MULTIPLE BUBBLING FOR THE EXPONENTIAL NONLINEARITY IN THE SLIGHTLY SUPERCRITICAL CASE

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Abstract. Let B denote the unit ball in $\mathbb{R}^2$. We consider the slightly super-critical Gelfand problem for the p-Laplacian operator $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$,

$$-\Delta_{2-\varepsilon} u = \lambda\, e^u \ \text{ in } B\,, \quad u = 0 \ \text{ on } \partial B\,,$$

for small $\varepsilon > 0$. We show that if $k \geq 1$ is given and $\lambda > 0$ is fixed and small, then there is a family of radial solutions exhibiting multiple blow-up as $\varepsilon \to 0$ in the form of a superposition of k bubbles of different blow-up orders and shapes. Similar phenomena is found for the same problem involving the operator $\Delta_{N-\varepsilon}$ in $\mathbb{R}^N$, $N \geq 3$.

1. **Introduction and statement of main results.** Let B denote the unit ball in $\mathbb{R}^N$, $N \geq 2$. This paper deals with the analysis of multiple-bubbling phenomena associated to *super-critical perturbations* of the well-known boundary value problem

$$-\Delta u = \lambda\, e^u \ \text{ in } B\,, \quad u = 0 \ \text{ on } \partial B\,. \tag{1}$$

This equation is used in stellar dynamics, combustion and chemotaxis models. It is often called the Emden-Fowler equation [14, 32, 43] or, more commonly, Gelfand's problem. Classical solutions of (1) are radially symmetric and positive. They correspond to solutions of the ODE

$$\begin{cases} u'' + \frac{n-1}{r}\, u' + \lambda\, e^u = 0\,, \quad r \in (0,1) \\[2ex] u'(0) = 0\,, \quad u(1) = 0\,. \end{cases} \tag{2}$$

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In the classical paper [34], Joseph and Lundgren performed a transformation of the above equation into a second order autonomous ODE, and described by a thorough phase-plane analysis the set of all pairs (u, λ) with u a solution of (1) and $\lambda > 0$. This set constitutes a curve emanating from $(0, 0)$ which extends up to a value $\lambda = \lambda_1^+$ so that no solutions exist for $\lambda > \lambda_1^+$. The eventual behavior of this curve depends strongly on dimension N. When $N = 2$ the branch has a single turning point at $\lambda = \lambda_1^+$ after which it goes left and blows-up as λ approaches 0, see Fig. 5. Solutions can be found in this case explicitly [5], taking into account that all classical radial solutions in $\mathbb{R}^2$ of the equation $\Delta w + \lambda\, e^w = 0$ are given by the one parameter family

$$w_\mu(|x|) \;=\; \log\left(\frac{8\mu^2}{\lambda\,(\,\mu^2\;+\;|x|^2\,)^2}\right), \quad \mu > 0\;, \tag{3}$$

and then $w_\mu(r)$ satisfies (2) provided that μ is chosen so that the boundary condition is satisfied. In particular exactly two solutions exist for small $\lambda > 0$, one large and one small, the large one behaving asymptotically as

$$u(x) \sim \log\left(\frac{1}{\left(\frac{\lambda}{8} + |x|^2\right)^2}\right)\;.$$

A solution of this type is commonly referred to as a *bubble*, in accordance with the geometric interpretation of e^u as the conformal factor to the euclidean metric in $\mathbb{R}^2$ corresponding to that of a sphere after stereographic projection. Analysis of blowing-up families of solutions to planar elliptic problems involving exponential nonlinearity has been a subject broadly treated in the literature in the last decade. For instance, from the analysis in [6, 35, 42], it follows that if B is replaced by an arbitrary domain $\Omega \subset \mathbb{R}^2$, blow-up of a family of solutions u_λ with $\int_\Omega \lambda\, e^{u_\lambda}\, dx$ bounded as $\lambda \to 0$ occurs in the form

$$u_\lambda(x) = \sum_{i=1}^{m} \log\left(\frac{1}{(c_i\lambda + |x-\xi_i|^2)^2}\right) \;+ O(1) \tag{4}$$

for some finite set of points ξ_i far apart one from each other and from the boundary, so that in particular

$$\lambda \int_\Omega e^{u_\lambda}\, dx \;\to\; 8\pi k \quad \text{as } \lambda \to 0\,. \tag{5}$$

Solutions of this type have been constructed for instance in [3, 15, 24, 26, 48].

It was recently observed in [47] that suitable perturbations of Gelfand's problem trigger *multiple bubbling*, case in which points ξ_i in (4) may accumulate around a single point. This is the case for the equation $-\mathrm{div}\,\big(a(x)\nabla u\big) \;=\; \lambda\, a(x)\, e^u$ under Dirichlet boundary conditions in a domain Ω, around an isolated maximum point of the (positive, smooth) function a. These solutions still satisfy $\lambda \int_\Omega e^u\, dx \to 8k\pi$.

Analogous bubbling phenomena in higher dimensions $N \geq 3$ is present associated to Sobolev's critical exponent $\frac{N+2}{N-2}$. Let us consider for instance the Brezis-Nirenberg problem in a bounded, smooth domain $\Omega \subset \mathbb{R}^N$,

$$-\Delta u = u^p + \lambda\, u \;\text{ in } \Omega\,, \quad u = 0 \;\text{ on } \partial\Omega\,. \tag{6}$$

When $p = \frac{N+2}{N-2}$ and $N \geq 4$, it is established in [7] that Problem (6) has a positive solution whenever $0 < \lambda < \lambda_1$. When $\Omega = B$ this solution is unique and blows-up as $\lambda \to 0$ in the form of a *bubble* at the origin. Large, bounded energy solutions u

as $\lambda \to 0$ arise as one or more isolated bubbles, so is the case for exponent slightly subcritical $p = \frac{N+2}{N-2} - \varepsilon$ as $\varepsilon \to 0^+$, see [1, 8, 2, 44]. By a bubble we mean now a solution which for a small $\mu > 0$ looks like

$$w_\mu(x) = c_N \left(\frac{\mu}{\mu^2 + |x|^2} \right)^{\frac{N-2}{2}}, \quad \mu > 0, \quad c_N = (N(N-2))^{\frac{N-2}{2}}$$

corresponding to radial positive solutions of $-\Delta w = w^{\frac{N+2}{N-2}}$ in $\mathbb{R}^N$. Bubbling solutions from the slightly supercritical side $p = \frac{N+2}{N-2} + \varepsilon$ have been analyzed for instance in [22, 23, 19, 21, 31, 45]. An interesting feature of this case is that multiple bubbling in the form of *towers of bubbles* may appear, a phenomenon first described in the radial case by the authors in [19]. Problem (6) for $p = \frac{N+2}{N-2} + \varepsilon$, $N \geq 4$, $\Omega = B$ was considered in [19], and it was shown that for $\lambda > \lambda_{k,\varepsilon} \sim 0$, there is a solution of the form

$$u_\varepsilon(x) \;=\; \sum_{i=1}^{k} c_N \left(\frac{\delta_{i,\varepsilon}}{\delta_{i,\varepsilon}^2 + |x|^2} \right)^{\frac{N-2}{2}} \;+\; o(1)\;, \tag{7}$$

where $0 < \delta_{i+1,\varepsilon} << \delta_{i,\varepsilon} << 1$ as $\varepsilon \to 0^+$, which again in the interpretation of $u^{4/(N-2)}$ as a conformal factor for the euclidean metric, one may regard geometrically as a finite string of shrinking spheres. This type of multiple bubbling, is also present in the non-radial case and in dimension 3, see [19, 21, 27, 31].

This phenomenon is linked to the asymptotic behavior of the bifurcation branch of radial positive solutions (λ, u) for problem (6) stemming from $(\lambda_1, 0)$ to the left. Rather than blowing-up at $\lambda = 0$ as it happens for $p = \frac{N+2}{N-2}$, as soon as $p > \frac{N+2}{N-2}$ the branch oscillates as a damped sinusoidal around a value $\lambda = \lambda_*$ close to $\lambda = 0$ at which a singular solution and infinitely many regular solutions exist. The k-towers found correspond to solutions lying near the k-th turning point to the right. This qualitative feature of the branch in fact prevails for all $p > \frac{N+2}{N-2}$ if $N \leq 10$, see [11, 38, 25].

Coming back to Gelfand's problem, it turns out that the above eventual behavior of the branch is also exactly the same in (1) if $3 \leq N \leq 9$. By analogy, we may then regard the nonlinearity e^u as *supercritical* for dimension $N > 2$.

It is natural to ask what type of *slightly supercritical* perturbations of e^u in dimension $N = 2$ could yield tower-bubbling. As we will see, it is still present, however it takes a more complex form. In geometric terms, rather than a string of spheres, the metrics found corresponding to e^u may be visualized as a string of a sphere as an endpoint, glued with pieces of manifolds with cusp singularities.

We formulate next two model problems, perturbations of (1) for $N = 2$, in which this phenomena occurs. The first of them is that of finding radial solutions for

$$-\mathrm{div}\left(|x|^\varepsilon \nabla u \right) \;=\; \lambda\, |x|^\varepsilon\, e^u \;\text{ in } B \subset \mathbb{R}^2\,, \quad u = 0 \;\text{ on } \partial B\,, \tag{8}$$

in dimension $N = 2$, where $\varepsilon > 0$ is a small parameter. Let us observe that this problem corresponds to the ODE

$$\begin{cases} u'' + \frac{1+\varepsilon}{r}\, u' + \lambda\, e^u = 0\,, \quad r \in (0,1) \\[2ex] u'(0) = 0\,, \quad u(1) = 0 \end{cases}$$

so that the problem can be regarded as a problem parametrized by a real parameter n with $n-1=1+\varepsilon$. Roughly spoken, this is Gelfand's problem in "dimension" slightly larger than 2.

The nonlinearity e^u is also supercritical if the operator is taken to be p-Laplacian with p slightly less than 2. Thus we also consider

$$-\mathrm{div}\left(|\nabla u|^{-\varepsilon}\nabla u\right) = \lambda\, e^u \ \text{ in } B \subset \mathbb{R}^2\,, \quad u=0 \ \text{ on } \partial B\,, \tag{9}$$

which is known as the Bratu-Gelfand problem, see *e.g.* [33].

Theorem 1.1. *Let k be a positive integer and assume that $N=2$. Then there exists a number λ_k^+ such that the following holds: if $0<\lambda<\lambda_k^+$ and $(\varepsilon_m)_{m\in\mathbb{N}}$ is a sequence converging to 0^+, then passing to a subsequence there is a radial solution u_{ε_m} of Problems* (8) *and* (9) *of the form*

$$u_{\varepsilon_m}(x) = \log\left[\sum_{j=1}^k \frac{8\,\alpha_j^2\,\delta_{j,\varepsilon_m}^2\,|x|^{2(\alpha_j-1)}}{\lambda\left(|x|^{2\alpha_j} + \delta_{j,\varepsilon_m}^2\right)^2}\right] + o(1)\,,$$

where the α_j are positive constants, independent of λ, with

$$\alpha_1<\alpha_2<\cdots<\alpha_k=1,$$

and the positive numbers δ_{j,ε_m} satisfy

$$\delta_{k,\varepsilon_m} << \delta_{k-1,\varepsilon_m} << \cdots << \delta_{2,\varepsilon_m} \to 0$$

as $m\to\infty$, while $\delta_{1,\varepsilon_m}=\delta_1$ is constant and it is determined from α_1 and λ by the boundary condition $u_{\varepsilon_m}(|x|=1)=0$ at main order, namely

$$\frac{8\,\alpha_1^2\,\delta_1^2}{\lambda\left(1+\delta_1^2\right)^2} = 1\,.$$

Numerical evidence suggests that the numbers α_j are independent of k, and of the particular sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ chosen but we have no proof of this fact. The main difference with the the tower-bubbling in the slightly supercritical problem (6) as in (7) is that now the elements of the tower have different shapes, they do not correspond to scalings of a single function. In problem (6), all parameters δ_{j,ε_m} in (7) are explicitly known. Here we do not have precise estimates of them.

On the other hand, we observe that the functions

$$w(x) = \log\frac{8\,\alpha^2\,\delta^2\,|x|^{2(\alpha-1)}}{\lambda\left(|x|^{2\alpha} + \delta^2\right)^2}$$

correspond to solutions of $\Delta w + \lambda\, e^w = 8\pi(\alpha-1)\,\delta_0(x)$. In geometric terms, the functions $\lambda\, e^w$ no longer represent conformal factors for euclidean metrics of spheres, but of manifolds with cusp singularities if $\alpha<1$. Asymptotic analysis of problems locally of this type has been a subject of a series of works, see *e.g.* [46] and references therein. Let us observe also that for λ small u_{ε_m} satisfies

$$\lambda\int_\Omega e^{u_{\varepsilon_m}}\,dx \sim 8\pi\sum_{j=1}^k \alpha_j.$$

More generally than equations (8) and (9), we shall consider radial solutions of an equation including slightly supercritical perturbations of Gelfand's problem for the n-Laplacian in "dimension" n. Thus we consider the problem

$$\begin{cases} (|u'|^{p-2}u')' + \frac{n-1}{r}\,|u'|^{p-2}u' + \lambda\, e^u = 0\;, \quad r \in (0,1) \\ \\ u'(0) = 0\;, \quad u(1) = 0 \end{cases} \tag{10}$$

where $n \geq 2$, $p > 1$. We assume in what follows that $\varepsilon = n - p > 0$, where both n and p can now are real parameters. Our main result for this problem, which includes that of Theorem 1.1, states as follows.

THEOREM 1.2. *Let k be a positive integer. Then there exists a number λ_k^+ such that the following holds. Given $0 < \lambda < \lambda_k^+$ and a sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ which converges to 0^+, and passing to a subsequence, there is a radial solution u_{ε_m} of Problem* (10) *with $n - p = \varepsilon_m \to 0^+$ of the form*

$$\lambda\,|x|^n\,e^{u_{\varepsilon_m}(x)} = \left[\sum_{j=1}^{k} w_j\Big(\log|x| + \mu_{j\varepsilon_m}\Big)\right]\Big(1 + o(1)\Big)\,.$$

The numbers μ_{j,ε_m} satisfy

$$\lim_{m\to\infty}(\mu_{j+1,\varepsilon_m} - \mu_{j,\varepsilon_m}) = +\infty \quad \forall\, j = 1,\, 2, \ldots k-1\,.$$

The functions w_j are even and positive with $w_j(s) \to 0$ as $s \to \pm\infty$. If $m_j =:$ $\max w_j$, then $m_1 < m_2 < \cdots < m_k$, and $w_j = -y'$, where y solves

$$y' + m_j = \frac{|y|^{n^*}}{n^*} + n\,y + n^{n-1}\;, \quad y(0) = -n^{n-1}$$

with $n^* = \frac{n}{n-1}$.

The proof actually yields that there exists a strictly decreasing sequence $(\bar\lambda_j^+)_{j\geq 1}$ such that $\bar\lambda_j^+ \leq m_j$ for $j = 1,\, 2, \ldots\; k$. We conjecture that w_j depends neither on k nor on the particular sequence $\varepsilon = \varepsilon_m \to 0^+$. We observe that the characterization of w_j implies in the special case $p = 2$,

$$w_j(s) = m_j\left[1 - \Big(\tanh\Big(\sqrt{m_j/2}\,s\Big)\Big)^2\right]\,.$$

which gives rise to the expressions in Theorem 1.1. The fact that $\alpha_k = 1$ corresponds to the fact that the solution is bounded up to the origin.

The proof of Theorem 1.2 is based on phase plane analysis. By means of a generalized Emden-Fowler change of variables [28], Problem (10) reduces to an autonomous ODE system whose qualitative behavior can be described in fairly precise terms. This allows us to describe how the supercritical regime $p < n$ approaches the critical case $p = n$. The proof of the theorem reduces to describing asymptotically a trajectory corresponding to a heteroclinic orbit connecting two equilibria, which is very degenerate in the critical limit. This is done via the analysis of a non-standard energy suitably associated to the problem.

It is illustrative to mention that as a consequence of phase-plane analysis, one gets a clear picture of the bifurcation diagram of problem (10) in different ranges of $p \leq n$, very similar to the one for $p = 2$ found in [34], see Fig. 1 below and [33] for a thorough discussion.

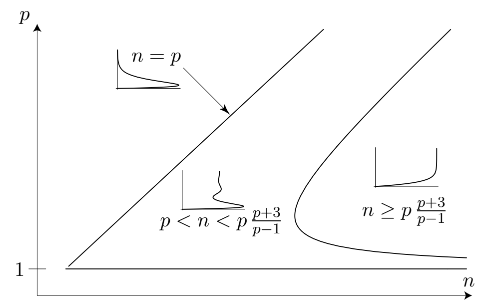

FIGURE 1. *Types of bifurcation diagrams in terms of n and p.*

The results of this paper have been announced without proofs (to be precise, without the estimates of Section 3) in [20], where the emphasis was put on the analogy with the Brezis-Nirenberg problem in the slightly supercritical case.

Finally, we should mention that after [32], Gelfand's problem in higher dimensions has received great attention on issues beyond the scope of this paper, in particular analysis of the first turning point of the branch of positive solutions in higher dimensions. We can cite for instance the works [9, 12, 17, 29, 37, 39, 40, 41] For the case of the p-Laplacian we refer the reader to [16, 30, 36, 33].

2. **Preliminary results.** We are going to state a series of very elementary results, which provides the proof of Theorem 1.2, up to the two properties which are summarized in Lemma 2.6 and whose proof is the subject of Section 3.

2.1. **The generalized Emden-Fowler change of variables.** Since (10) is invariant under rotations, for bounded solutions it makes sense to restrict the study to the case of radial solutions. See [18] and [10] for some recent result on the symmetry properties of the solutions. Let u be a solution of Equation (10). For $r = e^s$, $s \in (-\infty, 0]$, define $v(s) := u(r)$. Then (10) is equivalent to

$$\begin{cases} (p-1)\,|v'|^{p-2}\,v'' + (n-p)\,|v'|^{p-2}\,v' + \lambda\,e^{v+ps} = 0\;, \quad s \in (-\infty, 0) \\ \lim\limits_{s\to-\infty} v(s) > 0\;, \quad \lim\limits_{s\to-\infty} e^{-s}\,v'(s) = 0\;, \quad v(0) = 0 \end{cases}$$

where $v' = \frac{dv}{ds}$. Note that the change of variables means that

$$\lim_{s\to-\infty} v(s) = u(0)\;.$$

The equation for v can be reduced to an *autonomous* ODE system as follows. Let

$$x(s) = \lambda\,e^{v(s)+ps} \quad \text{and} \quad y(s) = |v'(s)|^{p-2}\,v'(s)\;.$$

Then

$$\begin{cases} x' = x\,(v' + p) \\ y' = (p-1)\,|v'|^{p-2}\,v'' \end{cases}$$

and (10) is finally equivalent to the system

$$\begin{cases} x' = x\,(|y|^{p^*-2}\,y + p)\;, & x(0) = \lambda \\ \\ y' = (p-n)\,y - x\;, & \displaystyle\lim_{s\to-\infty} e^{-s}\,|y(s)|^{p^*-2}\,y(s) = 0 \end{cases} \tag{11}$$

where $p^* = \frac{p}{p-1}$ so that $y = |v'|^{p-2}\,v' \iff v' = |y|^{p^*-2}\,y$. This change of coordinates is well-known [5, 28, 14, 32, 34, 41, 4, 33], at least for $p = 2$. We shall use it in order to understand the limit $n - p = \varepsilon \to 0$, $\varepsilon > 0$.

2.2. **Parametrization of the solutions.** The behaviour of the solutions easily follows from the study of the vector field and a linearization around the two fixed points: $P^- = (0,0)$ and $P^+ = p^{p-1}(n-p,-1)$. The linearization of (11) at P^- is

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} p & 0 \\ -1 & -(n-p) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with eigenvalues p and $-(n-p)$, and, at P^+,

$$\begin{pmatrix} X \\ Y \end{pmatrix}' = \begin{pmatrix} 0 & p\,(n-p)/(p-1) \\ -1 & -(n-p) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

with eigenvalues

$$\frac{1}{2}\sqrt{n-p}\,\left(-\sqrt{n-p} \pm i\,\sqrt{p\,(p+3)/(p-1)-n}\right)\;,$$

as long as $n < p\,(p+3)/(p-1)$.

Lemma 2.1. *Assume that $p < n < p\,(p+3)/(p-1)$. Then the following properties hold:*
(i) *Any trajectory of (11) is such that $x(s)$ does not change sign. Any trajectory with $x > 0$ enters the lower quadrant corresponding to $x > 0$, $y < 0$.*
(ii) *P^+ (resp. P^-) is attracting all trajectories with $x > 0$ as $s \to +\infty$ (resp. all bounded trajectories with $x > 0$ as $s \to -\infty$).*
(iii) *There exists a bounded trajectory $s \mapsto (x(s), y(s))$ such that*

$$\lim_{s\to\pm\infty} (x(s), y(s)) = P^\pm\;.$$

This heteroclinic trajectory is unique, up to any translation in s.

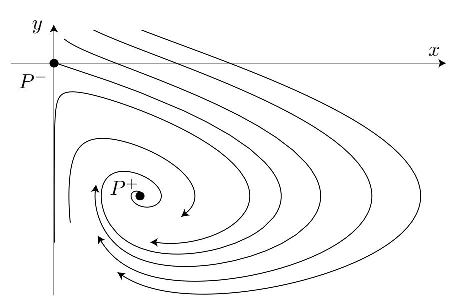

Figure 2. *Phase portrait in the supercritical case $p < n < p\,\frac{p+3}{p-1}$ (here $n = 2$, $p = 1.5$).*

Note that for $n \geq p\,(p+3)/(p-1)$, to the linearization of (11) at P^+ corresponds two negative eigenvalues, so that the trajectory connecting P^- to P^+ is unique, up to any translation in s, and monotone in y. As a consequence, we recover for instance that for $p = 2$, $n \geq 10$, the branch of the solutions of (10) in $L^\infty(\Omega)$ is monotone. From now on we assume that $p \leq n < p\,(p+3)/(p-1)$. Let $(\bar{x}, \bar{y})$ be the unique trajectory such that $\lim_{s\to-\infty}(\bar{x}(s), \bar{y}(s)) = P^-$ and $\bar{x}(0) = \max_{s\in\mathbb{R}} \bar{x}(s)$. In order to emphasize the dependence on ε, we shall write $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ whenever needed. The proofs of Lemma 2.1 and of the next result are standard (see Fig. 3) and therefore left to the reader.

LEMMA 2.2. *Assume that $p \leq n < p\,(p+3)/(p-1)$. For a given λ, to any solution v of* (10) *it corresponds a unique $s_0 \in \mathbb{R}$ such that*

$$\lambda\, e^{v(s)+ps} = \bar{x}(s+s_0)$$

for any $s \leq 0$. Reciprocally, for any $\lambda \in (0, \lambda_1^+]$, where $\lambda_1^+ := \max_{s\in\mathbb{R}} \bar{x}(s) = \bar{x}(0)$, the equation $\bar{x}(s_0) = \lambda$ has at least one solution and

$$v(s) = \log\left(\frac{1}{\lambda}\,\bar{x}(s+s_0)\right) - p\,s$$

is a solution of (10).

Note that with the change of variables $s = t - s_0$,

$$v(t-s_0) = \log\left(\frac{\bar{x}(t)}{\bar{x}(s_0)}\right) - p\,t + p\,s_0 \quad \forall\, t \in (-\infty, s_0)\,.$$

The corresponding solution u of (10) is fully determined by $\lambda = \bar{x}(s_0)$, $u'(0) = 0$ and

$$u(0) = \lim_{t\to-\infty} v(t-s_0) = \lim_{t\to-\infty} \log\left(\frac{\bar{x}(t)\,e^{-p\,t}}{\bar{x}(s_0)\,e^{-p\,s_0}}\right)\,.$$

2.3. **The supercritical case: $p < n$.** The parametrization in Lemma 2.2 is a straightforward consequence of the Emden-Fowler change of coordinates. The next result only involves an elementary phase plane analysis which is described in Fig. 3. Details of the proof are left to the reader.

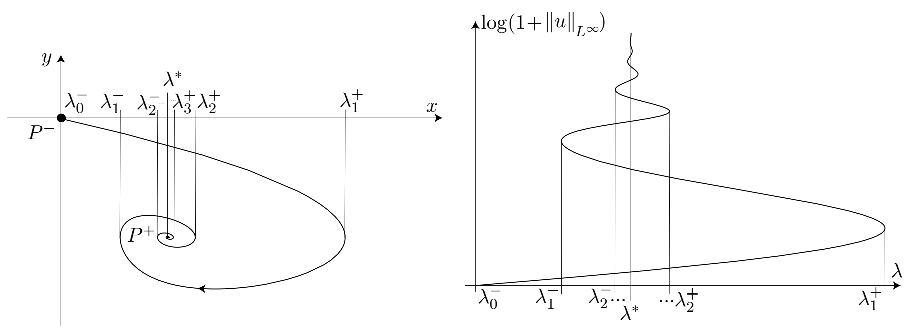

FIGURE 3. *Parametrization of the solutions of (10) in the supercritical case ($n = 2$, $p = 1.5$).*
Left: $(\bar{x}, \bar{y})$ in the phase space. Right: the bifurcation diagram for (10).

Lemma 2.3. *Let $\lambda^* = p^{p-1}(n-p)$. Assume that $p < n < p\,(p+3)/(p-1)$. There exists two sequences $(\lambda_k^-)_{k\ge1}$ and $(\lambda_k^+)_{k\ge1}$ such that:*
(i) *$(\lambda_k^-)_{k\ge1}$ is increasing and $\lim_{k\to+\infty}\lambda_k^- = \lambda^*$.*
(ii) *$(\lambda_k^+)_{k\ge1}$ is decreasing and $\lim_{k\to+\infty}\lambda_k^+ = \lambda^*$.*
(iii) *Equation (10) has no solutions if $\lambda > \lambda_1^+$, $2k-1$ solutions if $\lambda = \lambda_k^+$ or $\lambda \in (\lambda_{k-1}^-, \lambda_k^-)$ with the convention $\lambda_0^- = 0$, and $2k$ solutions if $\lambda = \lambda_k^-$ or $\lambda \in (\lambda_{k+1}^+, \lambda_k^+)$, $k \ge 1$.*
(iv) *Equation (10) has infinitely many solutions if and only if $\lambda = \lambda^*$.*

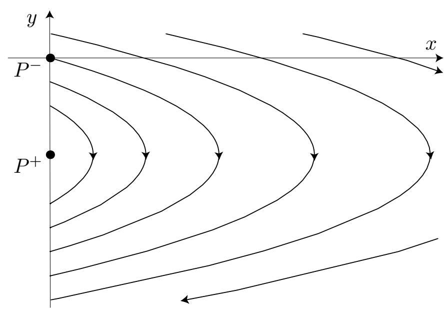

Figure 4. *Phase portrait in the critical case $n = p$ (here $n = 2$).*

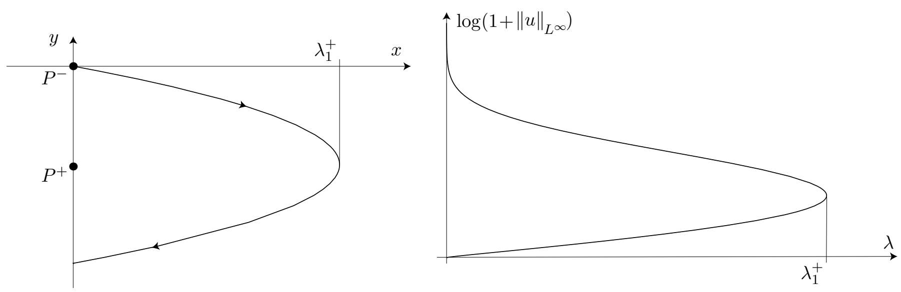

Figure 5. *Parametrization of the solutions of (10) for $n = p = 2$.*
Left: $(\bar x, \bar y)$ in the phase space. Right: the bifurcation diagram for (10).

2.4. **The critical case: $p = n$.** In the limit case $p = n$, (11) becomes an Hamiltonian system:

$$x' = x\,(|y|^{p^*-2}y + p)\;, \quad y' = -x\;, \tag{12}$$

which is explicitely solvable in the case $p = 2$ [5]: $u(r) = 2\log(a^2+1) - 2\log(a^2+r^2)$ is a solution of (10) for any $a > 0$ such that $\lambda = 8\,a^2\,(a^2+1)^{-2}$. See Fig. 4. The counterpart of Lemma 2.3 in the critical case is the

Lemma 2.4. *Assume that $p = n$ and let $\lambda_1^+ := \sup_{s\in\mathbb{R}} \bar x(s)$. Then Equation (10) has no solutions if $\lambda > \lambda_1^+$, one and only one solution if $\lambda = \lambda_1^+$ and two and only two solutions if $\lambda \in (0, \lambda_1^+)$.*

2.5. **Description of the critical limit.** This regime corresponds to the limit $\varepsilon = n - p \to 0$, $\varepsilon \geq 0$. For any $\varepsilon > 0$ (resp. $\varepsilon = 0$), define by $s_k(\varepsilon)$ (resp. $s_1(0)$) the sequence of the points of local maximum of $\bar{x}^\varepsilon$ (resp. the unique point of maximum of $\bar{x}^0$), where $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ is the unique trajectory such that

$$\lim_{s \to \pm\infty} (\bar{x}^\varepsilon(s), \bar{y}^\varepsilon(s)) = P^\pm$$

and $\bar{x}^\varepsilon(0) = \max_{s\in\mathbb{R}} \bar{x}^\varepsilon(s) =: \lambda_1^+(\varepsilon)$. Note that as a consequence, $\bar{y}^\varepsilon(0) = -p^{p-1}$. By definition of $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$,

$$s_1(\varepsilon) = 0 \quad \forall\, \varepsilon \in [0, p\,\frac{p+3}{p-1} - n)\ .$$

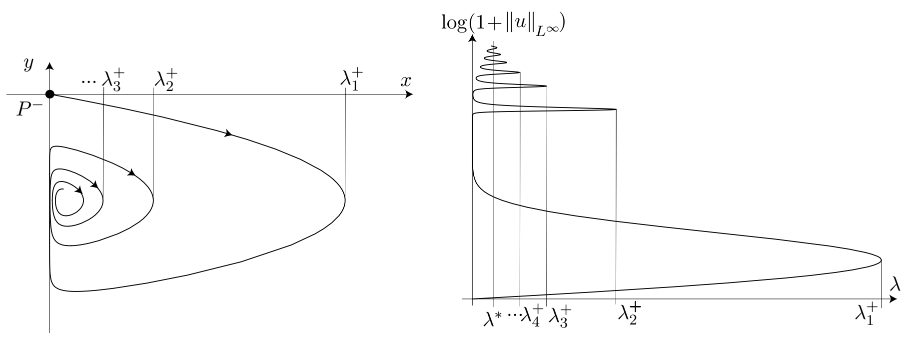

FIGURE 6. *Left: the solution $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ in the slightly supercritical regime $\varepsilon > 0$, $\varepsilon \to 0$.*
Right: the corresponding bifurcation diagram for (10). Here $n = 2$, $\varepsilon = 0.05$.

LEMMA 2.5. *For any $k \geq 1$, $\lim_{\varepsilon\to 0}[s_{k+1}(\varepsilon) - s_k(\varepsilon)] = +\infty$.*

The proof easily follows from the properties of the phase plane (see Fig. 6). To study the critical limit, we emphasize the dependence on ε. Let $\lambda_k^{\varepsilon,+} = \bar{x}^\varepsilon(s_k(\varepsilon))$. According to Lemma 2.3, $(\lambda_k^{\varepsilon,+})_{k\geq 1}$ is a positive decreasing sequence. Define $\bar{\lambda}_k^+ := \lim_{\varepsilon\to 0} \lambda_k^{\varepsilon,+}$. It is not clear that for any sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i > 0$, $\lim_{i\to\infty} \varepsilon_i = 0$, the limit of $\lambda_k^{\varepsilon_i,+}$ is unique and well defined so that one should consider a special sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ and potentially extract subsequences. For the sake of simplicity, we will speak of "the limit $\varepsilon \to 0$" in the rest of this section.

LEMMA 2.6. *For any $k \geq 1$,*

$$\bar{\lambda}_k^+ > 0 \tag{P_k}$$

and $\bar{\lambda}_1^+ = \lambda_1^{0,+}$. Moreover $(\bar{\lambda}_k^+)_{k\in\mathbb{N}}$ is a strictly decreasing sequence.

As seen above, Property (P_1) is always satisfied: the property $\bar{\lambda}_1^+ = \lambda_1^{0,+}$ is easy. In Section 3 we will prove the rest of Lemma 2.6.

It is now possible to give a precise description of the asymptotic behaviour of the solutions of (10) as $\varepsilon \to 0$. Let (x_k, y_k) be the solution of (12) with

$$x_k(0) = \bar{\lambda}_k^+ \quad \text{and} \quad y_k(0) = -p^{p-1}\ ,$$

for any $k \geq 1$. With these notations, $(x_1, y_1) = (\bar{x}^0, \bar{y}^0)$ but $(x_k, y_k) \neq (\bar{x}^0, \bar{y}^0)$ for any $k \geq 2$.

COROLLARY 2.1. *For any* $j \geq 1$, $\bar{x}^\varepsilon(\cdot + s_j(\varepsilon))$ *converges to* x_j *uniformly on compact subset in* $\mathbb{R}$.

Corollary 2.1 can be rephrased into

COROLLARY 2.2. *For any* $k \geq 1$, *as* $\varepsilon \to 0$,

$$\bar{x}^\varepsilon(s) \to \sum_{j=1}^{k} x_j(s - s_j(\varepsilon))$$

uniformly on any interval $(-\infty, a(\varepsilon)) \in \mathbb{R}$ *such that* $s_k(\varepsilon) < a(\varepsilon) < s_{k+1}(\varepsilon)$ *with* $\liminf_{\varepsilon\to 0}(s_{k+1}(\varepsilon) - a(\varepsilon)) = \liminf_{\varepsilon\to 0}(a(\varepsilon) - s_k(\varepsilon)) = +\infty$.

Let $\lambda \in (0, \bar{\lambda}_k^+)$ and define $s_k^\pm(\lambda) \in \mathbb{R}$ as the two solutions of $x_k(s_k^\pm(\lambda)) = \lambda$, $\pm s_k^\pm(\lambda) > 0$. A careful rewriting of the Emden-Fowler change of variables then allows to see the solution of (10) as a superposition of bubbles.

LEMMA 2.7. *Let* $\lambda \in (0, \bar{\lambda}_k^+]$ *for some* $k \geq 1$. *Then there exist two solutions* $u^\pm$ *of Problem* (10) *which take the form*

$$\lambda\, r^p \;\; e^{u^\pm(r)} = \left[\sum_{j=1}^{k} x_j\Big(\log r + s_k(\varepsilon) - s_j(\varepsilon) + s_k^\pm(\lambda)\Big)\right](1 + o(1)) \quad \forall\, r \in (0,1)$$

as $\varepsilon \to 0$.

This actually amounts to saying that there is a *k-bubble solution.* Note that we have to assume that $\varepsilon > 0$ is small enough so that with the notations of Corollary 2.2,

$$a(\varepsilon) > s_k(\varepsilon) + s_k^\pm(\lambda)\ .$$

Also note that the Property (P_k) is implicitly assumed in the statement of Lemma 2.7.

Proof. According to Lemma 2.4, for $\varepsilon > 0$ sufficiently small, we can define $s_{\varepsilon,k}^\pm(\lambda)$ as the two solutions of

$$\bar{x}^\varepsilon(s) = 0$$

which minimize $\pm\big(s_{\varepsilon,k}^\pm(\lambda) - s_k(\varepsilon)\big) > 0$. Then $\lim_{\varepsilon\to 0}\big(s_{\varepsilon,k}^\pm(\lambda) - s_k(\varepsilon)\big) = s_k^\pm(\lambda)$ and the statement is a consequence of Corollary 2.2. □

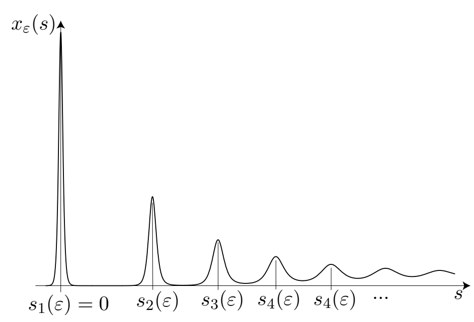

FIGURE 7. *Bubbles in the logarithmic scale, after the Emden-Fowler transformation.*

Proof. The proof of Theorem 1.2 is now straightforward with $w_j = x_j$. Note that $\mu_{i,j}(\varepsilon) = s^{\pm}_{\varepsilon_i,k}(\lambda) - s_j(\varepsilon_i)$, where $(\varepsilon_i)_{i\in\mathbb{N}}$ is a sequence of positive numbers with $\lim_{i\to+\infty} \varepsilon_i = 0$. □

3. **Multiple-bubble solutions.** This section is devoted to the proof of Lemma 2.6. We divide it in two steps. First, we prove that for any $k \geq 1$, $\bar\lambda^+_k$ is positive: *multi-bubbling* occurs, with an arbitrarily large number of bubbles provided $\lambda \in (0, \bar\lambda^+_k)$. Then we show that the bubbles do not have the same height, *i.e.*, $(\bar\lambda^+_k)_{k\in\mathbb{N}}$ is strictly decreasing.

3.1. **Multiple bubbling.** This section is devoted to the proof of Property (P_k) for $k \geq 2$. With the notations of Section 2.5, this means

$$\bar\lambda^+_k > 0 \; . \tag{P_k}$$

Before proving this result and actually more precise estimates, we start with some energy and angular velocity estimates in a new system of coordinates.

Consider

$$\left\{ \begin{array}{ll} x' = x\,(|y|^{p^*-2} y + p)\;, & x(0) = \lambda_1^{\varepsilon,+} \\ & \\ y' = -\varepsilon\, y - x\;, & y(-\infty) = 0 \end{array} \right. \tag{11}$$

with $\varepsilon = n - p > 0$. To simplify the notations, we will omit the index ε. In the new coordinates

$$V = \log x\;, \quad U = -y\;,$$

System (11) becomes

$$\left\{ \begin{array}{ll} U' = e^V - \varepsilon\, U\;, & U(-\infty) = 0 \\ & \\ V' = U_*^{p^*-1} - U^{p^*-1}\;, & V(0) = \log \lambda_1^{\varepsilon,+} \end{array} \right. \tag{13}$$

where

$$U_* := p^{p-1} \Longleftrightarrow U_*^{p^*-1} = p\;.$$

Note that to the trajectory (x, y) such that $\lim_{s\to-\infty}(x(s), y(s)) = (0, 0)$ now

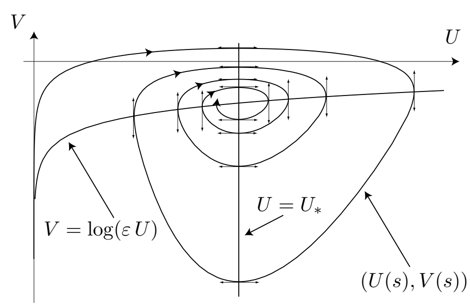

FIGURE 8. *The trajectory in the (U, V) coordinates.*

corresponds a trajectory such that $\lim_{s\to-\infty}(U(s), V(s)) = (0, -\infty)$ and such that $U(s) > 0$ for any $s \in \mathbb{R}$. With the notations of Section 2.1, this means

$$U(s) = -e^{(p-1)\,s}\,|u'(e^s)|^{p-2}\,u'(e^s)\;,$$

$$V(s) = \log\lambda + u(e^s) + p\,s\;.$$

Let

$$(U_*, V_* = \log(\varepsilon\, U_*)) = \lim_{s\to+\infty} (U(s), V(s)) \ .$$

The condition

$$x(0) = \max_{s\in\mathbb{R}} x(s) = \lambda_1^{\varepsilon,+}$$

now means

$$V(0) = \log \lambda_1^{\varepsilon,+} \ .$$

Consider the two following quantities, which are functions of s:

1. *energy:*

$$E = e^V - e^{V_*} - e^{V_*}(V - V_*) + \frac{1}{p^*}\left(U^{p^*} - U_*^{p^*}\right) - U_*^{p^*-1}(U - U_*) \ .$$

2. *angle:* let $\theta = \theta(s)$ be such that

$$\cos\theta = \frac{U - U_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} \quad \text{and} \quad \sin\theta = \frac{V - V_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} \ .$$

   We take the convention $\theta(0) = -\frac{\pi}{2}$ and assume that θ is continuous, which determines θ in a unique way.

LEMMA 3.1. *With the above notations, if (U, V) is a solution of equation (13), then U is uniformly bounded on $(0, +\infty)$ and there exists a constant $\nu > 0$ such that*

$$0 \geq \frac{dE}{ds} \geq -\varepsilon\, \nu\, E \quad \forall\, s \in \mathbb{R} \ . \tag{14}$$

As a consequence

$$E(s) \geq E(0)\, e^{-\varepsilon\, \nu\, s} \quad \forall\, s \geq 0 \ .$$

Note that the bound on U is also uniform in terms on $\varepsilon \in (0, 1)$, including in the limit $\varepsilon \to 0$.

Proof. A direct computation of $\frac{dE}{ds}$ gives

$$\begin{aligned}\tfrac{dE}{ds} &= (e^V - e^{V_*})\, V' + \left(U^{p^*-1} - U_*^{p^*-1}\right) U' \\ &= (e^V - e^{V_*})\,(U_*^{p^*-1} - U^{p^*-1}) + \left(U^{p^*-1} - U_*^{p^*-1}\right) (e^V - \varepsilon\, U)\end{aligned}$$

$$\frac{dE}{ds} = -\varepsilon\,(U - U_*) \left(U^{p^*-1} - U_*^{p^*-1}\right) \tag{15}$$

using $e^{V_*} = \varepsilon\, U_*$. The function $V \mapsto e^V - e^{V_*} - e^{V_*}(V - V_*)$ is nonnegative for any $V \in \mathbb{R}^+$, which means that

$$\frac{1}{p^*}\left(U^{p^*}(s) - U_*^{p^*}\right) - U_*^{p^*-1}(U(s) - U_*) \leq E(s) \leq E(0) \tag{16}$$

for any $s \geq 0$. Note that at $s = 0$, $V'(0) = 0$ so that $U(0) = U_*$ and

$$E(0) = \lambda_1^{\varepsilon,+} - \varepsilon\, U_*\, \log\left(\frac{e\, \lambda_1^{\varepsilon,+}}{\varepsilon\, U_*}\right) \ .$$

Since $\lambda_1^{\varepsilon,+} \to \bar\lambda_1^+$ as $\varepsilon \to 0$, $E(0)$ itself is uniformly bounded as $\varepsilon \to 0$. Combined with Inequality (16), this means that $U(s)$ is uniformly bounded in $s \in \mathbb{R}^+$, for $\varepsilon > 0$ fixed. Moreover, this bound is uniform as $\varepsilon \to 0$.

Independently of the uniform estimate on U, there exists a constant $\nu > 0$ such that

$$(U - U_*) \left(U^{p^*-1} - U_*^{p^*-1} \right) \leq \nu \left[\frac{1}{p^*} \left(U^{p^*} - U_*^{p^*} \right) - U_*^{p^*-1}(U - U_*) \right] \qquad (17)$$

for any $s \in \mathbb{R}^+$. This, using again (16), ends the proof of Lemma 3.1. Let us prove (17). Define

$$F(t) := \frac{1}{p^*}(t^{p^*} - 1) - (t - 1) - \kappa\,(t - 1)(t^{p^*-1} - 1)\ .$$

Then for $\kappa = 1/p^*$,

$$F'(t) = (1 - \kappa\, p^*)\, t^{p^*-1} + \kappa\,(p^* - 1)\, t^{p^*-2} + \kappa - 1$$

has the sign of $t - 1$ if $p^* \geq 2 \Longleftrightarrow p \in (1, 2)$. If $p^* \in (1, 2)$, since

$$\frac{1}{p^* - 1}\, F''(t) = (1 - \kappa\, p^*)\, t^{p^*-2} + \kappa\,(p^* - 2)\, t^{p^*-3}$$

changes sign only once in $(0, +\infty) \ni t$, the condition $F(0) = 0$ together with $\kappa < 1/p^*$ means $\kappa = (p^* - 1)/p^* = 1/p$. Thus $F(t)$ is nonnegative for any $t \in (0, +\infty)$ if $\kappa = \min(1/p, 1/p^*)$, so that (17) holds with $\nu = \max(p, p^*)$. □

Note that the exponential decay of E is not sufficient to assert that the multi-bubbling phenomenon occurs since the interval in s between two bubbles can be of a larger order than the scale $1/\varepsilon$. If this was the case, the height of the second bubble could converge to 0, *i.e.*, $\bar\lambda_2^+ = 0$. It is the purpose of the rest of Section 3.1 to prove that this is not the case.

Lemma 3.2. *Given $C > 0$, there exists a constant $\omega > 0$ such that, if*

$$V(s) \geq \log(\varepsilon\, U_*) - C \quad \forall\, s \in [s_1, s_2] \subset \mathbb{R}^+\ ,$$

then for $\varepsilon > 0$ small enough and any $s \in [s_1, s_2]$,

$$\frac{d\theta}{ds} \geq \varepsilon\, \omega\ . \qquad (18)$$

Proof. Let us remark that we can write

$$\tan\theta = -\frac{b}{a} \quad \text{where } a(s) := U(s) - U_* \text{ and } b(s) := V(s) - V_* = V(s) - \log(\varepsilon\, U_*)$$

for any $s \in \mathbb{R}$ such that $U(s) \neq U_*$. Differentiating with respect to s, we get

$$\frac{a^2 + b^2}{a^2}\, \frac{d\theta}{ds} = (1 + \tan^2\theta)\, \frac{d\theta}{ds} = \frac{a'b - a\, b'}{a^2}\ ,$$

$$\frac{d\theta}{ds} = \frac{a'b - a\, b'}{a^2 + b^2}\ .$$

On the one hand, $-a\, b' = (U - U_*)\, (U^{p^*-1} - U_*^{p^*-1}) \geq C_1\, (U - U_*)^2$ for some positive constant C_1. If $p^* < 2$, one has to use the fact that, according to Lemma 3.1, U is bounded. On the other hand

$$a'b = (e^V - \varepsilon\, U)(V - V_*) = (e^V - e^{V_*})(V - V_*) - \varepsilon\,(U - U_*)(V - V_*)\ .$$

Thus

$$\frac{d\theta}{ds} \geq \frac{1}{a^2 + b^2}\, \left[C_1\, a^2 - \varepsilon\, a\, b + C_2\, \varepsilon\, b^2 \right]\ ,$$

where

$$C_2 = \frac{1}{\varepsilon}\,\min_{s\in\mathbb{R}}\left(\frac{e^{V(s)}-e^{V_*}}{V(s)-V_*}\right)\,.$$

By assumption,

$$\frac{e^{V(s)}-e^{V_*}}{V(s)-V_*} \geq e^{V_*-C} = \varepsilon\,U_*\,e^{-C}\,,$$

which gives a lower estimate for C_2 which is independent of ε. It is now easy to prove that (18) holds for some $\omega > 0$. Namely we can estimate

$$C_1\,a^2 - \varepsilon\,a\,b + C_2\,\varepsilon\,b^2 = \left(\frac{1}{2}\,\sqrt{C_1}\,a - \frac{\varepsilon}{\sqrt{C_1}}\,b\right)^2 + \left(C_2\,\varepsilon - \frac{\varepsilon^2}{C_1}\right)\,b^2 + \frac{3}{4}\,C_1\,a^2$$

from below by

$$\frac{1}{2}\,C_2\,(a^2+b^2)\,\varepsilon$$

provided

$$\varepsilon \leq \frac{1}{2}\,C_1\,\min(C_2, 3\,C_2^{-1})\,.$$

This ends the proof with $\omega = C_2/2$. □

Corollary 3.1. *Let C be a positive constant and consider s_0, s_1 with $0\leq s_0 < s_1$. Assume that*

(i) *either* $s_0 = 0$ *or* $s_0 > 0$ *and* $V(s_0) = \log(\varepsilon\,U_*) - C$

(ii) $V(s) \geq \log(\varepsilon\,U_*) - C \quad \forall\, s\in[s_0, s_1]\,.$

Then

$$E(s) \geq E(s_0)\,e^{-\frac{\nu}{\omega}\,[\theta(s)-\theta(s_0)]}$$

where ν is the constant of Lemma 3.1.

Proof. It is an easy consequence of (14) and (18):

$$\frac{1}{E}\,\frac{dE}{ds} \geq -\nu\,\varepsilon \geq -\frac{\nu}{\omega}\,\frac{d\theta}{ds}\,.$$

□

Lemma 3.3. *Let K be a positive constant and assume that*

$$V \leq \log(\varepsilon\,U) - K\,. \tag{19}$$

Then

$$U' \leq -(1-e^{-K})\,\varepsilon\,U\,. \tag{20}$$

Proof. Condition (19) can be rephrased into

$$e^V \leq e^{-K}\,\varepsilon\,U$$

and the result easily follows. □

Lemma 3.4. *Let C be a positive constant and consider s_1, $s_2\in\mathbb{R}$, with $0 < s_1 < s_2$, such that*

$$V(s_i) = \log(\varepsilon\,U_*) - C \qquad i = 1, 2\,,$$

$$V(s) < \log(\varepsilon\,U_*) - C \qquad \forall\, s\in(s_1, s_2)\,.$$

Then there exists a constant $\kappa > 0$, which is independent of ε in the limit $\varepsilon \to 0$, such that

$$E(s_2) \geq \kappa \, E(s_1)$$

holds uniformly with respect to ε.

Proof. First of all, we may apply Lemma 3.3 with $K = C/2$. Since V' has the same sign as $U_* - U$, it is straightforward that

$$U(s_2) < U_* < U(s_1)$$

and that

$$s(K) = \inf\{s > s_1 \ : \ V(s) \geq \log(\varepsilon \, U(s)) - K\}$$

is such that

$$U(s(K)) < U_* \, .$$

Exactly as in Corollary 3.1, for any $s \in (s_1, s(K))$,

$$\frac{1}{E} \, \frac{dE}{ds} \geq -\nu \, \varepsilon \geq \frac{\nu}{(1 - e^{-K})} \, \frac{1}{U(s)} \, \frac{dU}{ds} \ ,$$

so that

$$E(s) \geq E(s_1) \, \left(\frac{U(s)}{U(s_1)} \right)^{\nu/(1-e^{-K})} \quad \forall \, s \in (s_1, s(K)) \ .$$

Let us argue by contradiction. If for $\varepsilon > 0$ small enough, $E(s_2)/E(s_1)$ can be taken arbitrarily small and if $s(K) < s_2$, then

$$U_* - U(s_2) \geq U_* - U(s) \geq U_* - U(s(K)) \geq 0 \quad \forall \, s \in (s(K), s_2)$$

can also be taken arbitrarily small, which contradicts the fact that $s(K) < s_2$. The condition

$$\log \Big(\varepsilon \, U(s_2) \Big) - \frac{C}{2} \leq \log \Big(\varepsilon \, U(s(K)) \Big) - \frac{C}{2} = V(s(K)) < \log \Big(\varepsilon \, U_* \Big) - C$$

indeed means that

$$U(s_2) < e^{-C/2} \, U_* \ ,$$

which is impossible if $\frac{1}{p^*} \, (U^{p^*}(s_2) - U_*^{p^*}) - U_*^{p^*-1}(U(s_2) - U_*) \leq E(s_2)$ is taken arbitrarily small. Remind indeed that $E(s_1) \leq E(0)$ is uniformly bounded in terms of ε.

Thus if $E(s_2)/E(s_1)$ can be taken arbitrarily small, then $s(K) \geq s_2$. This means that $U_* - U(s)$ is either negative or positive but small on $(s_1, s_2) \ni s$:

$$U' = e^V - \varepsilon \, U \leq \varepsilon \, U \, e^{-K} - \varepsilon \, U \quad \forall \, s \in (s_1, s_2) \subset (s_1, s(K))$$

is therefore at most of the order of $-\varepsilon \, U_* \, (1 - e^{-K})$ and there exists a constant $\mu > 0$, uniform in ε such that

$$U' \leq -\mu \, \varepsilon \quad \forall \, s \in (s_1, s_2) \ .$$

Combined with (14), this means that

$$\frac{1}{E} \, \frac{dE}{ds} \geq \frac{\nu}{\mu} \, U'$$

which by integration gives

$$E(s_2)/E(s_1) \geq e^{\frac{\nu}{\mu} (U(s_2) - U(s_1))}$$

and again provides a contradiction with the assumption that $E(s_2)/E(s_1)$ can be taken arbitrarily small. ∎

For any $k \geq 1$, let $s_k(\varepsilon)$ be such that

$$\theta(s_k(\varepsilon)) = -\frac{\pi}{2} + (k-1)\, 2\pi \ .$$

With the definition of $\bar\lambda_k^+$ given in Section 2.5, we get the following result.

PROPOSITION 3.1. *Consider a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} \varepsilon_i = 0$. Then up to the extraction of a subsequence,*

$$\lim_{i\to+\infty} E(s_k(\varepsilon_i)) = \bar\lambda_k^+$$

is positive. Moreover, there exists a constant $\kappa_0 \in (0,1)$ such that

$$\forall\, k \geq 1\ , \quad \bar\lambda_k \geq \kappa_0^{k-1}\, \bar\lambda_1\ .$$

Proof. The fact that $\bar\lambda_k$ is positive is a consequence of Corollary 3.1 and Lemma 3.4. Looking more carefully into the proofs, it holds that

$$E(s_1) \geq e^{-2\pi\,\nu/\omega}\, E(s_0) =: \kappa_1\, E(s_0)$$

in the case of Corollary 3.1 and $E(s_2) \geq \kappa_2\, E(s_1)$ for some $\kappa_2 > 0$ in the case of Lemma 3.4, so that the Proposition holds with $\kappa_0 = \kappa_1 \cdot \kappa_2$. $\square$

REMARK 1. (i) Note that $\bar\lambda_k^+$ may depend on the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$. It is an open question to prove that for each $k \in \mathbb{N}$, $k \geq 2$, the limit as $i \to +\infty$ is actually unique, and to identify the value of $\bar\lambda_k^+$.

(ii) We will see in the next Section that $(\bar\lambda_k^+)_{k\in\mathbb{N}}$ is decreasing and converges to 0. This means that a different phenomenon occurs, compared to multi-bubbling in the slightly supercritical Brezis-Nirenberg problem, where all bubbles are identical up to a scaling factor.

3.2. **Bubbles have different heights.**

LEMMA 3.5. *The sequence $(\bar\lambda_k^+)_{k\geq1}$ is a strictly decreasing sequence of positive numbers.*

Proof. Assume by contradiction that

$$\bar\lambda_{k+1}^+ = \bar\lambda_k^+ =: \bar\lambda \tag{21}$$

for some $k \geq 1$. On the one hand, according to (15),

$$\frac{dE_\varepsilon}{ds} = -\varepsilon\,(|\bar y^\varepsilon(s)| - |y_*|)\, \left(|\bar y^\varepsilon(s)|^{p^*-1} - |y_*|^{p^*-1}\right)$$

for some positive constant $\nu > 0$, where $y_* = -p^{p-1}$ and

$$E_\varepsilon(s) := \bar x^\varepsilon(s) - \varepsilon\, p^{p-1}\, \left[1 - \log\left(\frac{\bar x^\varepsilon(s)}{\varepsilon\, p^{p-1}}\right)\right] + \frac{1}{p^*}\left(|\bar y^\varepsilon(s)|^{p^*} - |y_*|^{p^*}\right) + p\,(\bar y^\varepsilon(s) - y_*)\ .$$

On the other hand, (21) means that there exists sequences $(\varepsilon_i)_{i\in\mathbb{N}}$, $(s_i^1)_{i\in\mathbb{N}}$ and $(s_i^2)_{i\in\mathbb{N}}$ such that:

1. For any $i \in \mathbb{N}$, $\varepsilon_i > 0$, and $\lim_{i\to+\infty} \varepsilon_i = 0$.
2. For any $i \in \mathbb{N}$, $s_i^1 < s_i^2$, and

$$\frac{dy_{\varepsilon_i}}{ds}(s_i^j) = 0 \quad \text{and} \quad \lim_{i\to+\infty} (\bar x_{\varepsilon_i}(s_i^j), \bar y_{\varepsilon_i}(s_i^j)) = (0, \bar y_j)\ , \quad j = 1, 2\ ,$$

where $\bar{y} = \bar{y}_1$, $\bar{y}_2$ are the two solutions of

$$\frac{1}{p^*}\,|\bar{y}|^{p^*} + p\;\bar{y} = \bar{\lambda} - p^{p-1}$$

such that $\bar{y}_1 < -p^{p-1} < \bar{y}_2 \leq 0$. Here we use the conservation of the energy along the limiting trajectory corresponding to $\varepsilon = 0$: if $\frac{dx}{ds} = x\,(|y|^{p^*-2}y + p)$, $\frac{dy}{ds} = -x$, then $\frac{d}{ds}(x + \frac{1}{p^*}\,|y|^{p^*} + p\,y) = 0$.

3. Asymptotically, the energy does not decay on (s_i^1, s_i^2):

$$\lim_{i\to+\infty}\left[E_{\varepsilon_i}(s_i^2) - E_{\varepsilon_i}(s_i^1)\right] = 0\;. \tag{22}$$

Let $\delta := \bar{y}_2 - \bar{y}_1 > 0$. Since

$$\bar{y}'_{\varepsilon_i} = -\varepsilon_i\,\bar{y}_{\varepsilon_i} - \bar{x}_{\varepsilon_i} \leq -\varepsilon_i\,\bar{y}_{\varepsilon_i}\;,$$

it is straightforward to see that

$$\bar{y}_{\varepsilon_i}(s) \leq \bar{y}_{\varepsilon_i}(s_i^1)\,e^{\varepsilon_i\,(s_i^1 - s)} \quad \forall\; s \geq s_i^1\;,$$

which implies that

$$\bar{y}_{\varepsilon_i}(s_i^2) - \bar{y}_{\varepsilon_i}(s_i^1) \leq \bar{y}_{\varepsilon_i}(s_i^1)\,\left(e^{\varepsilon_i\,(s_i^1 - s_i^2)} - 1\right)\;.$$

Since $\lim_{i\to+\infty}\bar{y}_{\varepsilon_i}(s_i^1) = -|\bar{y}_1|$ and $\lim_{i\to+\infty}\left(\bar{y}_{\varepsilon_i}(s_i^2) - \bar{y}_{\varepsilon_i}(s_i^1)\right) = \delta$, this means that asymptotically as $i \to +\infty$,

$$\delta \leq |\bar{y}_1|\,\left(1 - e^{\varepsilon_i\,(s_i^1 - s_i^2)}\right)(1 + o(1))\;,$$

$$\varepsilon_i\,(s_i^2 - s_i^1) \geq \kappa\,(1 + o(1))\;,$$

where $\kappa := -\log\left(1 - \frac{\delta}{|\bar{y}_1|}\right) > 0$.

On $(s_i^2 - s_i^1) \ni s$, if $|y_{\varepsilon_i}(s) - y_*| > \delta/4$, then $E_{\varepsilon_i}(s)$ compares with $E_{\varepsilon_i}(s_i^j)$ which is itself of the same order as $\frac{1}{\varepsilon_i}\,\frac{d}{ds}\,E_{\varepsilon_i}(s)$ since

$$\bar{x}^\varepsilon(s_i^j) - \varepsilon\,p^{p-1}\left[1 - \log\left(\frac{\bar{x}^\varepsilon(s_i^j)}{\varepsilon\,p^{p-1}}\right)\right] = -\varepsilon\,\bar{y}^\varepsilon(s_i^j) - \varepsilon\,p^{p-1}\left[1 - \log\left(\frac{-\bar{y}^\varepsilon(s_i^j)}{p^{p-1}}\right)\right] \to 0$$

as $\varepsilon \to 0$.

Summarizing these estimates, this means that

$$E_{\varepsilon_i}(s_i^2) \leq E_{\varepsilon_i}(s_i^1)\,e^{-\mu} \quad \text{as } i \to +\infty$$

for some $\mu > 0$, a contradiction with (22), since $\bar{\lambda} > 0$ implies $\liminf\limits_{i\to+\infty} E_{\varepsilon_i}(s_i^1) > 0$. ☐

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