{\lambda}{2}\int_{\mathbb{R}^d} |\phi - \phi_\infty|^2\, dx\ . \end{aligned}$$

Because of the term proportional to $|\phi - \phi_\infty|^2$, such a functional is not convex with respect to $\phi - \phi_\infty$, so we shall introduce $\rho = |\phi|^2$ and $\rho_\infty = |\phi_\infty|^2$ and measure the stability in terms of $\rho - \rho_\infty$. Let

$$\mathcal{F}[\rho] = \frac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2\, dx + \frac{B}{p+1}\int_{\mathbb{R}^d} \rho^{\frac{p+1}{2}}\, dx + \frac{1}{2}\int_{\mathbb{R}^d} V\rho\, dx - \frac{\lambda}{2}\int_{\mathbb{R}^d} \rho\, dx$$

and consider $\rho_t = t\rho_1 + (1-t)\rho_2$. Since

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} \frac{|\nabla\rho_t|^2}{\rho_t}\, dx = \int_{\mathbb{R}^d} \frac{2}{\rho_t^3}\, |\rho_t\nabla(\rho_2 - \rho_1) - (\rho_2 - \rho_1)\nabla\rho_t|^2\, dx > 0\ .$$

As a consequence, $\mathcal{F}$ is a strictly convex functional whose unique minimizer ρ_∞ solves

$$-A\,\frac{\Delta(\sqrt{\rho_\infty})}{\sqrt{\rho_\infty}} + B\,\rho_\infty^{\frac{p-1}{2}} + V(x) - \lambda = 0\ .$$

Next,

$$\begin{aligned}
\mathcal{G}[\rho] &= \mathcal{F}[\rho] - \mathcal{F}[\rho_\infty] \\
&= \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2\,dx + \tfrac{B}{p+1}\int_{\mathbb{R}^d} \rho^{\frac{p+1}{2}}\,dx \\
&\qquad - \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}_\infty|^2\,dx - \tfrac{B}{p+1}\int_{\mathbb{R}^d} \rho_\infty^{\frac{p+1}{2}}\,dx \\
&\qquad + \tfrac{1}{2}\int_{\mathbb{R}^d} (V-\lambda)(\rho-\rho_\infty)\,dx \\
&= \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2\,dx + \tfrac{B}{p+1}\int_{\mathbb{R}^d} \rho^{\frac{p+1}{2}}\,dx \\
&\qquad - \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}_\infty|^2\,dx - \tfrac{B}{p+1}\int_{\mathbb{R}^d} \rho_\infty^{\frac{p+1}{2}}\,dx \\
&\qquad + \tfrac{1}{2}\int_{\mathbb{R}^d} \left(A\,\tfrac{\Delta\sqrt{\rho_\infty}}{\sqrt{\rho_\infty}} - B\,\rho_\infty{}^{\frac{p-1}{2}}\right)(\rho-\rho_\infty)\,dx \\
&= \tfrac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho} - \sqrt{\tfrac{\rho}{\rho_\infty}}\,\nabla\sqrt{\rho_\infty}|^2\,dx \\
&\qquad + \tfrac{B}{p+1}\int_{\mathbb{R}^d} \left(\rho^{\frac{p+1}{2}} - \rho_\infty^{\frac{p+1}{2}} - \tfrac{p+1}{2}\rho_\infty^{\frac{p-1}{2}}(\rho-\rho_\infty)\right)dx\,.
\end{aligned}$$

This is because

$$\begin{aligned}
|\nabla\sqrt{\rho}|^2 - |\nabla\sqrt{\rho_\infty}|^2 - \nabla\sqrt{\rho_\infty}\cdot\nabla\left(\tfrac{\rho-\rho_\infty}{\sqrt{\rho_\infty}}\right) \\
&= |\nabla\sqrt{\rho}|^2 - 2\sqrt{\tfrac{\rho}{\rho_\infty}}\,\nabla\sqrt{\rho}\cdot\nabla\sqrt{\rho_\infty} + \tfrac{\rho}{\rho_\infty}|\nabla\sqrt{\rho_\infty}|^2 \\
&= |\nabla\sqrt{\rho} - \sqrt{\tfrac{\rho}{\rho_\infty}}\,\nabla\sqrt{\rho_\infty}|^2 \\
&= \rho_\infty\,|\nabla(\sqrt{\tfrac{\rho}{\rho_\infty}})|^2
\end{aligned} \tag{10}$$

Since $\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2\,dx \le \int_{\mathbb{R}^d} |\nabla\phi|^2\,dx$ for $\rho = |\phi|^2$ (*i.e.* $\int_{\mathbb{R}^d} (\nabla|\phi|\,)^2\,dx \le \int_{\mathbb{R}^d} |\nabla\phi|^2\,dx$), then $F[\phi] \ge \mathcal{F}[\rho]$, which means that

$$\mathcal{H}[\phi] := F[\phi] - \mathcal{F}[\rho_\infty] \ge \mathcal{G}[\rho]\ .$$

Up to now, we did not establish an existence result for ϕ_∞ or ρ_∞, but this also follows from the study of the energy functionals. Let

$$\mathcal{E}[\rho] = \frac{A}{2}\int_{\mathbb{R}^d} |\nabla\sqrt{\rho}|^2\,dx + \frac{B}{p+1}\int_{\mathbb{R}^d} \rho^{\frac{p+1}{2}}\,dx + \frac{1}{2}\int_{\mathbb{R}^d} V\rho\,dx\ .$$

Theorem 3 *Assume that A and B are positive constants, $d \ge 1$, $p \in [1, 2^*)$. Assume that V is a nonnegative potential defined on $\mathbb{R}^d$ which satisfies (8). Then there exists a unique up to a constant phase factor (resp. unique) minimizer of E (resp. $\mathcal{E}$) under the constraint $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$ (resp. $\rho \ge 0$, $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$) that we shall denote by ϕ_∞ (resp. ρ_∞).*

Moreover, for any ϕ such that $\|\phi\|_{L^2(\mathbb{R}^d)} = 1$ (resp. for any nonnegative ρ such that $\|\rho\|_{L^1(\mathbb{R}^d)} = 1$),

$$\begin{aligned} E[\phi] - E[\phi_\infty] \geq \;& \mathcal{E}[\rho] - \mathcal{E}[\rho_\infty] \\ &= \tfrac{A}{2} \int_{\mathbb{R}^d} |\nabla\sqrt{\rho} - \sqrt{\tfrac{\rho}{\rho_\infty}}\,\nabla\sqrt{\rho_\infty}|^2\,dx \\ &\quad + \tfrac{B}{p+1} \int_{\mathbb{R}^d} \left(\rho^{\frac{p+1}{2}} - \rho_\infty^{\frac{p+1}{2}} - \tfrac{p+1}{2}\rho_\infty^{\frac{p-1}{2}}(\rho - \rho_\infty)\right) dx \end{aligned} \tag{11}$$

where the above estimate holds provided $\rho = |\phi|^2$.

Proof. The existence is proved by a direct minimization method. For a minimizing sequence $(\phi_n)_{n\in\mathbb{N}}$, $\rho_n = |\phi_n|^2$ weakly converges in $L^1(\mathbb{R}^d)$ because of Dunford-Pettis' criterion. Concentration is indeed forbidden by the term $\int_{\mathbb{R}^d} |\nabla\sqrt{\rho_n}|^2\,dx$ provided $p < 2^*$ and vanishing is avoided because of the growth of V as $|x| \to +\infty$. Uniqueness is a consequence of the strict convexity of $\mathcal{F}$. □

The limiting cases: $A = 0$, $B > 0$ and $A > 0$, $B = 0$, also enter in this framework.

Proposition 1 *Consider the two limiting cases of Theorem 3.*

Case $A = 0$, $B > 0$: *let ϕ_∞ and ρ_∞ be defined by*

$$\phi_\infty(x) = \sqrt{\rho_\infty(x)} = \left[\frac{1}{B}(\lambda - V(x))\right]^{\frac{2}{p-1}} .$$

Case $A > 0$, $B = 0$: *let $\phi_\infty = \sqrt{\rho_\infty}$ be the first eigenfunction of the operator $(-A\,\Delta + V)$.*

In both cases, with the same notations as above, Estimate (11) also holds.

3.2 Csiszár-Kullback inequalities

In the case $B > 0$, the term involving

$$\int_{\mathbb{R}^d} \left(\rho^{\frac{p+1}{2}} - \rho_\infty^{\frac{p+1}{2}} - \tfrac{p+1}{2}\,\rho_\infty^{\frac{p-1}{2}}(\rho - \rho_\infty)\right)\,dx$$

gives a control on $\rho - \rho_\infty$ by the mean of a second order Taylor development (see [6, 11, 1, 7, 8]).

Consider first the case $\sigma(s) = \frac{1}{p+1} s^{\frac{p+1}{2}}$ for any $s \geq 0$, $p > 1$. On $L^1 \cap L^{(p+1)/2}(\mathbb{R}^d)$, let us define the functional

$$\Sigma[\rho] = \int_{\mathbb{R}^d} \left[\sigma(\rho) - \sigma(\rho_\infty) - \sigma'(\rho_\infty)\,(\rho - \rho_\infty)\right] d\xi$$

where

$$\rho_\infty(\xi) = \left(\lambda - \frac{1}{2}|\xi|^2\right)_+^{\frac{2}{p-1}} \tag{12}$$

with $\lambda = \lambda(M) > 0$ chosen such that $M = \|\rho_\infty\|_{L^1(\mathbb{R}^d)} = \|\rho\|_{L^1(\mathbb{R}^d)}$:

$$\lambda(M) = \left[\frac{1}{2}\,\frac{M}{(2\pi)^{d/2}}\,\frac{\Gamma\left(\frac{d}{2}+\frac{p}{p-1}\right)}{\Gamma\left(\frac{p}{p-1}\right)}\right]^{(\frac{1}{p-1}+\frac{d}{2})^{-1}}.$$

The function σ is nonnegative convex with nonincreasing second derivative whenever $p \leq 3$. We may also consider the limit case $p = 1$, which will be usefull for the case of the logarithmic NLS in Section 4. In that case, let $\sigma(s) = s\log s$, $\rho_\infty(\xi) = \exp(\lambda - \frac{1}{2}|\xi|^2)$ with $\lambda = (2\pi)^{-d/2}M$. With these notations, we have the following version of the Csiszár-Kullback inequality, which is more or less standard.

Lemma 2 *Assume that $d \geq 1$, $1 < p < 3$, $\max(1, \frac{2}{4-p}) \leq s < 2$, and let $q = s(3-p)/(2-s)$. With the above notations, for any $\rho \in L^1 \cap L^{(p+1)/2}(\mathbb{R}^d)$,*

$$\left\|\rho - \rho_\infty\right\|^2_{L^s(\mathbb{R}^d)} \leq \frac{2^{2(1+s)/s}}{p-1}\,K_q^{(3-p)/2}\Sigma[\rho]$$

where ρ_∞ is given by (12) with $\lambda = \lambda(\|\rho\|_{L^1(\mathbb{R}^d)})$ and

$$K_q = K_q[\rho] = \max\left\{\|\rho\|_{L^{q/2}(\mathbb{R}^d)}, \|\rho_\infty\|_{L^{q/2}(\mathbb{R}^d)}\right\}.$$

As a special case, if $s = (p+1)/2$ then $q = p+1$. In the limit case $p = 1$, $q = 2$, $K_2[\rho] = \|\rho\|_{L^1}$ and

$$\left\|\rho - \rho_\infty\right\|^2_{L^1(\mathbb{R}^d)} \leq 4\,\|\rho\|_{L^1}\,\Sigma[\rho]\,.$$

Proof. Using a Taylor development at order 2, we may write:

$$\Sigma[\rho] = \frac{1}{2}\int_{\mathbb{R}^d}\sigma''(\gamma)(\rho-\rho_\infty)^2\,d\xi$$

for some function $\gamma = \theta\rho_\infty + (1-\theta)\rho$, where θ takes its values in $[0,1]$. Since σ'' is nonincreasing we have $\sigma''(\gamma) \geq \sigma''(\rho_\infty)$ in $A = \{\xi : |\rho_\infty| > |\rho|\}$ and $\sigma''(\gamma) \geq \sigma''(\rho)$ in A^c. Using Hölder's inequality: $\int f^{p-3}g^2\,dx \geq \|g\|^2_{L^s}\,\|f\|^{p-3}_{L^{s(3-p)/(2-s)}}$, we get

$$\Sigma \geq \frac{p-1}{2^{2(1+s)/s}}\,K_q^{(p-3)/2}\left\|\rho-\rho_\infty\right\|^2_{L^s(\mathbb{R}^d)}.$$

For $p = 1$, the inequality is the standard Csiszár-Kullback inequality, which can be seen as the derivation of the above inequality with respect to p at $p = 1$. □

The energy E (resp. $\mathcal{E}$) gives a control in $L^{\frac{p+1}{2}}(\mathbb{R}^d)$ of $|\phi|^2 - |\phi_\infty|^2$ (resp. $\rho - \rho_\infty$).

Corollary 1 *Under the same assumptions as in Theorem 3, and with the same notations as above ($|\phi|^2 = \rho$, $|\phi_\infty|^2 = \rho_\infty$),*

$$\frac{A}{2}\int_{\mathbb{R}^d}\left|\nabla\Big(\sqrt{\frac{\rho}{\rho_\infty}}\Big)\right|^2\rho_\infty\,dx + \frac{C}{p+1}\|\rho-\rho_\infty\|^2_{L^{\frac{p+1}{2}}} \leq \mathcal{E}[\rho]-\mathcal{E}[\rho_\infty] \leq E[\phi]-E[\phi_\infty]$$

where $C = B\,\frac{p^2-1}{8}\,\|\phi_\infty\|^{-\frac{3-p}{p+1}}_{L^{p+1}(\mathbb{R}^d)}$.

Proof. The term involving $\rho_\infty\,|\nabla(\sqrt{\frac{\rho}{\rho_\infty}})|^2$ follows from (10). The other estimate is a straightforward consequence of Lemma 2. □

Remark 1 *It is also possible to obtain estimates in a weighted norm using the following estimate [7, 8]. Assume that Ω is a domain in $\mathbb{R}^d$ and that σ is a convex nonnegative function on $\mathbb{R}^+$ such that $\sigma(1) = 0$ and $\sigma'(1) = 0$. If μ is a nonnegative measure on Ω and if f and g are nonnegative measurable functions on Ω with respect to μ, then*

$$\int_\Omega \sigma(\frac{f}{g})\, g\, d\mu \geq \frac{K}{\max\{\int_\Omega f\, d\mu, \int_\Omega g\, d\mu\}} \cdot \|f-g\|^2_{L^1(\Omega,d\mu)} \tag{13}$$

where $K = \frac{1}{4}\cdot\min\{\min_{\eta\in]0,1[}\sigma''(\eta), \min_{\theta\in]0,1[,\, h>0}\sigma''(1+\theta h)(1+h)\}$, provided that all the above integrals are finite.

We may apply this estimate with $f=\rho$, $g=\rho_\infty$ and $d\mu(x)=\rho_\infty^{(p-1)/2}\,dx$:

$$\begin{aligned}&\tfrac{A}{2}\int_{\mathbb{R}^d}|\nabla(\sqrt{\tfrac{\rho}{\rho_\infty}})|^2\rho_\infty\,dx + \tfrac{C}{p+1}\left(\int_{\mathbb{R}^d}|\rho-\rho_\infty|\,\rho_\infty^{\frac{p-1}{2}}\,dx\right)^2\\ &\quad\leq \mathcal{E}[\rho]-\mathcal{E}[\rho_\infty]\leq E[\phi]-E[\phi_\infty]\end{aligned}$$

where $C = B\,\frac{p^2-1}{16}\left(\max\{\int_{\mathbb{R}^d}\rho_\infty^{\frac{p+1}{2}}\,dx\int_{\mathbb{R}^d}\rho\,\rho_\infty^{\frac{p-1}{2}}\,dx\}\right)^{-1}$.

3.3 Nonlinear stability, asymptotic nonlinear stability

Such estimates provide a straightforward stability result for the evolution problem corresponding to the nonlinear Schrödinger equation:

$$i\frac{\partial\phi}{\partial t} = -A\,\Delta\phi + B\,|\phi|^{p-1}\phi + V(x)\,\phi \tag{14}$$

with an initial data ϕ_0, that we shall assume appropriately chosen so that the solution globally exists (see [4] for more details).

Proposition 2 *Let $d\geq 1$, $p\in(1, 2+4/(d-2))$ if $d\geq 3$, $p\in(1,+\infty)$ if $d = 1$ or 2. Assume that V is a nonnegative potential defined on $\mathbb{R}^d$ which satisfies (8) and consider a global in time solution of (14) with initial condition $\phi_0\in H^1(\mathbb{R}^d)$ such that $\sqrt{V}\,\phi_0\in L^2(\mathbb{R}^d)$. Then for any $\epsilon>0$, there exists $\delta>0$ such that*

$$E[\phi_0]-E[\phi_\infty]<\delta \quad\Longrightarrow\quad \|\,|\phi(\cdot,t)|^2-|\phi_\infty|^2\|_{L^{\frac{p+1}{2}}(\mathbb{R}^d)}<\epsilon \quad \forall\, t>0\,.$$

¿From now on, we shall only consider the subcritical case with $V(x) = \frac{1}{2}|x|^2$, $A = R^{c_p-2}$ and $B = R^{c_p-d\,(p-1)/2}$, which corresponds to the coefficients obtained by rescaling in Section 2.

According to Lemma 1, the solution of (6), namely

$$i\phi_\tau = -R^{c_p-2}\Delta\phi + |\phi|^{p-1}\phi + \frac{1}{2}|\xi|^2\phi \tag{15}$$

is such that the energy functional

$$E(\tau) = \frac{R^{c_p-2}}{2}\int_{\mathbb{R}^d}|\nabla\phi|^2\,d\xi + \frac{1}{4}\int_{\mathbb{R}^d}|\xi|^2|\phi|^2\,d\xi + \frac{1}{p+1}\int_{\mathbb{R}^d}|\phi|^{p+1}\,d\xi$$

is decreasing.

Lemma 3 *With evident notations, if ϕ is a global solution of (15) in the subcritical case, then $\Sigma[|\phi|^2]+\frac{1}{2}R^{c_p-2}\int_{\mathbb{R}^d}|\nabla\phi|^2$ is a decaying function of τ, which is therefore bounded by $\Sigma[|\psi_0|^2]+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla\psi_0|^2$.*

Proof. This is a straightforward consequence of Lemma 1. □

Combining Lemma 2, Lemma 3 and going back to the original variables, we have the following asymptotic stability result, which is a precised version of Theorem 2. The proof goes as for Proposition 2, except that the coefficient A and B now depend on time, and that we keep track of the constants.

Theorem 4 *Under the same assumptions as in Theorem 1, if ψ is a global solution of (1) in the subcritical case with $p<3$, then, for all $t\geq 0$,*

$$\left\|\,|\psi|^2-|\psi_\infty|^2\right\|^2_{L^{(p+1)/2}}\leq C^2R(t)^{-d(p-1)/(p+1)}$$

where $\psi_\infty(t,x)=R^{-d/2}\phi_\infty(x/R)$ with $R=R(t)$ has been defined in Proposition 1, and, with $\kappa=\frac{2^{2\,(p+3)/(p+1)}}{p-1}$,

$$C=\kappa\,\max\{(p+1)\,E_0,\|\phi_\infty\|_{L^{p+1}}\}^{(3-p)/2}\left(\Sigma[|\psi_0|^2]+\frac{1}{2}\int_{\mathbb{R}^d}|\nabla\psi_0|^2\right)\,.$$

The proof of Theorem 2 uses similar arguments and interpolations like in the proof of Theorem 1. It is left to the reader. The constant can be chosen arbitrarily small if both $\|\nabla\psi_0\|_{L^2}$ and $\Sigma[|\psi_0|^2]$ are small.

4 Dispersion for the logarithmic NLS

The logarithmic NLS appears as a limit case of NLS when $p\to 1$. Time-dependent rescalings apply and provide dispersion results that are apparently new. The estimates use in a crucial way Jensen's inequality and the method is similar to the one that has been used in [9].

4.1 Preliminary computations

Notice first that $mK^{-1/m}\leq\varepsilon$ is equivalent to

$$\log m-\frac{\log K}{m}-\log\varepsilon=0\,,$$

which (for $\log K<0$) means that as $\varepsilon\to 0_+$,

$$m(\varepsilon)=\frac{\log K}{|\log\varepsilon|}\left(1+\frac{\log(|\log\varepsilon|)}{|\log\varepsilon|}\right)+o\left(\frac{\log(|\log\varepsilon|)}{|\log\varepsilon|^2}\right)\,. \qquad (16)$$

The equation

$$\ddot{R}=\frac{2}{R}\,,\quad R(0)=1\,,\quad R(1)=0 \qquad (17)$$

can easily be integrated once

$$\dot{R}=2\sqrt{\log R}\,,$$

and it is then not difficult to see that

$$R(t)\sim 2t\,\sqrt{\log t}\quad\text{as } t\to+\infty\,. \qquad (18)$$

4.2 A Lyapunov functional

Consider the logarithmic NLS

$$i\psi_t = -\Delta\psi + \log(|\psi|^2)\psi \ , \quad \psi_{|t=0} = \psi_0 \ . \tag{19}$$

As in the power nonlinearity case (with $\tau(t) = t$), the wave function

$$\phi(t,\xi) = R^{d/2}\, e^{-\frac{i}{4}R\dot{R}\,|\xi|^2}\psi(t, R\,\xi)$$

with R given by (17) is a solution of

$$i\phi_t = -\frac{1}{R^2}\,\Delta\phi + \log\,(\frac{|\phi|^2}{R^d})\,\phi + \frac{1}{2}\,|\xi|^2\phi \ , \quad \phi_{|t=0} = \psi_0 \ . \tag{20}$$

Assume for simplicity that $\|\psi_0\|_{L^2(\mathbb{R}^d)} = 1$. The energy

$$E(t) = \frac{1}{2R^2}\int_{\mathbb{R}^d}|\nabla\phi|^2\,d\xi + \frac{1}{2}\int_{\mathbb{R}^d}|\phi|^2\log\left(|\phi|^2\right)\,d\xi + \frac{1}{4}\int_{\mathbb{R}^d}|\xi|^2\,|\phi|^2\,d\xi$$

decays according to $\frac{dE}{dt} = -\frac{\dot{R}}{R^3}\int_{\mathbb{R}^d}|\nabla\phi|^2\,d\xi$, which for any $t > 0$ gives the estimate

$$\frac{1}{2R^2}\int_{\mathbb{R}^d}|\nabla\phi|^2\,d\xi + \frac{1}{2}\int_{\mathbb{R}^d}|\phi|^2\log\left(|\phi|^2\right)\,d\xi + \frac{1}{4}\int_{\mathbb{R}^d}|\xi|^2\,|\phi|^2\,d\xi \leq E(0) \ ,$$

where $E(0) = \frac{1}{2}\|\nabla\psi_0\|^2_{L^2} + \frac{1}{p+1}\|\psi_0\|^{p+1}_{L^{p+1}} + \frac{1}{2}\|\,|x|\,\psi_0\|^2_{L^2}$. Written in terms of ψ, this estimate gives the

Lemma 4 *Let $d \geq 1$ and consider a solution of (19) corresponding to an initial data $\psi_0 \in H^1(\mathbb{R}^d)$ such that $x \mapsto |x|\,\psi_0(x)$ belongs to $L^2(\mathbb{R}^d)$. Then, with the above notations,*

$$\frac{1}{2}\int_{\mathbb{R}^d}|(i\nabla - \tfrac{\dot{R}}{R}x)\,\psi|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^d}|\psi|^2\log\left(R^d|\psi|^2\right)\,dx + \frac{1}{4R^2}\int_{\mathbb{R}^d}|x|^2\,|\psi|^2\,dx \leq E(0) \ . \tag{21}$$

4.3 Rate of dispersion

Let Ω be a bounded open set in $\mathbb{R}^d$ and consider a nonnegative function ρ defined on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d}\rho\,dx = 1$. Assume that ρ satisfies the condition

$$\int_{\mathbb{R}^d}\rho\log\rho\,dx + \int_{\mathbb{R}^d}\rho\,\frac{|x|^2}{2}\,dx \leq \log\left(\frac{e^{1/e}C}{(2\pi)^{d/2}R^d}\right) \tag{22}$$

for some positive constant $C = e^K$. We want to evaluate $m = \int_\Omega \rho\,dx$ in terms of R in the limit $R \to +\infty$.
1) On $\mathbb{R}^d \setminus \Omega = \Omega^c$, one can write

$$\int_{\Omega^c}\rho\left(\log\rho + \frac{|x|^2}{2R^2}\right)dx = \int_{\mathbb{R}^d}\bar{\rho}\log\bar{\rho}\,dx - \int_{\mathbb{R}^d}\bar{\rho}\log\left(\frac{(1-m)\,e^{-\frac{|x|^2}{2R^2}}}{(2\pi R^2)^{d/2}}\right)dx + (1-m)\log\left(\frac{1-m}{(2\pi R^2)^{d/2}}\right)$$

with $\bar\rho = \rho \mathbb{1}_{\Omega^c}$: $\int_{\mathbb{R}^d} \bar\rho\, dx = 1 - m$. By Jensen's inequality the integral on the r.h.s. is nonnegative and $\min_{m\in(0,1)}(1-m)\log(1-m) = -\frac{1}{e}$, which proves the inequality:

$$\int_{\Omega^c} \rho \left(\log\rho + \frac{|x|^2}{2R^2} \right) dx \geq -\frac{1}{e} - \frac{d}{2}\log(2\pi) - d(1-m)\log R \,. \tag{23}$$

2) Using Jensen's inequality on Ω, we get

$$m \log\left(\frac{m}{|\Omega|}\right) \leq \int_\Omega \rho\log\rho\, dx \,.$$

A combination with (22) and (23) then gives

$$m \log\left(\frac{m}{|\Omega|}\right) \leq \log C - md\log R \,, \tag{24}$$

which also means

$$m \log\left(\frac{m\,K^{-1/m}}{|\Omega| R^{-d}}\right) \leq 0 \,,$$

where $K = \log C$. This is possible if and only if

$$m\,K^{-1/m} \leq |\Omega|\,R^{-d} \,. \tag{25}$$

The function $\varphi(m) = m\,K^{-1/m}$ is strictly increasing on $(0,+\infty)$ and $\varphi(0)=0$ so that m goes to 0 as $R\to+\infty$. Using the preliminary computation (16), Equation (25) is satisfied if and only if, as $R\to+\infty$,

$$m \leq \varphi^{-1}\left(\frac{|\Omega|}{R^d}\right) = \frac{\log K}{\log\left(\frac{R^d}{|\Omega|}\right)} \left(1 + \frac{\log\left(\log\left(\frac{R^d}{|\Omega|}\right)\right)}{\log\left(\frac{R^d}{|\Omega|}\right)} \left[1+o(1)\right] \right) \,. \tag{26}$$

Theorem 5 *Consider a global in time solution of (19) corresponding to an initial data $\psi_0 \in H^1(\mathbb{R}^d)$ such that $x\mapsto |x|\,\psi_0(x)$ belongs to $L^2(\mathbb{R}^d)$. Assume that both $|\psi|^2\log|\psi|^2$ and $|x|^2\,|\psi|^2$ are in $C^0(\mathbb{R}^+, L^1(\mathbb{R}^d))$. Let Ω be a bounded open set in $\mathbb{R}^d$. Then, with the above notations,*

$$\int_\Omega |\psi|^2\, dx \leq \int_{\mathbb{R}^d} |\psi_0|^2\, dx \cdot \varphi^{-1}\left(\frac{|\Omega|}{R^d}\right) = O\left(\frac{1}{\log t}\right)$$

as $t\to+\infty$, where $K = \frac{d}{2}\log(2\pi) + \frac{2}{e}\,E(0)$ and $t\mapsto R(t)\sim t\sqrt{\log t}$ is a solution of (17).

Proof. We can certainly assume that $\|\psi_0\|_{L^2}=1$, so that $\|\psi(\cdot,t)\|_{L^2}=1$ for any $t\in\mathbb{R}^+$ (the general case, only the normalisations in the computations are cchanges and details are left to the reader). It is then sufficient to apply the above estimates with $\rho=|\psi|^2$, $C=(2\pi)^{d/2}e^{2E(0)-\frac{1}{e}}$ and

$$E(0) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla\psi_0|^2\, dx + \frac{1}{2}\int_{\mathbb{R}^d} |\psi_0|^2\log\left(|\psi_0|^2\right)\, dx + \frac{1}{4}\int_{\mathbb{R}^d} |x|^2\,|\psi_0|^2\, dx \,.$$

□

Remark 2 *The method exposed above in the power law case for getting asymptotic stability also applies to the logarithmic NLS.*

This paper has been completed after Carlos' disaparition. Our collaboration has been short but friendly and intellectually stimulating. My contribution is dedicated to Carlos' memory. J.D.

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