CORRIGENDUM:
ON THE EIGENVALUES OF OPERATORS WITH GAPS. APPLICATION TO DIRAC OPERATORS

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ABSTRACT. In this *corrigendum*, we address some closability issues that were ignored in [1].

In [8], L. Schimmer, J.P. Solovej, and S. Tokus construct a distinguished self-adjoint extension of a general symmetric operator with a gap and give a variational characterization of its eigenvalues, thus connecting the extension problem considered in [3, 4] to the min-max principle for eigenvalues of self-adjoint operators studied in [1]. They also point out and solve several questions of closability and domain invariance that were not properly addressed in these papers. The min-max result of [8, Theorem 1.1], when applied to already self-adjoint operators, is analogous to the result of [1, Theorem 1.1], but with different assumptions. On the one hand, L. Schimmer, J.P. Solovej, and S. Tokus do not assume that the subspace F, in which the min-max principle is defined, is a core. On the other hand, they strengthen an assumption denoted (i) in [1, 8] and add a condition that we call $(\mathscr{C})$ in this corrigendum. Moreover they suggest that $(\mathscr{C})$ is also needed for [1, Theorem 1.1]. They are completely right: the proof in [1] overlooks several closability issues, but can be completed under Condition $(\mathscr{C})$, and this is done in [8]. However, under minor corrections which are exposed below, the result of [1, Theorem 1.1] also holds without assuming $(\mathscr{C})$.

1. MIN-MAX CHARACTERIZATION OF THE EIGENVALUES IN A GAP

We first recall (up to a minor change that will be commented below) the assumptions of [1, Theorem 1.1].

Let $\mathscr{H}$ be a Hilbert space with norm $\|\cdot\|_{\mathscr{H}}$ and scalar product $(\cdot,\cdot)$ and let A be a self-adjoint operator on $\mathscr{H}$ with domain $D(A)$. Let $\mathscr{H}_+$ and $\mathscr{H}_-$ be two orthogonal Hilbert subspaces of $\mathscr{H}$ such that $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$. Let Λ_+ and Λ_- be the projectors on $\mathscr{H}_+$ and $\mathscr{H}_-$. We assume the existence of a core F, *i.e.*, a dense subspace of $D(A)$, such that :

(i) $F_+ = \Lambda_+ F$ and $F_- = \Lambda_- F$ are two subspaces of $D(A)$,

(ii) $a = \sup_{x_- \in F_- \setminus \{0\}} \dfrac{(x_-, A x_-)}{\|x_-\|^2_{\mathscr{H}}} < +\infty$,

(iii) $\lambda_1 > a$.

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Here λ_1 is the first of the min-max levels

$$\lambda_k = \inf_{\substack{V \text{ subspace of } F_+ \\ \dim V = k}} \sup_{x \in (V \oplus F_-) \setminus \{0\}} \frac{(x, Ax)}{\|x\|^2_{\mathscr{H}}}, \quad k \geq 1.$$

Let $b = \inf\big(\sigma_{\rm ess}(A) \cap (a, +\infty)\big) \in [a, +\infty]$. For $k \geq 1$, μ_k denotes the $k^{\rm th}$ eigenvalue of A in the interval (a, b), counted with multiplicity, *if this eigenvalue exists.* If there is no $k^{\rm th}$ eigenvalue, we take $\mu_k = b$.

Theorem 1. *With the above notations, and under assumptions* (i)-(ii)-(iii)*, we have*

$$\lambda_k = \mu_k, \quad \forall\, k \geq 1$$

and, as a consequence, $b = \lim_{k \to +\infty} \lambda_k = \sup_{k \geq 1} \lambda_k > a$.

Compared to [1, Theorem 1.1], the only change is that in (i) one has to assume that F_+ and F_- are subspaces of $D(A)$ and not of the form domain $\mathscr{F}(A)$. This is weaker than Condition $(\mathscr{C})$

The operator $\Lambda_- A_{|F_-} : F_- \to \mathscr{H}_-$ *is essentially self-adjoint,*

which is called assumption (iii) in [8, Theorem 1.1]. Note that a similar change is also needed in the assumptions of the continuation principle of [1, Theorem 3.1]: in hypothesis (j), $\mathscr{F}(A_0)$ has to be replaced by $\mathscr{D}(A_0)$.

Note that [1, Theorem 1.1] supposedly applies to Dirac-Coulomb operators $A = \alpha \cdot \mathbf{p} + \beta\, m - \frac{\nu}{r}$ with $F = C^\infty_c(\mathbb{R}^3, \mathbb{C}^4)$ for any $0 \leq \nu < 1$. Such a claim is incorrect, since F is not a core of A when $\sqrt{3}/2 < \nu < 1$. This issue was pointed out to us by several researchers, and a partial solution, that requires the replacement of C^∞_c by $H^{1/2}$, was given by S. Morozov and D. Müller in [5, 6]. However, the issue can also be solved in C^∞_c by first applying the theorem to regularized Dirac-Coulomb operators $A_\varepsilon = \alpha \cdot \mathbf{p} + \beta\, m - \frac{\nu}{r+\varepsilon}$, and then passing to the limit in the norm-resolvent sense as $\varepsilon \to 0$, as done in [2]. Of course, another possibility is to apply directly the result of [8], which does not make use of the assumption that F is a core.

2. Changes in the proof of Theorem 1.1 of [1]

As pointed out in [8], several closability issues were overlooked in the proof of [1, Theorem 1.1]. Let us list the changes that have to be done in the proof to make it correct. The numbering of the formulae refers to [1].

In [1] we introduced the norm

$$N(y_-) = \sqrt{(a+1)\|y_-\|^2_{\mathscr{H}} - (y_-, Ay_-)}$$

on F_- and claimed without proof that the completion of F_- for this norm can be identified with a subspace of $\mathscr{H}_-$. In other words, we claimed that the quadratic form $q_-(y_-) = -(y_-, Ay_-)$ is closable in $\mathscr{H}_-$. Unfortunately we cannot prove this closability under the assumption $F_- \subset \mathscr{F}(A)$, but this becomes a standard fact if one assumes that $F_- \subset \mathscr{D}(A)$ as in [8]. Indeed, the operator $-\Lambda_- A_{|F_-}$ is then a symmetric and bounded from below operator defined on $\mathscr{H}_-$ with domain F_-. One can use the Friedrichs extension theorem [7, Theorem X.23] to establish that the form q_- is closable in $\mathscr{H}_-$. Denoting by $\overline{F}_-^N$ the form-domain of its closure $\overline{q}_-$, there is a unique self-adjoint operator B with domain $\mathscr{D}(B) \subset \overline{F}_-^N$ such that for any $y_- \in \mathscr{D}(B)$, $\overline{q}_-(y_-) = (y_-, By_-)$. The operator B is the Friedrichs extension of $-\Lambda_- A_{|F_-}$.

The operator $L_E : F_+ \to \mathscr{H}_-$ is defined by

$$L_E x_+ = (B+E)^{-1} \Lambda_- A x_+ .$$

In order to define L_E without the assumption $F_- \subset \mathscr{F}(A)$, a delicate construction was needed in [1], while the definition of L_E is now straightforward.

Then we considered the completion X of F_+ for the norm $n_E(x_+) = \|x_+ + L_E x_+\|_{\mathscr{H}}$. We claimed without proof that X can be identified with a subspace of $\mathscr{H}_+$. In other words, we claimed that the operator L_E is closable in $\mathscr{H}_+$. As pointed out in [8], this is far from obvious, and probably wrong without an additional assumption. In [8], the authors prove that L_E is closable assuming that Condition ($\mathscr{C}$) holds. The main goal of the present *corrigendum* is to explain that this additional assumption is not needed in the proof of Theorem 1.

Without condition ($\mathscr{C}$), we cannot claim that L_E is closable, but instead of X, we can consider the closure $\overline{\Gamma}_E$ of its graph represented by $\Gamma_E = \{x_+ + L_E x_+ : x_+ \in F_+\}$ in the Hilbert space $\mathscr{H}$ identified with $\mathscr{H}_+ \times \mathscr{H}_-$. The closed subspace $\overline{\Gamma}_E$ does not necessarily represent a graph over $\mathscr{H}_+$, but this does not affect too much the remainder of the proof. We just need to modify the definitions of some mathematical objects. Several expressions that were defined as functions of $x_+ \in F_+$ are now considered as functions of $x_+ + L_E x_+ \in \Gamma_E$ and their extensions by density become functions of $x \in \overline{\Gamma_E}$. In particular, n_E (resp. $\overline{n}_E$) becomes the restriction of $\|\cdot\|_{\mathscr{H}}$ to Γ_E (resp. $\overline{\Gamma_E}$); in formula (7) and everywhere else in the sequel, Q_E becomes a map from Γ_E to $\mathbb{R}$ while $Q_E(x_+)$ has to be replaced by

$$Q_E\big(x_+ + L_E x_+\big) = \big((A-E)x_+, x_+\big) + (L_E x_+, (B+E) L_E x_+\big);$$

in formula (8), $Q_E(\Lambda_+ x)$ becomes $Q_E(\Lambda_+ x + L_E \Lambda_+ x)$. Formulae (10) and (11) have to be rewritten as

$$\|x_+\|_{\mathscr{H}} \le \|x_+ + L_{E'} x_+\|_{\mathscr{H}} \le \|x_+ + L_E x_+\|_{\mathscr{H}} \le \frac{E'-a}{E-a} \|x_+ + L_{E'} x_+\|_{\mathscr{H}} \tag{10'}$$

and

$$(E'-E)\, \|x_+ + L_{E'} x_+\|^2_{\mathscr{H}} \le Q_E\big(x_+ + L_E x_+\big) - Q_{E'}\big(x_+ + L_{E'} x_+\big) \le (E'-E)\, \|x_+ + L_E x_+\|^2_{\mathscr{H}} . \tag{11'}$$

Up to these changes, [1, Lemma 2.1] remains as stated previously. Note that formula (10') implies that the map $x_+ + L_E x_+ \mapsto x_+ + L_{E'} x_+$ is an isomorphism between Γ_E and $\Gamma_{E'}$ which extends to an isomorphism between $\overline{\Gamma}_E$ and $\overline{\Gamma}_{E'}$. The replacement of $x_+ \in F_+$ by $x_+ + L_E x_+ \in \Gamma_E$ has also to be done in the definition of the norm $\mathscr{N}_E$, which becomes

$$\mathscr{N}_E(x_+ + L_E x_+) = \sqrt{Q_E(x_+ + L_E x_+) + (K_E + 1)\|x_+ + L_E x_+\|^2_{\mathscr{H}}} .$$

As a consequence, formula (13) takes a slighly different form, since $\mathcal{N}_E$ and $\mathcal{N}_{E'}$ are now defined on different spaces. It becomes

$$c(E,E')\,\mathcal{N}_E(x_+ + L_E x_+) \le \mathcal{N}_{E'}(x_+ + L_{E'} x_+) \le C(E,E')\,\mathcal{N}_E(x_+ + L_E x_+)\,, \quad \forall\, x_+ \in F_+\,. \tag{13'}$$

Let $\Pi_E : \mathcal{H} \to \overline{\Gamma}_E$ be the orthogonal projector on $\overline{\Gamma}_E$. Note that for any $x \in \Gamma_E$ we have $Q_E(x) = (x, S_E x)$ where

$$S_E x = \Pi_E\big(\Lambda_+(A-E)\Lambda_+ x + (B+E)\Lambda_- x\big)\,.$$

It is clear that S_E is a symmetric operator bounded from below on the Hilbert space $\overline{\Gamma_E}$, with domain Γ_E. So Q_E is closable in $\overline{\Gamma_E}$. We denote its closure $\overline{Q}_E$. Its domain is denoted by G_E and the corresponding extended norm is denoted by $\overline{\mathcal{N}}_E$. Note that G_E is a subspace of $\overline{\Gamma_E}$, so it depends on E, while in [1] its analogue G was considered as a subspace of $\mathcal{H}_+$, independent of E and, as a consequence, the norms $\overline{\mathcal{N}}_E$ were all equivalent. Now (13') implies that any two normed spaces $(G_E, \overline{\mathcal{N}}_E)$ and $(G_{E'}, \overline{\mathcal{N}}_{E'})$ are isomorphic.

We denote by T_E the Friedrichs extension of S_E. It is a self-adjoint operator in $\overline{\Gamma_E}$ with domain $\mathcal{D}(T_E)$ and we have $\Gamma_E \subset \mathcal{D}(T_E) \subset G_E \subset \overline{\Gamma_E} \subset \mathcal{H}$. In [1], $\mathcal{D}(T_E)$ was a subspace of $\mathcal{H}_+$, but this is no longer true with our new definition. We do not know whether $\mathcal{D}(T_E)$ is a graph over $\mathcal{H}_+$, but the important point is that the graph Γ_E is a form-core of T_E and G_E is its form-domain. In formula (14) for the min-max levels, G should be replaced by G_E and the notation x_+ should be replaced by x, since this variable no longer belongs to $\mathcal{H}_+$. Moreover, for any $x \in \overline{\Gamma}_E$, the extended norm $\overline{n}_E(x)$ considered in [1] is replaced by $\|x\|_{\mathcal{H}}$. Formula (14) thus becomes

$$\ell_k(T_E) = \inf_{\substack{V \text{ subspace of } G_E \\ \dim V = k}} \; \sup_{x \in V\setminus\{0\}} \frac{\overline{Q}_E(x)}{\|x\|^2_{\mathcal{H}}}\,. \tag{14'}$$

In formula (15) of [1, Lemma 2.2], x_+ should be replaced by $x_+ + L_\lambda x_+$. The lemma remains otherwise unchanged. In part (a) of its proof, the notation $Q_\lambda(x_+)$ is used repeatedly. It should be replaced everywhere by $Q_\lambda(x_+ + L_\lambda x_+)$, but no other change has to be done.

Similarly, in the sequel of the proof of [1, Theorem 1.1], $T_{\lambda_k} x_+$ should be replaced by $T_{\lambda_k}(x_+ + L_{\lambda_k} x_+)$ and G' by G'_{λ_k}; $\mathcal{A}$ is now the polar form of the quadratic form $\Gamma_{\lambda_k} \ni \tilde{y} \mapsto Q_{\lambda_k}(\tilde{y}) + \lambda_k \|\tilde{y}\|^2_{\mathcal{H}}$; one should replace $Q_{\lambda_k}(\Lambda_+ y)$ by $Q_{\lambda_k}(\tilde{y})$ in formula (20) and its proof.

Moreover we recall that Γ_E is a form-core of T_E and G_E is its form-domain, so that (14') is equivalent to

$$\ell_k(T_E) = \inf_{\substack{V_+ \text{ subspace of } F_+ \\ \dim V_+ = k}} \; \sup_{x_+ \in V_+\setminus\{0\}} \frac{Q_E(x_+ + L_E x_+)}{\|x_+ + L_E x_+\|^2_{\mathcal{H}}}\,.$$

3. An example

The additional assumption $F_\pm \subset \mathcal{D}(A)$ seems harmless: in all examples of the literature we are aware of, it is satisfied. Condition $(\mathscr{C})$ is satisfied in many situations of interest in physics, as explained in [8]. But in this short section we give an example where $(\mathscr{C})$ is not satisfied. Let V be an electric potential in $\mathbb{R}^3$ of the form

$$V(x) = -\frac{\nu_1}{|x|} + \frac{\nu_2}{|x - x_0|}$$

with $0 < \nu_1 < \sqrt{3}/2$, $3/4 < \nu_2 < \sqrt{3}/2$ and $x_0 \in \mathbb{R}^3 \setminus \{0\}$. On the Hilbert space $\mathscr{H} = \mathrm{L}^2(\mathbb{R}^3, \mathbb{C}^4)$, consider the associated Dirac-Coulomb operator

$$A = \alpha \cdot \mathbf{p} + \beta m + V$$

with domain $\mathrm{H}^1(\mathbb{R}^3, \mathbb{C}^4)$. Since $0 \le \nu_1, \nu_2 < \sqrt{3}/2$, A is self-adjoint and admits the subspace $F = C_c^\infty(\mathbb{R}^3, \mathbb{C}^4)$ as a core. Let $\Lambda_\pm = \mathbf{1}_{\mathbb{R}_\pm}(\alpha \cdot \mathbf{p} + \beta m)$. As was proved by C. Tix in [10], the Brown-Ravenhall operators $\Lambda_+ A_{|\Lambda_+ F}$ and $-\Lambda_- A_{|\Lambda_- F}$ are both positive, so that one can apply Theorem 1 with $a = 0$, but as shown in [9, Corollary 3], $\Lambda_- A_{|\Lambda_- F}$ is not essentially self-adjoint, since $\nu_2 > 3/4$.

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