A nonlinear fourth-order parabolic equation and related logarithmic Sobolev inequalities*

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Abstract

A nonlinear fourth-order parabolic equation in one space dimension with periodic boundary conditions is studied. This equation arises in the context of fluctuations of a stationary nonequilibrium interface and in the modeling of quantum semiconductor devices. The existence of global-in-time non-negative weak solutions is shown. A criterion for the uniqueness of non-negative weak solutions is given. We prove that the solution converges exponentially fast to its mean value in the "entropy norm" using a new optimal logarithmic Sobolev inequality for higher derivatives. The rate is therefore independent of the solution and the constant depends only on the initial value of the entropy.

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1 Introduction

This paper is concerned with the study of some properties of weak solutions to a nonlinear fourth-order equation with periodic boundary conditions and related logarithmic Sobolev inequalities. More precisely, we consider the problem

$$u_t + (u(\log u)_{xx})_{xx} = 0, \quad u(\cdot, 0) = u_0 \geq 0 \quad \text{in } S^1, \tag{1}$$

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where S^1 is the one-dimensional torus parametrized by a variable $x \in [0, L]$.

Recently equation (1) has attracted the interest of many mathematicians since it possesses some remarkable properties. For instance, it is a one-homogeneous equation which is a simple example of a generalization of the heat equation to higher order operators. The solutions are non-negative and there are several Lyapunov functionals. For instance, a formal calculation shows that the *entropy* is non-increasing:

$$\frac{d}{dt}\int_{S^1} u(\log u - 1)dx + \int_{S^1} u\left|(\log u)_{xx}\right|^2 dx = 0. \tag{2}$$

Another example of a Lyapunov functional is $\int_{S^1}(u - \log u)dx$ which formally yields

$$\frac{d}{dt}\int_{S^1} (u - \log u)dx + \int_{S^1} \left|(\log u)_{xx}\right|^2 dx = 0. \tag{3}$$

This last estimate is used to prove that solutions to (1) are non-negative. Indeed, a Poincaré inequality shows that $\log u$ is bounded in $H^2(S^1)$ and hence in $L^\infty(S^1)$, which implies that $u \geq 0$ in $S^1 \times (0, \infty)$. We prove this result rigorously in section 2. Notice that the equation is of higher order and no maximum principle argument can be employed. For more comments on Lyapunov functionals of (1) we refer to [4, 5].

Equation (1) has been first derived in the context of fluctuations of a stationary non-equilibrium interface [8]. It also appears as a zero-temperature zero-field approximation of the so-called quantum drift-diffusion model for semiconductors [1] which can be derived by a quantum moment method from a Wigner-BGK equation [7]. The first analytical result has been presented in [4]; there the existence of local-in-time classical solutions with periodic boundary conditions has been proved. A global-in-time existence result with homogeneous Dirichlet-Neumann boundary conditions has been obtained in [11]. However, up to now, no global-in-time existence result is available for the problem (1). Although it is essentially an adaptation of the method of [11], we give a proof for completeness.

The long-time behavior of solutions has been studied in [5] using periodic boundary conditions under restrictive regularity conditions on the initial data, in [13] with homogeneous Dirichlet-Neumann boundary conditions and finally, in [10] employing non-homogeneous Dirichlet-Neumann boundary conditions. In particular, it has been shown that the solutions converge exponentially fast to their steady state in various norms and even, in [11], in terms of the entropy. The decay rate has been numerically computed in [6]. We also mention the work [12] in which a positivity-preserving numerical scheme for the quantum drift-diffusion model has been proposed.

In the last years the question of non-negative or positive solutions of fourth-order parabolic equations has also been investigated in the context of lubrication-type equations, like the thin film equation

$$u_t + (f(u)u_{xxx})_x = 0$$

(see, e.g., [2, 3]), where typically, $f(u) = u^\alpha$ for some $\alpha > 0$. This equation is of degenerate type which makes the analysis easier than for (1), at least concerning the positivity property.

In this paper we show the following results. First, the existence of global-in-time weak solutions is shown under a rather weak condition on the initial datum u_0. We only assume that $u_0 \geq 0$ is measurable and such that $\int_{S^1}(u_0 - \log u_0)dx < \infty$. Compared to [4], we do not impose any smallness condition on u_0. We are able to prove that the solution is non-negative. Compared to [5], the H^1 regularity of the initial data is not required, which is consistent with the type of *a priori* asymptotic estimates we obtain later.

Our second result is concerned with uniqueness issues. If u_1 and u_2 are two non-negative solutions to (1) satisfying some regularity assumptions (see Theorem 5) then $u_1 = u_2$. A uniqueness result has already been obtained in [4] in the class of mild *positive* solutions; however, our result allows for all *non-negative* solutions satisfying only a few additional assumptions. It is slightly stronger than the one stated in [11].

The third and main result of this paper is the exponential time decay of the solutions, i.e., we show that the solution constructed in Theorem 1 converges exponentially fast to its mean value $\bar{u} = \int u(x,t)dx/L$:

$$\int_{S^1} u(x,t) \log\left(\frac{u(x,t)}{\bar{u}}\right) dx \leq e^{-Mt} \int_{S^1} u_0 \log\left(\frac{u_0}{\bar{u}}\right) dx \quad \forall\, t > 0, \tag{4}$$

where $M = 32\pi^4/L^4$. This constant is easily obtained by linearization in the asymptotic regime. It shows up in [5], but in the by far more restrictive context of the H^1 setting. Our proof is based on the entropy–entropy production method. For this, we show that the entropy production term $\int u\,|(\log u)_{xx}|^2\,dx$ in (2) can be bounded from below by the entropy itself yielding

$$\frac{d}{dt}\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx + M \int_{S^1} u \log\left(\frac{u}{\bar{u}}\right) dx \leq 0.$$

Then Gronwall's inequality gives (4). This argument is formal since we only have weak solutions; we refer to Theorem 9 for details of the rigorous proof. Notice that an exponential convergence rate in the L^p norm is given in [5], but it involves a $(p-1)$ factor which vanishes in the limit $p = 1$ corresponding to the entropy setting. In [13], the exponential decay of the relative entropy is established, but with a rate which depends on the initial data. Here M is independent of the solution and the constant in the right hand side of (4) is optimal: it is simply the initial value of the relative entropy.

The lower bound for the entropy production is obtained through a logarithmic Sobolev inequality in S^1. We show (see Theorem 6) that any function $u \in H^n(S^1)$ ($n \in \mathbb{N}$) satisfies

$$\int_{S^1} u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(S^1)}}\right) dx \leq 2\left(\frac{L}{2\pi}\right)^{2n} \int_{S^1} \left|u^{(n)}\right|^2 dx, \tag{5}$$

where $\|u\|^2_{L^2(S^1)} = \int u^2 dx/L$, and the constant is *optimal*. As already mentioned in the case $n = 2$, the proof of this result uses the entropy–entropy production method.

Entropy estimates are interesting for the following reason. The L^1 norm of a solution u to (1) is preserved by the evolution. It is therefore natural to look for a convergence of

u to its average $\bar{u}$ measured in L^1 rather than in L^p, $p > 1$. As noted in many papers, the limit of such L^p estimates as $p \to 1$, $p > 1$, is the entropy rather than the L^1 norm itself. The convergence of u to $\bar{u}$ is then a consequence of the standard Csiszar-Kullback inequality. Exactly as for the heat equation, $p = 1$ looks as a threshold from the point of view of the existence theory and for the optimality of the estimates on the asymptotic behaviour. This work is a step towards a deeper understanding of both entropy methods and higher order equations.

The paper is organized as follows. In section 2 the existence of solutions is proved. Section 3 is concerned with the uniqueness result. Then section 4 is devoted to the proof of the optimal logarithmic Sobolev inequality (5). Finally, in section 5, the exponential time decay (4) is shown.

2 Existence of solutions

Theorem 1. *Let $u_0 : S^1 \to \mathbb{R}$ be a nonnegative measurable function such that $\int_{S^1}(u_0 - \log u_0)dx < \infty$. Then there exists a global weak solution u of (1) satisfying*

$$u \in L^{5/2}_{\mathrm{loc}}(0,\infty; W^{1,1}(S^1)) \cap W^{1,10/9}_{\mathrm{loc}}(0,\infty; H^{-2}(S^1)),$$
$$u \geq 0 \quad \text{in } S^1 \times (0,\infty), \quad \log u \in L^2_{\mathrm{loc}}(0,\infty; H^2(S^1)),$$

and for all $T > 0$ and all smooth test functions ϕ,

$$\int_0^T \langle u_t, \phi\rangle_{H^{-2},H^2} dt + \int_0^T \int_{S^1} u(\log u)_{xx}\phi_{xx} dx dt = 0.$$

The initial datum is satisfied in the sense of $H^{-2}(S^1) := (H^2(S^1))^*$.

Proof. We first transform (1) by introducing the new variable $u = e^y$ as in [11]. Then (1) becomes

$$(e^y)_t + (e^y y_{xx})_{xx} = 0, \quad y(\cdot, 0) = y_0 \quad \text{in } S^1, \tag{6}$$

where $y_0 = \log u_0$. In order to prove the existence of solutions to this equation, we semi-discretize (6) in time. For this, let $T > 0$, and let $0 = t_0 < t_1 < \cdots < t_N = T$ with $t_k = k\tau$ be a partition of $[0, T]$. Furthermore, let $y_{k-1} \in H^2(S^1)$ with $\int \exp(y_{k-1})dx = \int u_0 dx$ and $\int(\exp(y_{k-1}) - y_{k-1})dx \leq \int(u_0 - \log u_0)dx$ be given. Then we solve recursively the elliptic equations

$$\frac{1}{\tau}(e^{y_k} - e^{y_{k-1}}) + (e^{y_k}(y_k)_{xx})_{xx} = 0 \quad \text{in } S^1. \tag{7}$$

Lemma 2. *There exists a solution $y_k \in H^2(S^1)$ to (7).*

Proof. Set $z = y_{k-1}$. We consider first for given $\varepsilon > 0$ the equation

$$(e^y y_{xx})_{xx} - \varepsilon y_{xx} + \varepsilon y = \frac{1}{\tau}(e^z - e^y) \quad \text{in } S^1. \tag{8}$$

In order to prove the existence of a solution to this approximate problem we employ the Leray-Schauder theorem. For this, let $w \in H^1(S^1)$ and $\sigma \in [0,1]$ be given, and consider

$$a(y,\phi) = F(\phi) \quad \text{for all } \phi \in H^2(S^1), \tag{9}$$

where

$$\begin{aligned} a(y,\phi) &= \int_{S^1} (e^w y_{xx}\phi_{xx} + \varepsilon y_x \phi_x + \varepsilon y \phi)dx, \\ F(\phi) &= \frac{\sigma}{\tau}\int_{S^1} (e^z - e^w)\phi dx, \quad y,\phi \in H^2(S^1). \end{aligned}$$

Clearly, $a(\cdot,\cdot)$ is bilinear, continuous and coercive on $H^2(S^1)$ and F is linear and continuous on $H^2(S^1)$. (Here we need the additional ε-terms.) Therefore, the Lax-Milgram lemma provides the existence of a solution $y \in H^2(S^1)$ to (9). This defines a fixed-point operator $S : H^1(S^1) \times [0,1] \to H^1(S^1)$, $(w,\sigma) \mapsto y$. It holds $S(w,0) = 0$ for all $w \in H^1(S^1)$. Moreover, the functional S is continuous and compact (since the embedding $H^2(S^1) \subset H^1(S^1)$ is compact). We need to prove a uniform bound for all fixed points of $S(\cdot,\sigma)$.

Let y be a fixed point of $S(\cdot,\sigma)$, i.e., $y \in H^2(S^1)$ solves for all $\phi \in H^2(S^1)$

$$\int_{S^1} (e^y y_{xx}\phi_{xx} + \varepsilon y_x\phi_x + \varepsilon y\phi)dx = \frac{\sigma}{\tau}\int_{S^1} (e^z - e^y)\phi dx. \tag{10}$$

Using the test function $\phi = 1 - e^{-y}$ yields

$$\int_{S^1} y_{xx}^2 dx - \int_{S^1} y_{xx}y_x^2 dx + \varepsilon \int_{S^1} e^{-y}y_x^2 dx + \varepsilon \int_{S^1} y(1-e^{-y})dx = \frac{\sigma}{\tau}\int_{S^1} (e^z - e^y)(1-e^{-y})dx.$$

The second term on the left-hand side vanishes since $y_{xx}y_x^2 = (y_x^3)_x/3$. The third and fourth term on the left-hand side are non-negative. Furthermore, with the inequality $e^x \geq 1 + x$ for all $x \in \mathbb{R}$,

$$(e^z - e^y)(1-e^{-y}) \leq (e^z - z) - (e^y - y).$$

We obtain

$$\frac{\sigma}{\tau}\int_{S^1} (e^y - y)dx + \int_{S^1} y_{xx}^2 dx \leq \frac{\sigma}{\tau}\int_{S^1} (e^z - z)dx.$$

As z is given, this provides a uniform bound for y_{xx} in $L^2(S^1)$. Moreover, the inequality $e^x - x \geq |x|$ for all $x \in \mathbb{R}$ implies a (uniform) bound for y in $L^1(S^1)$ and for $\int y dx$. Now we use the Poincaré inequality

$$\Big\| u - \int_{S^1} u \frac{dx}{L} \Big\|_{L^2(S^1)} \leq \frac{L}{2\pi}\|u_x\|_{L^2(S^1)} \leq \left(\frac{L}{2\pi}\right)^2 \|u_{xx}\|_{L^2(S^1)} \quad \text{for all } u \in H^2(S^1).$$

Recall that $\|u\|_{L^2(S^1)}^2 = \int_{S^1} u^2 dx/L$. Then the above estimates provide a (uniform in ε) bound for y and y_x in $L^2(S^1)$ and thus for y in $H^2(S^1)$. This shows that all fixed points of the operator $S(\cdot,\sigma)$ are uniformly bounded in $H^1(S^1)$. We notice that we even obtain a uniform bound for y in $H^2(S^1)$ which is independent of ε. The Leray-Schauder fixed-point theorem finally ensures the existence of a fixed point of $S(\cdot,1)$, i.e., of a solution $y \in H^2(S^1)$ to (8).

It remains to show that the limit $\varepsilon \to 0$ can be performed in (8) and that the limit function satisfies (7). Let y_ε be a solution to (8). The above estimate shows that y_ε is bounded in $H^2(S^1)$ uniformly in ε. Thus there exists a subsequence (not relabeled) such that, as $\varepsilon \to 0$,

$$\begin{aligned} y_\varepsilon &\rightharpoonup y \qquad \text{weakly in } H^2(S^1), \\ y_\varepsilon &\to y \qquad \text{strongly in } H^1(S^1) \text{ and in } L^\infty(S^1). \end{aligned}$$

We conclude that $e^{y_\varepsilon} \to e^y$ in $L^2(S^1)$ as $\varepsilon \to 0$. In particular, $e^{y_\varepsilon}(y_\varepsilon)_{xx} \rightharpoonup e^y y_{xx}$ weakly in $L^1(S^1)$. The limit $\varepsilon \to 0$ in (10) can be performed proving that y solves (7). Moreover, using the test function $\phi \equiv 1$ in the weak formulation of (7) shows that $\int \exp(y_k)dx = \int \exp(y_{k-1})dx = \int u_0 dx$. □

For the proof of Theorem 1 we need further uniform estimates for the finite sequence $(y^{(N)})$. For this, let $y^{(N)}$ be defined by $y^{(N)}(x,t) = y_k(x)$ for $x \in S^1$, $t \in (t_{k-1}, t_k]$, $1 \le k \le N$. Then we have shown in the proof of Lemma 2 that there exists a constant $c > 0$ depending neither on τ nor on N such that

$$\|y^{(N)}\|_{L^2(0,T;H^2(S^1))} + \|y^{(N)}\|_{L^\infty(0,T;L^1(S^1))} + \|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(S^1))} \le c. \tag{11}$$

To pass to the limit in the approximating equation, we need further compactness estimates on $e^{y^{(N)}}$. Here we proceed similarly as in [10].

Lemma 3. *The following estimates hold:*

$$\|y^{(N)}\|_{L^{5/2}(0,T;W^{1,\infty}(S^1))} + \|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(S^1))} \le c, \tag{12}$$

where $c > 0$ does not depend on τ and N.

Proof. We obtain from the Gagliardo-Nirenberg inequality and (11):

$$\begin{aligned} \|y^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} &\le c\|y^{(N)}\|^{3/5}_{L^\infty(0,T;L^1(S^1))}\|y^{(N)}\|^{2/5}_{L^1(0,T;H^2(S^1))} \le c, \\ \|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} &\le c\|y^{(N)}\|^{1/5}_{L^\infty(0,T;L^1(S^1))}\|y^{(N)}\|^{4/5}_{L^2(0,T;H^2(S^1))} \le c. \end{aligned}$$

This implies the first bound in (12). The second bound follows from the first one and (11):

$$\begin{aligned} \|e^{y^{(N)}}\|_{L^{5/2}(0,T;W^{1,1}(S^1))} &\le c\left(\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(S^1))} + \|(e^{y^{(N)}})_x\|_{L^{5/2}(0,T;L^1(S^1))}\right) \\ &\le c\|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^1(S^1))} + c\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(S^1))}\|y_x^{(N)}\|_{L^{5/2}(0,T;L^\infty(S^1))} \\ &\le c. \end{aligned}$$

The lemma is proved. □

We also need an estimate for the discrete time derivative. We introduce the shift operator σ_N by $(\sigma_N(y^{(N)}))(x,t) = y_{k-1}(x)$ for $x \in S^1$, $t \in (t_{k-1}, t_k]$.

Lemma 4. *The following estimate holds:*

$$\|e^{y^{(N)}} - e^{\sigma_N(y^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(0,1))} \leq c\tau, \tag{13}$$

where $c > 0$ does not depend on τ and N.

Proof. From (7) and Hölder's inequality we obtain

$$\begin{aligned}\tfrac{1}{\tau}\|e^{y^{(N)}} - e^{\sigma_N(y^{(N)})}\|_{L^{10/9}(0,T;H^{-2}(S^1))} &\leq \|e^{y^{(N)}} y_{xx}^{(N)}\|_{L^{10/9}(0,T;L^2(S^1))} \\ &\leq \|e^{y^{(N)}}\|_{L^{5/2}(0,T;L^\infty(S^1))} \|y_{xx}^{(N)}\|_{L^2(0,T;L^2(S^1))},\end{aligned}$$

and the right-hand side is uniformly bounded by (11) and (12) since $W^{1,1}(0,1) \hookrightarrow L^\infty(0,1)$. □

Now we are able to prove Theorem 1, i.e. to perform the limit $\tau \to 0$ in (7). From estimate (11) the existence of a subsequence of $y^{(N)}$ (not relabeled) follows such that, as $N \to \infty$ or, equivalently, $\tau \to 0$,

$$y^{(N)} \rightharpoonup y \quad \text{weakly in } L^2(0,T;H^2(S^1)). \tag{14}$$

Since the embedding $W^{1,1}(S^1) \subset L^1(S^1)$ is compact it follows from the second bound in (12) and from (13) by an application of Aubin's lemma [15, Thm. 5] that, up to the extraction of a subsequence, $e^{y^{(N)}} \to g$ strongly in $L^1(0,T;L^1(S^1))$ and hence also in $L^1(0,T;H^{-2}(S^1))$.

We claim that $g = e^y$. For this, we observe that, by (11),

$$\begin{aligned}\|e^{y^{(N)}} - g\|_{L^2(0,T;H^{-2}(S^1))}^2 &\leq \|e^{y^{(N)}} - g\|_{L^\infty(0,T;H^{-2}(S^1))} \|e^{y^{(N)}} - g\|_{L^1(0,T;H^{-2}(S^1))} \\ &\leq c\left(\|e^{y^{(N)}}\|_{L^\infty(0,T;L^1(S^1))} + \|g\|_{L^\infty(0,T;L^1(S^1))}\right) \\ &\qquad \times \|e^{y^{(N)}} - g\|_{L^1(0,T;H^{-2}(S^1))} \\ &\leq c\|e^{y^{(N)}} - g\|_{L^1(0,T;H^{-2}(S^1))} \to 0 \quad \text{as } N \to \infty.\end{aligned}$$

Now let z be a smooth function. Since $e^{y^{(N)}} \to g$ strongly in $L^2(0,T;H^{-2}(S^1))$ and $y^{(N)} \rightharpoonup y$ weakly in $L^2(0,T;H^2(S^1))$, we can pass to the limit $N \to \infty$ in

$$0 \leq \int_0^T \langle e^{y^{(N)}} - e^z, y^{(N)} - z\rangle_{H^{-2},H^2} dt$$

to obtain the inequality

$$0 \leq \int_0^T \int_{S^1} (g - e^z)(y - z) dx dt.$$

The monotonicity of $x \mapsto e^x$ finally yields $g = e^y$.

In particular, $e^{y^{(N)}} \to e^y$ strongly in $L^1(0,T;L^1(S^1))$. The second uniform bound in (12) implies that, up to the possible extraction of a subsequence again, $e^{y^{(N)}} \rightharpoonup e^y$ weakly* in $L^{5/2}(0,T;\,L^\infty(S^1))$. Thus, Lebesgue's convergence theorem gives

$$e^{y^{(N)}} \to e^y \quad \text{strongly in } L^2(0,T;L^2(S^1)). \tag{15}$$

Furthermore, the uniform estimate (13) implies, for a subsequence,

$$\frac{1}{\tau}\left(e^{y^{(N)}} - e^{\sigma_N(y^{(N)})}\right) \rightharpoonup (e^y)_t \quad \text{weakly in } L^{10/9}(0,T;H^{-2}(S^1)). \tag{16}$$

We can pass to the limit $\tau \to 0$ in (7), using the convergence results (14)-(16), which concludes the proof of Theorem 1. ☐

3 Uniqueness of solutions

To get a uniqueness result, we need an additional regularity assumption.

Theorem 5. *Let u_1, u_2 be two weak solutions to* (1) *in the sense of Theorem 1 with the same initial data such that $u_1, u_2 \in C^0([0,T];L^1(S^1))$ and $\sqrt{u_1/u_2}$, $\sqrt{u_2/u_1} \in L^2(0,T;$ $H^2(S^2))$ for some $T>0$. Then $u_1 = u_2$ in $S^1 \times (0,T)$.*

Bleher et al. have showed the uniqueness of solutions to (1) in the class of mild solutions, i.e. $C^0([0,T];H^1(S^1))$, which are *positive*. We allow for the more general class of *non-negative* solutions satisfying the above regularity assumptions.

Proof. We use a similar idea as in [11]. Employing the test function $1-\sqrt{u_2/u_1}$ in equation (1) for u_1 and the test function $\sqrt{u_1/u_2}-1$ in equation (1) for u_2 and taking the difference of both equations yields

$$\begin{aligned}
&\int_0^t \left\langle (u_1)_t, 1-\sqrt{\frac{u_2}{u_1}}\right\rangle_{H^{-2},H^2} dt - \int_0^t \left\langle (u_2)_t, \sqrt{\frac{u_1}{u_2}}-1\right\rangle_{H^{-2},H^2} dt \\
&= \int_0^t \left\langle (u_1(\log u_1)_{xx})_{xx}, \sqrt{\frac{u_2}{u_1}}\right\rangle_{H^{-2},H^2} dt + \int_0^t \left\langle (u_2(\log u_2)_{xx})_{xx}, \sqrt{\frac{u_1}{u_2}}\right\rangle_{H^{-2},H^2} dt \\
&= I_1 + I_2.
\end{aligned}$$

The left-hand side can be *formally* written as

$$\begin{aligned}
&\int_0^t \left\langle (u_1)_t, 1-\sqrt{\frac{u_2}{u_1}}\right\rangle_{H^{-2},H^2} dt - \int_0^t \left\langle (u_2)_t, \sqrt{\frac{u_1}{u_2}}-1\right\rangle_{H^{-2},H^2} dt \\
&= 2\int_0^t\int_{S^1} \left[(\sqrt{u_1})_t(\sqrt{u_1}-\sqrt{u_2}) - (\sqrt{u_2})_t(\sqrt{u_1}-\sqrt{u_2})\right] dxdt \\
&= \int_{S^1}\left(\sqrt{u_1(t)}-\sqrt{u_2(t)}\right)^2 dx.
\end{aligned}$$

As the first and the last equation hold rigorously, it is possible to make the computation rigorous by approximating u_1 and u_2 by suitable smooth functions and then passing to the limit in the first and the last equation by a standard procedure.

We claim now that $I_1 + I_2$ is non-positive. For this we compute *formally* as follows.

$$\begin{aligned} I_1 &= 2\int_0^t \left\langle (\sqrt{u_1})_{xxxx} - \frac{1}{\sqrt{u_1}}\left|(\sqrt{u_1})_{xx}\right|^2, \sqrt{u_2} \right\rangle_{H^{-2},H^2} dt \\ &= -2\int_0^t \int_{S^1} \left[-(\sqrt{u_1})_{xx}(\sqrt{u_2})_{xx} + \left|(\sqrt{u_1})_{xx}\right|^2 \sqrt{\frac{u_2}{u_1}} \right] dxdt. \end{aligned}$$

A similar result can be obtained for I_2. Thus

$$I_1 + I_2 = -2\int_0^t \int_{S^1} \left| \sqrt[4]{\frac{u_2}{u_1}}(\sqrt{u_1})_{xx} - \sqrt[4]{\frac{u_1}{u_2}}(\sqrt{u_2})_{xx} \right|^2 \leq 0.$$

This calculation can be made rigorous again by an approximation argument. We conclude that

$$\int_{S^1} \left| \sqrt{u_1(t)} - \sqrt{u_2(t)} \right|^2 dx \leq 0,$$

which gives $u_1(t) = u_2(t)$ in S^1 for all $t \leq T$. ☐

4 Optimal logarithmic Sobolev inequality on S^1

The main goal of this section is the proof of a logarithmic Sobolev inequality for periodic functions. The following theorem is due to Weissler and Rothaus (see [9, 14, 16]). We give a simple proof using the entropy–entropy production method. Recall that S^1 is parametrized by $0 \leq x \leq L$.

Theorem 6. *Let* $\mathcal{H}_1 = \{u \in H^1(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$ *and* $\|u\|^2_{L^2(S^1)} = \int_{S^1} u^2 dx/L$. *Then*

$$\inf_{u\in\mathcal{H}_1} \frac{\int_{S^1} u_x^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|^2_{L^2(S^1)})dx} = \frac{2\pi^2}{L^2}. \tag{17}$$

We recall that the optimal constant in the usual Poincaré inequality is $L/2\pi$, i.e.

$$\inf_{v\in\mathcal{H}_1} \frac{\int_{S^1} v_x^2 dx}{\int_{S^1} (v-\bar{v})^2 dx} = \frac{4\pi^2}{L^2}, \tag{18}$$

where $\bar{v} = \int_{S^1} v dx/L$.

Proof. Let I denote the value of the infimum in (17). Let $u \in \mathcal{H}_1$ and define v by setting $u = 1 + \varepsilon(v - \bar{v})$. Then, if we can prove that

$$I \leq \frac{1}{2} \inf_{v\in\mathcal{H}_1} \frac{\int_{S^1} v_x^2 dx}{\int_{S^1} (v-\bar{v})^2 dx}, \tag{19}$$

we obtain the upper bound $I \leq 2\pi^2/L^2$ from (18). Without loss of generality, we may replace $v - \bar{v}$ by v such that $\int_{S^1} v dx = 0$. Then $u^2 = 1 + 2\varepsilon v + \varepsilon^2 v^2$ and the expansion $\log(1+x) = x + x^2/2 + O(x^3)$ for $x \to 0$ yield for $\varepsilon \to 0$

$$\begin{aligned}
\int_{S^1} u^2 \log(u^2) dx &= \int_{S^1} (1 + 2\varepsilon v + \varepsilon^2 v^2) \log(1 + 2\varepsilon v + \varepsilon^2 v^2) dx \\
&= 3\varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^3), \\
\int_{S^1} u^2 dx \log\left(\frac{1}{L}\int_{S^1} u^2 dx\right) &= \int_{S^1} (1 + \varepsilon^2 v^2) dx \log\left(\frac{1}{L}\int_{S^1} (1 + \varepsilon^2 v^2) dx\right) \\
&= \varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^4).
\end{aligned}$$

Taking the difference of the two expansions gives

$$\int_{S^1} u^2 \log\left(\frac{u^2}{\int_{S^1} u^2 dx/L}\right) dx = 2\varepsilon^2 \int_{S^1} v^2 dx + O(\varepsilon^3).$$

Therefore, using $\int_{S^1} u_x^2 dx = \varepsilon^2 \int_{S^1} v_x^2 dx$,

$$\frac{\int_{S^1} u_x^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|^2_{L^2(S^1)}) dx} = \frac{1}{2} \frac{\int_{S^1} v_x^2 dx}{\int_{S^1} v^2 dx} + O(\varepsilon).$$

In the limit $\varepsilon \to 0$ we obtain (19).

In order to prove the lower bound for the infimum we use the entropy–entropy production method. For this we consider the heat equation

$$v_t = v_{xx} \quad \text{in } S^1 \times (0, \infty), \quad v(\cdot, 0) = u^2 \quad \text{in } S^1$$

for some function $u \in H^1(S^1)$. We assume for simplicity that $\|u\|^2_{L^2(S^1)} = \int_{S^1} u^2 dx/L = 1$. Then

$$\frac{d}{dt} \int_{S^1} v \log v dx = -4 \int_{S^1} w_x^2 dx,$$

where the function $w := \sqrt{v}$ solves the equation $w_t = w_{xx} + w_x^2/w$. Now, the time derivative of

$$f(t) = \int_{S^1} w_x^2 dx - \frac{2\pi^2}{L^2} \int_{S^1} w^2 \log(w^2) dx$$

equals

$$f'(t) = -2 \int_{S^1} \left(w_{xx}^2 + \frac{w_x^4}{3w^2} - \frac{4\pi^2}{L^2} w_x^2 \right) dx \leq -\frac{2}{3} \int_{S^1} \frac{w_x^4}{w^2} dx \leq 0,$$

where we have used the Poincaré inequality

$$\int_{S^1} w_x^2 dx \leq \frac{L^2}{4\pi^2} \int_{S^1} w_{xx}^2 dx. \tag{20}$$

This shows that $f(t)$ is non-increasing and moreover, for any $u \in H^1(S^1)$,

$$\int_{S^1} u_x^2 dx - \frac{2\pi^2}{L^2} \int_{S^1} u^2 \log(u^2/\|u\|_{L^2(S^1)}^2) dx = f(0) \geq f(t).$$

As the solution $v(\cdot,t)$ of the above heat equation and hence $w(\cdot,t)$ converges to zero in appropriate Sobolev norms as $t \to +\infty$, we conclude that $f(t) \to 0$ as $t \to +\infty$. This implies $I \geq 2\pi^2/L^2$. □

Remark 7. Similar results as in Theorem 6 can be obtained for the so-called convex Sobolev inequalities. Let $\sigma(v) = (v^p - \bar{v}^p)/(p-1)$, where $\bar{v} = \int_{S^1} v dx/L$ for $1 < p \leq 2$. We claim that

$$\inf_{v \in \mathcal{H}_1} \frac{\int_{S^1} \sigma''(v) v_x^2 dx}{\int_{S^1} \sigma(v) dx} = \frac{8\pi^2}{L^2}.$$

As in the logarithmic case, the lower bound is achieved by an expansion around 1 and the usual Poincaré inequality. On the other hand, let v be a solution of the heat equation. Then

$$\frac{d}{dt} \int_{S^1} \sigma(v) dx = -\frac{4}{p} \int_{S^1} w_x^2 dx$$

where $w = v^{p/2}$ solves

$$w_t = w_{xx} + \left(\frac{2}{p} - 1\right) \frac{w_x^2}{w}, \tag{21}$$

and, using (20),

$$\begin{aligned} \frac{d}{dt} \int_{S^1} \left(w_x^2 - \frac{2\pi^2 p}{L^2} \sigma(v) \right) dx &= -2 \int_{S^1} \left(w_{xx}^2 - \frac{4\pi^2}{L^2} w_x^2 + \left(\frac{2}{p} - 1\right) \frac{w_x^4}{3w^2} \right) dx \\ &\leq -\frac{2}{3} \left(\frac{2}{p} - 1\right) \int_{S^1} \frac{w_x^4}{w^2} dx \leq 0. \end{aligned}$$

This proves the upper bound

$$\frac{p}{4} \int_{S^1} \sigma''(v) v_x^2 dx = \int_{S^1} w_x^2 dx \geq \frac{2\pi^2 p}{L^2} \int_{S^1} \sigma(v) dx.$$

With the notation $v = u^{2/p}$ this result takes the more familiar form

$$\frac{1}{p-1} \left[\int_{S^1} u^2 dx - L \left(\frac{1}{L} \int_{S^1} u^{2/p} dx \right)^p \right] \leq \frac{L^2}{2\pi^2 p} \int_{S^1} u_x^2 dx \quad \text{for all } u \in H^1(S^1). \tag{22}$$

The logarithmic case corresponds to the limit $p \to 1$ whereas the case $p = 2$ gives the usual Poincaré inequality.

We may notice that the method gives more than what is stated in Theorem 6 since there is an integral remainder term. Namely, for any $p \in [1,2]$, for any $v \in H^1(S^1)$, we have

$$\frac{p}{4} \int_{S^1} \sigma''(v) v_x^2 dx + \mathcal{R}[v] \geq \frac{2\pi^2 p}{L^2} \int_{S^1} \sigma(v) dx$$

with

$$\mathcal{R}[v] = 2\int_0^\infty \int_{S^1} \left(w_{xx}^2 - \frac{4\pi^2}{L^2} w_x^2 + \left(\frac{2}{p} - 1\right) \frac{w_x^4}{3w^2} \right) dx\,dt,$$

where $w = w(x,t)$ is the solution to (21) with initial datum $u_0^{p/2}$. Inequality (22) can also be improved with an integral remainder term for any $p \in [1,2]$, where in the limit case $p = 1$, one has to take $\sigma(v) = v \log(v/\bar{v})$. As a consequence, the only optimal functions in (17) or in (22) are the constants.

Corollary 8. *Let $n \in \mathbb{N}$, $n > 0$ and let $\mathcal{H}_n = \{u \in H^n(S^1) : u_x \not\equiv 0 \text{ a.e.}\}$. Then*

$$\inf_{u \in \mathcal{H}_n} \frac{\int_{S^1} \left|u^{(n)}\right|^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|^2_{L^2(S^1)}) dx} = \frac{1}{2}\left(\frac{2\pi}{L}\right)^{2n}. \tag{23}$$

Proof. We obtain a lower bound by applying successively Theorem 6 and the Poincaré inequality:

$$\int_{S^1} u^2 \log\left(\frac{u^2}{\|u\|^2_{L^2(S^1)}}\right) dx \le \frac{L^2}{2\pi^2} \int_{S^1} u_x^2 dx \le 2\left(\frac{L}{2\pi}\right)^{2n} \int_{S^1} \left|u^{(n)}\right|^2 dx$$

The upper bound is achieved as in the proof of Theorem 6 by expanding the quotient for $u = 1 + \varepsilon v$ with $\int_{S^1} v dx = 0$ in powers of ε,

$$\frac{\int_{S^1} \left|u^{(n)}\right|^2 dx}{\int_{S^1} u^2 \log(u^2/\|u\|^2_{L^2(S^1)}) dx} = \frac{1}{2} \frac{\int_{S^1} \left|v^{(n)}\right|^2 dx}{\int_{S^1} v^2 dx} + O(\varepsilon),$$

and using the Poincaré inequality

$$\inf_{u \in \mathcal{H}_n} \frac{\int_{S^1} \left|v^{(n)}\right|^2 dx}{\int_{S^1} |v - \bar{v}|^2 dx} = \left(\frac{2\pi}{L}\right)^{2n}.$$

The best constant $\omega = (2\pi/L)^{2n}$ in such an inequality is easily recovered by looking for the smallest positive value of ω for which there exists a nontrivial periodic solution of $(-1)^n v^{(2n)} + \omega v = 0$. □

5 Exponential time decay of the solutions

We show the exponential time decay of the solutions of (1). Our main result is contained in the following theorem.

Theorem 9. *Assume that u_0 is a nonnegative measurable function such that $\int_{S^1}(u_0 - \log u_0)dx$ and $\int_{S^1} u_0 \log u_0 dx$ are finite. Let u be the weak solution of (1) constructed in Theorem 1 and set $\bar{u} = \int_{S^1} u_0(x)dx/L$. Then*

$$\int_{S^1} u(\cdot,t) \log\left(\frac{u(\cdot,t)}{\bar{u}}\right) dx \le e^{-Mt} \int_{S^1} u_0 \log\left(\frac{u_0}{\bar{u}}\right) dx,$$

where
$$M = \frac{32\pi^4}{L^4}.$$

Proof. Since we do not have enough regularity of the solutions to (1) we need to regularize the equation first. For this we consider the semi-discrete problem

$$\frac{1}{\tau}(u_k - u_{k-1}) + (u_k(\log u_k)_{xx})_{xx} = 0 \quad \text{in } S^1 \tag{24}$$

as in the proof of Theorem 1. The solution $u_k \in H^2(S^1)$ of this problem for given u_{k-1} is strictly positive and we can use $\log u_k$ as a test function in the weak formulation of (24). In order to simplify the presentation we set $u := u_k$ and $z := u_{k-1}$. Then we obtain as in [13]

$$\frac{1}{\tau}\int_{S^1}(u\log u - z\log z)dx + \int_{S^1} u\left|(\log u)_{xx}\right|^2 dx \leq 0. \tag{25}$$

From integration by parts it follows

$$\int_{S^1}\frac{u_x^2 u_{xx}}{u^2}dx = \frac{2}{3}\int_{S^1}\frac{u_x^4}{u^3}dx.$$

This identity gives

$$\begin{aligned}\int_{S^1} u\left|(\log u)_{xx}\right|^2 dx &= \int_{S^1}\left(\frac{u_{xx}^2}{u} + \frac{u_x^4}{u^3} - 2\frac{u_{xx}u_x^2}{u^2}\right)dx = \int_{S^1}\left(\frac{u_{xx}^2}{u} - \frac{1}{3}\frac{u_x^4}{u^3}\right)dx \\ &= \int_{S^1}\left(\frac{u_{xx}^2}{u} + \frac{1}{3}\frac{u_x^4}{u^3} - \frac{u_{xx}u_x^2}{u^2}\right)dx = 4\int_{S^1}|(\sqrt{u})_{xx}|^2 dx + \frac{1}{12}\int_{S^1}\frac{u_x^4}{u^3}dx.\end{aligned}$$

Thus, (25) becomes

$$\frac{1}{\tau}\int_{S^1}\left(u\log\left(\frac{u}{\bar{u}}\right) - z\log\left(\frac{z}{\bar{u}}\right)\right)dx + 4\int_{S^1}\left|(\sqrt{u})_{xx}\right|^2 dx \leq 0. \tag{26}$$

Now we use Corollary 8 with $n = 2$:

$$\int_{S^1} u\log\left(\frac{u}{\bar{u}}\right)dx \leq \frac{L^4}{8\pi^4}\int_{S^1}\left|(\sqrt{u})_{xx}\right|^2 dx.$$

From this inequality and (26) we conclude

$$\frac{1}{\tau}\int_{S^1}\left(u\log\left(\frac{u}{\bar{u}}\right) - z\log\left(\frac{z}{\bar{u}}\right)\right)dx + \frac{32\pi^4}{L^4}\int_{S^1} u\log\left(\frac{u}{\bar{u}}\right)dx \leq 0.$$

This is a difference inequality for the sequence

$$E_k := \int_{S^1} u_k\log\left(\frac{u_k}{\bar{u}}\right)dx,$$

yielding

$$(1 + \tau M)E_k \leq E_{k-1} \quad \text{or} \quad E_k \leq E_0(1 + \tau M)^{-k},$$

where M is as in the statement of the theorem. For $t \in ((k-1)\tau, k\tau]$ we obtain further

$$E_k \leq E_0(1 + \tau M)^{-t/\tau}.$$

Now the proof as exactly as in [13]. Indeed, the functions $u_k(x)$ converge a.e. to $u(x,t)$ and $(1 + \tau M)^{-t/\tau} \to e^{-Mt}$ as $\tau \to 0$. This implies the assertion. □

Remark 10. The decay rate M is not optimal since in the estimate (26) we have neglected the term $\frac{1}{12} \int (u_x^4/u^3) dx$.

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