STABILITY FOR THE GRAVITATIONAL VLASOV-POISSON SYSTEM IN DIMENSION TWO

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Abstract. *We consider the two dimensional gravitational Vlasov-Poisson system. Using variational methods, we prove the existence of stationary solutions of minimal energy under a Casimir type constraint. The method also provides a stability criterion of these solutions for the evolution problem.*

Key-words. Vlasov-Poisson system – stellar dynamics – polytropic gas spheres – gravitation – mass – energy – kinetic energy – potential energy – interpolation – Hardy-Littlewood-Sobolev inequality – optimal constants – symmetric nonincreasing rearrangements – Riesz' theorem – bounded solutions – direct variational methods – minimization – scalings – solutions with compact support – minimizers – Lagrange multiplier – Semilinear elliptic equations – Uniqueness – Dirichlet boundary conditions – dynamical stability

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1. Introduction

The n-dimensional gravitational Vlasov-Poisson system describes the evolution of a nonnegative distribution function $f : (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ according to Vlasov's equation, under the action of a self-consistent attractive force determined by Poisson's equation:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U_f \cdot \nabla_v f = 0 \ , \\ \Delta U_f = \rho_f \ , \\ \rho_f(x) = \int f(x,v) \, dv \ . \end{cases} \tag{1}$$

In dimension $n = 3$, this system is used to modelize the evolution of a large ensemble of particles subject to their own gravity, under the assumption that both the relativistic effects and the collisions between particles can be neglected. In this case the Newtonian potential U_f is given in terms of ρ_f by the mean field equation

$$U_f(t,x) = -\frac{1}{4\pi} \int \frac{1}{|x-y|} \, \rho_f(y) \, dy$$

under the natural asymptotic condition $\lim_{|x|\to\infty} U_f = 0$, which has to be understood in an average sense.

The system (1) also makes sense in dimension $n = 2$. In the context of gravitation, it modelizes a system with translation invariance along a direction, giving rise for instance to solutions with cylindrical symmetry which can be expected to be close, at least heuristically, to 3-dimensional solutions with cigar-like shapes (see [2, 1]).

At the level of the characteristics, the energy can be defined as

$$\mathsf{e}_f(x,v,t) = \frac{1}{2}\,|v|^2 + U_f(x,t)\;.$$

Newton's equations are

$$\frac{dx}{dt} = \nabla_v\,\mathsf{e}_f\;,\quad \frac{dv}{dt} = -\nabla_x\,\mathsf{e}_f\;,$$

and $t \mapsto \mathsf{e}_f(x(t),v(t),t)$ is therefore constant if U_f does not depend on t. Since the Vlasov equation takes the form

$$0 = \partial_t f + \nabla_v\,\mathsf{e}_f\cdot\nabla_x f - \nabla_x\,\mathsf{e}_f\cdot\nabla_v f\;,$$

any function of the form

$$f(x,v) = \phi(\mathsf{e}_f(x,v))$$

will therefore be a stationary solution of (1).

A wide literature (see [9, 11, 12, 13, 14, 16, 21]) has been devoted to the characterization by variational methods of the steady states of (1) and the study of their stability properties. For this purpose, the basic tool is the total energy

$$\mathcal{E}(f) := \frac{1}{2}\iint |v|^2\,f\;dx\,dv + \frac{1}{2}\int U_f\,\rho_f\;dx\;.$$

If (f,U_f) is a solution of (1) with U_f independent of t, then

$$\frac{d}{dt}\mathcal{E}(f(\cdot,\cdot,t)) = \iint \mathsf{e}_f(x,v,t)\,\partial_t f\;dx\,dv = 0\;.$$

Observe that the so-called Casimir functionals

$$\mathcal{C}(f) := \iint Q(f)\;dx\,dv$$

are also preserved along time evolution, as well as the total mass $M = M(f) := \iint f\,dx\,dv$. These quantities are therefore appropriate to study the dynamical stability of the solutions of (1) with respect to the special stationary solutions which are characterized as the minimizers of $\mathcal{E}(f)$ under Casimir and mass constraints.

Concerning the minimization of $\mathcal{E}(f)$ under the Casimir constraint $\mathcal{C}(f) \le K$ for some given positive constant K in the 3-dimensional case, Guo and Rein proved in [14] under the additional assumption

$$Q(f) \ge f + f^{1+1/k}\;,\quad k \in (0,7/2)$$

that minimizers exist and are compactly supported stationary solutions of (1). Afterwards Schaeffer showed in [17] that, if f_1 and f_2 are two minimizers, either there exists an $a \in \mathbb{R}^3$ such that $f_2(x,v) = f_1(x+a,v)$ almost everywhere, or $M(f_1) \neq M(f_2)$. The combination of these two results gives the following dynamical stability result. With

$$d(f,g) := \iint (f-g)\,\mathsf{e}_g\;dx\,dv\;,$$

if f_∞ is a minimizer, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for any solution of (1) with initial data f_0, if $d(f_0, f_\infty) < \delta$, then $d(f(\cdot,\cdot,t), f_\infty) < \epsilon$.

The goal of this paper is to adapt such results to the 2-dimensional case. Difficulties arise from the fact that the two-dimensional Newtonian potential

$$U_f := \frac{1}{2\pi} \log|\cdot| * \rho_f \ ,$$

behaves like $\frac{M}{2\pi} \log|x|$ as $|x| \to \infty$, and that its gradient is not bounded in $L^2(\mathbb{R}^2)$. The dimension also plays a role in the scaling properties of the system. This is reflected by interpolation estimates which differ significantly from the 3-dimensional ones. As a result, the dependence of the minimization problem in parameters like the total mass is completely different. See Section 8 for some considerations on the normalization of the potential, the mass of the minimizer and the size of its support, which are specific to dimension 2, and [8] for results concerning stationary states and dynamical stability of the 2-dimensional Vlasov-Poisson system in the electrostatic case with confinement, which are far simpler to obtain.

This paper is organized as follows. In Section 2 we introduce the notations and state our main results. Section 3 is devoted to *a priori* estimates, which are used in Section 4 to prove the existence of minimizers. Their properties are studied in Section 5. Section 6 is devoted to the statement and the proof of a dynamical stability result. In Section 7, we state some uniqueness properties which apply in the special case of the polytropic states.

2. Notations and main results

We consider the two-dimensional time-dependent Vlasov-Poisson system (1). To any nonnegative, measurable function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, we associate the spatial density

$$\rho_f(x) := \int f(x,v)\, dv \ ,$$

and consider the Newtonian potential associated to ρ_f,

$$U_f := \frac{1}{2\pi} \log|\cdot| * \rho_f \ .$$

The distribution function f may, or not, depend on the time $t \in \mathbb{R}^+$. In the sequel all integrals are taken on the whole space $\mathbb{R}^2$ unless it is explicitly specified. We will omit the subscript f in the density and the potential whenever there is no ambiguity. Moreover, when working with a sequence of functions (f_n), we will denote the corresponding densities and potentials by (ρ_n) and (U_n) respectively.

The kinetic energy, potential energy and total energy associated to f are respectively

$$E_{\mathrm{kin}}(f) := \frac{1}{2}\iint |v|^2 f(x,v)\,dx\,dv\ ,$$
$$E_{\mathrm{pot}}(f) := \frac{1}{2}\int U_f(x)\rho(x)\,dx =$$
$$\frac{1}{4\pi}\iint \log|x-y|\,\rho(x)\rho(y)\,dx\,dy\ ,$$
$$\mathcal{E}(f) := E_{\mathrm{kin}}(f) + E_{\mathrm{pot}}(f)\ .$$

The main problem we are going to consider in this paper is the minimization problem

$$h_K := \inf_{f\in\mathcal{F}_K} \mathcal{E}(f)\ . \qquad (\mathcal{I}_K)$$

Here the set $\mathcal{F}_K$ is defined by

$$\mathcal{F}_K := \left\{ f\in L^1(\mathbb{R}^2\times\mathbb{R}^2) \mid f\geq 0\ ,\quad E_{\mathrm{kin}}(f)<\infty\ ,\quad \mathcal{C}(f)\leq K \right\}\ ,$$

K is a prescribed positive constant and

$$\mathcal{C}(f) := \iint Q(f(x,v))\,dx\,dv$$

is the Casimir constraint based on a function Q for which we assume :

(Q1) *The function Q is of class C^1 on $[0,\infty)$ and such that $Q(0)=0$.*

(Q2) *There exist two positive constants C_0 and k such that*

$$Q(f) \geq C_0\ f^{1+\frac{1}{k}} \quad \forall\ f\geq 0\ .$$

(Q3) *Q is convex.*

Under these assumptions, our first main result is concerned with proving that h_K is in fact a minimum .

Theorem 1. *Let $K>0$. There exists a function $f_\infty\in\mathcal{F}_K$, with symmetric and nonincreasing density ρ_{f_∞}, such that the support of ρ_{f_∞} is contained in $B(0,1)$, and*

$$\mathcal{E}(f_\infty) = \inf_{\mathcal{F}_K} \mathcal{E}(f) = h_K\ .$$

Stability results for the minimizers of $(\mathcal{I}_K)$ will be established in the framework of the results of Ukai and Okabe, [20], for which we need some additional definitions and assumptions. Let $B^1(\mathbb{R}^2\times\mathbb{R}^2)$ be the class of all bounded C^1 functions with bounded first derivatives. A function g on $D\subset\mathbb{R}^n$, with values in a Banach space B equipped with a norm $|\cdot|$, is uniformly Hölder continuous of exponent $\sigma\in(0,1)$ if and only if

$$\sup_{x,y\in D,\,x\neq y} \frac{|g(x)-g(y)|}{|x-y|^\sigma} < \infty \quad \Longleftrightarrow: \quad g\in C^{0,\sigma}_{\mathrm{unif}}(D;B)\ ,$$

and it is said to be uniformly Lipschitz continuous if $\sigma = 1$. When $D = [0,T]$ and $B = L^1(\mathbb{R}^2)$, we will denote the corresponding functional space by $C^{0,\sigma}_{\rm unif}([0,T]; L^1(\mathbb{R}^2))$.

Definition. *A function $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an* admissible initial data *if and only if*

(I1) $f \in B^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

(I2) *There are two positive constants κ and γ such that*

$$|f(x,v)| \le \kappa\,(1+|x|)^{-2\gamma}(1+|v|)^{-2\gamma} \quad \forall\,(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2\,.$$

(I3) $\nabla_x f,\ \nabla_v f \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$.

(I4) *There are two positive constants $\eta > 0$ and $\gamma > 2$ such that*

$$|\nabla_x f| + |\nabla_v f| \le \eta\,(1+|v|)^{-\gamma} \quad \forall\,(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2\,.$$

Definition. *Let $0 \le \sigma \le 1$ and $T > 0$. The* class of solutions $\mathcal{A}^\sigma(T)$ *is made of pairs of functions (f,U) with $f : [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ and $U : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ such that*

(A1) *The function $f \in C^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ is bounded.*

(A2) *The function ρ_f belongs to $C^{0,\sigma}_{\rm unif}([0,T]; L^1(\mathbb{R}^2))$.*

(A3) *The function ρ_f belongs to $C^{0,\sigma}_{\rm unif}([0,T] \times \mathbb{R}^2; \mathbb{R}^+)$.*

(A4) *The function $(t,x) \mapsto |\nabla_x U|$ belongs to $C^{0,1}_{\rm unif}([0,T] \times \mathbb{R}^2; \mathbb{R}^+)$.*

Theorem 2 (Ukai and Okabe, [20])**.** *Let f_0 be an admissible initial data, $T > 0$ and $0 < \sigma < 1$. Then there exists a solution (f,U) of (1) with $f(0,\cdot,\cdot) = f_0(\cdot,\cdot)$, which is unique in the class of functions $\mathcal{A}^\sigma(T)$, up to the addition to U of any function of t.*

Dynamical stability with respect to energy minimizing stationary states requires an assumption of isolation of the solutions, see [14]. Such a property has been proved in [17] in dimension $n = 3$, under an additional mass constraint. In this paper we will state a similar result in dimension $n = 2$: see Theorem 26 in Section 6. Now let us state a stability result without mass constraint, provided the nonlinear Poisson equation has a unique solution. Let $\phi = (Q')^{-1}$ on $(Q'(0),\infty)$ and extend it by 0 to $(-\infty, Q'(0))$. Denote by $\psi(s) := 2\pi \int_0^s \phi(\sigma)\,d\sigma$ a primitive. By uniqueness, we mean that the equation

$$\begin{cases} -\Delta V = \psi(V) & \text{in} \quad B(0,1) \\ V > 0 & \text{in} \quad B(0,1) \\ V = 0 & \text{on} \quad \partial B(0,1) \end{cases}$$

has at most one bounded solution. Denote this assumption by (U). Notice that by our assumptions, ψ is a C^1 function on $\mathbb{R}$, and by the theorem of Gidas, Ni and Nirenberg, V is known to be a radial function.

Theorem 3. *Let $K > 0$ and suppose that assumption (U) holds. Assume that $Q \in C^2(0,\infty)$ satisfies the following additional assumption: There exists a $p \in [1,2]$ such that $\inf_{s>0} s^{2-p}\,Q''(s) =: A > 0$. Consider* *the unique minimizer* (f_∞, U_∞) *for* $(\mathcal{I}_K)$*. Then, for any* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that the following property holds. Let* $f_0 \in \mathcal{F}_K$ *be an admissible data and consider the corresponding solution of (1). If* $\mathcal{E}(f_0) - \mathcal{E}(f_\infty) < \delta$*, then*

$$\|f^{*x}(t) - f_\infty\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} + \|f^{*x}(t) - f_\infty\|_{L^{1+1/k}(\mathbb{R}^2\times\mathbb{R}^2)} < \epsilon \quad \forall\, t > 0\,.$$

Here f^{*x} denotes the symmetric rearrangement of the function f in the x variable. See Section 3 for more details.

We will prove later that the minimizer (f_∞, U_∞) is such that $V = \lambda_K^{-1} U_\infty$ satisfies $-\Delta V = \psi(V)$ on $B(0,1)$, $V \equiv 0$ and $2\pi\,|\nabla V| \equiv M$ on $\partial B(0,1)$. The uniqueness assumption amounts to require that $M = M(f_\infty)$ is uniquely determined for any given K. Of course, if M is fixed to some value for which there exists a minimizer (f_∞, U_∞), then a stability result holds without Assumption (U): see Theorem 26.

It is out of the scope of this paper to give optimal conditions on Q which imply (U). A huge literature has indeed been devoted to this question. As an example, let us simply mention the following sufficient conditions, which can be deduced from [19]:

(U1) $Q'(0) = 0$.
(U2) The function $s \mapsto \frac{s\,\phi(s)}{\psi(s)}$ is nonincreasing on $\mathbb{R}^+$.

We will come back to this important example which covers the polytropic case in Section 7.

Notations. Throughout this paper, C will denote a generic positive constant which is independent of f. Its value may change from line to line. The characteristic function of a measurable set A will be noted $\mathbb{1}_A$.

3. A priori estimates

We first prove that the total energy $\mathcal{E}(f)$ is bounded from below in $\mathcal{F}_K$. Define

$$E_{\rm pot}^-(f) = \iint_{|x-y|<1} \log|x-y|\,\rho(x)\rho(y)\,\frac{dx\,dy}{4\pi} \le 0\,.$$

Lemma 4. *Take* $m = 1+k$, $k > 0$*. Then for any* $f \in \mathcal{F}_K$ *the following inequalities hold:*

(i) $\int \rho_f^{1+1/m}(x)\,dx \le C\|f\|_{L^{1+1/k}}\ E_{\rm kin}(f)^{\frac{1}{m}} \le C\,K^{\frac{m-1}{m}}\,E_{\rm kin}(f)^{\frac{1}{m}}$.
(ii) $\iint \rho_f(x)\ |\log|x-y||\ \rho_f(y)\,dx\,dy \le \mathcal{E}(f) + 2\,|E_{\rm pot}^-(f)|$.
(iii) $E_{\rm pot}^-(f) \ge -C\|f\|_{L^{1+1/k}}^{\frac{2m}{m+1}}\ E_{\rm kin}(f)^{\frac{2}{m+1}} \ge -C\,K^{\frac{2(m-1)}{m+1}}\,E_{\rm kin}(f)^{\frac{2}{m+1}}$.

Proof. According to (Q2) and the definition of $\mathcal{F}_K$, $\|f\|_{L^{1+1/k}} < \infty$. By writing

$$\rho(x) \le \int_{|v|<R} f(x,v)\,dv + \frac{1}{R^2}\int_{|v|\ge R} |v|^2 f(x,v)\,dv\,,$$

using Hölder's inequality,

$$\rho \leq \left(\pi R^2\right)^{\frac{1}{k+1}} \left(\int_{|v|<R} f^{1+1/k}\, dv\right)^{\frac{k}{k+1}} + \frac{1}{R^2}\int_{|v|\geq R} |v|^2 f\, dv\ ,$$

and then optimizing in $R>0$, for x fixed, we get the inequality

$$\rho \leq C\left(\int f^{1+1/k}\, dv\right)^{\frac{k}{k+2}} \left(\int |v|^2 f\, dv\right)^{\frac{1}{k+2}} .$$

Taking the power $1+1/m$ and integrating in the x variable, we see that

$$\int \rho^{1+1/m}\, dx \leq C\int \left(\int f^{1+1/k}\, dv\right)^{\frac{k}{k+1}} \left(\int |v|^2 f\right)^{\frac{1}{k+1}} dx\ ,$$

and it is enough to apply Hölder's inequality to prove (i).

To prove (ii) and (iii), observe that

$$E_{\rm pot}(f) = \iint \log|x-y|\, \rho(x)\rho(y)\, \frac{dx\, dy}{4\pi} \geq \iint \kappa(x-y)\, \rho(x)\rho(y)\, \frac{dx\, dy}{4\pi}\ .$$

With $\kappa(x) := \log|x|\, \mathbb{1}_{B(0,1)}(x)$, using successively Hölder's and Young's inequalities, we get

$$E^-_{\rm pot}(f) \geq -\frac{1}{4\pi}\, \|\rho\|_{L^{1+1/m}}\, \|\kappa * \rho\|_{L^{m+1}} \geq -C\, \|\kappa\|_{L^{(m+1)/2}}\, \|\rho\|^2_{L^{1+1/m}}\ .$$

By (i), this completes the proof. □

Lemma 5. *Let $f\in\mathcal{F}_K$ and consider for any $\alpha>0$ the scaled distribution function*

$$f^{(\alpha)}(x,v) := f(\alpha\, x, \alpha^{-1}\, v) \quad \forall\, (x,v)\in\mathbb{R}^2\times\mathbb{R}^2\ .$$

Then $\|f^{(\alpha)}\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} = \|f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$, $\mathcal{C}(f^{(\alpha)}) = \mathcal{C}(f)$ and

$$\mathcal{E}(f) \geq \inf_{\alpha>0} \mathcal{E}(f^{(\alpha)}) = E_{\rm pot}(f) + \frac{\|f\|^2_{L^1}}{8\pi}\, [1-2\log\bar\alpha] = \mathcal{E}(f^{(\bar\alpha)})$$

with $\bar\alpha^2 := \frac{\|f\|^2_{L^1}}{8\pi E_{\rm kin}(f)}$, and

$$\mathcal{E}(f) = \mathcal{E}(f^{(\bar\alpha)}) \Longleftrightarrow \bar\alpha = 1 \Longleftrightarrow E_{\rm kin}(f) = \frac{1}{8\pi}\, \|f\|^2_{L^1}\ .$$

Proof. An easy calculation shows that, as a function of α,

$$\mathcal{E}(f^{(\alpha)}) = \alpha^2 E_{\rm kin}(f) + E_{\rm pot}(f) - \frac{1}{4\pi}\|f\|^2_{L^1}\log\alpha\ .$$

achieves its minimum for $\alpha=\bar\alpha$. □

Corollary 6. *For any $f\in\mathcal{F}_K$,*

$$\mathcal{E}(f) \geq E_{\rm kin}(f) - C\, K^{\frac{2(m-1)}{m+1}}\, E_{\rm kin}(f)^{\frac{2}{m+1}}$$

As a consequence, the total energy $\mathcal{E}$ is bounded from below in $\mathcal{F}_K$ and

$$h_K := \inf\{\mathcal{E}(f)\ |\ f\in\mathcal{F}_K\} \in (-\infty, 0)\ .$$

Proof. The lower bound on $\mathcal{E}$ is a consequence of Lemma 4, (iii). Using $2/(m+1) < 1$, it follows that $h_K > -\infty$. To prove that $h_K < 0$, it is enough to find a function $f \in \mathcal{F}_K$ such that $\mathcal{E}(f) < 0$. For $\beta > 0$, $\delta > 0$, consider therefore

$$f(x,v) := \beta\, 1\!\!1_{B(0,1/2)}(x)\ 1\!\!1_{B(0,\delta)}(v)\ .$$

Such a distribution function f satisfies

$$\|f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} = \tfrac{1}{4}\,\pi^2\,\delta^2\,\beta\ , \qquad \mathcal{C}(f) = \tfrac{1}{4}\,\pi^2\,\delta^2\,Q(\beta)\ ,$$

$$E_{\rm kin}(f) = \tfrac{1}{16}\,\pi^2\,\delta^4\,\beta\ , \qquad E_{\rm pot}(f) < 0\ .$$

and belongs to $\mathcal{F}_K$ if $\pi^2\,\delta^2\,Q(\beta)/4 \leq K$. If additionally, with the notations of Lemma 5, $\bar\alpha > \mathsf{e}$, which amounts to $(8\pi E_{\rm kin}(f))^{-1}\|f\|^2_{L^1} = \pi\,\beta/8 \geq \sqrt{\mathsf{e}}$, then $\mathcal{E}(f^{(\bar\alpha)}) < 0$. Taking $\beta = 8\,\sqrt{\mathsf{e}}/\pi$ and $\delta = \frac{2}{\pi}\sqrt{K/Q(\beta)}$ therefore shows that $h_K < 0$. □

Remark. The 2- and 3-dimensional gravitational Vlasov-Poisson systems differ on many points.
(i) In the case $n = 3$, the Hardy-Littlewood-Sobolev inequality is

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}\,dx\,dy \leq C\,\|\rho\|^2_{L^{6/5}(\mathbb{R}^3)}\ .$$

To control $\|\rho\|_{L^{6/5}(\mathbb{R}^3)}$ by interpolation between $\|\rho\|_{L^1(\mathbb{R}^3)} = \|f\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$ and

$$\|\rho\|_{L^q(\mathbb{R}^3)} \leq C\,\|f\|^{\theta}_{L^{1+1/k}(\mathbb{R}^3\times\mathbb{R}^3)}\left(\iint |v|^2 f\ dx\,dv\right)^{1-\theta}\ ,$$

where $\theta = 2(k+1)/(2k+5)$ and $q = (2k+5)/(2k+3)$, one has to require that $q \geq 6/5$, which means $k \leq 7/2$. Such a restriction is not required in dimension $n = 2$. As seen above, due to the change of sign of the logarithm we only need to control the term involving the convolution kernel $\kappa(x) := \log|x|\,1\!\!1_{B(0,1)}(x)$, which is bounded in L^p for any $p > 1$.
(ii) From Lemma 4, (iii), if $n = 2$, for any $f \in \mathcal{F}_K$, $E_{\rm kin}(f)$, $E_{\rm pot}(f)$ and $\iint \rho(x)\ |\log|x-y||\,\rho(y)\,dx\,dy$ are bounded in terms of $\mathcal{E}(f)$ independently of $M := \|f\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$, while for $n = 3$, the bounds on $E_{\rm kin}(f)$ and $E_{\rm pot}(f)$ also depend on M.
(iii) In dimension $n = 2$, if $M = \iint f(x,v)\ dx\,dv \neq 0$, $U_f(x) \equiv \frac{M}{2\pi}\log|x| \to \infty$ as $|x| \to \infty$, and $|\nabla U_f|^2 \equiv \frac{M}{2\pi\,|x|}$ is not integrable (see [18] for more details). H^1_0 estimates can however be established by working on differences of distribution functions with same mass. Moreover, the following estimate on the mass in terms of $\mathcal{E}$ can be shown :

Corollary 7. *There exists a positive constant* A *such that, for any* $K > 0$ *and for any* $f \in \mathcal{F}_K$*, if* $M := \|f\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$*, then*

$$\mathcal{E}(f) \geq \mathcal{E}(f^{(\bar\alpha)}) \geq \frac{1}{8\pi}\,M^2 - AK^{\frac{2k}{k+2}}M^{\frac{4}{k+2}} =: g(M,K)\;.$$

In other words, $M \in [M_1(K,\mathcal{E}(f)), M_2(K,\mathcal{E}(f))]$ *where* $M_i(K,\mathcal{E}(f))$*,* $i = 1,\,2$*, are the two values of* m *for which* $g(m,K) = \mathcal{E}(f)$*. These bounds are positive except for the case* $\mathcal{E}(f) \geq 0$*, where the lower bound can be taken* $M_1(K,\mathcal{E}(f)) = 0$*.*

Proof. With the notations of Lemma 5, $\mathcal{E}(f) \geq \mathcal{E}(f^{(\bar\alpha)}) \geq g(M,K)$. Using $E_{\rm kin}(f^{(\bar\alpha)}) = (8\pi)^{-1}\,\|f\|_{L^1}^2$, we get $\mathcal{E}(f) \geq g(M,K)$ by Lemma 4, (iii). □

Proposition 8. *Let* ρ *be a nonnegative function in* $L^1(\mathbb{R}^n)$*,* $M := \|\rho\|_{L^1(\mathbb{R}^n)}$ *and* $P := \iint_{|x-y|>1}\rho(x)\,\log|x-y|\,\rho(y)\,dx\,dy < \infty$*. Then*

$$\int \rho(x)\,\log(1+|x|^2)\,dx < 9\,M\,\log(1+R^2) + 32\,\frac{P}{M}\;,$$

where $R \geq 1$ *is chosen such that* $\int_{|x|>R/2}\rho(x)\,dx \leq M/2$*.*

Proof. First of all, $\int_{|x|\leq R}\rho(x)\,\log\left(1+|x|^2\right)dx \leq M\,\log(1+R^2)$. Using the inequality $\log\left(1+t\right) \leq 4\,\log\left(1+t/4\right)$ for any $t \geq 0$, we get

$$\int_{|x|>R}\rho(x)\,\log\left(1+|x|^2\right)\,dx \leq 4\int_{|x|>R}\rho(x)\,\log\left(1+\frac{1}{4}\,|x|^2\right)\,dx\;.$$

Notice now that for any $(x,y) \in \mathbb{R}^2\times\mathbb{R}^2$ such that $2\,|y| \leq R < |x|$, we have $|x-y|^2 \geq |x|^2(1-2\,t+t^2) \geq \frac{1}{4}\,|x|^2$ since $t := \frac{|y|}{|x|}$ is in the interval $(0,\frac{1}{2})$. Thus

$$\int_{|x|>R}\rho(x)\,\log\left(1+|x|^2\right)\,dx \leq \frac{8}{M}\iint_{2\,|y|\leq R<|x|}\rho(x)\,\rho(y)\,\log\left(1+|x-y|^2\right)\,dx\,dy$$

and the conclusion holds by observing that

$$\log\left(1+|x-y|^2\right) \leq \left\{\begin{array}{ll}\log(1+R^2) & \text{if}\quad |x-y| \leq R\;,\\ 4\log|x-y| & \text{if}\quad |x-y| > R\;.\end{array}\right.$$

□

From Lemma 4, Corollary 7 and Proposition 8, we deduce the

Corollary 9. *There exists a continuous function* $b : (0,\infty)^2 \to \mathbb{R}^+$ *such that*

$$\int \rho_f(x)\,\log(1+|x|^2)\,dx < b\big(\mathcal{E}(f),K\big)\;.$$

Additional estimates can be obtained for radial spatial densities, which motivates the introduction of radially symmetric nonincreasing rearrangements. For any measurable set $A \in \mathbb{R}^n$ with Lebesgue measure $|A| = \mathrm{meas}(A)$, we set $\mathbb{1}_A^* := \mathbb{1}_{B(0,n|A|/|S^{n-1}|)}$. Then, to any measurable nonnegative function g on $\mathbb{R}^n$ which vanishes at infinity, we can associate

$$g^*(x) := \int_0^\infty \mathbb{1}^*_{\{g>t\}}(x)\;dt\;.$$

It is straightforward to check that $\int q(g^*)\,dx = \int q(g)\,dx$ if q is a continuous function with nonnegative values, and as a special case, we get $\|g^*\|_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)}$ for any $p \geq 1$, see [15] for more details. Symmetric nonincreasing rearrangements with respect to only one of the variables can also be defined for any nonnegative measurable function h defined on $\mathbb{R}^n \times \mathbb{R}^n$ and vanishing at infinity as follows:

$$h^{*_x}(x,v) := \int_0^\infty \mathbb{1}^*_{\{h(x,v)>t\}}(x)\;dt$$

Observe that for almost every $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ this function is well defined, see *e.g.* [9] for more details. The following rearrangement inequality holds for the logarithmic kernel.

Lemma 10. [6] *If g is a nonnegative function in $L^1(\mathbb{R}^2)$ such that $\int g(x)\log(1+|x|^2)\,dx < \infty$, then*

$$\iint g^*(x)\log|x-y|\,g^*(y)\;dx\,dy \leq \iint g(x)\log|x-y|\,g(y)\;dx\,dy\;,$$

with equality if and only if $g = g^*$ up to a translation.

Corollary 11. *If $f \in \mathcal{F}_K$ is such that $\mathcal{E}(f) < \infty$, then $f^{*_x} \in \mathcal{F}_K$, ρ_f is radially symmetric nonincreasing and*

$$\mathcal{E}(f^{*_x}) \leq \mathcal{E}(f)\;.$$

Proof. The result follows from the basic properties of symmetric nonincreasing rearrangements and from Corollary 9. Note that the value of the kinetic energy is unchanged by a rearrangement with respect to the x variable. □

For radially symmetric densities the potential has a simple expression.

Lemma 12. [7] *Let w be a nonnegative bounded, radial measure on $\mathbb{R}$, such that $\int_0^\infty \log(1+s^2)\,w(s)\,ds < \infty$. Let*

$$U_w(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,w(|y|)\;dy\;.$$

Then

(i) $$U_w(x) = \frac{1}{2\pi}\log|x|\int_{|y|\leq|x|} w(|y|)\;dy + \frac{1}{2\pi}\int_{|y|>|x|} \log|y|\,w(|y|)\;dy\;,$$

(ii) $$\nabla U_w(x) = \frac{x}{2\pi|x|^2}\int_{|y|\leq|x|} w(|y|)\;dy\;.$$

Proof. With $u(r) := \log r \int_0^r w(s)\, s\, ds + \int_r^\infty \log s\, w(s)\, s\, ds$, which is well defined because of the integrability conditions, an elementary computation shows that

$$r\,\frac{du}{dr} = \int_0^r w(s)\, s\, ds \quad \text{and} \quad \frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = w\ .$$

□

Let $f \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ be a nonnegative distribution function such that $\mathcal{E}(f) < \infty$. Because of Proposition 8, if ρ_f is radially symmetric, then one can use Lemma 12, (i), to compute U_f in terms of ρ_f.

Corollary 13. *For any radially symmetric $f \in \mathcal{F}_K$ such that $\mathcal{E}(f)$ is finite, the function $\bar{f}(x,v) := f(x,v) \cdot 1\!\!1_{B(0,1)}(x)$ also belongs to $\mathcal{F}_K$ and*

$$\mathcal{E}(\bar{f}) \leq \mathcal{E}(f)\ .$$

Proof. Since $\bar{f} \leq f$, $E_{\rm kin}(\bar{f}) - E_{\rm kin}(f) \leq 0$. According to Lemma 12, (i),

$$E_{\rm pot}(\bar{f}) - E_{\rm pot}(f) = -\frac{1}{2\pi} \iint_{|x|>1,\,|y|<|x|} \log|x|\, \rho_f(x)\, \rho_f(y)\, dx\, dy$$

is therefore also nonpositive. □

Inspired by Corollary 13, we can state a more precise result.

Corollary 14. *Consider a radially symmetric function $f \in \mathcal{F}_K$. With the notations of Corollary 13, for any $\epsilon > 0$,*

$$\mathcal{E}(\bar{f}) + \frac{1}{2\pi} \log(1+\epsilon) \iint_{|x|>1+\epsilon,\,|y|<|x|} \rho_f(x)\, \rho_f(y)\, dx\, dy \leq \mathcal{E}(f)\ .$$

Remark. It may look surprising that for any $K > 0$, the support of a radial minimizer has to be contained in the unit ball. Also notice that the mass of the minimizer is determined by the minimization problem, at least when the minimizer is unique. This is one of the major differences compared to the 3-dimensional case. See Section 8 for more comments.

4. Existence of minimizers

We are going to prove Theorem 1 by considering an appropriate minimizing sequence. More general sequences will be considered in Section 6.

Theorem 15. *Let $K > 0$. There exists a minimizing sequence (f_n) of $\mathcal{E}$ in $\mathcal{F}_K$ with a sequence (ρ_n) of symmetric and nonincreasing associated densities, such that (f_n) converges weakly in $L^1 \cap L^{1+1/k}(\mathbb{R}^2 \times \mathbb{R}^2)$ to a function $f_\infty \in \mathcal{F}_K$, with symmetric and nonincreasing density ρ_{f_∞}, such that the support of ρ_{f_∞} is contained in $B(0,1)$, and*

$$\mathcal{E}(f_\infty) = \inf_{\mathcal{F}_K} \mathcal{E}(f) = \lim_{n\to\infty} \mathcal{E}(f_n) = h_K\ .$$

Summarizing the estimates of Section 3, we can choose a minimizing sequence (f_n) in $\mathcal{F}_K$ with radially symmetric associated densities (ρ_n), and such that the supports of f_n and ρ_n are contained for any $n \in \mathbb{N}$ in $B(0,1) \times \mathbb{R}^2_v$ and $B(0,1) \subset \mathbb{R}^2_x$ respectively. For simplicity, we decompose the proof of Theorem 15 into three intermediate results (Proposition 16, Corollary 18 and Proposition 19).

Proposition 16. *Let (f_n) be a minimizing sequence as in Theorem 15. Then there exists a function $f_\infty \in L^1 \cap L^{1+1/k}$ such that, up to the extraction of a subsequence,*

(i) $f_n \rightharpoonup f_\infty$ *in* $L^1 \cap L^{1+1/k}(\mathbb{R}^2 \times \mathbb{R}^2)$,

(ii) $\rho_n \rightharpoonup \rho_\infty := \int f_\infty(x,v)\,dv$ *in* $L^1 \cap L^{1+1/m}(\mathbb{R}^2)$, *with* $m = k+1$,

(iii) $\mathrm{supp}(\rho_\infty) \subset B(0,1)$.

Proof. The weak convergence in L^1 is obtained thanks to the Dunford-Pettis criterion. By Corollary 7, we already know that (f_n) is uniformly bounded in L^1. Concentration is forbidden by the inequality

$$\iint_A f_n(x,v)\,dx\,dv \le \|f_n\|_{L^{1+1/k}} \cdot |A|^{\frac{1}{k+1}}$$

for any measurable set $A \subset \mathbb{R}^2 \times \mathbb{R}^2$ with Lebesgue measure $|A|$. Note that $\|f_n\|_{L^{1+1/k}} \le (K/C_0)^{k/(k+1)}$ by Assumption $(Q2)$, which proves the relative compactness in $L^{1+1/k}$ with respect to the weak convergence. Vanishing is impossible because $\mathrm{supp}(f_n) \subset B(0,1) \times \mathbb{R}^2$ and

$$\iint_{|v|>R} f_n(x,v)\,dx\,dv \le \frac{1}{R^2}\,E_{\mathrm{kin}}(f_n)$$

which is uniformly bounded by Corollary 6. This achieves the proof of (i).

The relative compactness of (ρ_n) in L^1 holds for the same reasons, while the relative compactness in $L^{1+1/m}$ is a consequence of Lemma 4, (i). After extraction of subsequences, (f_n) and (ρ_n) weakly converge to functions f_∞ and ρ_∞, and the fact that $\rho_\infty = \int f_\infty\,dv$ easily follows using appropriate test functions. □

The following result on products of sequences (see for instance [10]) allows to pass to the limit in the potential energy term.

Lemma 17. *If (g_n) and (h_n) are respectively a sequence which converges weakly to some g in L^1, and a bounded sequence in L^∞ which converges almost everywhere to some function h, then*

$$g_n\,h_n \rightharpoonup g\,h \quad in \quad L^1\ .$$

Corollary 18. *Let (f_n) and (ρ_n) be as in Proposition 16, and define $U_n := \frac{1}{2\pi}\log|\cdot| * \rho_n$, $U_\infty := \frac{1}{2\pi}\log|\cdot| * \rho_\infty$. Then*

$$\int U_n\,\rho_n\,dx \to \int U_\infty\,\rho_\infty\,dx \quad as \quad n \to \infty\ .$$

Proof. Because of the assumption on the support of ρ_n and by Young's inequality,

$$U_n(x) = \frac{1}{2\pi}\int \log|x-y|\,\rho_n(y)\,dy$$

is bounded in $L^\infty(B(0,1))$ by $\frac{1}{2\pi}\,\|\,\log|x-y|\,\|_{L^{(m+1)}(B(0,1))}\,\|\rho_n\|_{L^{1+1/m}(B(0,1))}$. By weak convergence of (ρ_n) in $L^{1+1/m}$ we have that

$$\int_{|y|\le|x|}\rho_n(y)\,dy \to \int_{|y|\le|x|}\rho_\infty(y)\,dy\;,$$

$$\int_{|x|\le|y|\le 1}\log|y|\,\rho_n(y)\,dy \to \int_{|x|\le|y|\le 1}\log|y|\,\rho_\infty(y)\,dy\;.$$

Using Lemma 12, this proves the pointwise convergence of U_n to U_∞ almost everywhere, and the result then holds by Lemma 17. □

Proposition 19. *If (f_n) is as in Proposition 16, then*

(i) $E_{\rm kin}(f_\infty) \le \liminf_{n\to\infty} E_{\rm kin}(f_n) < \infty$,
(ii) $\mathcal{C}(f_\infty) \le \liminf_{n\to\infty}\mathcal{C}(f_n) \le K$.

Proof. The proof of (i) follows by weak convergence. By Assumption $(Q3)$, the functional $\mathcal{C}$ is convex and therefore lower semi-continuous by Mazur's Lemma. □

The minimizer of Theorem 15 saturates the constraint $\mathcal{C}(f) \le K$.

Proposition 20. *If $f_\infty \in \mathcal{F}_K$ is a minimizer of $\mathcal{E}$, i.e. if $\mathcal{E}(f_\infty) = h_K$, then*

$$\mathcal{C}(f_\infty) = K\;.$$

Proof. Take $f \in \mathcal{F}_K$ and define the rescaled distribution function

$$f^\mu(x,v) = f(\,x, \mu\, v)\;,\quad \mu > 0\;.$$

Then

$$\mathcal{C}(f^\mu) = \mu^{-2}\,\mathcal{C}(f)\;,$$
$$E_{\rm kin}(f^\mu) = \mu^{-4}\,E_{\rm kin}(f)\quad\text{and}\quad E_{\rm pot}(f^\mu) = \mu^{-4}\,E_{\rm pot}(f)\;.$$

If $\mathcal{C}(f_\infty) < K$, then for $\mu = \sqrt{\mathcal{C}(f_\infty)/K} < 1$,

$$\mathcal{E}(f^\mu_\infty) = \mu^{-4}\,\mathcal{E}(f_\infty) < \mathcal{E}(f_\infty) < 0$$

since $h_K = \mathcal{E}(f_\infty)$ is negative by Corollary 6. Thus $f^\mu_\infty \in \mathcal{F}_K$ and $\mathcal{E}(f^\mu_\infty) < h_K$, a contradiction. □

As a simple consequence of the above scaling, we obtain the dependence of h_K in terms of K.

Corollary 21. *If $h_K := \inf\{\mathcal{E}(f) \mid f \in \mathcal{F}_K\}$ for any $K > 0$, then*

$$h_K = K^2\,h_1\;.$$

Proof. With the above notations, $f \in \mathcal{F}_K$ if and only if $f^{\sqrt{K}} \in \mathcal{F}_1$, and $\mathcal{E}(f^{\sqrt{K}}) = K^{-2}\,\mathcal{E}(f)$. The result easily follows. □

5. Properties of the minimizers

Minimizers of $\mathcal{E}$ on $\mathcal{F}_K$ are steady states of the Vlasov-Poisson system. The proof in dimension $n=2$ is almost the same as in dimension $n=3$. For completeness, we give the main steps of the method in the spirit of [14]. First of all, we need some additional notations. Assume that U_∞ is fixed and let

$$\mathsf{e}_\infty(x,v) := \frac{1}{2}\,|v|^2 + U_\infty(x)\;,$$

and define the function

$$\begin{aligned} \phi\ &: \mathbb{R} \to \mathbb{R}^+ \\ \phi(t)\ &:= \begin{cases} (Q')^{-1}(t) & \text{if}\quad t > Q'(0)\;, \\ 0 & \text{otherwise}\;. \end{cases} \end{aligned}$$

For any $f \in \mathcal{F}_K$, let

$$\lambda[f] := \frac{\iint \mathsf{e}_\infty(x,v)\, f\; dx\, dv}{\iint Q'(f)\, f\; dx\, dv}\;,$$

and

$$\mathsf{e}_0(\lambda) := \lambda\, Q'(0)\;.$$

We can notice that for any $\lambda < 0$, the function

$$f_\infty(x,v) = \begin{cases} \phi\left(\frac{\mathsf{e}_\infty(x,v)}{\lambda}\right) & \text{if}\quad \mathsf{e}_\infty(x,v) < \mathsf{e}_0(\lambda)\;, \\ \\ 0 & \text{otherwise}\;, \end{cases} \tag{2}$$

is a stationary solution of the Vlasov equation:

$$\partial_t f + v\cdot\nabla_x f - \nabla_x U_\infty \cdot \nabla_v f = 0$$

such that $\lambda[f] = \lambda$. In the next result we summarize some properties of the minimizers and prove that they take this form.

Theorem 22. *If $f_\infty \in \mathcal{F}_K$ is such that $\mathcal{E}(f_\infty) = h_K$, then*

$$8\pi\, E_{\rm kin}(f_\infty) = \|f_\infty\|^2_{L^1}\;, \quad \mathcal{C}(f_\infty) = K\;.$$

Up to a translation, $\rho_\infty = \int f_\infty\, dv$ is symmetric and nonincreasing, with $\mathrm{supp}(\rho_\infty) \subset B(0,1)$. The potential $U_\infty = U_{f_\infty}$ is of class C^1 on $\mathbb{R}^n/\{0\}$ and (f_∞, U_∞) is a stationary solution of the Vlasov-Poisson system which takes the form (2) with $\lambda = \lambda[f_\infty] < 0$.

Proof. The expression of $E_{\rm kin}(f_\infty)$ corresponds to the case $\bar\alpha = 1$ in Lemma 5. $\mathcal{C}(f_\infty) = K$ has been proved in Proposition 20. The fact that ρ_∞ is symmetric and nonincreasing is a consequence of Corollary 13 and of the equality case in Lemma 10. The assertion on the support of ρ_∞ is given in Corollary 13 (also see Proposition 16, (iii)). The regularity of U_∞ follows from a general result on solutions $\Delta U_\infty = \rho_\infty$ when ρ_∞ is compactly supported (see ([7]).

We still have to derive the expression of f_∞. Take $\epsilon > 0$ small and define

$$D_\epsilon := \{(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \epsilon \le f_\infty(x,v) \le \epsilon^{-1}\} .$$

Obviously, D_ϵ is a set of finite, positive measure. Let w be a compactly supported function in $L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, such that w is nonnegative on $D_\epsilon^c = \mathbb{R}^2 \times \mathbb{R}^2 \setminus D_\epsilon$. Define

$$G(\sigma,\tau) := \iint Q\left(f_\infty + \sigma\, \mathbb{1}_{D_\epsilon} + \tau\, w\right) dx\, dv$$

and observe that for $(\sigma,\tau) \in \mathbb{R} \times \mathbb{R}^+$, $|\sigma|$, τ small, $f_\infty + \sigma \mathbb{1}_{D_\epsilon} + \tau w$ is bounded in D_ϵ and nonnegative on $\mathbb{R}^2 \times \mathbb{R}^2$. The function G is therefore continuously differentiable with respect to σ and τ on a neighborhood of $(0,0)$ in $\mathbb{R} \times \mathbb{R}^+$. Moreover, $G(0,0) = K$ and

$$\partial_\sigma G(0,0) = \iint Q'(f_\infty)\, dx\, dv \neq 0 ,$$

so by the implicit function theorem there exists a continously differentiable function $\tau \to \sigma(\tau)$ with $\sigma(0) = 0$ defined for any τ small enough, and such that $G(\sigma(\tau),\tau) = K$. This means that

$$f^{(\tau)} := f_\infty + \sigma(\tau)\, \mathbb{1}_{D_\epsilon} + \tau\, w \in \mathcal{F}_K ,$$

and we have

$$\sigma'(0) = -\frac{\partial_\tau G(0,0)}{\partial_\sigma G(0,0)} = -\frac{\iint Q'(f_\infty)\, w\; dx\, dv}{\iint_{D_\epsilon} Q'(f_\infty)\, dx\, dv} . \tag{3}$$

$\mathcal{E}\big(f^{(\tau)}\big)$ reaches its minimum at $\tau = 0$, so a Taylor expansion implies

$$0 \le \mathcal{E}\big(f^{(\tau)}\big) - \mathcal{E}(f_\infty) = \tau \iint \mathsf{e}_\infty(x,v)\, (\sigma'(0)\, \mathbb{1}_{D_\epsilon} + w)\; dx\, dv + o(\tau)$$

for $\tau \ge 0$, small. Using (3), we see that

$$\iint \left(-\lambda_\epsilon\, Q'(f_\infty) + \mathsf{e}_\infty(x,v)\right) w\; dx\, dv \ge 0 , \tag{4}$$

where

$$\lambda_\epsilon := \frac{\iint_{D_\epsilon} \mathsf{e}_\infty(x,v)\, dx\, dv}{\iint_{D_\epsilon} Q'(f_\infty)\, dx\, dv} .$$

The choice of the test function w being arbitrary on D_ϵ, (4) implies that $\mathsf{e}_\infty(x,v) = \lambda_\epsilon Q'(f_\infty)$ a.e. on D_ϵ, and $\mathsf{e}_\infty(x,v) \ge \lambda_\epsilon Q'(f_\infty)$ D_ϵ^c. This also means that $\lambda_\epsilon = \lambda$ does not depend on ϵ, and if we let $\epsilon \to 0$, it follows that

$$\mathsf{e}_\infty(x,v) = \lambda\, Q'(f_\infty) \text{ a.e. on } f_\infty^{-1}((0,\infty)) , \tag{5}$$

$$\mathsf{e}_\infty(x,v) \ge \lambda\, Q'(0) = \mathsf{e}_0 \text{ a.e. on } f_\infty^{-1}(0) . \tag{6}$$

To get the expression for λ given in the theorem, we only need to multiply (5) by f_∞ and then integrate with respect to x and v. The fact that $\lambda < 0$ follows from

$$\iint \mathsf{e}_\infty(x,v)\, f_\infty\, dx\, dv = E_{\rm kin}(f_\infty) + 2\, E_{\rm pot}(f_\infty) < \mathcal{E}(f_\infty) < 0\ .$$

To finish the proof we need to invert (5). First of all, observe that by Assumptions $(Q1)$–$(Q3)$

$$Q' : [0,\infty) \to [Q'(0),\infty)$$

is a continous, increasing and onto function.

For any $\eta \geq 0$, the set $\{(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 \,|\, \mathsf{e}_\infty(x,v) = \eta\}$ has zero Lebesgue measure, since for any fixed x, it is made of a sphere in v-space, so almost everywhere in $\mathbb{R}^2 \times \mathbb{R}^2$ equation (5) can therefore be inverted to yield $f_\infty(x,v) = \phi(\mathsf{e}_\infty(x,v)/\lambda)$. □

Following the ideas of Schaeffer in [17], we assign an explicit value to the parameter λ. Notice that λ is the Lagrange multiplier associated to the constraint $\mathcal{C}(f) \leq K$ in the definition of $\mathcal{F}_K$. As we shall see below, the value of λ does not depend on the minimizer.

Proposition 23. *If $f_\infty \in \mathcal{F}_K$ is such that $\mathcal{E}(f) = h_K$, then*

$$\lambda[f_\infty] = 2\,\frac{h_K}{K} = 2\,K\,h_1 =: \lambda_K\ .$$

Proof. For $\alpha > 0$, consider the rescaled distribution function

$$g^{(\alpha)}(x,v) := \alpha f_\infty(\gamma(\alpha)\,x, \delta(\alpha)\,v)\ , \quad q(\alpha) = \iint Q(\alpha f_\infty(x,v))\, dx\, dv$$

with $\gamma(\alpha) := \sqrt{\alpha}$, $\delta(\alpha) := \sqrt{\frac{q(\alpha)}{K\alpha}}$. Then

$$\int Q(g^{(\alpha)}(x,v)) = K \quad \text{and} \quad \mathcal{E}(g^{(\alpha)}) = \frac{K^2\alpha^2}{q(\alpha)^2}\left(\mathcal{E}(f_\infty) - \frac{\log\alpha}{8\pi}\|f_\infty\|_{L^1}^2\right).$$

Differentiating $\mathcal{E}(g^{(\alpha)})$ with respect to α at $\alpha = 1$ and using the fact that $g^{(1)} = f_\infty$ is a minimizer, we get

$$2\left(1 - \frac{1}{K}\,q'(1)\right) h_K - \frac{\|f_\infty\|_{L^1}^2}{8\pi} = 0\ ,$$

which allows to compute

$$q'(1) = \iint Q'(f_\infty)\, f_\infty\, dx\, dv = \frac{K}{2\,h_K}\left(2h_K - \frac{\|f_\infty\|_{L^1}^2}{8\pi}\right)\ .$$

On the other hand, using $\iint \mathsf{e}_\infty(x,v)\, f_\infty\, dx\, dv = E_{\rm kin}(f_\infty) + 2E_{\rm pot}(f_\infty)$, we obtain

$$\iint \mathsf{e}_\infty(x,v)\, f_\infty\, dx\, dv = \frac{\|f_\infty\|_{L^1}^2}{8\pi} + 2\left(h_K - \frac{\|f_\infty\|_{L^1}^2}{8\pi}\right) = 2\,h_K - \frac{\|f_\infty\|_{L^1}^2}{8\pi}$$

and, by Corollary 21 and Theorem 22, end up with

$$\lambda[f_\infty] = \frac{\iint \mathsf{e}_\infty(x,v)\, f_\infty\, dx\, dv}{\iint Q'(f_\infty) f_\infty\, dx\, dv} = 2\,\frac{h_K}{K} = 2\,K\,h_1\ .$$

□

Corollary 24. *For any $K > 0$ and $M > 0$, $\mathcal{E}$ has at most one minimizer $f_\infty \in \mathcal{F}_K$ such that $\|f_\infty\|_{L^1} = M$, up to space translations.*

Proof. By Proposition 8, any minimizer U_∞ is such that $\int \rho_\infty(x) \log(1+|x|^2)\, dx < \infty$. Because of Lemma 10, ρ_∞ is radially symmetric. From Corollary 14, we infer that ρ_∞ is supported in the unit ball. U_∞ is radial and by Lemma 12, $U_\infty(1) = 0$ and ${U_\infty}'(1) = \frac{M}{2\pi}$. With $\psi(u) := 2\pi \int_{-\infty}^{u} \phi(s)$, the Poisson equation is now reduced to an ODE

$$U''_\infty + \frac{1}{r} U'_\infty = |\lambda_K|\, \psi\left(\frac{U_\infty}{\lambda_K}\right)$$

which has a unique solution. □

6. Nonlinear stability under mass constraint

In this section we adapt a dynamical stability criterion for the three dimensional Vlasov-Poisson system ([9, 13, 14, 22]) to the two dimensional case. First of all we define an appropriate notion of “distance” as

$$d(f, f_\infty) := \iint (f - f_\infty) \left(\frac{1}{2}|v|^2 + U_\infty(x)\right)\, dv\, dx\ .$$

Lemma 25. *Let f_∞ be a minimizer of $\mathcal{E}$ on $\mathcal{F}_K$, with spatial density ρ_∞ and corresponding potential U_∞. For any $f \in \mathcal{F}_K$ such that $\|f\|_{L^1} = \|f_\infty\|_{L^1}$, $\mathcal{C}(f) = K$, ρ_f is radial and $\int \rho_f \log|x|\, dx$ is finite we have that*

$$d(f, f_\infty) = \mathcal{E}(f) - \mathcal{E}(f_\infty) + \frac{1}{2}\int |\nabla U_f - \nabla U_\infty|^2\, dx \tag{7}$$

and $d(f, f_\infty)$ is nonnegative.

Proof. Observe that

$$\begin{aligned}\mathcal{E}(f) - \mathcal{E}(f_\infty) - d(f, f_\infty) &= E_{pot}(f) + E_{pot}(f_\infty) - \iint f\, U_\infty\, dx\, dv \\ &= \frac{1}{2}\lim_{R\to\infty} \int_{B(0,R)} (\rho_f\, U_f + \rho_\infty\, U_\infty - 2\rho_f\, U_\infty)\, dx\ .\end{aligned}$$

By integration by parts we get

$$\int_{B(0,R)} \rho_f\, U_\infty\, dx = -\int_{B(0,R)} \nabla U_f\, \nabla U_\infty\, dx + \int_{|x|=R} U_\infty\, \nabla U_f \cdot \frac{x}{|x|}\, d\sigma(x)\ .$$

Applying Lemma 12, (ii), to the second term of the right hand side, we arrive at

$$\int_{B(0,R)} \rho_f\, U_\infty\, dx = -\int_{B(0,R)} \nabla U_f\, \nabla U_\infty\, dx + U_\infty(R) \int_{|y|\le R} \rho_f(y)\, dy\ ,$$

and in the same way

$$\int_{B(0,R)} \rho_\infty\, U_\infty\; dx = -\int_{B(0,R)} \nabla U_\infty\, \nabla U_\infty\; dx + U_\infty(R) \int_{|y|\le R} \rho_\infty(y)\; dy\;,$$

$$\int_{B(0,R)} \rho_f\, U_f\; dx = -\int_{B(0,R)} \nabla U_f\, \nabla U_f\; dx + U_f(R) \int_{|y|\le R} \rho_f(y)\; dy\;.$$

Thus, we can rewrite $\int_{B(0,R)}(\rho_f\, U_f + \rho_\infty\, U_\infty - 2\rho_f\, U_\infty)\, dx$ as (I) + (II) with

$$(\mathrm{I}) = -\int_{B(0,R)} |\nabla U_f - \nabla U_\infty|^2\, dx$$

$$(\mathrm{II}) = [U_f(R) - 2\, U_\infty(R)] \int_{|y|\le R} \rho_f(y)\; dy + U_\infty(R) \int_{|y|\le R} \rho_\infty(y)\; dy$$

Assume that $R > 0$ is large enough so that $\mathrm{supp}(f_\infty) \subset B(0,R)$.

$$\int_{|y|\le R} \rho_\infty(y)\; dy = M \quad \text{and} \quad \int_{|y|\le R} \rho_f(y)\; dy = M - \int_{|y|>R} \rho(y)\; dy\;.$$

By Lemma 12,

$$\begin{aligned} U_\infty(R) &= \frac{M}{2\pi}\log R\;, \\ U_f(R) &= \frac{M}{2\pi}\log R + \frac{1}{2\pi}\left[\int_{|y|>R} \log|y|\,\rho(y)\; dy - \log R \int_{|y|>R} \rho(y)\; dy\right]. \end{aligned}$$

Thus

$$\begin{aligned} (\mathrm{II}) &= \frac{1}{2\pi}\int_{|y|>R} \log|y|\,\rho(y)\; dy\; \left[M - \int_{|y|>R} \rho(y)\; dy\right] \\ &\quad + \frac{\log R}{2\pi}\left[\int_{|y|>R} \rho(y)\; dy\right]^2 \end{aligned}$$

Because of the integrability of $y \mapsto \log|y|\,\rho(y)$ and the estimate

$$\log R \int_{|y|>R} \rho(y)\; dy \le \int_{|y|>R} \log|y|\,\rho(y)\; dy \to 0 \quad \text{as} \quad R \to \infty\;,$$

(II) vanishes in the limit $R \to \infty$ and we obtain (7).

If $\mathcal{C}(f) = K$, *i.e.* $\int Q(f)\, dv = \int Q(f_\infty)\, dv$, following [17], we get

$$\begin{aligned} d(f, f_\infty) &= \iint \left[\tfrac{1}{2}\,|v|^2 + U_\infty(x) - \lambda\, Q'(f_\infty)\right](f - f_\infty)\, dx\, dv \\ &\quad -\lambda \iint \left[Q(f) - Q(f_\infty) - Q'(f_\infty)(f - f_\infty)\right]\, dx\, dv\;. \end{aligned}$$

Now, by (5), we have

$$\frac{1}{2}\,|v|^2 + U_\infty(x) - \lambda\, Q'(f_\infty) = 0$$

on the support of f_∞, so, by (6),

$$
\begin{aligned}
d(f, f_\infty) &= \iint \left[\tfrac{1}{2}\,|v|^2 + U_\infty(x) - \lambda\, Q'(f_\infty)\right]_+ f \; dx\, dv \\
&\quad - \lambda \iint \left[Q(f) - Q(f_\infty) - Q'(f_\infty)(f - f_\infty)\right] \, dx\, dv \,,
\end{aligned}
$$

which completes the proof because of the convexity of Q, Assumption $(Q3)$. □

Theorem 26. *Let f_∞ be a minimizer of $(\mathcal{I}_K)$ and $M = \|f_\infty\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}$. For any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that the following property holds. Let $f_0 \in \mathcal{F}_K$ be an admisible data in the sense of Ukai and Okabe (see Section 2) verifying $\|f_0\|_{L^1(\mathbb{R}^2\times\mathbb{R}^2)} = M$, $\mathcal{C}(f_0) = K$ and consider the corresponding solution of (1) given in Theorem 2.*

$$
\textit{If} \quad d(f_0, f_\infty) < \delta \,, \qquad \textit{then} \quad d(f^{*_x}(\cdot,\cdot,t), f_\infty) < \epsilon \quad \forall\, t > 0 \,.
$$

Here $f^{*_x}(t)$ denotes the symmetric rearrangement of f with respect to the x variable.

Proof. Assume by contradiction that we can find $\epsilon_0 > 0$, $t_n > 0$ and a sequence of solutions $f_n(t) = f_n(\cdot,\cdot,t)$ of (1) with admissible initial data $f_{0,n} \in \mathcal{F}_K$ such that

$$
\lim_{n\to\infty} d(f_{0,n}, f_\infty) \to 0 \quad \text{and} \quad d(f_n^{*_x}(t_n), f_\infty) \geq \epsilon_0 \quad \forall\, n \in \mathbb{N} \,.
$$

By Lemma 25, $(f_n^{*_x}(t_n))$ is a minimizing sequence for $(\mathcal{I}_K)$ since

$$
f_n^{*_x}(t_n) \in \mathcal{F}_K, \;\; \mathcal{E}(f_\infty) \leq \mathcal{E}(f_n^{*_x}(t_n)) \leq \mathcal{E}(f_n(t_n)) = \mathcal{E}(f_n(0)) \to \mathcal{E}(f_\infty) \,.
$$

Moreover, the associated densities $\rho_n^* := \rho_{f_n^{*_x}}$ are symmetric and nonincreasing. Define $M_n(r) := \int_{|x|>r} \rho_n^* \, dx$. Observe that from Corollary 14, $\lim_{n\to\infty} M_n(1) = 0$. Let us compute

$$
\int_{|x|>1} |\nabla U_n^* - \nabla U_\infty|^2 \, dx = \int_{|x|>1} (M_n(|x|))^2 \, \frac{dx}{4\pi^2 |x|^2} = \int_1^\infty (M_n(r))^2 \, \frac{dr}{2\pi\, r} \,,
$$

according to Lemma 12, (ii). By Corollary 9,

$$
M_n(r) \leq \frac{1}{\log(1+r^2)} \int_{|x|>r} \log(1+|x|^2)\, \rho_n^*(x)\, dx \leq \frac{b\big(\mathcal{E}(f_n), K\big)}{\log(1+r^2)} \,,
$$

so that for some constant C, as $n \to \infty$,

$$
\int_{|x|>1} |\nabla U_n^* - \nabla U_\infty|^2 \, dx \leq C \sqrt{M_n(1)} \int_1^\infty \frac{1}{[\log(1+r^2)]^{3/2}} \, \frac{dr}{r} \to 0 \,.
$$

By Lemma 25, to provide a contradiction, it suffices to prove that

$$
\int_{|x|\leq 1} |\nabla U_n^* - \nabla U_\infty|^2 \, dx \to 0 \,.
$$

Notice first that by Lemma 4, ρ_n^* is bounded in $L^{1+1/m}$. By Hölder's inequality,

$$\int_{|x|<r} \rho_n^* \, dx = O\left(r^{2/(m+1)}\right) \, .$$

As a consequence, by Lemma 12, (ii), $\int_{|x|<r} |\nabla U_n^* - \nabla U_\infty|^2 \, dx$ can be made uniformly small as $r \to 0$. The sequence (f_n^*) weakly converges to a minimizer of $(\mathcal{I}_K)$ in $L^1 \cap L^{1+1/k}$ for the same reasons as in the proof of Theorem 15. The L^1 weak convergence of the minimizing sequence implies that the mass of the minimizer is equal to M, thus it coincides with $\|f_\infty\|_{L^1}$. Moreover this minimizer is radial. It is therefore unique and equal to f_∞ by Theorem 22 and Corollary 24. It is compactly supported since it verifies (2) and the associated potential grows like a logarithm. Finally, one can establish that $\lim_{n\to\infty} \int_{r<|x|\leq 1} |\nabla U_n^* - \nabla U_\infty|^2 \, dx = 0$ by using again Lemma 12, (ii). □

Proof of Theorem 3. In the polytropic gas case $Q(f) = C_0 \, f^p$, $p = 1 + \frac{1}{k}$, we can simply use the fact that $\mathcal{C}(f_\infty) = \lim_{n\to\infty} \mathcal{C}(g_n)$ to deduce that $g_n := f_n^{*x}(t_n)$ strongly converges to f_∞.

In the general convex case, from the assumptions made on Q, we deduce by integrating $Q''(s)$ twice from $s = f_\infty$ to $s = g_n$, and then with respect to x and v, that

$$\begin{aligned} &\mathcal{C}(g_n) - \mathcal{C}(f_\infty) - \iint Q'(f_\infty)\,(g_n - f_\infty) \, dx\, dv \\ &\quad \geq \tfrac{A}{p(p-1)} \iint \left[g_n^p - {f_\infty}^p - p\, {f_\infty}^{p-1}(g_n - f_\infty)\right] \, dx\, dv \end{aligned}$$

if $p > 1$. In the limit case $p = 1$, the r.h.s. has to be replaced by $\iint [g_n \log(g_n/f_\infty) - (g_n - f_\infty)] \, dx\, dv$. Since $Q'(f_\infty)$ is bounded and $g_n \rightharpoonup f_\infty$ in L^1, then $\lim_{n\to\infty} \iint Q'(f_\infty)\,(g_n - f_\infty) \, dx\, dv = 0$. Thus, since $\lim_{n\to\infty} \mathcal{C}(g_n) = \mathcal{C}(f_\infty)$, the l.h.s. of the above inequality also converges to 0. Using the generalized Csiszár-Kullback inequality stated in [5], we get

$$\begin{aligned} &\iint \left[g_n^p - {f_\infty}^p - p\, {f_\infty}^{p-1}(g_n - f_\infty)\right] \, dx\, dv \\ &\quad \geq C \, \min\left(\|f_\infty\|_{L^p}^{p-2}, \|g_n\|_{L^p}^{p-2}\right) \|g_n - f_\infty\|_{L^p}^2 \end{aligned}$$

with $C := 2^{-2/p} A$. Thus $g_n \to f_\infty$ in L^p and a.e. By [4], $g_n \to f_\infty$ in $L^{1+1/k}$. Since f_∞ has compact support, by Corollary 14, the strong convergence also holds in L^1.

By assumption (U), the mass M is uniquely determined. The result is then a consequence of Corollary 24 and Theorem 26. □

Whether the uniqueness assumption (U) is justified or not is a difficult issue, which depends on the nonlinearity of Q. We are going to illustrate this question in the next Section, and come back to the mass normalization issues in Section 8.

7. Uniqueness and dynamical stability. Application to the polytropic case

Many results of uniqueness for the solutions of radial semilinear elliptic equations have been obtained during the last twenty years, and it is definitely out of the scope of this paper to give a review of the various cases which are now covered. We will however illustrate the kind of results which can be achieved with the following Theorem, which can be deduced from the results listed in [19]. We refer to this paper for a partial list of earlier results and do not pretend that this example is in any sense optimal.

Theorem 27 (Uniqueness, [19])**.** *Assume that* $\psi \in C^1[0,\infty)$*,* $\psi(0) = 0$ *and* $\psi(s) > 0$ *for any* $s > 0$*. If* $g(s) := \frac{s\,\psi'(s)}{\psi(s)}$ *is nonincreasing on* $\mathbb{R}^+$ *and such that either* $\lim_{s\to 0} g(s) < 1$ *or* $\lim_{s\to\infty} g(s) > 1$*, then*

$$\begin{cases} -\Delta V = \psi(V) & \text{in} \quad B(0,1) \subset \mathbb{R}^2 \\ V > 0 & \text{in} \quad B(0,1) \\ V = 0 & \text{on} \quad \partial B(0,1) \end{cases} \tag{8}$$

has a unique solution.

With the notations of Sections 1 and 2, let $\phi := (Q')^{-1}$ on $(Q'(0),\infty)$ and extend it by 0 to $(-\infty, Q'(0))$, $\psi(u) := 2\pi \int_{-\infty}^{u} \phi(s)$. Then $V = \lambda_K^{-1} U_\infty$ solves (8) if f_∞ is a minimizer for $\mathcal{F}_K$. If the solution of (8) is unique, then by Lemma 12, $M \equiv 2\pi\,|\nabla V|$ on $\partial B(0,1)$ is uniquely determined. Then, by Corollary 24, f_∞ is uniquely determined.

It is left to the reader to check that under assumptions (U1), (U2) of Section 2, Theorem 27 applies. To illustrate somewhat further this issue, let us reformulate in terms of V the main quantities which appear in our statements. Since

$$f_\infty(x,v) = \phi(\mathsf{e}(x,v)) = \phi\left(V(x) - \frac{|v|^2}{2\,|\lambda_K|}\right) ,$$

by the change of variables $s = |v|^2/(2\,|\lambda_K|)$, we get

$$M = |\lambda_K| \int \psi(V)\, dx\ .$$

Let $\Psi(u) := \int_0^u \psi(s)\, ds$. Using the relations

$$(Q\circ\phi)(u) = \int_0^u (Q\circ\phi)'\, ds = \int_0^u s\,\phi'(s)\, ds = u\,\phi(u) - \frac{1}{2\pi}\,\psi(u)\ ,$$
$$2\pi \int_0^u s\,\phi(s)\, ds = u\,\psi(u) - \Psi(u)\ ,$$
$$2\pi \int_0^u (Q\circ\phi)\, ds = u\,\psi(u) - 2\Psi(u)\ ,$$

we obtain

$$\mathcal{C}(f_\infty) = |\lambda_K| \int \left[V\,\psi(V) - 2\Psi(V) \right]\,dx\;,$$

$$E_{\rm kin}(f_\infty) = |\lambda_K|^2 \int \Psi(V)\,dx\;,$$

$$E_{\rm pot}(f_\infty) = -\frac{1}{2}\,|\lambda_K|^2 \int V\,\psi(V)\,dx\;.$$

Collecting these estimates, we obtain

$$\mathcal{E}(f_\infty) = \frac{1}{2}\,|\lambda_K|^2 \int \left[-V\,\psi(V) + 2\Psi(V) \right]\,dx\;,$$

which gives another proof of Proposition 23:

$$\mathcal{E}(f_\infty) = -\frac{1}{2}\,|\lambda_K|\,\mathcal{C}(f_\infty) = -\frac{1}{2}\,|\lambda_K|\,K\;.$$

In the case of *polytropic gases*, up to a multiplicative constant which plays no role, Q is assumed to take the form

$$Q(f) = \frac{k}{k+1}\,f^{1+1/k} \quad\Longleftrightarrow\quad \phi(\mathsf{e}) = \mathsf{e}_+^k$$

for some $k > 0$ (the case $k < 0$ can also be considered: see for instance [14]). All the above quantities can be rephrased as follows:

$$\psi(u) = 2\pi\,\frac{{u_+}^{k+1}}{k+1} \quad\text{and}\quad \Psi(u) = 2\pi\,\frac{{u_+}^{k+2}}{(k+1)(k+2)}\;,$$

$$\mathcal{C}(f_\infty) = 2\pi\,|\lambda_K|\,\frac{k}{(k+1)(k+2)} \int V_+^{k+2}\,dx = K\;.$$

Using Proposition 23, we get

$$\int V_+^{k+2}\,dx = \frac{(k+1)(k+2)}{4\pi\,k\,h_1}\;.$$

All the other quantities are therefore explicit:

$$E_{\rm kin}(f_\infty) = 2\pi\,\frac{|\lambda_K|^2}{(k+1)(k+2)} \int V_+^{k+2}\,dx = \frac{2}{k}\,K^2\,|h_1|\;,$$

$$E_{\rm pot}(f_\infty) = -\pi\,\frac{|\lambda_K|^2}{k+1} \int V_+^{k+2}\,dx = -\frac{k+2}{k}\,K^2\,|h_1|\;,$$

$$\mathcal{E}(f_\infty) = \frac{\pi\,|\lambda_K|^2\,k}{(k+1)(k+2)} \int V_+^{k+2}\,dx = -\,K^2\,|h_1|\;.$$

Moreover, using the fact that $E_{\rm kin}(f_\infty)$ is uniquely determined and Theorem 22, we find that for a fixed $K > 0$, M is given by

$$M = \sqrt{8\pi E_{\rm kin}(f_\infty)} = 4\sqrt{\frac{\pi\,|h_1|}{k}}\,K\;.$$

Since $M = 2\pi\,\frac{|\lambda_K|}{k+1}\|V_+\|_{L^{k+1}}^{k+1}$, this also allows to compute $\|V_+\|_{L^{k+1}}$.

Remark. On the above example, we see that assumption (U) is too strong. Indeed we need the uniqueness of the solutions to (8) only for solutions such that

$$\frac{1}{2}\,|\lambda_K|^2 \int [V\,\psi(V) - 2\Psi(V)]\;dx = -K^2\,h_1\;.$$

This uniqueness property is equivalent to assert that among all bounded positive solutions to (8) the value of $|\lambda_K| \int \psi(V)\;dx$ is uniquely defined under the above constraints.

Although we will not use it later, the minimization result in the case of *polytropic gases* amounts to optimal inequalities, exactly as in [9]. We state the result without proof, since it is only a

Corollary 28. *For any $k > 0$, if $m = k+1$, then*

$$\inf_{f \in L^{1+1/k}_+(\mathbb{R}^2\times\mathbb{R}^2)} \frac{\left(\iint f^{1+1/k}\;dx\,dv\right)^{\frac{2\,k}{2+k}} (E_{\rm kin}(f))^{\frac{2}{2+k}}}{|E_{\rm pot}(f)|} = C_1(k)\;,$$

$$\inf_{\rho \in L^{1+1/m}_+(\mathbb{R}^2)} \frac{\|\rho\|^2_{L^{1+1/m}_+(\mathbb{R}^2)}}{\left|\iint \rho(x)\,\rho(y)\,\log|x-y|\;dxdy\right|} = C_2(m)\;,$$

where the optimal constants are given by f_∞ and by ρ_∞:

$$C_1(k) = \frac{2^{\frac{2}{k+2}}\,k^{\frac{3k}{2+k}}}{(k{+}2)\,(k{+}1)^{\frac{2k}{k+2}}\,|h_1|^{\frac{k}{k+2}}} \quad \textit{and} \quad C_2(m) = \frac{2^{\frac{2}{m+1}}\,m^{\frac{4m}{m+1}}}{(m{+}1)\,(2\pi\,(m{-}1)\,|h_1|)^{\frac{m-1}{m+1}}}\;.$$

In other words, we have found the optimal constants in the interpolation of $E_{\rm pot}(f)$ between $E_{\rm kin}(f)$ and $\mathcal{C}(f)$, in Lemma 4. The value of h_1 is not explicitly known but can easily be computed numerically for any value of $k > 0$.

8. On Lagrange multipliers, mass, normalization of the potential and support of the minimizers

From the point of view of the dynamics, the potential U_f in (1) is defined up to an additive constant with respect to the x variable. In dimension $n = 3$, if one writes the potential energy as $E_{\rm pot}(f) = -\frac{1}{2}\int_{\mathbb{R}^3} |\nabla U_f|^2\,dx$, this additive constant does not play any role. In dimension $n = 2$, the function $x \mapsto |\nabla U_f(x)|^2$ is not integrable, and this is why we define the potential energy as $E_{\rm pot}(f) = \frac{1}{2}\int \rho_f\,U_f\,dx$. If one considers the minimization problem $(\mathcal{I}_K)$, the additive constant plays a role for the normalization of the minimizer in L^1.

Consider for $K > 0$ and $M > 0$ the minimization problem

$$\mathcal{I}_{K,M} = \inf_{f \in \mathcal{F}_{K,M}} \mathcal{E}(f)\;,$$

where $\mathcal{F}_{K,M} := \{f \in \mathcal{F}_K : \|f\|_{L^1} = M\}$. If $\mu_{K,M}$ is the Lagrange multiplier associated to the constraint $\|f\|_{L^1} = M$, the above minimization problem is equivalent

$$\mathcal{I}_{K,M} := \inf_{f \in \mathcal{F}_K} \Big(\mathcal{E}(f) - \mu_{K,M} \, \|f\|_{L^1} \Big) \, .$$

The mass which shows up in $(\mathcal{I}_K)$ is the one (the ones if the minimizers are not unique) for which $\mu_{K,M} = 0$. Requiring (U) is equivalent to ask that $M \mapsto \mu_{K,M}$ has only one zero. Proving the convergence of a minimizing sequence for $\mathcal{I}_{K,M}$ does not present any additional difficulty compared to the proof of Theorem 15.

To determine the range of $\mu_{K,M}$ when M varies, consider therefore for $\mu \in \mathbb{R}$, given, the minimization problem

$$\mathcal{I}_K^\mu := \inf_{f \in \mathcal{F}_K} \Big(\mathcal{E}(f) - \mu \, \|f\|_{L^1} \Big) \, .$$

Consider as in Proposition 23 the rescaling

$$g^{(\alpha)}(x,v) := \alpha f(\gamma(\alpha)\, x, \delta(\alpha)\, v) \, , \quad q(\alpha) = \iint Q(\alpha f(x,v)) \, dx \, dv$$

with $\gamma(\alpha) := \sqrt{\alpha}$, $\delta(\alpha) := \sqrt{\frac{q(\alpha)}{K\alpha}}$. If $f \in \mathcal{F}_K$, then $g^{(\alpha)} \in \mathcal{F}_K$ and

$$\mathcal{E}(g^{(\alpha)}) = \frac{K^2\alpha^2}{q(\alpha)^2} \left(\mathcal{E}(f) - \frac{\log \alpha}{8\pi} \, \|f\|_{L^1}^2 \right) .$$

Assume that M is the smallest (resp. largest) possible value of $M = M(f_\infty)$ for all $f_\infty \in \mathcal{F}_K$ which are a minimizer for $(\mathcal{I}_K)$ if $\mu > 0$ (resp. if $\mu < 0$), and choose $\alpha := \exp(8\pi \, \mu / M)$. It is then easy to prove that

$$\mathcal{I}_K^\mu = \frac{K^2\alpha^2}{q(\alpha)^2} \left(\mathcal{I}_K - \frac{\log \alpha}{8\pi} \, M^2 \right) .$$

Assume for simplicity that (U) holds. Then $\mathcal{I}_K^\mu$ makes sense for any $\mu \in \mathbb{R}$, and the mass of the corresponding minimizer is therefore parametrized by $\mu \mapsto M(\mu) = \frac{K\,\alpha}{q(\alpha)} \, M$. Since $\mathcal{I}_{K,M}$ has a minimizer for any M, the range of $\mu \mapsto M(\mu)$ is therefore $(0,\infty)$.

As a final remark, the above scaling explains why the support of ρ_∞ is $B(0,1)$. For the more general minimization problem $(\mathcal{I}_{K,M(\mu)}) = (\mathcal{I}_K^\mu)$, $\mu \in \mathbb{R}$, given, the support of the spatial density of a minimizer is contained in the ball $B(0,(\gamma(\alpha))^{-1})$. The special choice of U_f made in Section 1 corresponds to the choice of the Lagrange multiplier $\mu = 0$, which itself selects a minimizer with a fixed mass $M(\mu)$, at least if (U) holds. This minimizer has a spatial density ρ_∞ with support in $B(0,1)$ only because of the sign of the term $\log|x-y|$ in the potential energy.

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