Convexity Properties of the Free Boundary and Gradient Estimates for Quasi-linear Elliptic Equations

Jean Dolbeault
Ceremade, U.M.R. C.N.R.S. no. 7534,
Univ. Paris IX-Dauphine, Pl. de Lattre de Tassigny
75775 Paris Cédex 16, France
e-mail: dolbeaul@ceremade.dauphine.fr
and
Régis Monneau
CERMICS, Ecole Nationale des Ponts et Chaussées
6 et 8 avenue Blaise Pascal, Cité Descartes
Champs-sur-Marne, 77455 Marne-La-Vallée, France
e-mail: monneau@cermics.enpc.fr

October 15, 1999

Abstract

This paper is devoted to a proof of the convexity of a free boundary for a quasi-linear problem defined on a convex domain in $\mathbb{R}^2$ and to the obtention of L^∞-bounds on the gradient of a solution. The free boundary can be seen as the boundary of the coincidence set of an obstacle problem. The bound on the gradient explicitely depends on the curvature of the boundary of the domain. The main tool is an estimate of the maximum of the gradient on the level lines, which involves their curvature. The second result is valid in an analytical framework only. We indicate how to extend our results to dimensions higher than 2.

Key-words and phrases: Quasi-linear elliptic equations – Free boundary – Gradient estimates – Fréchet formula – Curvature of level sets – Obstacle problem – Coincidence set

1991 Mathematics Subject Classification: 35J25, 35J67, 35R35.

1 Introduction

Our first result is concerned with the following free boundary problem. Consider a solution of

$$\mathrm{div}\left(a(|\nabla u|^2)\nabla u\right) = f(u) \tag{1.1}$$

in $\Omega \setminus \Lambda^{\underline{t}}$ where $\Lambda^{\underline{t}}$ is a closed set strictly included in a given bounded domain Ω in $\mathbb{R}^2$ such that

$$Const = \underline{t} = u_{|\partial\Lambda^{\underline{t}}} < u < u_{|\partial\Omega} = \bar{t} = Const\,, \tag{1.2}$$

$$\partial_n u = 0 \quad \text{on} \quad \partial\Lambda^{\underline{t}}. \tag{1.3}$$

Here $\partial_n u$ is the normal (to $\partial\Lambda^{\underline{t}}$) outgoing derivative of u.

Theorem 1.1 *Assume that $a(0)$, $f(0) > 0$ and that $q \mapsto a(q)$, $u \mapsto f(u)$ are increasing functions of class C^1 and C^0 respectively. If Ω is convex and if u is a solution of (1.1) on $\Omega \setminus \Lambda^{\underline{t}}$ satisfying Conditions (1.2) and (1.3), then $\Lambda^{\underline{t}}$ is also convex.*

This theorem has been proved in the special case where $a \equiv a_0$ and $f \equiv f_0$ are constants by Friedman and Phillips [12] in two dimensions. Then the result of Friedman and Phillips has been extended to any dimension by Kawohl [15]. Similar results were also proved (in any dimensions) for $a \equiv a_0$, $f \equiv 0$ and $\partial_n u = const > 0$ in place of $\partial_n u = 0$ by Caffarelli and Spruck [6].

Here the notion of solution has to be understood as defined by variational inequalities (see for instance [3, 21]). Except for the final step of the proof of Theorem 1.1, based on approximation and uniqueness arguments, we will work in the framework of analytic solutions, unless it is explicitely specified.

This free boundary problem formally arises from the following obstacle problem. Consider the energy

$$\mathcal{E}(u) = \int_\Omega \left(\frac{1}{2}A(|\nabla u|^2) + G(u)\right) dx \tag{1.4}$$

where A, G are analytic convex functions on $[0, +\infty)$ such that $A'(0)$, $G'(0) > 0$. The Euler-Lagrange equation corresponding to a minimizer of (1.4) is nothing else than (1.1) with $a = A'$ and $f = G'$. Assume that u^0 is a solution of such a problem in Ω with the boundary condition $u^0_{|\partial\Omega} = \bar{t} > 0$ and such that $\min_\Omega u^0 = 0$ and consider for $\underline{t} \in (0, \bar{t})$ the minimizer $u^{\underline{t}}$ of

$$\inf_{\substack{u \in H^1_0(\Omega) + \bar{t} \\ u \geq \underline{t}}} \mathcal{E}(u)\,.$$

Then $u^{\underline{t}}$ formally satisfies the free boundary problem with $\Lambda^{\underline{t}} = \{x \in \overline{\Omega} : u^{\underline{t}}(x) = \underline{t}\}$.

Note that there is a natural way of generalizing Theorem 1.1 to higher dimensions and a formal result in that direction is stated in Section 3.4 (with a sketch of the main steps of the proof).

Our second result is an estimate on the gradient which holds for slightly more general equations than (1.1), but only for analytical solutions. Consider a solution u of

$$\operatorname{div}\left(a(u,|\nabla u|^2)\nabla u\right)=f(u,|\nabla u|^2) \tag{1.5}$$

with constant Dirichlet boundary conditions on a bounded domain Ω in $\mathbb{R}^2$. With the direct trigonometric orientation and for any $x_0\in\Omega$ such that $\nabla u(x_0)\neq 0$, we may define the normal and tangent unit vectors n and τ to the level set $\{x\in\Omega : u(x)=t\}$ respectively by $n=n(x_0)=\frac{\nabla u(x_0)}{|\nabla u(x_0)|}$ and $\tau=-n^\perp$, so that (τ,n) is a direct orthonormal basis in $\mathbb{R}^2$. In theses local coordinates, u is a solution of

$$\alpha(u,|\nabla u|^2)D_{\tau\tau}u+\beta(u,|\nabla u|^2)D_{nn}u=f_0(u,|\nabla u|^2) \tag{1.6}$$

where α, β and f_0 are given in terms of the functions a and f by

$$\alpha(u,q)=a(u,q)\,, \tag{1.7}$$

$$\beta(u,q)=a(u,q)+2a'_q(u,q)q\,, \tag{1.8}$$

$$f_0(u,q)=f(u,q)-2a'_u(u,q)\,.$$

Here (D^2u) is the Hessian of u and $D_{e_1e_2}u=(D^2u)e_1\cdot e_2$ for any e_1, $e_2\in S^1$. At this point, we may notice that if

$$\beta(u,0)\equiv\alpha(u,0) \tag{1.9}$$

on the set $\{x\in\Omega : \nabla u(x)=0\}$, Equation (1.6) is nothing else than

$$\alpha(u,0)\Delta u(x)=f(u(x),0)\,,$$

which still perfectly makes sense even if n and τ are not well defined. Condition (1.9) is of course satisfied if u is a solution of Equation (1.5) and α and β are given by (1.7) and (1.8). Our main assumption is the following ellipticity condition:

$$\inf_{x\in\Omega}\alpha(u,|\nabla u|^2)>0 \quad\text{and}\quad \inf_{x\in\Omega}\beta(u,|\nabla u|^2)>0\,. \tag{1.10}$$

We will not assume that the domain Ω is convex any more. An external sphere condition is indeed sufficient. To each point $x\in\partial\Omega$, we may associate the outgoing normal unit vector $n(x)$ and consider

$$\rho(x)=\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\max\{\rho\geq 0 : B(x,\epsilon)\cap B(x+\rho n(x),\rho)\subset\Omega^c\}\,.$$

The crucial assumption on Ω is

$$\rho_0 = \min_{x\in\partial\Omega} \rho(x) > 0\,, \tag{1.11}$$

and we may then define

$$K_0 = -\frac{1}{\rho_0}\,. \tag{1.12}$$

If Ω is convex, instead of (1.12), we shall consider

$$\tilde\rho(x) = \lim_{\substack{\epsilon\to 0\\ \epsilon>0}} \max\{\rho\geq 0 : B(x,\epsilon)\cap B(x-\rho n(x),\rho)\subset\Omega\}\,,\quad \tilde\rho_0 = \max_{x\in\partial\Omega}\tilde\rho(x)\geq 0$$

and define in that case

$$K_0 = \frac{1}{\tilde\rho_0} \tag{1.13}$$

(with the convention $1/\infty = 0$). With these definitions, K_0 is nothing else than the signed curvature of $\partial\Omega$ (which is positive for a ball).

Theorem 1.2 *Assume that α, β and f_0 are analytic. Consider an analytic solution u of Equation (1.6) in Ω such that $u_{|\partial\Omega} = \bar t > 0$ and $\min_{x\in\Omega} u = 0$ and assume that Conditions (1.9), (1.10) and (1.11) are satisfied. Then there exists a continuous function $M(t,K)$ such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leq M(\bar t, K_0)$$

where K_0 is defined either by (1.12), or by (1.13) if Ω is convex.

The heuristic idea of the proof of Theorems 1.1 and 1.2 is the following: use the value t of the solution u to evaluate quantities q_i related to u which are bounded by the solution $\bar q_i$ of a corresponding ordinary differential equation (given by the radial case for instance). A "first order development" along the level lines has been used in [9]. Here we perform a "second order development". Actually, we obtain a system of two ordinary differential equations, which involves $q_1(t) = \sup_{u(x)=t}|\nabla u(x)|$ and $q_2(t) = K(t)$, which is the curvature of the level line $\{x\in\Omega : u(x) = t\}$ at the point realizing the maximum for $q_1(t)$. Higher order developments could in principle be used, but the number of the quantities q_i increases in such a way that practically only the first and the second order developments are interesting.

This method is easy to understand in the radially symmetric case. Assume that $\Omega = B(0,R_0)$ and consider a radial solution of Equation (1.6). First let $a = 1$ ($\alpha = \beta = 1$) and define for any $t\in(\underline{t},\bar t)$ the function $K(t)$ such that

$$t = u\left(\frac{1}{K(t)}\right) \tag{1.14}$$

($\underline{t}=0$ in the case of Theorem 1.2). If $m(t)=u'\left(\frac{1}{K(t)}\right)$, a derivation of (1.14) with respect to t gives

$$\dot{K}=-\frac{K^2}{m}\,, \tag{1.15}$$

while Equation (1.6) becomes

$$\dot{m}=\frac{f(t,m^2)}{m}-K\,. \tag{1.16}$$

Here $(\dot{\ })$ and $(\ )'$ respectively denote the derivatives with respect to t and r. Integrating Equations (1.15)-(1.16) from $\underline{t}$ to $\bar{t}$ with the initial values $m(\underline{t})=0$ and $\lim_{t\to\underline{t}_+}\frac{1}{K(t)}=0$, we obtain the result of Theorem 1.2 with

$$M(\bar{t},K_0)\geq\max_{s\in[\underline{t},\bar{t}]}m_0(s)\geq m(t)\quad\forall\, t\in[\underline{t},\bar{t}]\,,\quad K_0=\frac{1}{R_0}>0>-1\,,$$

where $t\mapsto m_0(t)$ is the solution of

$$\dot{m}_0=\frac{f(t,m_0^2)}{m_0}+1\,,\quad m_0(\underline{t})=0\,. \tag{1.17}$$

Note that

$$\frac{1}{2}\,\frac{d}{dt}(m_0^2-m^2)>\frac{\partial f}{\partial q}(t,m_0^2)(m_0^2-m^2)+\frac{1}{m+m_0}(m_0^2-m^2)+o(m_0^2-m^2)$$

as $t\to\underline{t}$, thus proving that $m_0(t)>m(t)$ for $t-\underline{t}>0$, small. It is then easy to prove that $m_0(t)=m(t)$ is impossible for a larger t (see (2.24) for the dependence in K_0 in the non-radial case).

The general radial case ($a\not\equiv 1$) is not much more difficult. Consider a system of cartesian coordinates (x_1,x_2):

$$\tau=\frac{1}{r}\begin{pmatrix} x_2\\ -x_1\end{pmatrix}\,,\quad n=\frac{1}{r}\begin{pmatrix} x_1\\ x_2\end{pmatrix}\,,$$

$$(D^2u)=\frac{1}{r^3}\begin{pmatrix} (ru''x_1^2+u'x_2^2) & (ru''-u')x_1x_2\\ (ru''-u')x_1x_2 & (ru''x_2^2+u'x_1^2)\end{pmatrix}\,,$$

$D_{\tau\tau}u=\frac{u'}{r}=mK$, $D_{nn}u=u''=m\dot{m}$. The equation for K is still (1.15) while the equation for m has to be replaced by

$$\dot{m}=\frac{f_0(t,m^2)}{m\beta(t,m^2)}-\frac{\alpha(t,m^2)}{\beta(t,m^2)}K\,. \tag{1.18}$$

When Ω is not a ball, Equation (1.18) still holds while Equation (1.15) is replaced by an inequality, but $K(t)$ is still bounded from below by a decreasing function, which is itself bounded from below by K_0. The conclusion of Theorem 1.1 then holds because $\partial\Lambda^{\underline{t}}$ has a nonnegative curvature if Ω is convex: see Sections 2 and 3 for more details. Arguments based on analyticity have been rejected in Sections 4 and 5.

The method applies to more general equations than (1.6) for which the key argument (Theorem 2.1) is still true. For fully nonlinear isotropic homogenous equations like

$$\mathcal{G}(D_{nn}u, D_{\tau\tau}u, |\nabla u|^2, u) = 0\,,$$

α and β would then be defined by

$$\alpha = (\mathcal{G})'_{D_{nn}u} \quad \text{and} \quad \beta = (\mathcal{G})'_{D_{\tau\tau}u}$$

(with the ellipticity condition: $\alpha > 0$ and $\beta > 0$, similar to (1.10)).

2 Technical results, proof of Theorem 1.2

In this section, we assume that u is an analytic solution of Equation (1.6) on $\Omega \setminus \Lambda^{\underline{t}}$ (with eventually $\underline{t} = 0$ or $\Lambda^{\underline{t}} = \emptyset$). For $t > \underline{t}$, let

$$\Gamma^t = \{x \in \Omega : u(x) = t\}\,, \quad m(t) = \max_{y \in \Gamma^t} |\nabla u(y)|\,,$$

$$\text{and} \quad X^t = \{x \in \Gamma^t : |\nabla u(x)| = m(t)\}\,.$$

Note that $m(t) \geq 0$ for any $t \in [\underline{t}, \overline{t}]$ by definition of m. For any t such that $m(t) > 0$, Γ^t is an analytic curve near any $x \in X^t$ and m has the following properties.

Lemma 2.1 *Consider an analytic solution u of Equation (1.6) such that Conditions (1.9) and (1.10) are satisfied.*

i) $m(t) > 0$ for any $t > \underline{t}$ as soon as

$$f_0(u,q) \geq 0 \text{ for } (u,q) \in [\underline{t}, \overline{t}] \times [0, \delta] \text{ for some } \delta > 0\,. \tag{2.1}$$

ii) If α, β and f_0 are functions of $(u, q = v^2)$ such that

$$\min_{x \in \Omega} \mathcal{F}(u(x), |\nabla u(x)|, D_{nn}u(x)) \geq 0\,, \tag{2.2}$$

$$\text{with} \quad \mathcal{F}(u,v,p) = \frac{1}{v\alpha}\left(f_0 - \beta p\right)^2 - v\frac{\alpha'_u}{\alpha}\left(f_0 - \beta p\right) + v\,(f_0)'_u\,, \tag{2.3}$$

then $t \mapsto m(t)$ is non-decreasing and $v^2 = |\nabla u|^2$ reaches its maximum on $\partial\Omega$ (here $D_{nn}u = \partial_n v = n \cdot \nabla v$).

As mentioned in the introduction, the main estimate is given by the generalization to the non radial case of Equations (1.15) and (1.18).

Theorem 2.1 *Consider an analytic solution* u *of Equation (1.6) such that Conditions (1.9) and (1.10) are satisfied. With the above notations,* m *and* K *are solutions in the sense of distributions on* $(\underline{t},\bar{t})$ *of*

$$\dot{m} = \frac{f_0(t,m^2)}{m\beta(t,m^2)} - \frac{\alpha(t,m^2)}{\beta(t,m^2)} K \,, \tag{2.4}$$

$$\dot{K} \le -\frac{K^2}{m} - \frac{2}{m}\sqrt{\frac{\alpha(t,m^2)}{\beta(t,m^2)}}\,|\partial_\tau K| \,. \tag{2.5}$$

Theorem 2.1 will be proved in all generality in section 4. Nevertheless, we give at the end of this section a proof of this theorem in a particular case.

Notations. In the following, we denote by ∂_τ the derivative along the curvilinear coordinate, on the (local) curve Γ^t: for a function v defined on a neighbourhood of x_0 in Ω, $(\partial_\tau v)(x_0) = \nabla v(x_0)\cdot\tau$ for $\tau = \tau(x_0) = \frac{1}{|\nabla u(x_0)|}\begin{pmatrix} \frac{\partial u}{\partial x_2}(x_0) \\ -\frac{\partial u}{\partial x_1}(x_0) \end{pmatrix}$ if at least $\nabla u(x_0) \neq 0$. We can similarly define the normal derivative $\partial_n = n\cdot\nabla$ for $n = n(x_0) = \frac{1}{|\nabla u(x_0)|}\begin{pmatrix} \frac{\partial u}{\partial x_1}(x_0) \\ \frac{\partial u}{\partial x_2}(x_0) \end{pmatrix} = \tau^\perp$. With these notations, the Fréchet formula are

$$\partial_\tau n = K\tau \,, \quad \partial_\tau \tau = -Kn \,, \tag{2.6}$$

$$\partial_n n = \rho\tau \,, \quad \partial_n \tau = -\rho n \,, \tag{2.7}$$

where $K = \frac{1}{|\nabla u|} D_{\tau\tau}u = \frac{1}{|\nabla u|}(D^2u)\tau\cdot\tau$ is the curvature of the level line and $\rho = \frac{1}{|\nabla u|} D_{\tau n}u = \frac{1}{|\nabla u|}(D^2u)\tau\cdot n$. One may indeed write $\partial_\tau u = 0$ and derivations with respect to τ and n respectively provide

$$0 = \partial_\tau(\tau\cdot\nabla u) = -Kn\cdot\nabla u + \tau\cdot((D^2u)\tau) = -K|\nabla u| + D_{\tau\tau}u \,,$$

$$0 = \partial_n(\tau\cdot\nabla u) = -\rho n\cdot\nabla u + \tau\cdot((D^2u)n) = -\rho|\nabla u| + D_{\tau n}u \,.$$

Note that in general $\partial_\tau(\partial_\tau u) \neq D_{\tau\tau}u$ and $\partial_n(\partial_\tau u) \neq D_{n\tau}u$. With a straightforward abuse of notations, we may define $K(t) = K(x^t) = \frac{D_{\tau\tau}u}{|\nabla u|}(x^t)$ while the quantity $\partial_\tau K(t)$ in Inequality (2.5) is nothing else than $\partial_\tau(\frac{D_{\tau\tau}u}{|\nabla u|})_{|x=x^t}$.

With these notations, we can now prove Lemma 2.1 and Theorem 2.1. Note that only Property i) of Lemma 2.1 will be used in Section 3. The computations for proving ii) are however a good introduction to the last part of Section 2.

Proof of Lemma 2.1.
i) If (2.1) is satisfied, then $m(t) > 0$ follows from Hopf's lemma.
ii) Let us prove the monotonicity of $t \mapsto m(t)$. Consider $v = \partial_n u = n\cdot\nabla u = |\nabla u|$. The following computations are done in the set $\{x\in\Omega : v(x) > 0\}$. First,

$$\partial_n\Big(|\nabla u|^2\Big) = 2vD_{nn}u \,, \tag{2.8}$$

and using the Fréchet formula and $0=\partial_\tau u$, we also get

$$\partial_n v = D_{nn} u\,, \tag{2.9}$$

$$\partial_\tau v = D_{n\tau} u\,, \tag{2.10}$$

$$D_{nn} v = D_{nnn} u + \frac{1}{v}(D_{n\tau} u)^2\,, \tag{2.11}$$

$$D_{\tau\tau} v = D_{n\tau\tau} u + \frac{1}{v}(D_{\tau\tau} u)^2\,. \tag{2.12}$$

Using (2.8) and (2.9),

$$\partial_n \Big(|\nabla u|^2 \Big) = 2v\partial_n v\,,$$

while using the Fréchet formula again and Equations (2.10), (2.11) and (2.12), we obtain

$$\partial_n(D_{\tau\tau}u) = D_{\tau\tau}v - \frac{1}{v}(D_{\tau\tau}u)^2 - \frac{2}{v}(\partial_\tau v)^2\,, \quad \partial_n(D_{nn}u) = D_{nn}v + \frac{1}{v}(\partial_\tau v)^2\,.$$

Consider now Equation (1.6) and apply the operator ∂_n:

$$\begin{aligned} 0 = &\ \partial_n \Big[\alpha(u,v^2) D_{\tau\tau} u + \beta(u,v^2) D_{nn} u - f_0(u,v^2) \Big] \\ = &\ \alpha(u,v^2) D_{\tau\tau} v + \beta(u,v^2) D_{nn} v + B(u,v,\nabla v)\cdot\nabla v - \mathcal{F}(u,v^2,\partial_n v) \end{aligned}$$

where

$$B(u,v,\nabla v) = v\Big[\beta'_u + 2\alpha'_q D_{\tau\tau}u + 2\beta'_q \partial_n v - 2(f_0)'_q\Big] n + \frac{\partial_\tau v}{v}(\beta - 2\alpha)\tau\,,$$

$$\mathcal{F} = \frac{\alpha}{v}(D_{\tau\tau}u)^2 - v\alpha'_u D_{\tau\tau}u + v(f_0)'_u\,.$$

Using again Equation (1.6), we get

$$D_{\tau\tau}u = \frac{1}{\alpha}(f_0 - \beta\partial_n v)$$

which provides

$$B(u,v,\nabla v) = v\Big[\beta'_u + 2\frac{\alpha'_q}{\alpha}(f_0 - \beta\partial_n v) + 2\beta'_q\partial_n v - 2(f_0)'_q\Big] n + \frac{\partial_\tau v}{v}(\beta - 2\alpha)\tau$$

and Equation (2.3). When Condition (2.2) is satisfied, Hopf's Lemma applied to v leads to the conclusion. □

Proof of Theorem 2.1 in a particular case. In this proof we assume that locally in t, X^t is supported in an analytic curve $t \mapsto x^t$ such that $t = u(x^t)$. The justification of such an assumption will be given in Section 4.

Equation for m: By definition of x^t, $0=\partial_\tau(|\nabla u|^2)(x^t)=2\partial_n u(x^t)D_{n\tau}u(x^t)$ and according to Lemma 2.1, $m(t)=\partial_n u(x^t)>0$: for any $t>0$,

$$D_{n\tau}u(x^t)=0\,. \tag{2.13}$$

According to the assumption made on x^t: $t=u(x^t)$, a derivation with respect to t provides

$$1=\nabla u(x^t)\cdot\frac{dx^t}{dt}=\partial_n u(x^t)\cdot(n\cdot\frac{dx^t}{dt})\,,$$

$$n\cdot\frac{dx^t}{dt}=\frac{1}{m(t)}\,. \tag{2.14}$$

Then according to (2.13) and (2.14),

$$\begin{aligned} m\dot{m}=\frac{d}{dt}(\frac{1}{2}|\nabla u(x^t)|^2) \; &=\nabla u(x^t)\cdot(D^2u)\frac{dx^t}{dt}\\ &=n\cdot(D^2u)\left(n+m\,(\frac{dx^t}{dt}\cdot\tau)\tau\right)=D_{nn}u(x^t)\,. \end{aligned}$$

By definition of

$$K(t)=\frac{1}{m(t)}D_{\tau\tau}u(x^t)\,, \tag{2.15}$$

and using Equation (1.6), we obtain

$$\alpha mK+\beta m\dot{m}=f_0\,,$$

thus proving equation (2.4).

Inequation for K: Consider

$$h(t)=\frac{dx^t}{dt}\cdot\tau(x^t)\,.$$

We will use the notation $\partial_\tau K$ for $\partial_\tau(\frac{D_{\tau\tau}u}{|\nabla u|})_{|x=x^t}$ and $\partial_n K$ for $\partial_n(\frac{D_{\tau\tau}u}{|\nabla u|})_{|x=x^t}$ and we omit to specify that $x=x^t$. By definition of x^t,

$$0=\bar{F}(t)=\partial_\tau(\frac{1}{2}|\nabla u|^2)_{|x=x^t}=\partial_n u\,D_{n\tau}u\,,$$

so

$$0=\frac{1}{m}\,\frac{d(\bar{F}/m)}{dt}(t)=\frac{1}{m}\partial_n(D_{n\tau}u)+h\partial_\tau(D_{n\tau}u)\,, \tag{2.16}$$

and

$$0\geq\bar{G}(t)=(\partial_\tau)^2(\frac{1}{2}|\nabla u|^2)_{|x=x^t}=m\partial_\tau(D_{n\tau}u)\,,$$

$$\partial_\tau (D_{n\tau} u) = \frac{\bar{G}}{m} \,. \tag{2.17}$$

Combining (2.16) and (2.17), we get

$$h\bar{G} = -\partial_n (D_{n\tau} u) \,. \tag{2.18}$$

On the other hand,

$$\partial_\tau K = \frac{1}{m} D_{\tau\tau\tau} u$$

since $D_{n\tau} u = 0$,

$$D_{\tau\tau\tau} u = m \partial_\tau K \,, \tag{2.19}$$

and applying the operator ∂_τ to Equation (1.6), we get

$$\alpha D_{\tau\tau\tau} u + \beta D_{nn\tau} u = 0 \,,$$

which together with (2.19) gives

$$D_{nn\tau} u = -\frac{\alpha}{\beta} m \partial_\tau K \,. \tag{2.20}$$

Because of the Fréchet formula (2.7) and since $\rho = \frac{D_{n\tau} u}{|\nabla u|} = 0$, $\partial_n (D_{n\tau} u) = D_{nn\tau} u$: Equations (2.18) and (2.20) therefore imply

$$h\bar{G} = \frac{\alpha}{\beta} m \, \partial_\tau K \,. \tag{2.21}$$

Let us compute now

$$\dot{K} = \frac{1}{m} \partial_n K + h \partial_\tau K \,,$$

$$\partial_n K = \frac{1}{m} D_{n\tau\tau} u - \frac{1}{m^2} D_{nn} u D_{\tau\tau} u \,, \tag{2.22}$$

and with (2.17) again,

$$\frac{\bar{G}}{m} = D_{n\tau\tau} u + K (D_{\tau\tau} u - D_{nn} u) \,,$$

$$D_{n\tau\tau} u = \frac{\bar{G}}{m} - m K^2 + \frac{1}{m} (D_{\tau\tau} u)(D_{nn} u) \,,$$

which together with (2.22) gives

$$\partial_n K = \frac{\bar{G}}{m^2} - K^2 \,,$$

and therefore

$$\dot{K} = -\frac{K^2}{m} + \frac{\bar{G}}{m^3} + h \partial_\tau K \,. \tag{2.23}$$

The system of equations (2.21) and (2.23) is now equivalent to:

1) either

$$\partial_\tau K = 0 \quad \text{and} \quad \dot{K} = -\frac{K^2}{m} + \frac{\bar{G}}{m^3}\,, \quad h\bar{G} = 0\,,$$

2) or

$$\partial_\tau K \neq 0\,, \quad \bar{G} < 0 \quad \text{and} \quad \dot{K} = -\frac{K^2}{m} + \frac{\bar{G}}{m^3} + \frac{\alpha}{\beta} m(\partial_\tau K)^2 \cdot \frac{1}{\bar{G}}\,.$$

In this last case, an optimization on $\bar{G}$ gives

$$\dot{K} \leq -\frac{K^2}{m} - 2\sqrt{\frac{\alpha}{\beta}} \cdot \frac{|\partial_\tau K|}{m}$$

(which is of course also true in case 1)). □

The rest of the proof of Theorem 2.1 (*i.e.* when X^t is not locally supported in an analytic curve) will be given in Section 4. It is now easy to deduce Theorem 1.2 from Theorem 2.1 (here $\underline{t} = 0$).

Proof of Theorem 1.2. K is decreasing and $\liminf_{\substack{t \to \bar{t} \\ t < \bar{t}}} K(t) \geq K_0$. Then

1) either $K_0 \leq 0$ and

$$\dot{m} \leq \frac{f_0}{m\beta} + \frac{\alpha}{\beta}|K_0|\,,$$

2) or $K_0 \geq 0$ and $-\frac{\alpha}{\beta}K \leq 0$,

$$\dot{m} \leq \frac{f_0}{m\beta}\,.$$

In both cases, $m(0) = 0$, and if we denote by $(K_0)_-$ the negative part of K_0, m is bounded by the solution of

$$\frac{dm_0}{dt} = \frac{f_0(t, m_0^2)}{m_0 \beta(t, m_0^2)} + \frac{\alpha}{\beta}(t, m_0^2)(K_0)_-\,, \quad m_0(0) = 0\,, \tag{2.24}$$

since as $t \to 0$,

$$\frac{1}{2}\,\frac{d}{dt}(m_0^2 - m^2) > \frac{\partial f}{\partial q}(t, m_0^2)(m_0^2 - m^2) + \frac{(K_0)_-}{m + m_0}(m_0^2 - m^2) + o(m_0^2 - m^2)\,,$$

thus proving that $m_0(t) \geq m(t)$ for $t > 0$, small (if $K_0 \geq 0$, replace $(K_0)_-$ by ϵ and take the limit $\epsilon \to 0$, $\epsilon > 0$). It is then easy to prove that $m_0(t) = m(t)$ is impossible for a larger t: $m(t) < m_0(t)$ for any $t > 0$, which ends the proof. □

3 Proof of Theorem 1.1 and extensions

In this section we will prove Theorem 1.1. We first consider an extension of Problem (1.1)-(1.2)-(1.3) for which we state a list of results with sketches of the proofs (except for Proposition 3.3). Then we prove Theorem 1.1 and Proposition 3.3, and state the natural extension of Theorem 1.1 to dimensions higher than 2.

3.1 An Extension of Problem (1.1)-(1.2)-(1.3)

Consider

$$\begin{cases} \alpha(u,|\nabla u|^2)D_{\tau\tau}u+\beta(u,|\nabla u|^2)D_{nn}u=f_0(u,|\nabla u|^2) \text{ in } \Omega\backslash\Lambda^{\underline{t}} \\ \underline{t}=u_{|\partial\Lambda^{\underline{t}}}\leq u_{|\partial\Omega}=\overline{t} \\ \partial_n u=const=\lambda\geq 0 \text{ on } \partial\Lambda^{\underline{t}} \end{cases} \tag{3.1}$$

where $\partial\Omega$, $\partial\Lambda^{\underline{t}}$, α, β, f_0 are analytic and α, β satisfy Assumptions (1.9)-(1.10) of the introduction. We assume moreover that $f_0(\underline{t},0)>0$ in case $\lambda=0$.

We say that (3.1) has *analytic solutions* if $\partial\Lambda^{\underline{t}}$ is analytic and we denote by $u^{\underline{t}}$ the corresponding solution (see for instance [5]).

The exterior problem in $\Omega=\mathbb{R}^2\backslash\mathcal{O}$ with $\mathcal{O}$ convex and $\alpha\equiv\beta\equiv 1$ has been studied for $\lambda>0$ by Kawohl [19], Hamilton [13] and for $\lambda=0$ by Kawohl [18]. Let us also mention two results on convex rings [6, 7]. For questions on the convexity of the level sets, we refer to [16].

We start with a perturbation result.

Proposition 3.1 *If $u^{\underline{t}_0}$ is an analytic solution to the free boundary problem (3.1) for $\underline{t}=\underline{t}_0$, then there exists an $\eta>0$ such that (3.1) has analytic solutions for every $\underline{t}\in(\underline{t}_0-\eta,\underline{t}_0+\eta)$. Moreover the map $\underline{t}\mapsto\partial\Lambda^{\underline{t}}$ is C^∞ (and $\partial\Lambda^{\underline{t}}$ is of class C^∞).*

Remark 3.1 *A particular version of Proposition 3.1 is proved in [2] for problem (1.1)-(1.2)-(1.3) with $f(u)=f(0)+f'(0)\cdot u$.*

Sketch of the proof of Proposition 3.1. From the assumption of Proposition 3.1, we have $\partial\Omega$, $\partial\Lambda^{\underline{t}}$ of class C^∞, and we can apply the Nash-Moser inverse function theorem as in [2] to prove that (3.1) has a solution $u^{\underline{t}}$ for $\underline{t}$ in a neighbourhood of $\underline{t}_0$ with a smooth free boundary $\partial\Lambda^{\underline{t}}\in C^\infty$. We conclude with the help of the following result on the regularity of the free boundary due to Kinderleherer and Nirenberg [20]:

Lemma 3.1 *Under the previous assumptions on the analytic problem (3.1), if the free boundary $\partial\Lambda^{\underline{t}}$ is C^1 and $u^{\underline{t}}$ is C^2 up to the free boundary, then the free boundary $\partial\Lambda^{\underline{t}}$ is analytic.* □

Actually the perturbation result holds in a neighborhood of $\overline{t}$.

Proposition 3.2 *There exists $\eta>0$, such that for every $\underline{t}$ in $(\overline{t}-\eta,\overline{t}\ ]$, the free boundary problem (3.1) has an analytic solution. Moreover the map $\underline{t}\mapsto\partial\Lambda^{\underline{t}}$ is C^∞ (and $\partial\Lambda^{\underline{t}}$ is of class C^∞).*

Remark 3.2 *A particular version of Proposition 3.2 is also proved in [2] for problem (1.1)-(1.2)-(1.3) with $f(u)=f(0)+f'(0)\cdot u$.*

Sketch of the proof of Proposition 3.2. For $\underline{t} = \overline{t}$, the function $u^{\underline{t}} \equiv \overline{t}$ is a solution with $\Lambda^{\underline{t}} = \overline{\Omega}$. This problem is then degenerate in $\underline{t} = \overline{t}$. Nevertheless, as in [2], we can apply a Nash-Moser approach in this degenerate case which proves Proposition 3.2. □

The last result on Problem (3.1) is the following

Proposition 3.3 *Every analytic solution $u^{\underline{t}}$ to the free boundary problem (3.1) satisfies:*

$$\inf_{\partial\Lambda^{\underline{t}}} K \geq \inf_{\partial\Omega} K \,.$$

The proof of this result is deferred to Section 3.3.

Remark 3.3

(i) The "coincidence set" $\Lambda^{\underline{t}}$ may have several connected components.

(ii) In Proposition 3.3 we get a global bound from below on the curvature of the free boundary. We must cite the remarkable work of Schaeffer [26] where for an obstacle problem of the type $\Delta u = f$, he proves the existence of a local bound from below on the curvature $K(x_0)$ of the free boundary at a point x_0. For his proof the main tool is a quasiconformal mapping.

(iii) Propositions 3.1, 3.2 and 3.3 are true for the more general problem where λ in (3.1) is replaced by an analytic function $\lambda(\underline{t}, K) \geq 0$ such that $K \mapsto \lambda(\underline{t}, K)$ is nonincreasing (in the case $\sup_{\partial\Lambda^{\underline{t}}} \lambda(\underline{t}, K) = 0$ we have to do the additional assumption $f_0(\underline{t}, 0) > 0$).

Let $I = \{\underline{t} \in (0, \overline{t}\ ],$ *(3.1) has an analytic solution* $u^{\underline{t}}\}$. From Proposition 3.1, we know that I is an open set and the map $I \ni \underline{t} \mapsto \partial\Lambda^{\underline{t}}$ is smooth. From Proposition 3.2 we have $I \neq \emptyset$. By a classical argument of connexity, if we can prove that I is closed, then we would have $I = (0, \overline{t}\ ]$ and $\partial\Lambda^{\underline{t}}$ is diffeomorphic to $\partial\Omega$ for any $\underline{t} \in I$. In particular if Ω is convex we would conclude from Proposition 3.3 that $\Lambda^{\underline{t}}$ is convex too (and as a consequence, $\Lambda^{\underline{t}}$ has only one connected component).

The difficulty is now to prove that I is closed. In general this is not the case (see examples due to Schaeffer [25] for singular free boundaries for the obstacle problem in case Ω is not convex). Nevertheless for the obstacle problem ($\lambda = 0$) we will prove that I is closed if Ω is convex (Theorem 1.1).

Remark 3.4 *We will prove Theorem 1.1 in the context of analytic solutions but as soon as the solution is the limit of an approximating sequence of solutions in an analytical framework, the result holds as well. This argument applies for instance when existence and uniqueness results can be proved, which is true for the obstacle problem (1.1)-(1.2)-(1.3) with $a \in C^1$ and $f \in C^0$ but is not known in the most general case of Problem 3.1.*

3.2 Proof of Theorem 1.1

The main advantage of the obstacle problem (1.1)-(1.2)-(1.3) compared to the more general free boundary problem (3.1) is that there exists a unique weak solution (see [24, 11]), and this solution is bounded in $W^{2,\infty}$ (see [11, 10, 3, 1]). As a consequence of this uniqueness the map $\underline{t}\mapsto u^{\underline{t}}\in W^{2,p}$ is continuous for every $p\in(1,+\infty)$. Moreover from the nondegeneracy lemma (see Caffarelli [4], and for example [22]), we have

Lemma 3.2 *Consider a solution of Problem (1.1)-(1.2)-(1.3). Under the assumptions of Theorem 1.1, for every $t^*\in[0,\overline{t}]$,*

$$\lim_{\underline{t}\to\underline{t}^*}\Lambda^{\underline{t}}=\Lambda^{\underline{t}^*} \tag{3.2}$$

$$\textit{and}\quad |\partial\Lambda^{\underline{t}}|=0\,. \tag{3.3}$$

Let us prove that $\underline{t}^*$ defined by

$$\underline{t}^*=\inf\{\underline{t}_0\in(0,\overline{t}\,],\forall\underline{t}\in[\underline{t}_0,\overline{t}\,],\ \text{(1.1)-(1.2)-(1.3) has an analytic solution } u^{\underline{t}}\}$$

is actually 0. Assume by contradiction that $\underline{t}^*>0$. From Proposition 3.3 and Lemma 3.2, (3.2), we deduce that $\Lambda^{\underline{t}^*}$ is convex and $\inf_{\partial\Lambda^{\underline{t}^*}}K\geq\inf_{\partial\Omega}K$.

Case $Int(\Lambda^{\underline{t}^*})=\emptyset$ **:** $|\Lambda^{\underline{t}^*}|=|\partial\Lambda^{\underline{t}^*}|=0$ from Lemma 3.2, (3.3). In this case there is no really free boundary, i.e. the solution $u^{\underline{t}}$ satisfies the Euler-Lagrange equation (1.1) of the energy (1.4) without constraints. Consequently the uniqueness of the weak solution to the free boundary problem implies that $u^{\underline{t}^*}=u^0$, and because we assumed that $\min_\Omega u^0=0$, we get $\underline{t}^*=0$, a contradiction.

Case $Int(\Lambda^{\underline{t}^*})\neq\emptyset$ **:** We use the following result due to Caffarelli [4]:

Lemma 3.3 *Under the previous assumptions on the obstacle problem (1.1)-(1.2)-(1.3), if the coincidence set $\Lambda^{\underline{t}}$ is convex and if $Int(\Lambda^{\underline{t}^*})\neq\emptyset$, then $\partial\Lambda^{\underline{t}^*}$ is C^1 and $u^{\underline{t}^*}$ is C^2 up to the free boundary.*

Lemma 3.1 therefore implies that the free boundary is analytic. Finally $\underline{t}^*\in I$, and Proposition 3.1 gives a contradition with the definition of $\underline{t}^*$.

If a and f are not analytic but only of class C^1 and C^0 respectively, Remark 3.4 applies. This ends the proof of Theorem 1.1. □

3.3 Proof of Proposition 3.3

From Morrey [23], we see that $u=u^{\underline{t}}$ is analytic up to the free boundary $\partial\Lambda^{\underline{t}}$, because $\partial\Lambda^{\underline{t}}$ is analytic itself. Then we search for the points x^t where $|\nabla u|$ is maximum on $\Gamma^t=\{u=t\}$ as $t\to\underline{t}$.

Case $\lambda > 0$: Let $x_0 \in \partial\Lambda^{\underline{t}}$ and $\gamma_{x_0}(t)$ be the integral curve of the vector field n such that $\gamma_{x_0}(\underline{t}) = x_0$ and $u(\gamma_{x_0}(t)) = t$. Then

$$\gamma_{x_0}(t) = x_0 + \frac{t-\underline{t}}{\lambda} n(x_0) + o(t-\underline{t}) \text{ for } t \geq \underline{t},$$

$$|\nabla u(\gamma_{x_0}(t))| = \lambda + (t-\underline{t}) \frac{D^2_{nn} u(x_0)}{\lambda} + o(t-\underline{t}),$$

which gives, using (3.1),

$$|\nabla u(\gamma_{x_0}(t))| = \lambda + (t-\underline{t}) \left(\frac{f_0(\underline{t},0)}{\lambda \alpha(\underline{t},0)} - \frac{\beta(\underline{t},0)}{\alpha(\underline{t},0)} K(x_0) + \epsilon_{x_0}(t-\underline{t}) \right), \tag{3.4}$$

for some continous function ϵ_{x_0} which tends to 0 uniformly in $x_0 \in \partial\Omega$:

$$\epsilon(t-\underline{t}) = \sup_{x_0 \in \partial\Omega} |\epsilon_{x_0}(t-\underline{t})| \to 0 \quad \text{as} \quad t \to \underline{t}, \quad t > \underline{t}.$$

For t close enough to $\underline{t}$, the map $x_0 \mapsto \gamma_{x_0}(t)$ is a diffeomorphism from $\partial\Lambda^{\underline{t}}$ onto $\Gamma^t = \{u = t\}$. Then for every $x^t \in X^t$ such that $t \to \underline{t}$, $t > \underline{t}$, there exists a unique $x_0^t \in \partial\Lambda^{\underline{t}}$ such that $x^t = \gamma_{x_0^t}(t)$ and then:

$$\begin{aligned} \sup\nolimits_{x_0 \in \partial\Lambda^{\underline{t}}} |\nabla u(\gamma_{x_0}(t))| &= |\nabla u(\gamma_{x_0^t}(t))| \\ &= \lambda + (t-\underline{t}) \left(\tfrac{f_0(\underline{t},0)}{\lambda\alpha(\underline{t},0)} - \tfrac{\beta(\underline{t},0)}{\alpha(\underline{t},0)} K(x_0^t) + \epsilon_{x_0^t}(t-\underline{t}) \right). \end{aligned} \tag{3.5}$$

Thus

$$\sup_{x_0 \in \partial\Lambda^{\underline{t}}} |\nabla u(\gamma_{x_0}(t))| = \lambda + (t-\underline{t}) \left(\frac{f_0(\underline{t},0)}{\lambda\alpha(\underline{t},0)} - \frac{\beta(\underline{t},0)}{\alpha(\underline{t},0)} \inf_{x_0 \in \partial\Lambda^{\underline{t}}} K(x_0) + O(\epsilon(t-\underline{t})) \right). \tag{3.6}$$

Let $X^{\underline{t}} = \{x_0 \in \partial\Lambda^{\underline{t}} : K(x_0) = \inf_{\partial\Lambda^{\underline{t}}} K\}$. We will prove the

Lemma 3.4 *With the above notations,*

$$d(x_0^t, X^{\underline{t}}) \to 0 \text{ as } t \to \underline{t}.$$

Then $x_0^t \to x_0 \in X^{\underline{t}}$ and $x^t = \gamma_{x_0^t}(t) \to x_0$. Consequently $K(x^t) \to K(x_0) = \inf_{\partial\Lambda^{\underline{t}}} K$ and for $t > \underline{t}$, $K(t) = \inf_{x^t \in X^t} K(x^t) \to \inf_{\partial\Lambda^{\underline{t}}} K$. Now from Theorem 2.1, we have $\dot{K} \leq 0$, thus

$$K(t) \geq K(\overline{t}),$$

and passing to the limit $t \to \underline{t}$ we get

$$\inf_{\partial\Lambda^{\underline{t}}} K = K(\underline{t}) \geq K(\overline{t}) \geq \inf_{\partial\Omega} K,$$

which proves the proposition in case $\lambda > 0$.

Proof of Lemma 3.4. If the lemma was false, then up to extraction of some subsequence, x_0^t would tend to $x_1 \in \partial\Lambda^{\underline{t}}$ as $t \to \underline{t}^+$ with $x_1 \notin X^{\underline{t}}$, i.e.

$$K(x_1) > \inf_{\partial\Lambda^{\underline{t}}} K\,. \tag{3.7}$$

In this case
$$\frac{|\nabla u(\gamma_{x_0^t}(t))| - \lambda}{t - \underline{t}} = \frac{\sup_{x_0 \in \partial\Lambda^{\underline{t}}} |\nabla u(\gamma_{x_0}(t))| - \lambda}{t - \underline{t}}$$

and by (3.5)-(3.6) we get

$$\frac{f_0}{\lambda\alpha} - \frac{\beta}{\alpha} K(x_0^t) + \epsilon_{x_0^t}(t - \underline{t}) = \frac{f_0}{\lambda\alpha} - \frac{\beta}{\alpha} \inf_{\partial\Lambda^{\underline{t}}} K + O(\epsilon(t - \underline{t}))\,.$$

Taking the limit $t \to \underline{t}^+$ we prove that

$$K(x_1) = \inf_{\partial\Lambda^{\underline{t}}} K\,,$$

a contradiction with (3.7). This ends the proof of lemma 3.4. □

__Case $\lambda = 0$:__ Assume that $0 = \lambda = |\nabla u|$ on $\partial\Lambda^{\underline{t}}$. Let us remark that $n(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ is well defined for x close to $\partial\Lambda^{\underline{t}}$. But it is delicate to prove that the field n is analytic on $\Omega \backslash \Lambda^{\underline{t}}$ up to the boundary $\partial\Lambda^{\underline{t}}$. To avoid this difficulty we introduce the vector field n_0 only defined on $\partial\Lambda^{\underline{t}}$ as the exterior unit normal vector to $\Lambda^{\underline{t}}$. For x close enough to $\partial\Lambda^{\underline{t}}$, let $h = d(x, \partial\Lambda^{\underline{t}})$ and $x_0 \in \partial\Lambda^{\underline{t}}$ such that $x = \gamma_{x_0}(h)$ for $\gamma_{x_0}(h)$ defined by $\gamma_{x_0}(h) = x_0 + hn(x_0)$. For h small enough, the map $(x_0, h) \mapsto \gamma_{x_0}(h)$ is a local diffeomorphism. Then

$$\begin{aligned} u(\gamma_{x_0}(h)) - \underline{t} &= \int_0^h D_{n_0} u(x_0 + s n_0) ds \\ &= \int_0^h ds (D_{n_0} u(x_0) + s D^2_{n_0 n_0} u(x_0) + \tfrac{s^2}{2} D^3_{n_0 n_0 n_0} u(x_0) + O(s^3)) \end{aligned}$$

which gives

$$t - \underline{t} = \frac{h^2}{2} D^2_{n_0 n_0} u(x_0) + \frac{h^3}{6} D^3_{n_0 n_0 n_0} u(x_0) + O(h^4)$$

We have now to evaluate $D^2_{n_0 n_0} u(x_0)$ and $D^3_{n_0 n_0 n_0} u(x_0)$.

Lemma 3.5 *With the previous notations,*

$$D^2_{n_0 n_0} u(x_0) = \frac{f_0}{\alpha} \neq 0\,.$$

Then
$$h = \left(\frac{2}{D^2_{n_0 n_0} u}\right)^{\frac{1}{2}} (t - \underline{t})^{\frac{1}{2}} - \frac{1}{3} \frac{D^3_{n_0 n_0 n_0} u}{(D^2_{n_0 n_0} u)^2} (t - \underline{t}) + O((t - \underline{t})^{\frac{3}{2}})$$

and a computation (where repeated indices are summed) gives

$$\begin{aligned}|\nabla u(\gamma_{x_0}(h))|^2 &= |\nabla u(x_0)|^2 + 2\nabla_i u D^2_{ij} u \cdot y_j \\ &+ \tfrac{1}{2}\{D^2_{ij} u D^2_{ik} u + 2\nabla_i u D^3_{ijk} u\} y_j y_k \\ &+ \tfrac{1}{6}\{6 D^3_{ijk} u D^2_{il} u + 2\nabla_i u D^4_{ijkl} u\} y_j y_k y_l + O(|y|^4) \\ &= (D^2_{n_0 n_0} u)^2 h^2 + (D^2_{n_0 n_0} u)(D^3_{n_0 n_0 n_0} u) h^3 + O(|y|^4) \\ &= 2(t-\underline{t})(D^2_{n_0 n_0} u)^2 + \tfrac{4}{3}(t-\underline{t})^{\frac{3}{2}}(D^3_{n_0 n_0 n_0} u)(\tfrac{2}{D^2_{n_0 n_0} u})^{\frac{1}{2}} + O((t-\underline{t})^2)\end{aligned}$$

for $y = \gamma_{x_0}(h) - x_0$. Similarly to Lemma 3.5, we have the

Lemma 3.6 *With the same notations as above,*

$$D^3_{n_0 n_0 n_0} u(x_0) = -(D^2_{n_0 n_0} u(x_0)) K(x_0)\,.$$

Combining these computations, we may evaluate

$$|\nabla u(\gamma_{x_0}(h))|^2 = 2(t-\underline{t})(D^2_{n_0 n_0} u)^2 - \frac{4}{3}(t-\underline{t})^{\frac{3}{2}} K(x_0)(2 D^2_{n_0 n_0} u)^2)^{\frac{1}{2}} + O((t-\underline{t})^2)\,.$$

As in the first case we see that $K(t) \to \inf_{\partial\Lambda^{\underline{t}}} K$ as $t \to \underline{t}$, $t > \underline{t}$ and then $\inf_{\partial\Lambda^{\underline{t}}} K \geq \inf_{\partial\Omega} K$ because $\dot{K} \leq 0$. This ends the proof of Proposition 3.3 in the case $\lambda = 0$. $\square$

Proof of lemma 3.5. Let

$$n_0(x) = n_0(x_0(x)) \text{ with } x_0(x) \in \partial\Lambda^{\underline{t}} \text{ such that } x = \gamma_{x_0(x)}(d(x, \partial\Lambda^{\underline{t}}))\,. \tag{3.8}$$

We note $\tau_0 = (n_0)^\perp$:

$$0 = \partial_{\tau_0(x)}(\partial_{\tau_0(x)} u) = D^2_{\tau_0 \tau_0} u + \nabla_{\partial_{\tau_0}(\tau_0)} u = D^2_{\tau_0 \tau_0} u \text{ on } \partial\Lambda^{\underline{t}},$$

$$0 = \partial_{\tau_0(x)}(\partial_{n0(x)} u) = D^2_{\tau_0 n_0} u + \nabla_{\partial_{\tau_0}(n_0)} u = D^2_{\tau_0 n_0} u \text{ on } \partial\Lambda^{\underline{t}}.$$

From (3.1) we deduce that

$$D^2_{n_0 n_0} u(x_0) = \frac{f_0}{\alpha}\,. \qquad \square$$

Proof of Lemma 3.6. Let us introduce the function

$$v(x) = \partial_{n_0(x)} u(x)$$

where $n_0(x)$ is defined in (3.8). Then v is analytic in a neighbourhood of $\partial\Lambda^{\underline{t}}$ up to $\partial\Lambda^{\underline{t}}$. In particular it is easy to verify that

$$\partial_{n_0} v = D^2_{n_0 n_0} u \text{ in } \Omega \backslash \Lambda^{\underline{t}}. \tag{3.9}$$

Since $v = 0$ and $|\nabla v| = D^2_{n_0 n_0} u = const > 0$ on $\partial\Lambda^{\underline{t}}$, the curvature of $\partial\Lambda^{\underline{t}}$ is given at a point x_0 by

$$K(x_0) = \frac{D^2_{\tau_0\tau_0} v}{|\nabla v|}\,. \tag{3.10}$$

It is also easy to verify that on $\partial\Lambda^{\underline{t}}$

$$D^2_{n_0 n_0} v = D^3_{n_0 n_0 n_0} u\,, \quad D^2_{\tau_0\tau_0} v = D^3_{n_0 \tau_0 \tau_0} u$$

and we can derive Equation (3.1) relatively to the field $n_0(x)$. Denoting by α, β and f_0 the quantities $\alpha(u,q)$, $\beta(u,q)$, $f_0(u,q)$ for $q = |\nabla u|^2$, and $q' = 2|\nabla u| D^2_{n n_0} u$, we get

$$\alpha D^3_{n n n_0} u + \beta D^3_{\tau\tau n_0} u + (D^2_{nn} u)(\partial_u \alpha \cdot v + \partial_q \alpha \cdot q') + (D^2_{\tau\tau} u)(\partial_u \beta \cdot v + \partial_q \beta \cdot q') + 2J$$
$$= \partial_u f_0 \cdot v + \partial_q f_0 \cdot q'$$

where

$$J = \alpha(D^2_{\cdot n} u)\partial_{n_0} n + \beta(D^2_{\cdot\tau} u)\partial_{n_0}\tau\,.$$

An independent computation gives

$$\partial_{n_0} n = \Big(\frac{D^2_{\tau n_0} u}{|\nabla u|}\Big)\cdot\tau\,, \quad \partial_{n_0}\tau = -\Big(\frac{D^2_{\tau n_0} u}{|\nabla u|}\Big)\cdot n$$

and thus

$$J = \frac{\alpha - \beta}{|\nabla u|}(D^2_{n\tau} u)(D^2_{n_0 \tau})\,.$$

Let us recall that we have $\alpha(u,0) = \beta(u,0)$ and then $\alpha - \beta = O(|\nabla u|^2)$, $J = O(|\nabla u|)$, which implies

$$J = 0 \text{ on } \partial\Lambda^{\underline{t}}.$$

On $\partial\Lambda^{\underline{t}}$,

$$D^3_{n n n_0} u + D^3_{\tau\tau n_0} u = 0\,,$$

which gives $D^3_{n_0 n_0 n_0} u(x_0) = -D^3_{\tau_0 \tau_0 n_0} u = -D^2_{\tau_0\tau_0} v = -(D^2_{n_0 n_0} u(x_0)) K(x_0)$ from (3.9) and (3.10). This ends the proof of Lemma 3.6. □

3.4 Higher dimensions

In this subsection, we formally extend our approach to dimensions $d \geq 3$. The main difference is that the curvature has to be replaced by the arithmetic mean curvature. We will justify the derivation of this system only at a formal level by considering the generic case.

Let u be a solution of

$$\alpha(u, |\nabla u|^2) \sum_{i=1}^{d-1} D_{ii} u + \beta(u, |\nabla u|^2) D_{dd} u = f(u, |\nabla u|^2) \tag{3.11}$$

where D_{ii} is defined as follows. Consider

$$\Gamma^t = \{x \in \Omega : u(x) = t\} \subset \mathbb{R}^d$$

and $n(x) = \frac{\nabla u}{|\nabla u|}(x)$ the unitary normal vector, orthogonal to the hyperplane $\Pi = \Pi(x)$ tangent to Γ^t at x. For $i = 1, 2, ...d-1$, we may diagonalize $(D^2 u)_\Pi = P_\Pi (D^2 u) P_\Pi$ where P_Π is the projection on Π and define τ_i $(i = 1, 2, ...d-1)$ as the corresponding eigenvectors such that $(\tau_1, \tau_2, ...\tau_{d-1}, \tau_d = n)$ forms an orthonormalized basis in $\mathbb{R}^d$ (the derivative along the normal to the level hypersurface, *i.e.* along the direction n, corresponds to the index d. For $d = 2$, it was noted with the letter n). $\lambda_i = (\tau_i, (D^2 u)_\Pi \tau_i)$ are the eigenvalues of $(D^2 u)_\Pi$ and we define $\mu_i = (\tau_d, (D^2 u)\tau_i) = (\tau_i, (D^2 u)\tau_d)$ for $i = 1, 2, ...d$ and the curvatures $K_i = \frac{\lambda_i}{|\nabla u|}$ $(i = 1, 2, ...d-1)$. With the notations $D_{ij} u = (\tau_i, (D^2 u)\tau_j)$ the Fréchet formula are (as in Section 2, $\frac{d}{d\tau_i} = \tau_i \cdot \nabla$, so that two derivatives do not necessarily commute and $\frac{d\tau_i}{d\tau_i}$ can be different from 0):

$$\frac{d\tau_d}{d\tau_i} = \frac{dn}{d\tau_i} = \frac{\lambda_i}{|\nabla u|}\tau_i = K_i \tau_i$$

(without summation on $i = 1, 2, ...d-1$),

$$\frac{d\tau_d}{d\tau_d} = \frac{dn}{dn} = \sum_{i=1}^{d-1} \frac{D_{id} u}{|\nabla u|}\tau_i\,,$$

$$\frac{d\tau_i}{d\tau_d} = \frac{d\tau_i}{dn} = \sum_{\substack{j=1 \\ j \neq i}}^{d} a_{ij}\tau_j \quad (i = 1, 2, ...d-1)$$

where

$$a_{ij} = \frac{2\frac{\mu_i \mu_j}{|\nabla u|} - D^3_{dij} u}{|\nabla u|(K_j - K_i)} \quad (i, j = 1, 2, ...d-1)$$

(at least for $K_j \neq K_i$) and

$$a_{id} = -\frac{\mu_i}{|\nabla u|} \quad (i = 1, 2, ...d-1)\,,$$

and for $i, j = 1, 2, ...d-1$,

$$\frac{d\tau_i}{d\tau_j} = \sum_{\substack{k=1 \\ k \neq i}}^{d} a_{ijk}\tau_k$$

where

$$a_{ijk} = \frac{K_j(\mu_i \delta_{kj} + \mu_k \delta_{ij}) - D_{ijk} u}{|\nabla u|(K_k - K_i)} \quad (k = 1, 2, ...d-1,\ k \neq i)$$

and

$$a_{ijd} = -K_j \delta_{ij}\,.$$

As in Section 2, we denote by $x^t \in \Gamma^t$ a point which realizes the maximum of $|\nabla u|^2$ on Γ^t and assume that $t \mapsto x^t$ is an analytic curve. We assume moreover that $K_i(x^t) \neq K_j(x^t)$ for any $i,j = 1,2,...d-1$ $(j \neq i)$. By definition of x^t,

$$\frac{d}{d\tau_i}(|\nabla u|^2)(x^t) = 0 \quad (i = 1,2,...d-1),$$

thus proving that $\mu_{i|x=x^t} = 0$. Because $|\nabla u|^2$ restricted to Γ^t has a critical point at $x = x^t$, we may also define its Hessian as

$$\frac{d}{d\tau_i}(\frac{d}{d\tau_j}(|\nabla u|^2)) = \frac{d}{d\tau_j}(\frac{d}{d\tau_i}(|\nabla u|^2)) =: H(\tau_i,\tau_j) \leq 0,$$

where

$$H = (D^2(|\nabla u|^2))_\Pi - 2(D_{dd}u)(D^2u)_\Pi$$

and

$$(D^2(|\nabla u|^2))_\Pi = 2((D^2u)_\Pi)^2 + 2|\nabla u|(D_{d\cdot\cdot}u)_\Pi\,.$$

In the following, we shall assume for simplicity that H is actually negative definite. Let us compute now $\frac{d}{dt}(\sum_{i=1}^{d-1} K_i)$.

1) With notations similar to the ones of the 2-dimensional case, we have

$$\frac{1}{\delta t}(x^{t+\delta t} - x^t) = (\frac{1}{|\nabla u|} + b\,\delta t)n + (\vec{h} + \vec{B}\delta t) + O((\delta t)^2)$$

where $\vec{h} = (h_1, h_2, ...h_{d-1})$, $\vec{B} \in \Pi(x^t)$. With $\vec{\delta} = x^{t+\delta t} - x^t$,

$$\begin{aligned}\delta t = u(x^{t+\delta t}) - u(x^t) &= \vec{\delta}\cdot\nabla u + \frac{1}{2}(\vec{\delta}\cdot(D^2u)\vec{\delta}) + o(|\vec{\delta}|^2)\\ &= \delta t + (b\,|\nabla u| + \frac{1}{2}(\vec{h}\cdot(D^2u)\vec{h}) + \frac{1}{2}\frac{D_{dd}u}{|\nabla u|^2})(\delta t)^2 + o((\delta t)^2)\,,\end{aligned}$$

we get

$$b = -\frac{1}{2}(\vec{h}\cdot(D^2u)_\Pi\vec{h}) - \frac{1}{2}\frac{D_{dd}u}{|\nabla u|^3}\,.$$

2) Using the Taylor expansion of $|\nabla u(x^{t+\delta t})|^2 - |\nabla u(x^t)|^2$ and maximizing it with respect to $\vec{h}$, we get

$$\vec{h} = -\frac{1}{|\nabla u|}(H^{(-1)} \mathrm{o} P_\Pi)\left(D^2_{d\cdot}u(|\nabla u|^2)\right). \tag{3.12}$$

3) We compute $\sum_{i=1}^{d-1} \dot{K}_i$:

$$\sum_{i=1}^{d-1} \dot{K}_i = \frac{d}{dt}(\sum_{i=1}^{d-1} K_i(x^t)) = \frac{1}{|\nabla u|}\frac{d}{dn}(\sum_{i=1}^{d-1} K_i) + \vec{h}\cdot\nabla_\Pi(\sum_{i=1}^{d-1} K_i)\,. \tag{3.13}$$

Using the Fréchet formulas, we get in $x = x^t$

$$\frac{dK_i}{dn} = \frac{1}{|\nabla u|}(D_{dii}u - D_{dd}uK_i)\,, \tag{3.14}$$

$$\frac{dK_i}{d\tau_j} = \frac{1}{|\nabla u|}D_{iij}u\,. \tag{3.15}$$

4) On one hand let us remark that

$$0 \geq tr(H) = 2|\nabla u|\left\{\sum_{i=1}^{d-1} D_{dii}u + |\nabla u|K_i^2 - D_{dd}uK_i\right\},$$

so that

$$\frac{1}{|\nabla u|}\frac{d}{dn}(\sum_{i=1}^{d-1} K_i) \leq -\frac{1}{|\nabla u|}\sum_{i=1}^{d-1} K_i^2\,.$$

5) On the other hand, deriving Equation (3.11) with respect to τ_j proves that

$$\alpha\sum_i D_{iij}u + \beta D_{ddj}u = 0$$

Consequently $\frac{d}{d\tau_j}(\sum_{i=1}^{d-1} K_i) = -\frac{\beta}{\alpha}\frac{D_{ddj}u}{|\nabla u|}$. Since $D^2_{dj}(|\nabla u|^2) = 2|\nabla u|D_{ddj}u$, then

$$\vec{h}\cdot\nabla_\Pi(\sum_{i=1}^{d-1} K_i) = 2\frac{\beta}{\alpha}[P_\Pi(D_{dd\cdot}u)]H^{(-1)}[P_\Pi(D_{dd\cdot}u)] \leq 0$$

because $H < 0$. Therefore

$$\frac{d}{dt}(\sum_{i=1}^{d-1} K_i) \leq -\frac{1}{|\nabla u|}\sum_{i=1}^{d-1} K_i^2\,.$$

Remark 3.5 *Instead of Equation (3.11), we could consider a fully nonlinear isotropic homogenous equation.*

In view of the free boundary problem, we may simply quote that if the domain $\Omega \subset \mathbb{R}^d$ is convex, the mean curvature $\sum_{i=1}^{d-1} K_i$ of the free boundary at the limit point of the points that maximize the gradient, is positive. Concerning the estimates on the gradient, it is clear that Theorem 1.2 can be generalized to any dimension, thus providing an estimate taking the geometry of the domain into account. However a rigourous justification of these estimates would involve a tedious discussion of the various special cases, similar for the methods to the 2-dimensional case, but much longer. This is why we left it here at a formal level.

4 Analyticity and proof of Theorem 2.1

In this section we introduce analytic functions relevant to our problem and recall a simple result which gives the main motivation for introducing analytic functions and analytic sets. Then we give the proof of Theorem 2.1 in this framework.

Let us recall that $x^t \in X^t$ if and only if $u(x^t) = t$ and $|\nabla u(x^t)| = \max_{\{u(y)=t\}} |\nabla u(y)|$:

$$\frac{d}{d\tau}(|\nabla u|^2/2)_{|x=x^t} = 0\,. \tag{4.1}$$

Now let us define on $\{|\nabla u| > 0\}$ the analytic function:

$$F(x) = \frac{d}{d\tau}(|\nabla u|^2/2)_{|x}\,.$$

Let $X = \bigcup_{t\in[\underline{t},\overline{t}]} X^t$. From (4.1), we have $X \subset \{F = 0\}$. We define *analytic sets* as sets where analytic functions vanish.

4.1 Preliminaries on analytic sets

From [8] (chapter 8: Etude locale des fonctions et des ensembles analytiques; Propositions 4.2.5, 7.2, 7.7 and Theorem 1.2.2) we deduce the

Theorem 4.1 *For $N \geq 1$ let $F_i(x_1,x_2)$, $i = 1,2,...,N$ be real analytic functions of $(x_1,x_2) \in U$ where U is an open set of $\mathbb{R}^2$. We assume that $F_1 \not\equiv 0$ and $F_i(0) = 0$, $i = 1,2,...,N$. Then there exists a ball $B_r(0)$ and $k \in \mathbb{N}$ such that*

$$(\bigcap_{i=1}^{N}\{F_i = 0\}) \cap B_r(0) = \{0\} \cup (\bigcup_{j=1}^{k} \gamma_j)$$

for a disjoint union of analytic open curves $\gamma_j(s)$, $s \in (0,1)$ with

$$\begin{cases} \lim_{s\to 0}\gamma_j(s) = \gamma_j(0) = 0 \\ \lim_{s\to 1}\gamma_j(s) = \gamma_j(1) = x_j \in \partial B_r(0) \end{cases} \tag{4.2}$$

Moreover the same property is true for every ball $B_{r'}(0)$ with $r' < r$.

This result gives a precise description of the structure of analytic sets. In our proof of Theorem 2.1, we are interested in the following special situation. Let $F_1 \not\equiv 0$ be an analytic function with $F_1(0)=0$. Theorem 4.1 for $N=1$ gives the existence of an open curve $\gamma \subset \{F_1=0\}$ with $\gamma(0)=0$. Let F_0 be a second analytic function such that $F_0(0)=0$ and $\nabla F_0(0)\neq 0$. What can be said on $d(F_0\circ\gamma)/ds$? The answer to this question is given by the:

Corollary 4.1 *Consider a real analytic function F_0 of the variables $(x_1,x_2)\in U$, where U is an open set of $\mathbb{R}^2$, such that $F_0(0)=0$ and $\nabla F_0(0)\neq 0$. If $\gamma:(0,1)\to U$ is an analytic curve such that $\gamma(0)=0$, then for an $\epsilon>0$ small enough, on the interval $(0,\epsilon)$,*

i) either $d(F_0\circ\gamma)/ds\equiv 0$,

ii) or $\pm d(F_0\circ\gamma)/ds>0$.

The proof of this corollary is given in the appendix. It takes advantage of the following classical result.

Proposition 4.1 *Let g and h be two analytic functions defined on the interval $(-1,1)$. If 0 is an accumulation point of the set $\{s\in(-1,1),\ f(s)=g(s)\}$, then $f=g$ on $(-1,1)$.*

4.2 Proof of Theorem 2.1

With our notations, $X\subset\{F=0\}$.

Case A: $F\equiv 0$

Lemma 4.1 *If $F\equiv 0$, then Ω is a disk and the solution u is radially symmetric.*

Proof. If $F\equiv 0$, then $|\nabla u|=const=m(t)$ on each level line $\Gamma^t=\{u=t\}$. Let γ be a smooth curve such that $u(\gamma(t))=t$. Then $\frac{d}{dt}|\nabla u(\gamma(t))|=\frac{f_0}{m\beta}-\frac{\alpha}{\beta}K(\gamma(t))$. Because $|\nabla u(\gamma(t))|=m(t)$ and $\gamma(t)$ is arbitrarily chosen on each Γ^t, we see that $K=const=K(t)$ on Γ^t. Because Ω is bounded, the level lines of u are circles. Moreover $\partial_n n=(D_{n\tau}u)\tau/m(t)$, and $D_{n\tau}u=F/m(t)\equiv 0$, thus proving that: $\partial_n n=0$.

This implies that if $x_1\in\partial\Omega$ and γ_{x_1} is the integral curve of the vector field n such that $u(\gamma_{x_1}(t))=t$ and $\gamma_{x_1}(\overline{t})=x_1$, then $\frac{d}{dt}n(\gamma_{x_1}(t))=0$:

$$\gamma_{x_1}(t)=x_1+(t-\overline{t})n(x_1) \text{ and } u(\gamma_{x_1}(t))=t$$

Because this is true for every point x_1 in the circle $\partial\Omega$, we see that the circles Γ^t have the same center which is $x_0=x_1+(\underline{t}-\overline{t})n(x_1)$. In particular the solution is radial on the disk $\Omega=B_{1/K(\overline{t})}(x_0)$. □

The computations of Section 1 then apply without modifications.

<u>**Case B:** $F \not\equiv 0$</u>

Lemma 4.2 *For every $\epsilon > 0$, let (for $m(t) = \sup_{\{u=t\}} |\nabla u|$)*

$$\omega_\epsilon = \{x \in \Omega,\ \underline{t} + \epsilon < u(x) < \overline{t} - \epsilon,\ |\nabla u(x)| > \epsilon\ m(u(x))\}.$$

Then there exists an open set

$$\omega = \bigcup_{i=1}^{N} B_{r_i}(x_i) \subset \{|\nabla u| > 0\} \tag{4.3}$$

such that $\{F = 0\} \cap \omega_\epsilon \subset \omega$. Moreover the set $F_\omega = \{F = 0\} \cap \omega$ has the following property:

$$\forall i \in [1, N], \quad \exists k_i \in \mathbb{N}, \quad F_\omega \cap B_{r_i}(x_i) = \{x_i\} \cup (\bigcup_{j=1}^{k_i} \gamma_j^i) \tag{4.4}$$

where γ_j^i are analytic open curves which satisfy (4.2) with x_i as origin.

As a consequence of the lemma we get that for any $\epsilon < 1$, $X \cap \{\underline{t} + \epsilon < u < \overline{t} - \epsilon\}$ is contained in F_ω which has an analytic structure given by (4.4) and ω is defined by (4.3).

Proof of Lemma 4.2. The map $t \mapsto m(t)$ is continuous on $[\underline{t}, \overline{t}]$. For every $0 < \epsilon' < \epsilon$ we have $\overline{\omega_\epsilon} \subset \omega_{\epsilon'}$. Let $F_{\omega_\epsilon} = \{F = 0\} \cap \omega_\epsilon$. Then $\overline{F_{\omega_\epsilon}}$ is a compact set included in $\{|\nabla u| > 0\}$. At every point $x_0 \in \overline{F_{\omega_\epsilon}}$, the set $\{F = 0\}$ has the property given by Theorem 4.1:

$$\forall x_0 \in \overline{F_{\omega_\epsilon}}, \quad \exists r_{x_0} > 0, \quad \exists k_{x_0} \in \mathbb{N}, \quad \{F = 0\} \cap B_{r_{x_0}}(x_0) = \{x_0\} \cup (\bigcup_{j=1}^{k_{x_0}} \gamma_j^{x_0})$$

Because $\overline{F_{\omega_\epsilon}}$ is compact we can extract a finite subsequence from the recovering $\bigcup_{x_0 \in \overline{F_{\omega_\epsilon}}} B_{\frac{r_{x_0}}{2}}(x_0) \supset \overline{F_{\omega_\epsilon}}$, and for $r_i = \frac{r_{x_i}}{2}$,

$$\overline{F_{\omega_\epsilon}} \subset \bigcup_{i=1}^{N} B_{r_i}(x_i) = \omega. \qquad \square$$

Let us assume $\epsilon > 0$ fixed in all what follows.

If γ is a curve in the decomposition (4.4) close to a singular point x_i, then from Corollary 4.1, either $d(u \circ \gamma)/ds \equiv 0$, or (up to change of s into $-s$) $d(u \circ \gamma)/ds > 0$ locally near x_i. In this last case we can choose as a parameter $s = u(\gamma(s))$.

Let $\{x_i'\}$ be the set of points on curves γ such that $d(u \circ \gamma)/ds \not\equiv 0$ and $d(u \circ \gamma)/ds_{|\gamma(s) = x_i'} = 0$. Because the curves γ and the function u are analytic, we deduce from Proposition 4.1 that there are only a finite number N' of such points x_i'. Then let $\mathcal{G}$ be the set of curves γ such that $d(u\circ\gamma)/ds>0$. We will denote by $\mathcal{G}^*$ the set of curves γ such that $u(\gamma)=const$. Then

$$\{F=0\}\cap\omega=\mathcal{G}\cup\mathcal{G}^*\cup\{x_i\}_{1\le i\le N}\cup\{x_i'\}_{1\le i\le N'}$$

We can rewrite the discret set

$$\{u(x_i)\}_{1\le i\le N}\cup\{u(x_i')\}_{1\le i\le N'}\cup\{u(\gamma)\}_{\gamma\in\mathcal{G}^*}$$

as an increasing finite sequence $t_k^*\in(\underline{t}+\epsilon,\overline{t}-\epsilon), k=1,2,...,M$ of critical values. Let $\mathcal{G}_k$ be the set of curves $\gamma\in\mathcal{G}$ which are defined on (t_k^*,t_{k+1}^*). Then on (t_k^*,t_{k+1}^*) we have:

$$m(t)=\sup_{\gamma\in\mathcal{G}_k}|\nabla u(\gamma(t))|\,.$$

Because each map $t\mapsto|\nabla u(\gamma(t))|$ is analytic, we deduce from Proposition 4.1 that the *sup* is analytic possibly except on a discret set $\{t_{k,n}^*\}_{p_k^-<n<p_k^+}$ with $p_k^-,p_k^+\in\mathbb{Z}\cup\{-\infty\}\cup\{+\infty\}$ which has no accumulation point in (t_k^*,t_{k+1}^*). Only t_k^* and t_{k+1}^* are possible accumulation points. In particular there exists $\gamma_{t_{k,n}}\in\mathcal{G}_k$ such that

$$m(t)=|\nabla u(\gamma_{t_{k,n}}(t))| \text{ on } (t_{k,n},t_{k,n+1})\,.$$

Then the proof of Theorem 2.1, given in Section 2 in the special case where X^t is supported in an analytic curve $t\mapsto x^t=\gamma_{t_{k,n}}(t)$, applies here and gives the equations written in Theorem 2.1 for

$$K(t)=K(\gamma_{t_{k,n}}(t)) \text{ on } (t_{k,n},t_{k,n+1})\,.$$

Although the map $t\mapsto m(t)$ is continuous, the map $t\mapsto K(t)$ can be discontinuous in $t_{k,n}$; in other words we can have $K(\gamma_{t_{k,n-1}}(t_{k,n}))\neq K(\gamma_{t_{k,n}}(t_{k,n}))$. Nevertheless we have the

Lemma 4.3 *With the above notations,*

$$K(\gamma_{t_{k,n}}(t_{k,n}))\le K(\gamma_{t_{k,n-1}}(t_{k,n}))\,.$$

Consequently we can define $K(t_{k,n}):=K(\gamma_{t_{k,n}}(t_{k,n}))$ and then (2.5) is true in the sense of distribution on $(t_{k,n-1},t_{k,n+1})$:

$$\dot{K}\le-\frac{K^2}{m}-\frac{2}{m}\sqrt{\frac{\alpha(t,m^2)}{\beta(t,m^2)}}\,|\partial_\tau K| \tag{4.5}$$

Proof of Lemma 4.3. Let us recall that for each curve $\gamma=\gamma_{t_{k,n-1}}$ and $\gamma=\gamma_{t_{k,n}}$ we have

$$\dot{m}=\frac{f_0(t,m^2)}{m\beta(t,m^2)}-\frac{\alpha(t,m^2)}{\beta(t,m^2)}K \tag{4.6}$$

where $m(t)=|\nabla u(\gamma(t))|$ and $K(t)=K(\gamma(t))$. Let $m^+(t)=|\nabla u(\gamma_{t_{k,n}}(t))|$ and $m^-(t)=|\nabla u(\gamma_{t_{k,n-1}}(t))|$. Then

$$m^{\pm}(t)=m(t_{k,n})+l^{\pm}\cdot(t-t_{k,n})+o(|t-t_{k,n}|)$$

where $l^{\pm}=\frac{dm^{\pm}(t)}{dt}_{|t=t_{k,n}}$. Note that

$$l^+-l^-=-\frac{\alpha}{\beta}\left(K(\gamma_{t_{k,n}}(t_{k,n}))-K(\gamma_{t_{k,n-1}}(t_{k,n}))\right).$$

Because

$$m(t)=\sup(m^+(t),m^-(t))=\begin{cases} m^+(t) \text{ on } (t_{k,n},t_{k,n+1}) \\ m^-(t) \text{ on } (t_{k,n-1},t_{k,n}) \end{cases}$$

we deduce that $l^+\geq l^-$ and then $K(\gamma_{t_{k,n}}(t_{k,n}))\leq K(\gamma_{t_{k,n-1}}(t_{k,n}))$ which ends the proof of Lemma 4.3. □

More generally (4.6) and (4.5) are true on (t_k^*,t_{k+1}^*) and K satisfies

$$K(t)=\inf_{x^t\in X^t}K(x^t).$$

We now want to prove that these equations stay true in a neighbourhood of a critical value t_k^*. It is easy to prove that (4.6) is true in (t_{k-1}^*,t_{k+1}^*) in the sense of distributions, because the map $t\mapsto m(t)$ is continuous. Now we will prove it for (4.5). Let $\phi\in C_0^\infty(t_{k-1}^*,t_{k+1}^*)$, $\phi\geq 0$:

$$\begin{aligned}
<\dot{K},\phi> &= -\int_{t_{k-1}^*}^{t_{k+1}^*}K\dot{\phi} \\
&= -\lim_{\delta\to 0}\{\int_{t_{k-1}^*}^{t_k^*-\delta}K\dot{\phi}+\int_{t_k^*+\delta}^{t_{k+1}^*}K\dot{\phi}\} \\
&= -\lim_{\delta\to 0}\left\{\begin{array}{l}\int_{t_{k-1}^*}^{t_k^*-\delta}-\dot{K}\phi+\int_{t_k^*+\delta}^{t_{k+1}^*}-\dot{K}\phi \\ +[K\phi]_{t_{k-1}^*}^{t_k^*-\delta}+[K\phi]_{t_k^*+\delta}^{t_{k+1}^*}\end{array}\right\} \\
&\leq \int_{t_{k-1}^*}^{t_{k+1}^*}(-\frac{K^2}{m}-\frac{2}{m}\sqrt{\frac{\alpha}{\beta}}|\partial_\tau K|)\phi+\phi(t_k^*)\lim_{\delta\to 0}(K(t_k^*+\delta)-K(t_k^*-\delta)) \\
&\leq \quad <-\frac{K^2}{m}-\frac{2}{m}\sqrt{\frac{\alpha}{\beta}}|\partial_\tau K|,\phi>
\end{aligned}$$

because $\phi(t_k^*)\geq 0$ and the

Lemma 4.4 *With the above notations,*

$$\lim_{\delta\to 0}(K(t_k^*+\delta)-K(t_k^*-\delta))\leq 0. \tag{4.7}$$

Lemma 4.4 is a kind of generalization of Lemma 4.3, and will be proved later.

(4.5) and (4.6) are satisfied in the sense of distributions on (t^*_{k-1}, t^*_{k+1}) and then on $(\underline{t}+\epsilon, \overline{t}-\epsilon)$. Taking the limit $\epsilon \to 0$, we end up with the proof of Theorem 2.1.

Before proving Lemma 4.4, we need the following result.

Lemma 4.5 *Let $x_0 \in X$ and $t_0 = u(x_0)$. Then*

i) either there exists (at least) two curves $\gamma_+, \gamma_- \in \mathcal{G}$ with the same extremity x_0 such that $\gamma_+ \subset \{u > t_0\}$, $\gamma_- \subset \{u < t_0\}$,

ii) or $|\nabla u|$ is constant on the connected component of $\{u = t_0\}$ which contains x_0.

Proof. Let us remark that the map $x \mapsto |\nabla u(x)|$ is analytic in a neighbourhood of x_0, and in particular on the analytic curve $\Gamma^{t_0} = \{u = t_0\}$. Then

i) either there exists an $\epsilon > 0$ s. t. $0 < |\nabla u| < |\nabla u(x_0)|$ on $\Gamma^{t_0} \cap (B_\epsilon(x_0) \setminus \{x_0\})$,

ii) or there exists a sequence of points of Γ^{t_0} which converges to x_0 and such that the modulus of the gradient is equal to $|\nabla u(x_0)|$, which implies that the real analytic function $|\nabla u|_{|\Gamma^{t_0}}$ is constant on the connected component of Γ^{t_0} which contains x_0 (see Proposition 4.1).

In case i), it is easy to prove by a perturbation argument that there exists a sequence of local maxima of $|\nabla u|$ on the level lines Γ^t for t close to t_0. Moreover as $t \to t_0$, this sequence tends to x_0. We can restrict our analysis on $\{t > t_0\}$ (resp. on $\{t < t_0\}$). Then with the help of the local structure of $\{F = 0\}$ (Lemma 4.2), we deduce the existence of at least one curve $\gamma_+ \in \mathcal{G}$ with $\gamma_+ \subset \{u > t_0\}$ (resp. one curve $\gamma_- \in \mathcal{G}$ with $\gamma_- \subset \{u < t_0\}$). □

Proof of Lemma 4.4. To prove (4.7), we now consider a point $x_0 \in X^{t^*_k}$ and discuss the problem according to the two possible cases of Lemma 4.5.

In case i), for $\gamma = \gamma_- \cup \{x_0\} \cup \gamma_+$, we see as previously that $|\nabla u(\gamma(t))| = m(t^*_k) + l(t - t^*_k) + o(t - t^*_k)$ where $l = \frac{d}{dt}|\nabla u(\gamma(t))|_{|t=t^*_k} = \frac{f_0}{m\beta} - \frac{\alpha}{\beta} K(x_0)$. Then as in the proof of Lemma 4.3, we see that

$$\limsup_{\delta \to 0^+} K(t^*_k + \delta) \leq K(x_0)\,, \quad \liminf_{\delta \to 0^+} K(t^*_k - \delta) \geq K(x_0)\,,$$

and thus (4.7) is satisfied.

In case ii), we have $|\nabla u| = \mathit{const}$ on the connected component $\mathcal{C}_{x_0}$ of $\{u = u(x_0)\}$ which contains x_0. Let us recall that $m(t) = \sup_{\gamma \in \mathcal{G}_k} |\nabla u(\gamma(t))|$ for $t \in (t^*_k, t^*_{k+1})$. Then

$$\frac{1}{\delta}\Big(m(t^*_k + \delta) - m(t^*_k)\Big) = \sup_{\gamma \in \mathcal{G}_k} \frac{1}{\delta}\Big(|\nabla u(\gamma(t^*_k + \delta))| - |\nabla u(\gamma(t^*_k))|\Big)\,.$$

On the other hand we have

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\Big(|\nabla u(\gamma(t_k^*+\delta))|-|\nabla u(\gamma(t_k^*))|\Big)=\lim_{\delta\to 0^+}\frac{1}{\delta}\int_0^\delta\Big(\frac{f_0}{|\nabla u(\gamma)|\beta}-\frac{\alpha}{\beta}K(\gamma)\Big)_{|t_k^*+\eta}\,d\eta$$
$$=\Big(\frac{f_0}{|\nabla u(\gamma)|\beta}-\frac{\alpha}{\beta}K(\gamma)\Big)(t_k^*)$$

by continuity of the map $x\mapsto|\nabla u(x)|$, $x\mapsto K$ and $\delta\mapsto\gamma(t_k^*+\delta)$ as $\delta\to 0$. Because $\mathcal{G}_k$ is finite we have: $\lim_\delta\sup_{\mathcal{G}_k}=\sup_{\mathcal{G}_k}\lim_\delta$. Then

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\Big(m(t_k^*+\delta)-m(t_k^*)\Big)=\frac{f_0}{m\beta}-\frac{\alpha}{\beta}\inf_{\gamma\in\mathcal{G}_k}K(\gamma(t_k^*))$$

and

$$\lim_{\delta\to 0^+}K(t_k^*+\delta)=\inf_{\gamma\in\mathcal{G}_k}K(\gamma(t_k^*))\,. \tag{4.8}$$

Let $x_1\in\mathcal{C}_{x_0}$. We will prove that

$$\lim_{\delta\to 0^+}K(t_k^*+\delta)\le K(x_1)\,. \tag{4.9}$$

To this end, let us consider a smooth curve γ_0 defined for $t\in(t_k^*-\eta,t_k^*+\eta)$ for some small $\eta>0$, and such that $\gamma_0(t_k^*)=x_1$ and $u(\gamma_0(t))=t$. Then as previously: $\frac{d}{dt}|\nabla u(\gamma_0(t))|_{|t=t_k^*}=\frac{f_0}{m\beta}-\frac{\alpha}{\beta}K(x_1)$. On the other hand, by definition of $t\mapsto m(t)$, we have

$$\frac{1}{\delta}\Big(m(t_k^*+\delta)-m(t_k^*)\Big)\ge\frac{1}{\delta}\Big(|\nabla u(\gamma_0(t_k^*+\delta))|-|\nabla u(\gamma_0(t_k^*))|\Big)\ \text{ for } \delta\ge 0$$

and then as $\delta\to 0^+$:

$$\frac{f_0}{m\beta}-\frac{\alpha}{\beta}\inf_{\gamma\in\mathcal{G}_k}K(\gamma(t_k^*))\ge\frac{f_0}{m\beta}-\frac{\alpha}{\beta}K(x_1)\,.$$

Because of (4.8), this imply (4.9).

Similarly we get $\lim_{\delta\to 0^+}K(t_k^*-\delta)\ge K(x_1)$, which with (4.9) implies (4.7). This ends the proof of Lemma 4.4. □

5 Appendix: Proof of Corollary 4.1

Consider an analytic function F_1 defined on U such that $\gamma(0,1)=\{F_1=0\}$. Let l be the smallest integer such that $\gamma\subset\cap_{j=0}^{l}\{D^jF_1=0\}$ and $\gamma\not\subset\{D^{l+1}F_1=0\}$, where D^jF_1 denotes the set of all partial derivatives of total order j: $\{\partial_1^{j_1}\partial_2^{j_2}F_1\}_{j_1+j_2=j}$. We know that $F_1\not\equiv 0$, so l is finite and there exists $j_1,j_2\ge 0$, $j_1+j_2=l$ such that for $\tilde F_1=\partial_1^{j_1}\partial_2^{j_2}F_1$ we have

$$\tilde F_1\circ\gamma\equiv 0\,,\quad(\nabla\tilde F_1)\circ\gamma\not\equiv 0\,. \tag{5.1}$$

Now let τ be the unitary vector field tangent to level lines of F_0 $(\partial_\tau F_0 = 0)$. We now study different cases.

Case 1: $\tau \cdot \nabla \tilde{F}_1(0) \neq 0$**:** γ is an analytic curve in a neighbourhood of 0 up to $s=0$. In particular we can chose the curvilinear abscissa s as a parametrization up to $s=0$ and

$$\frac{dF_0(\gamma(s))}{ds} = \frac{d\gamma(s)}{ds} \cdot \nabla F_0 = -|\nabla F_0| \, (\frac{d\gamma(s)}{ds})^\perp \cdot \tau$$

because $\tau = -\frac{(\nabla F_0)^\perp}{|\nabla F_0|}$. We know that $(\frac{d\gamma(s)}{ds})^\perp$ is colinear to $\nabla \tilde{F}_1$, and consequently

$$\frac{dF_0(\gamma(s))}{ds} = \pm \frac{|\nabla F_0|}{|\nabla \tilde{F}_1|} (\tau \cdot \nabla \tilde{F}_1) \,, \tag{5.2}$$

so we deduce that $\pm \frac{dF_0(\gamma(s))}{ds} > 0$ in a neighbourhood of 0.

Case 2: $\tau \cdot \nabla \tilde{F}_1(0) = 0$**:** If $\tau \cdot \nabla \tilde{F}_1 \equiv 0$ on U, then of course $(\tau \cdot \nabla \tilde{F}_1) \circ \gamma \equiv 0$ and $d(F_0 \circ \gamma)/ds \equiv 0$.

If $\tau \cdot \nabla \tilde{F}_1 \not\equiv 0$ on U, then from Theorem 4.1 we have

$$(\{\tilde{F}_1 = 0\} \cap \{\tau \cdot \nabla \tilde{F}_1 = 0\}) \cap B_r(0) = \{0\} \cup (\bigcup_{j=1}^{k} \gamma_j)$$

for some $r > 0$ small enough. In that case, either for any j, $\gamma_j \neq \gamma$ and then $\pm(\tau \cdot \nabla \tilde{F}_1)_{|\gamma} > 0$: as in Case 1, from Equation (5.2) we get $\pm d(F_0 \circ \gamma)/ds > 0$ in a neighbourhood of 0, or $\exists j$, $\gamma_j = \gamma$. In that case, $\tau \cdot \nabla \tilde{F}_1(\gamma(s)) \equiv 0$ on a neighborhood of 0. From (5.1) we know that $\nabla \tilde{F}_1(\gamma(s)) \neq 0$ except in a decreasing sequence of points $(s_n)_n \in (0,1)$. Because the map $(0,1) \ni s \mapsto \nabla \tilde{F}_1(\gamma(s))$ is analytic, the only possible accumulation point of the sequence $(s_n)_n$ is 0 according to Proposition 4.1. Away from these points s_n we can apply the implicit function theorem which proves that $F_0 \circ \gamma = const = C_n$ on (s_{n+1}, s_n). By continuity at s_n we get $C_n = C_{n+1} = F_0(0) = 0$ and consequently $\gamma \subset \{F_0 = 0\}$, $d(F_0 \circ \gamma)/ds \equiv 0$ on $(0, \epsilon)$ for $\epsilon > 0$ small enough. □

Acknowledgements. The authors thank H. Berestycki for introducing them to the question of the convexity of the coincidence set for obstacle problems. They also thank B. Teissier for his comments on the properties of the analytic functions and for his suggestions.

References

[1] Alt H.W., Phillips D., *A free boundary problem for semilinear elliptic equations*, J. Reine Angew. Math., 368, 63-107, (1986).

[2] Bonnet A., Monneau R., *Distribution of vortices in a type II superconductor as a free boundary problem: Existence and regularity via Nash-Moser theory*, work in preparation.

[3] Brézis H., Kinderlehrer D., *The Smoothness of Solutions to Nonlinear Variational Inequalities*, Indiana Univ. Math. J., 23 (9), 831-844 (1974).

[4] Caffarelli L.A., *Compactness Method in free boundary problems*, Comm. P.D.E., 5 (4), 427-448, (1980).

[5] Caffarelli L.A., Rivière N.M., *Smoothness and Analyticity of Free Boundaries in Variational Inequalities*, Ann. Scuola Norm. Sup. Pisa, serie IV, 3, 289-310, (1975).

[6] Caffarelli L.A., Spruck J., *Convexity properties of solutions to some classical variational problems*, Comm. P.D.E., 7 (11), 1337-1379, (1982).

[7] Diaz J.I., Kawohl B., *On convexity and Starshapedness of level Sets for Some Nonlinear Elliptic and Parabolic Problems on Convex Rings*, J. Math. Anal. and Applic., 177, 263-286, (1993).

[8] Dolbeault P., *Analyse complexe*, Collection Maîtrise de Mathematiques Pures, ed. Masson, (1990).

[9] Dolbeault J., Poupaud F., *A remark on the critical explosion parameter for a semilinear elliptic equation in a generic domain using an explosion time of an ordinary differential equation*, Nonlinear Analysis, Theory, Methods & Applications, 24 (8), 1149-1162, (1995).

[10] Frehse J., *On the Regularity of the Solution of a Second Order Variational Inequality*, Boll. U.M.I., (4) 6, 312-315, (1972).

[11] Friedman A., *Variational Principles and Free Boundary Problems*, Pure and Applied Mathematics, ISSN 0079-8185, Wiley-Interscience, (1982).

[12] Friedman A., Phillips D., *The free boundary of a semilinear elliptic equation*, Trans. Amer. Math. Soc., 282, 153-182, (1984).

[13] Hamilton R.S., *The inverse function theorem of Nash and Moser*, Bull. A.M.S., 7, 65-222, (1982).

[14] Kaup L., Kaup B., *Holomorphic Functions of Several Variables*, Walter de Gruyter, Berlin, New York, (1983).

[15] Kawohl B., *When are solutions to nonlinear elliptic boundary value problems convex ?*, Comm. P.D.E., 10, 1213-1225, (1985).

[16] Kawohl B., *Rearrangements and Convexity of Level Sets in PDE*, Springer Lecture Notes in Math., 1150, (1985).

[17] Kawohl B., *Geometrical properties of level sets of solutions to elliptic problems*, Proc. Symp. Pure Math., 45, Part 2, Amer. Math. Soc., Providence, 25-36, (1986).

[18] Kawohl B., *On the convexity and symmetry of solutions to an elliptic free boundary problem*, J. Austral. Math. Soc. (Series A) 42, 57-68, (1987).

[19] Kawohl B., *On the convexity of level sets for elliptic and parabolic exterior boundary value problems*, Potential theory (Prague, 1987), 153-159, Plenum, New York-London, 1988.

[20] Kinderlehrer D., Nirenberg L., *Regularity in Free Boundary Problems*, Bull. A.M.S., 7, 65-222, (1982).

[21] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.

[22] Monneau R., *Problèmes de frontières libres, EDP elliptiques non linéaires et applications en combustion, supraconductivité et élasticité*, Thèse de doctorat de l'Université de Paris VI, (1999).

[23] Morrey C.B., *Multiple Integrals in the Calculus of Variations*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Vol. 130, Springer-Verlag, N.Y., (1966).

[24] Rodrigues J.F., *Obstacle problems in Mathematical Physics*, North-Holland, (1987).

[25] Schaeffer D.G., *Some Examples of Singularities in a Free Boundary*, Ann. Scuola Norm. Sup. Pisa, 4 (4), 131-144, (1977).

[26] Schaeffer D.G., *One-sided Estimates for the Curvature of the Free Boundary in the Obstacle Problem*, Advances in Math., 24, 78-98, (1977).