Nonlinear diffusions as limit of kinetic equations with relaxation collision kernels

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Abstract

Kinetic transport equations with a given confining potential and non-linear relaxation type collision operators are considered. General (monotone) energy dependent equilibrium distributions are allowed with a chemical potential ensuring mass conservation. Existence and uniqueness of solutions is proven for initial data bounded by equilibrium distributions. The diffusive macroscopic limit is carried out using compensated compactness theory. The result are drift-diffusion equations with nonlinear diffusion. The most notable examples are of the form $\partial_t \rho = \nabla \cdot (\nabla \rho^m + \rho \nabla V)$, ranging from porous medium equations to fast diffusion, with the exponent satisfying $0 < m < 5/3$ in $\mathbb{R}^3$.

Key words Kinetic equation, Macroscopic limit, Diffusion limit, Boltzmann equation, Equilibrium distribution function, Gibbs state, Porous medium equation, Fast diffusion equation, Relaxation time approximation, Compensated compactness

MSC classification (2000): Primary: 82C40, 35B40; Secondary: 35A05, 35K55, 45K05, 82D05, 85A05

1. Introduction

We consider the scaled kinetic equation

$$\varepsilon^2 \, \partial_t f + \varepsilon \, [v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f] = Q(f) \,, \tag{1.1}$$

$$Q(f) := G_f - f \,, \quad G_f := \gamma\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t))\Big) \,, \tag{1.2}$$

where the distribution function $f = f(x,v,t)$ depends on position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$, and time $t > 0$. The collision model is a simple relaxation kernel towards a generalized local Gibbs state G_f. The chemical potential $\bar{\mu}(\rho_f)$ is implicitly determined by the condition $\int_{\mathbb{R}^3} G_f \, dv = \rho_f := \int_{\mathbb{R}^3} f \, dv$, in the sense that the function $\bar{\mu}$ is given by the equation

$$\rho = \int_{\mathbb{R}^3} \gamma \left(\frac{1}{2} \, |v|^2 - \bar{\mu}(\rho) \right) \,. \tag{1.3}$$

We are interested in the diffusion limit $\varepsilon \to 0$ which corresponds to a large time scale and a high collision frequency limit. We prove that in the limit $\varepsilon \to 0$, the distribution function f is a local Gibbs state: $f = G_f$, whose density is subject to a nonlinear diffusion equation

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \, \nu(\rho) + \rho \, \nabla_x V(x))$$

with $\nu(\rho) := \int_0^\rho \tilde{\rho} \, \bar{\mu}'(\tilde{\rho}) \, d\tilde{\rho}$. The main modelling ingredient is the energy dependent equilibrium profile $\gamma(E) \geq 0$, which is assumed to be nonincreasing. The given external potential $V(x)$ will be assumed to be 'confining'. An appropriate definition of this property depends on the profile γ and will be given below. Functions of the total energy $|v|^2/2 + V(x)$ constitute the kernel of the transport operator on the left hand side of (1.1). Therefore, the definition of the quasi Fermi potential $\mu_\rho(x,t) := \bar{\mu}(\rho(x,t)) + V(x)$ will be convenient. In particular, equilibrium distributions with constant quasi Fermi potential are steady state solutions of (1.1).

The equation (1.1) is considered subject to initial conditions

$$f(x,v,0) = f_I(x,v) \tag{1.4}$$

with $f_I \in L^1_+(\mathbb{R}^3 \times \mathbb{R}^3)$. The total mass $M := \int_{\mathbb{R}^6} f_I(x,v)\,dx\,dv$ is formally preserved by the evolution, i.e., $\int_{\mathbb{R}^6} f(x,v,t)\,dx\,dv = M$ for all $t > 0$. For notational convenience, we restrict our attention to the three-dimensional problem. Generalizations of our results to other dimensions are straightforward.

1.1. Modelling and references

Reducing kinetic equations to macroscopic equations is a standard procedure, at least from a formal point of view. At the kinetic level, it is easy to relate the parameters with simple physical quantities since characteristics in the phase space can be interpreted in terms of particle dynamics. The price to pay is the high dimensionality of the phase space. On the other hand, hydrodynamical equations or parabolic models are in principle simpler to compute, but their direct derivation is far less intuitive. This motivates the study of hydrodynamics or diffusion limits, with the idea that the models are easier to build at the kinetic level, but that one is mostly interested only in the macroscopic observables.

Nonlinear diffusion equations have attracted a lot of attention over the last few years, because of their deep mathematical properties and the physical interpretation of their naturally associated Lyapunov functionals. A very striking point of view is to consider these equations as the gradient flow of their Lyapunov functionals with respect to some appropriate notion of distance. However the derivation of such nonlinear diffusion equations in a physical context up to now looks rather unclear, although some properties, like finite diffusion speed in the porous media case, make them very appealing from a modelling point of view. It is the purpose of this paper to provide a justification of nonlinear diffusion equations as limits of appropriate simple kinetic models. Notice that it is also possible to proceed the other way around and to reconstruct the Gibbs state at the kinetic level knowing the explicit form of the diffusion coefficient at the macroscopic level.

The appropriate way to design general collision kernels and on which, experimental or theoretical, physical grounds they should be established is completely out of the scope of this paper. What we intend to do here relies on a much more pragmatic approach which can be decomposed as follows:

(1) If the kinetic equilibrium or Gibbs states, are known, or equivalently the energy profiles of such equilibria, then local equilibria, or local Gibbs states, are easily derived. From a physical point of view, such local equilibria make sense, whenever local relaxation phenomena occur on a faster time scale than the global evolution of the solution, thus giving rise to solutions at local equilibrium in the velocity space. This is particularly the case in models with collisions. Transfer of momentum during a collision is generally assumed to occur on a faster time scale than transport related effects like the ones induced by mean field forces, and are usually considered to be instantaneous. Thus local or global Gibbs states will be considered in our approach as basic input for the modeling. This is a very standard assumption for instance in semiconductor theory when one speaks of Fermi-Dirac distributions, or when one considers polytropic distribution functions in stellar dynamics.

(2) Non monotone energy profiles result in various pathologies like linear and nonlinear instabilities. On the opposite, a monotonically decreasing energy profile provides a consistent way of finding a convex Lyapunov functional which, under appropriate constraints like mass conservation, allows to characterize the global Gibbs state as its unique minimizer. From a mathematical point of view, the Lyapunov functional is the sum of the total energy and of a convex nonlinear entropy based on the Legendre transform of a primitive (up to a sign) of the energy profile.

(3) In our approach, we say nothing about the physical phenomena responsible for the relaxation towards the local Gibbs state and, on the long time range, towards the global Gibbs state. We only derive the nonlinear diffusion limit in a way which is consistent with the Gibbs state. This is why we introduce at the kinetic level a caricature of a collision kernel, which is simply a 'projection' onto the local Gibbs state with the same spatial density, thus introducing a local Lagrange multiplier which will be referred to as the chemical potential. In the mathematical literature, such a collision kernel is known as a relaxation-time kernel. At least in the long time asymptotics, such a kernel is generally believed to be a reasonable approximation of more realistic physical kernels.

Diffusion limits appear when collision effects become dominant, but also when one is interested in long time effects so that phenomena due to convection, which would be essential in the hydrodynamical regime, are also dominated by diffusion. This can be derived in physical situations by an adimensionalization of the equation and a proper scaling. To stick to our purpose and since this is by many aspects standard, we refer for instance to [26] for the equations with correct physical parameters in the semiconductor context. Here ε will simply be a small positive parameter and we are interested in the singular limit $\varepsilon \to 0$.

To come back to the two fields of applications quoted above, let us mention that in astrophysics, power law Gibbs states are well known (see, *e.g.*, [7], and [23] for some mathematical properties of such equilibrium states), thus justifying on reasonable grounds how diffusion models can be introduced starting from more fundamental models (see for instance [10] for recent results in astrophysics and two dimensional turbulence, and references therein for earlier papers). Note by the way that nonlinear diffusions with vanishing diffusion coefficients, like for the porous media model, cure one of the major problem of linear diffusion, namely the non-existence of global Gibbs states with finite mass coupled with gravitational interaction in a Euclidean space. Lyapunov functionals corresponding to porous media in an astrophysical context are usually called Tsallis entropies, referring to [33].

In the mathematical study of diffusion limits for semiconductor physics, more results are known, starting with [17,18]. The reference paper has been written by Goudon and Poupaud [21], but many other works deal with some more specific cases [29,1,4,6,14,27,20,19]. Some of the results of this paper were written at a formal level in [5].

The derivation of porous medium type equations is certainly a question of general interest which can be undertaken from various points of view. This has been done for instance in the theory of radiative transfer, in the so-called Rosseland approximation, [2,3]. A probabilistic approach can be found for instance in [22]. Discrete velocities approaches of nonlinear diffusion equations have also been studied, see [8,24].

Concerning the relation of the energy profile γ of the Gibbs state with the entropy generating function (see Section 3), apart from [5], one can refer to [32,13] for some general considerations, and to [11] for the examples given in this paper. More specifically in view of applications in astrophysics, we refer to [12] for a formal derivation based on a Chapman-Enskog expansion which is closely related to our method. In such a setting, the nonlinear diffusion equation is referred to as the generalized Smoluchowski-Poisson equation. Also see [30,31] for some recent contributions in this direction and, from a physics point of view, we refer to a series of recent papers by Chavanis et al. which goes far beyond the scope of our study.

1.2. Formal macroscopic limit

Consider formal asymptotic expansions $f = f^0+\varepsilon f^1+O(\varepsilon^2)$, $G_f = G^0+\varepsilon G^1+O(\varepsilon^2)$, $\mu_{\rho_f} = \mu^0+O(\varepsilon)$, and $\rho_f = \rho^0 + O(\varepsilon)$. Then, by going to the limit $\varepsilon \to 0$ in (1.1), we obtain, at lowest order in ε,

$$f^0(x,v,t) = G^0(x,v,t) = \gamma(|v|^2/2 + V(x) - \mu^0(x,t)) \quad \text{and} \quad \rho^0 = \int_{\mathbb{R}^3} f^0 \, dv \ .$$

Notice that $\mu^0 = \bar\mu(\rho^0) + V$. The $O(\varepsilon)$-terms in (1.1) give

$$v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1 \ ,$$

which can be rewritten as

$$f^1 = \nabla_v \gamma(|v|^2/2 + V - \mu^0) \cdot \nabla_x \mu^0 + G^1 \ . \tag{1.5}$$

Now we pass to the limit in the continuity equation

$$\partial_t \rho_f + \frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^3} v f \, dv = 0 \ ,$$

and obtain

$$\partial_t \rho^0 + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 \, dv = 0 \ .$$

For the evaluation of the flux, we use (1.5). Since G^1 is an even function of v, it does not contribute, and we end up with

$$\partial_t \rho^0 = \nabla_x \cdot (\rho^0 \, \nabla_x \mu^0) \ .$$

Equivalently, since $\mu^0 = \bar\mu(\rho^0) + V$, we may write

$$\partial_t \rho^0 = \nabla_x \cdot (D(\rho^0) \, \nabla_x \rho^0 + \rho^0 \, \nabla_x V) = \nabla_x \cdot (\nabla_x \, \nu(\rho^0) + \rho^0 \, \nabla_x V) \ , \tag{1.6}$$

with

$$D(\rho) = \rho \, \bar\mu'(\rho) \quad \text{and} \quad \nu(\rho) := \int_0^\rho D(\tilde\rho) \, d\tilde\rho \ . \tag{1.7}$$

This equation has to be supplemented with the initial condition

$$\rho^0(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\, dv \ . \tag{1.8}$$

Formally, this can be derived considering an initial layer governed by the equation $\partial_\tau f = Q(f)$, $\tau = t/\varepsilon^2$, and using mass conservation. We refer to [12] for further details on Chapman-Enskop type expansions of this type. The main result of this paper is a rigorous justification of this formal asymptotic expansion.

1.3. Reconstruction of energy profiles

Consider the inverse problem of finding an equilibrium profile γ producing a given macroscopic equation of the form (1.6). The macroscopic model is determined by the chemical potential function $\bar\mu(\rho)$, where $\bar\mu^{-1}$ is given by

$$\bar\mu^{-1}(\theta) := \int_{\mathbb{R}^3} \gamma\left(\frac{1}{2}\,|v|^2 - \theta\right) = 4\pi\sqrt{2}\int_{-\theta}^{\infty} \gamma(q)\sqrt{\theta+q}\; dq$$

so that (1.3) amounts to solving $\rho = \bar\mu^{-1}(\theta)$. Differentiation leads to the Abelian equation

$$(\bar\mu^{-1})'(\theta) = 2\pi\sqrt{2}\int_{-\theta}^{\infty} \frac{\gamma(q)}{\sqrt{\theta+q}}\; dq \ . \tag{1.9}$$

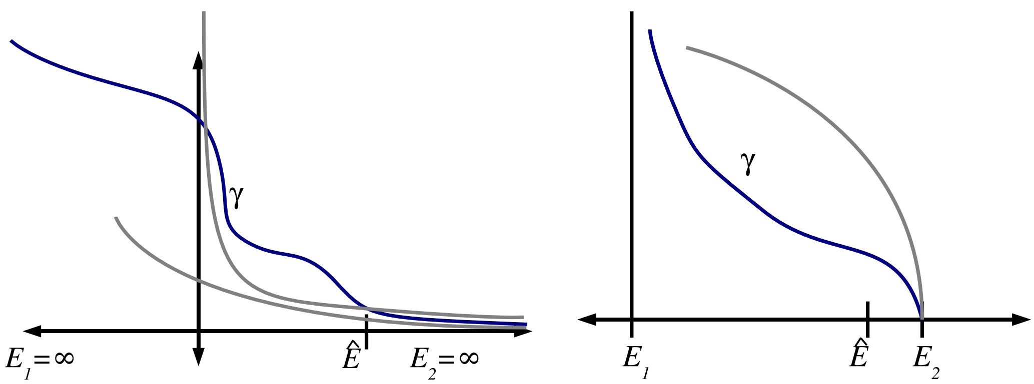
This identity can be inverted (see [28], pp. 9, 10) and gives an explicit expression for γ in terms of $\bar\mu^{-1}$, which can be written as

$$\gamma(E) = \frac{1}{2\pi^2\sqrt{2}}\frac{d^2}{dE^2}\int_0^{\infty} \bar\mu^{-1}(-\theta - E)\;\frac{d\theta}{\sqrt{\theta}} \ .$$

We refer to [16] for a generalization to other space dimensions. Note that convexity of $\bar\mu^{-1}$ is sufficient for obtaining a nonnegative equilibrium profile. A precise classification of all $\bar\mu^{-1}$ corresponding to nonnegative γ is not known (see [16] for a discussion).

1.4. Assumptions and main result

Our rigorous justification of the macroscopic limit covers all the examples of equilibrium profiles described in the last section. This involves quite a range of different qualitative behaviours including equilibrium profiles with compact support. This generality comes at the expense of a number of technical assumptions.



(a) γ in the case $E_2 = \infty$ satisfying (1.10). (b) γ with $-\infty < E_1 < E_2 < \infty$.

Fig. 1. Exemplary graphs of the energy profile γ.

Assumption **1** *The energy profile, or Gibbs state, $\gamma \in C((E_1,\infty),\mathbb{R}_+)$ with $E_1 \geq -\infty$ is nonincreasing and nonnegative. It is continuously differentiable on its support (E_1,E_2) with $\gamma'(E) < 0$ for $E_1 < E < E_2$. There exists a $\hat E < E_2$, such that γ is either convex or concave on the interval $(\hat E, E_2)$.*

The following two assumptions refer to the cases of bounded and unbounded supports separately.

Assumption **2** *If $E_2 = \infty$, there are constants $\overline{E}$ and $\tau > 0$ such that*

$$\text{either} \quad \frac{-\gamma'(E)}{\gamma(E)} \le \frac{1}{\tau} \quad \text{for} \quad E > \overline{E}\,, \tag{1.10}$$

$$\text{or} \quad \frac{-\gamma'(E)}{\gamma(E)} \ge \frac{1}{\tau} \quad \text{for} \quad E > \overline{E}\,. \tag{1.11}$$

Moreover, in case (1.10) (cf. figure 1(a)), there exists a $\delta > 0$ such that

$$\gamma(E) = O(E^{-5/2-\delta}) \quad \text{as} \quad E \to \infty\,. \tag{1.12}$$

By (1.10), (1.11), the decay is bounded by an exponential either from above or from below, and by (1.12) it is fast enough to ensure the existence of second order velocity moments. As a map from (E_1, E_2) to $(0, \gamma(E_1))$, γ is invertible. The inverse will be used below. Assumptions 1 and 2 also guarantee that the function $\bar{\mu}(\rho)$ defined by (1.3) is invertible as a map from $(0, \bar{\rho})$ to $(-E_2, -E_1)$, where

$$\lim_{\rho\to 0} \bar{\mu}(\rho) = -E_2 \quad \text{and} \quad \bar{\rho} := \lim_{-\bar{\mu}\to E_1+} \int_{\mathbb{R}^3} \gamma(|v|^2/2 - \bar{\mu})\, dv \tag{1.13}$$

can be finite for $E_1 > -\infty$.

Assumption **3** *If $E_2 < \infty$ (cf. figure 1(b)), there are constants $k > 0$ and $C > 0$ such that*

$$\gamma(E) \le C(E_2 - E)^k \quad \text{for} \quad \hat{E} < E < E_2\,. \tag{1.14}$$

For the initial data we assume boundedness by a stationary solution of (1.1).

Assumption **4** *There is a constant quasi Fermi energy $\mu^* < -E_1$ such that*

$$0 \le f_I(x,v) \le f^*(x,v) := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^*\Big)\,, \quad \forall (x,v) \in \mathbb{R}^6\,. \tag{1.15}$$

It will be shown in the following section that this bound is propagated by (1.1). Finally, we collect our assumptions on the confining potential V.

Assumption **5** *The potential V is nonnegative and satisfies*

$$V \in C^{1,1}(\mathbb{R}^3)\,. \tag{1.16}$$

It is 'confining' in the sense that the upper bound f^* for the initial data has finite mass and energy:

$$\iint_{\mathbb{R}^6} \Big(1 + \frac{1}{2}|v|^2 + V(x)\Big) f^*(x,v)\, dv\, dx < \infty\,. \tag{1.17}$$

Remark 1 *If γ has compact support, then $f^*(x,v)$ has compact support as a function of v. If additionally $V(x) \ge E_2 + \mu^*$ outside of a compact set in $\mathbb{R}^3_x$, then f^* has compact support in $\mathbb{R}^6$. Obviously,* (1.17) *is satisfied in this situation.*

As a consequence of the assumptions on the equilibrium distribution, we prove either an upper or a lower bound on the diffusivity.

Lemma 1 *Let $0 \le \rho < \rho^{max} = \bar{\mu}^{-1}(\mu^*)$ and denote by $\mathrm{Id}^{3\times 3}$ the identity matrix in $\mathbb{R}^{3\times 3}$. Then the function $\nu(\rho)$ as defined in* (1.7) *represents the second order moments of the local equilibrium:*

$$\int_{\mathbb{R}^3} v \otimes v\, \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\right) dv = \nu(\rho)\, \mathrm{Id}^{3\times 3}\,.$$

Moreover, on the closed interval $[0, \rho^{max}]$, the diffusivity $D(\rho) = \nu'(\rho)$ has either an upper bound or a strictly positive lower bound.

Proof. Observe that by Assumption 2 the second order moments exist and are uniformly bounded as $\rho \to 0$, and that the off-diagonal elements of the pressure tensor vanish. By an integration by parts with respect to v_i, where i is one of $1, 2, 3$ we get

$$\int_{\mathbb{R}^3} v_i^2\,\gamma'\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\Big)(-\bar{\mu}'(\rho))\,dv = \bar{\mu}'(\rho)\int_{\mathbb{R}^3} -v_i^2\,\gamma'\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\Big)\,dv = \rho\,\bar{\mu}'(\rho) = \nu'(\rho)\ .$$

By (1.13) the integrand $\gamma(|v|^2 - \bar{\mu}(\rho))$ tends to zero as $\rho \to 0$ pointwise in v. The uniform boundedness mentioned above then guarantees $\nu(0) = 0$ and the fact that (1.7) holds.

Now consider an interval $[\delta, \rho^{\max}]$ for some $\delta > 0$. If $\bar{\mu}'$ does not behave 'badly' in this interval, the same must be true for $(\bar{\mu}^{-1})'(\theta)$ on $[\bar{\mu}(\delta), \bar{\mu}(\rho^{\max})]$. As (1.9) can be written as

$$(\bar{\mu}^{-1})'(\theta) = 2\pi\sqrt{2}\int_0^\infty \frac{\gamma(q-\theta)}{\sqrt{q}}\,dq$$

we infer for $\bar{\mu}(\delta) \le \theta \le \bar{\mu}(\rho^{\max})$

$$0 < \frac{1}{m(\delta)} := (\bar{\mu}^{-1})'(\bar{\mu}(\delta)) \le (\bar{\mu}^{-1})'(\theta) \le (\bar{\mu}^{-1})'(\bar{\mu}(\rho^{\max})) =: \frac{1}{M} < \infty\ .$$

Hence $0 \le M \le \bar{\mu}'(\rho) < m(\delta) < \infty$ where $\rho \in [\delta, \rho^{\max}]$. Consequently also the product $\rho\,\bar{\mu}'(\rho)$ is well behaved in this sense and the values 0 and ∞ may only be approached as $\rho \to 0$.

As far as the behaviour at $\rho = 0$ is concerned we have to distinguish the case where $E_2 < \infty$ and the two cases (1.10) and (1.11) corresponding to energy profiles which are, compared to an exponential decay, converging to zero not faster, or respectively, not slower. Let $E_2 < \infty$, then

$$\begin{aligned}
\lim_{\rho\to 0}\nu'(\rho) &= \lim_{\theta\to -E_2}\frac{\bar{\mu}^{-1}(\theta)}{(\bar{\mu}^{-1})'(\theta)} = \lim_{\theta\to -E_2}\frac{\int_0^{E_2+\theta}\gamma(p-\theta)\sqrt{p}\,dp}{\frac{1}{2}\int_0^{E_2+\theta}\gamma(p-\theta)\frac{1}{\sqrt{p}}\,dp}\\
&= \lim_{\theta\to -E_2}\frac{\int_0^{E_2+\theta}\gamma(p-\theta)\frac{p}{\sqrt{p}}\,dp}{\frac{1}{2}\int_0^{E_2+\theta}\gamma(p-\theta)\frac{1}{\sqrt{p}}\,dp} \le \lim_{\theta\to -E_2}\frac{(E_2+\theta)\int_0^{E_2+\theta}\gamma(p-\theta)\frac{1}{\sqrt{p}}\,dp}{\frac{1}{2}\int_0^{E_2+\theta}\gamma(p-\theta)\frac{1}{\sqrt{p}}\,dp} = 0\ .
\end{aligned}$$

In the case (1.10) we obtain

$$\begin{aligned}
\liminf_{\rho\to 0}\nu'(\rho) = \liminf_{\theta\to -\infty}\frac{\bar{\mu}^{-1}(\theta)}{(\bar{\mu}^{-1})'(\theta)} &= \liminf_{\theta\to -\infty}\frac{\int_0^\infty\gamma(p-\theta)\sqrt{p}\,dp}{\int_0^\infty -\gamma'(p-\theta)\sqrt{p}\,dp}\\
&\ge \liminf_{\theta\to -\infty}\frac{\int_0^\infty\gamma(p-\theta)\sqrt{p}\,dp}{\frac{1}{\tau}\int_0^\infty\gamma(p-\theta)\sqrt{p}\,dp} = \tau\ .
\end{aligned}$$

In the case (1.11) we obtain by an analogous computation

$$\limsup_{\rho\to 0}\nu'(\rho) \le \tau\ ,$$

which concludes the proof for all possible cases.

Our main result is the following

Theorem 2 *Under Assumptions 1–5, for any $\varepsilon > 0$, the problem* (1.1)–(1.4) *has a unique weak solution $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$. As $\varepsilon \to 0$, f^ε weakly converges to a local Gibbs state f^0 given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho(x,t))\right) \quad \forall\,(x,v,t) \in \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+\ ,$$

where ρ is a solution of the nonlinear diffusion equation

$$\partial_t\rho = \nabla_x\cdot(\nabla_x\,\nu(\rho) + \rho\,\nabla_x V(x)) \tag{1.18}$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$ and where ν is given by (1.7). *Moreover, $\int_{\mathbb{R}^3} f^\varepsilon\,dv$ strongly converges to ρ in L^p_{loc} as $\varepsilon \to 0$.*

Remark 2 *Notice that as a consequence of Theorem 2, there exists a global weak solution of (1.18) with initial data ρ_I. We are however not aware of a result of uniqueness under such general assumptions. Uniqueness is guaranteed in certain cases, e.g. for linear diffusion, when $\nu(\rho) = D\,\rho$. Since we are using compactness arguments in the proof, the diffusion limit has to be understood up to the extraction of a sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ converging to zero, whenever uniqueness is not guaranteed. For simplicity, we will however abusively speak of the convergence as '$\varepsilon \to 0$'.*

The rest of this article is organized as follows. In Section 2, existence and uniqueness of solutions of the initial value problem for (1.1) is proven with some additional details. The main part is Section 3, where the macroscopic limit is rigorously justified. Finally, Section 4 contains a number of examples satisfying the above assumptions. Unless we are explicitly considering the limit $\varepsilon \to 0$, we will simply write f instead of f^ε.

2. Existence and uniqueness

Proposition 3 *Let Assumptions 1, 4, and 5 be satisfied. Then the problem* (1.1)–(1.4) *has a unique weak solution $f \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p \in [1,\infty)$ satisfying $f \in L^\infty(\mathbb{R}^6 \times (0,\infty))$, $\rho_f \in L^\infty(\mathbb{R}^3 \times (0,\infty))$, with $0 \le f(\cdot,\cdot,t) \le f^*$ and $0 \le \rho(\cdot,t) \le \rho^* := \int_{\mathbb{R}^3} f^* dv$ for all $t > 0$.*

Proof. The result is an adaption of a result stated in [29], where, due to a different choice of the scattering operator, the inhomogenities of the linear problems involved in the proof take a different form and therefore have to be treated with different arguments. For a given $f \in \mathcal{V} := \{f \in C(0,T; L^1(\mathbb{R}^6)) : 0 \le f \le f^*\}$, let $g = \Gamma(f)$ be the solution of the linear transport problem

$$\varepsilon^2\,\partial_t g + \varepsilon\Big(v\cdot\nabla_x g - \nabla_x V\cdot\nabla_v g\Big) = G_f - g\ ,\ \text{with}\ \ G_f = \gamma\left(\frac{1}{2}|v|^2 - \bar\mu(\rho_f)\right)\ ,$$
$$g(t=0) = f_I\ ,$$

constructed by the method of characteristics. Obviously, fixed points of Γ correspond to solutions of (1.1), (1.4). First we show that Γ maps $\mathcal{V}$ into itself, and then that it is a contraction for sufficiently small time intervals.

As a consequence of the nonnegativity of f_I and G_f, g is nonnegative by the maximum principle. The function $r := f^* - g$ solves the linear transport problem

$$\varepsilon^2\,\partial_t r + \varepsilon\Big(v\cdot\nabla_x r - \nabla_x V\cdot\nabla_v r\Big) + r = f^* - G_f =: S\ ,$$
$$r(t=0) = f^* - f_I \ge 0\ ,$$

where $S = G_{f^*} - G_f$ is nonnegative because γ is decreasing, $\bar\mu$ is increasing and

$$\rho^* = \int_{\mathbb{R}^3} f^*\,dv \ge \int_{\mathbb{R}^3} f\,dv = \rho_f$$

by assumption. Therefore also r is nonnegative and $\mathcal{V}$ is stable under the action of Γ.

In order to prove the contraction property, consider two functions f_1, f_2 in $\mathcal{V}$ and let $w := \Gamma(f_2) - \Gamma(f_1)$. Then w is a solution of the problem

$$\varepsilon^2\,\partial_t w + \varepsilon\,v\cdot\nabla_x w - \varepsilon\,\nabla_x V\cdot\nabla_v w + w = G_{f_2} - G_{f_1} =: U\ , \tag{2.1}$$
$$w(t=0) = 0\ . \tag{2.2}$$

A multiplication by $\operatorname{sign}(w)$ transforms (2.1) into an equation for $|w|$ with the inhomogeneity replaced by $\operatorname{sign}(w)U$. The integration of this equation with respect to x and v implies

$$\varepsilon^2\,\frac{d}{dt}\|w(.,.,t)\|_{L^1(\mathbb{R}^6)} + \|w(.,.,t)\|_{L^1(\mathbb{R}^6)} \le \|U(.,.,t)\|_{L^1(\mathbb{R}^6)}\ . \tag{2.3}$$

As γ and $\bar\mu$ are respectively monotonically decreasing and increasing functions, the sign of $G_{f_2} - G_{f_1}$ equals the sign of $\rho_{f_2} - \rho_{f_1}$ and does not depend on v. Hence for any $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$,

$$\int_{\mathbb{R}^3} \big|G_{f_2} - G_{f_1}\big|\,dv = |\rho_{f_2} - \rho_{f_1}|\ ,$$

and therefore

$$\|U(.,.,t)\|_{L^1(\mathbb{R}^6)} = \int_{\mathbb{R}^3} |\rho_{f_2} - \rho_{f_1}| \, dx \le \|f_2 - f_1\|_{L^1(\mathbb{R}^6)} \, .$$

Application of the Gronwall Lemma to (2.3) leads to

$$\|\Gamma(f_2) - \Gamma(f_1)\|_{C(0,T;L^1(\mathbb{R}^6))} \le \big(1 - \exp(-T/\varepsilon^2)\big) \, \|f_2 - f_1\|_{C(0,T;L^1(\mathbb{R}^6))} \, ,$$

implying that Γ is a contraction on $\mathcal{C}(0,T;L^1(\mathbb{R}^6))$ for any $T > 0$. The same is true for the spaces $\mathcal{C}(0,T;L^p(\mathbb{R}^6))$, $1 \le p < \infty$ by interpolation using the L^∞-bound. The global $L^\infty \cap L^1$ bound implies that the solution exists globally.

3. The Drift-Diffusion limit

Let the free energy be defined by

$$\mathcal{F}(f) := \int_{\mathbb{R}^6} \left[\Big(\frac{1}{2}|v|^2 + V \Big) f + \beta_\gamma(f) \right] dv \, dx \, , \quad \text{where} \quad \beta_\gamma(f) := -\int_0^f \gamma^{-1}(s) \, ds \, .$$

As $-\gamma^{-1}$ is monotonically increasing, β_γ is a convex function. The definition of β_γ implies that $\beta_\gamma(0) = 0$, that it achieves its minimum on $\mathbb{R}_+$ at

$$\bar f := \gamma(\max\{0, E_1\})$$

and that it is negative on $(0, \bar f)$. Here we recall and extend some of the results stated on a formal level in [5].

Let the microscopic energy associated to a distribution function f be denoted by

$$E_f(x,v,t) := \frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t) = \frac{1}{2}|v|^2 - \bar\mu\big(\rho_f(x,t)\big) \, .$$

If f is a solution of (1.1), an elementary computation shows that

$$\varepsilon^2 \frac{d}{dt} \mathcal{F}\big(f(.,.,t)\big) = \int_{\mathbb{R}^6} \big(\gamma(E_f) - f)(E_f - \gamma^{-1}(f)\big) \, dv \, dx := -D(f) \, . \tag{3.1}$$

This holds provided f has a sufficient decay at infinity to justify the integrations by parts. Moreover we used the fact that $\int_{\mathbb{R}^3} (G_f - f) \, \mu_{\rho_f} \, dv = 0$, which is a straightforward consequence of mass conservation. Since $-\gamma^{-1}$ is monotonically increasing, $D(f)$ is nonnegative. However, in the case when γ has compact support, $D(f)$ does not provide a good control of the distance between f and the local equilibrium G_f. This will be a major difficulty in our analysis below.

We first establish some *a priori* estimates on the free energy. By integration of the entropy production (3.1) on the time interval $(0,T)$ we obtain

$$\varepsilon^2 \Big[\mathcal{F}\big(f(,.,.,T)\big) - \mathcal{F}\big(f_I\big) \Big] = -\int_0^T D(f)(t) \, dt \, , \tag{3.2}$$

which proves that

$$\mathcal{F}\big(f(.,.,T)\big) \le \mathcal{F}(f_I) \quad \text{for any } T > 0 \, .$$

If $\bar f = 0$, β_γ is increasing on $\mathbb{R}_+$ and we conclude $\mathcal{F}(f_I) \le \mathcal{F}(f^*)$. As we shall see below, $\mathcal{F}(f^*) < \infty$. In the case $\bar f > 0$ we use $\text{meas}_{\mathbb{R}^6}(\{f^* > \bar f\}) \le (\bar f)^{-1} \int_{\mathbb{R}^6} f^* \, dv \, dx$ and obtain

$$\begin{aligned} \mathcal{F}(f_I) \le & \int_{\mathbb{R}^6} \Big(\frac{1}{2}|v|^2 + V(x) \Big) f_I \, dv \, dx + \int_{f^I > \bar f} \beta_\gamma(f_I) \, dv \, dx \\ & \qquad \le \int_{\mathbb{R}^6} \Big(\frac{1}{2}|v|^2 + V(x) \Big) f^* \, dv \, dx + \beta_\gamma\big(\|f^*\|_{L^\infty}\big) \frac{1}{\bar f} \int_{\mathbb{R}^6} f^* \, dv \, dx < \infty \end{aligned}$$

by (1.17).

It remains to prove that $\mathcal{F}(f)$ is bounded from below. For fixed $(x,t) \in \mathbb{R}^3 \times \mathbb{R}_+$, the functional $\mathcal{F}_{\rm loc}$:

$$f(x,.,t) \mapsto \int_{\mathbb{R}^3} \Big[E_f(x,v,t) f(x,v,t) + \beta_\gamma\big(f(x,v,t)\big) \Big] \, dv =: \mathcal{F}_{\rm loc}\big(f(x,.,t)\big)$$

is convex and achieves its unique critical point if and only if

$$0 = E_f(x,v,t) + \beta_\gamma'(f) = E_f(x,v,t) - \gamma^{-1}(f) \;,$$

i.e., if $f = G_f$ is a local equilibrium distribution function. As a consequence:

$$\mathcal{F}_{\rm loc}(f(x,.,t)) \geq \mathcal{F}_{\rm loc}(G_f(x,.,t)) \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R}_+ \;,$$

which, after an integration with respect to x, proves that

$$\mathcal{F}(f) \geq \mathcal{F}(G_f) \;, \tag{3.3}$$

using $\int_{\mathbb{R}^3} f\, dv = \int_{\mathbb{R}^3} G_f\, dv$ again.

In the case of a local equibrium $g(x,v,t) = \gamma\big(|v|^2/2 + V(x) - \mu(x,t)\big)$, we may integrate by parts twice with respect to v_i, $i \in \{1,2,3\}$. Using $(\beta_\gamma \circ \gamma)'(E) = -E\,\gamma'(E)$, we get

$$\int_{\mathbb{R}^3} \beta_\gamma(g)\; dv = \int_{\mathbb{R}^3} v_i\Big(\frac{1}{2}|v|^2 + V - \mu\Big)\partial_{v_i}\gamma\Big(\frac{1}{2}|v|^2 + V - \mu\Big)\; dv = -\int_{\mathbb{R}^3}\Big(\frac{1}{2}|v|^2 + v_i^2 + V - \mu\Big) g\; dv \;.$$

This yields

$$\mathcal{F}(g) = \int_{\mathbb{R}^6} \gamma\Big(\frac{1}{2}|v|^2 + V - \mu\Big)\Big(\mu - \frac{|v|^2}{3}\Big)\; dv\; dx = \int_{\mathbb{R}^3}\big(\mu\rho_g - \nu(\rho_g)\big)\; dx \;,$$
$$\text{with} \quad \rho_g = \bar{\mu}^{-1}(\mu - V) \;. \tag{3.4}$$

As a consequence, by (3.3) we obtain that

$$\mathcal{F}(f) \geq \overline{\mathcal{F}}(\rho_f)$$

where $\overline{\mathcal{F}}$ is again a convex functional defined on $L^1_+(\mathbb{R}^3)$ by

$$\rho \mapsto \int_{\mathbb{R}^3}\big(\mu_\rho\rho - \nu(\rho)\big)\; dx = \int_{\mathbb{R}^3}\big(\rho\bar{\mu}(\rho) + \rho V(x) - \nu(\rho)\big)\; dx \;.$$

Under the constraint $\int_{\mathbb{R}^3} \rho\, dx = M$, $\overline{\mathcal{F}}$ achieves its global minimum at ρ^∞ such that

$$\bar{\mu}(\rho^\infty(x)) + V(x) = \mu^\infty \;,$$

where μ^∞ is the Lagrange multiplier implicitly defined by the constraint

$$\int_{\mathbb{R}^3} \bar{\mu}^{-1}\big(\mu^\infty - V(x)\big)\; dx = \int_{\mathbb{R}^6} f^\infty\; dv\; dx = M$$

and

$$f^\infty := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^\infty\Big) \;.$$

Observe that $\overline{\mathcal{F}}(\rho^\infty) > -\infty$ by (3.4) and (1.17). We summarize all these observations in the following result.

Lemma 4 *Let Assumptions 1-5 hold, and let f be the solution of* (1.1), (1.4), *constructed in Proposition 3. Then*

(i) the mass is preserved along the evolution

$$\int_{\mathbb{R}^6} f(x,v,t)\; dv\; dx = \int_{\mathbb{R}^6} f_I(x,v)\; dv\; dx \quad \forall\, t \in \mathbb{R}_+ \;,$$

(ii) The following estimates hold

$$-\infty < \mathcal{F}(f^\infty) \leq \mathcal{F}(G_f(.,.,t)) \leq \mathcal{F}(f(.,.,t)) \leq \mathcal{F}(f_I) < \infty \quad \forall\, t \in \mathbb{R}_+ \;.$$

Proof. All these estimates can be justified by standard regularizations of the initial data f_I and of the potential V. Passing to the limit in the regularization parameter, the estimates on the free energy hold by semi-continuity, due to the convexity of the functional (μ has to be considered as a Lagrange multiplier for fixed (x,t) and strongly converges by standard averaging lemmata).

The conservation of mass is then a consequence of the Dunford-Pettis lemma.

Notice that as a consequence of the Dunford-Pettis lemma, any solution f is contained in a relatively weakly compact set of $L^1(\mathbb{R}^6 \times \mathbb{R}_{+,\mathrm{loc}})$. From now on, weak convergence at least means convergence in this sense.

We continue by using the entropy production (3.2) and a technical lemma based on the assumptions on the energy profile γ to prove uniform estimates on the first and second moments of the solution f. Let $f \le f^*$. Consider a partition of the support of f^* given by

$$\begin{aligned} \Omega_+(f) &:= \left\{ (x,v,t) \in \operatorname{supp} f^* \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar\mu(\rho_f(x,t)) < E_2 \right\} , \\ \Omega_0(f) &:= \left\{ (x,v,t) \in \operatorname{supp} f^* \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar\mu(\rho_f(x,t)) \ge E_2 \right\} , \end{aligned} \tag{3.5}$$

and define

$$\Omega_+^{x,t}(f) := \{ v \in \mathbb{R}^3 : (x,v,t) \in \Omega_+(f) \} \quad \text{and} \quad \Omega_0^{x,t}(f) := \{ v \in \mathbb{R}^3 : (x,v,t) \in \Omega_0(f) \} . \tag{3.6}$$

Lemma 5 *Let Assumptions 1–3 hold. Then, for any nonnegative function $f(x,v,t) \le f^*(x,v)$ there exists a constant $\mathcal{M}$, which does not depend on x and t, such that*

$$\int_{\Omega_+^{x,t}} |v|^{2m} \frac{\gamma(E_f) - f}{\gamma^{-1}(f) - E_f} \, dv \le \mathcal{M} ,$$

for $m = 1, 2$, and $\Omega_+^{x,t} := \Omega_+^{x,t}(f)$.

Proof. The proof relies on the sign of γ'' in $(\hat E, E_2)$ (see Assumption 1). Observe that $f^*(x,v) > \gamma(\hat E)$ implies $|v|^2 < 2(\hat E + \mu^*)$.

Assume first that γ is convex on $(\hat E, E_2)$. We first use the mean value theorem:

$$\begin{aligned} - \int_{\Omega_+^{x,t} \cap \{ f^* > \gamma(\hat E)\}} |v|^{2m} \gamma'(\tilde E) \, dv &\le \sup_{-\mu^* \le E \le \hat E} \big(-\gamma'(E) \big) \int_{\Omega_+^{x,t} \cap \{ f^* > \gamma(\hat E)\}} |v|^{2m} \, dv \\ &\le \sup_{-\mu^* \le E \le \hat E} \big(-\gamma'(E) \big) \, \frac{4\pi}{3} \big(2(\hat E + \mu^*) \big)^{\frac{3}{2}+m} = C_1 . \end{aligned}$$

On $(\hat E, E_2)$, $-\gamma'$ is decreasing, and $\tilde E \ge \gamma^{-1}(f^*) = |v|^2/2 + V(x) - \mu^*$ implies

$$\begin{aligned} & - \int_{\Omega_+^{x,t} \cap \{ f^* \le \gamma(\hat E)\}} v_i^{2m} \gamma'(\tilde E) \, dv \\ & \qquad \le - \int_{\mathbb{R}^3} v_i^{2m} \gamma' \left(|v|^2/2 + V(x) - \mu^* \right) \, dv = (2m-1) \int_{\mathbb{R}^3} v_i^{2m-2} f^* \, dv \le C_2 . \end{aligned}$$

Consider now the case where γ is concave on $(\hat E, E_2)$. For fixed x and t, let $\Lambda^{x,t} := \{ v \in \mathbb{R}^3 : \max\big(f, G_f\big)(x,v,t) \le \gamma(\hat E) \}$. The function $v \mapsto -\gamma'(\gamma^{-1}(f(x,v,t)))$ is bounded on $\Omega_+^{x,t} \setminus \Lambda^{x,t}$, because γ is of class $\mathcal{C}^1$ on (E_1, E_2). We infer

$$- \int_{\Omega_+^{x,t} \setminus \Lambda^{x,t}} |v|^{2m} \gamma'(\tilde E) \, dv \le \frac{4\pi}{3} \big(2(\hat E + \mu^*) \big)^{\frac{3}{2}+m} \sup_{\Omega_+^{x,t} \setminus \Lambda^{x,t}} \big(-\gamma'(\tilde E(x,.,t)) \big) = C_1 .$$

Observe that if γ is concave on $(\hat E, E_2)$, then E_2 is finite. The function

$$s \mapsto \chi(s) := \frac{E_2 - \gamma^{-1}(s)}{s}$$

is an increasing and positive function on $(0, \gamma(\hat E)) \subset \mathbb{R}$. This allows us to write

$$\begin{aligned} \frac{\gamma^{-1}(f) - E_f}{G_f - f} &= \frac{E_2 - E_f}{G_f} + \frac{f}{G_f - f} \left(\frac{E_2 - E_f}{G_f} - \frac{E_2 - \gamma^{-1}(f)}{f} \right) \\ &= \chi(G_f) + f \, \frac{\chi(G_f) - \chi(f)}{G_f - f} \ge \chi(G_f) > 0 . \end{aligned}$$

We conclude by making use of Assumptions 3, 4, and 5:

$$\begin{aligned}\int_{\Omega_+^{x,t}\cap\Lambda^{x,t}} v_i^{2m}\frac{G_f-f}{\gamma^{-1}(f)-E_f}\,dv &\le \int_{\Omega_+^{x,t}\cap\Lambda^{x,t}} v_i^{2m}\frac{G_f}{E_2-E_f}\,dv\\ &\le C\int_{\Omega_+^{x,t}\cap\Lambda^{x,t}} v_i^{2m}\frac{(E_2-E_f)^k}{E_2-E_f}\,dv = C\int_{\Omega_+^{x,t}\cap\Lambda^{x,t}} v_i^{2m-1}\frac{(-1)}{k}\frac{\partial}{\partial v_i}\left[(E_2-E_f)^k\right]\,dv\\ &\le C\int_{\mathbb{R}^3} v_i^{2m-2}(E_2-E_f)_+^k\,dv \le C(E_2+\mu^*)^{k+\frac12+m}\le C_2\ ,\end{aligned}$$

where we used C to represent various constants.

Starting to investigate the limit $\varepsilon\to 0$, we denote the solution of (1.1) by f^ε from now on. With the definition (3.5), the equation (3.2) can be rewritten as

$$O(\varepsilon^2) = \int_{\Omega_+(f^\varepsilon)} \underbrace{\big(\gamma(E_{f^\varepsilon})-f^\varepsilon\big)\big(\gamma^{-1}(f^\varepsilon)-E_{f^\varepsilon}\big)}_{\ge 0}\,dx\,dv\,dt + \int_{\Omega_0(f^\varepsilon)} f^\varepsilon\big(\underbrace{E_{f^\varepsilon}-E_2}_{\ge0}+\underbrace{E_2-\gamma^{-1}(f^\varepsilon)}_{\ge0}\big)\,dx\,dv\,dt\ . \tag{3.7}$$

Let us define the scaled flux and nonequilibrium part of the stress tensor,

$$j^\varepsilon := \frac{1}{\varepsilon}\int_{\mathbb{R}^3} vf^\varepsilon\,dv = \int_{\mathbb{R}^3} v\,\frac{f^\varepsilon-G_{f^\varepsilon}}{\varepsilon}\,dv \quad\text{and}\quad \kappa^\varepsilon := \int_{\mathbb{R}^3} v\otimes v\,\frac{f^\varepsilon-G_{f^\varepsilon}}{\varepsilon}\,dv\ . \tag{3.8}$$

Lemma 6 *Let Assumptions 1-5 hold and let $U\subset\mathbb{R}^3\times[0,T)$ be open and bounded. Then there are two constants $\mathcal{M}_U^1$ and $\mathcal{M}_U^2$, which do not depend on ε, such that*

$$\|j^\varepsilon\|_{L^2_{x,t}(U)}\le\mathcal{M}_U^1 \quad\text{and}\quad \|\kappa^\varepsilon\|_{L^2_{x,t}(U)}\le\mathcal{M}_U^2\ .$$

Proof. We have to verify that, for $m\in\{1,2\}$,

$$\int_U\left(\int_{\mathbb{R}^3}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2 dx\,dt \tag{3.9}$$

is bounded uniformly in ε. Depending on the function f^ε we split the domains of integration by making use of the respective sets defined in (3.6):

$$\begin{aligned}&\int_U\left(\int_{\mathbb{R}^3}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2 dx\,dt\\ &\qquad\le 2\int_U\left(\int_{\Omega_+^{x,t}}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2+\left(\int_{\Omega_0^{x,t}}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2 dx\,dt\ .\end{aligned} \tag{3.10}$$

We then use Lemma 5 and (3.7) to estimate the first term:

$$\begin{aligned}&\int_U\left(\int_{\Omega_+^{x,t}}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2 dx\,dt\\ &\quad\le\int_U\left(\int_{\Omega_+^{x,t}}|v|^{2m}\frac{G_{f^\varepsilon}-f^\varepsilon}{\gamma^{-1}(f^\varepsilon)-E_{f^\varepsilon}}\,dv\right)\left(\frac{1}{\varepsilon^2}\int_{\Omega_+^{x,t}}(G_{f^\varepsilon}-f^\varepsilon)(\gamma^{-1}(f^\varepsilon)-E_{f^\varepsilon})\,dv\right)dx\,dt\le C_1\ .\end{aligned} \tag{3.11}$$

On Ω_0 the local equilibrium satisfies $G_{f^\varepsilon}=0$ by definition, and Ω_0 is nonempty only if $E_2<\infty$, implying that suppf^* and, thus, Ω_0 is bounded in the velocity direction:

$$|v|^2\le 2(E_2+\mu^*)\quad\text{for } (x,v,t)\in\Omega_0\ .$$

Hence we need to prove that

$$\int_U\left(\int_{\Omega_0^{x,t}}|v|^m\frac{|f^\varepsilon-G_{f^\varepsilon}|}{\varepsilon}\,dv\right)^2 dx\,dt\le C\int_U\left(\int_{\Omega_0^{x,t}}\frac{f^\varepsilon}{\varepsilon}\,dv\right)^2 dx\,dt \tag{3.12}$$

is bounded uniformly in ε. Let us define

$$g(x,t) := \int_{\Omega_0^{x,t}} (E_{f^\varepsilon} - E_2)\, f^\varepsilon(x,v,t)\; dv\;,$$

such that and $0 < g(x,t)$ and $\int_U g\, dx\, dt = O(\varepsilon^2)$ by (3.7) and, for some $A > 0$ that we will finally choose large enough, let

$$U_A := \big\{(x,t) \in U : g(x,t) < A\big\}\;.$$

We split again the domain of integration in (3.12) according to

$$\int_U \left(\int_{\Omega_0^{x,t}} f^\varepsilon\; dv\right)^2 dx\; dt \le \int_{U_A} \left(\int_{\Omega_0^{x,t}} f^\varepsilon\; dv\right)^2 dx\; dt + \int_{U\setminus U_A} \left(\int_{\Omega_0^{x,t}} f^\varepsilon\; dv\right)^2 dx\; dt\;.$$

Let $(x,t) \in U_A$ and let $R > 0$ be some radius, for which will choose a specific value later on. We estimate

$$\int_{\Omega_0^{x,t}} f^\varepsilon\; dv \le \int_{0<E_{f^\varepsilon}-E_2<R} \|f^\varepsilon\|_{L^\infty}\; dv + \frac{1}{R}\int_{\Omega_0^{x,t}} (E_{f^\varepsilon} - E_2)\, f^\varepsilon\; dv\;. \tag{3.13}$$

The volume of the set $\{v \in \mathbb{R}^3 : 0 < E_{f^\varepsilon} - E_2 < R\}$ is given by

$$\begin{aligned}\phi_{\rho^\varepsilon}(R) &:= \mathrm{meas}\Big\{v \in \mathbb{R}^3 : 0 < E_{f^\varepsilon} - E_2 < R\Big\}\\ &= \mathrm{meas}\left\{v \in \mathbb{R}^3 : \sqrt{2\big(E_2 + \bar\mu(\rho_{f^\varepsilon})\big)} < |v| < \sqrt{2\big(R + E_2 + \bar\mu(\rho_{f^\varepsilon})\big)}\right\}\;.\end{aligned}$$

Observe that for fixed $R > 0$ $\rho \mapsto \phi_{\rho^\varepsilon}(R)$ is a monotone function and conclude

$$\phi_{\rho^\varepsilon}(R) \le \phi_{\bar\mu^{-1}(\mu^*)}(R) = \frac{4\pi}{3}\Big(2(E_2+\mu^*)\Big)^{3/2}\left(\Big(1 + \frac{R}{E_2+\mu^*}\Big)^{3/2} - 1\right) =: \psi(R)\;,$$

where that function $\psi(R)$ is given by $\psi(R) = C_1((1 + C_2 R)^{3/2} - 1)$ with constants $C_1 > 0$ and $C_2 > 0$ since $E_2 + \mu^* > 0$. Otherwise the support of f^* would indeed be empty.

From (3.13) we obtain

$$\int_{\Omega_0^{x,t}} f^\varepsilon\; dv \le C\frac{\psi(R)}{R}\, R + \frac{1}{R}\, g(x,t) = h(R)\;, \tag{3.14}$$

for a positive constant C where the function h is given by

$$h(r) := C\frac{\psi(R)}{R}\, r + \frac{1}{r}\, g(x,t)\;.$$

This convex function assumes its minium at $\bar r = \sqrt{\frac{g(x,t)\, R}{C\, \psi(R)}}$. The minimal value is given

$$h\,(\bar r) = 2\sqrt{\frac{g(x,t)\, \Psi(R)\, C}{R}}\;. \tag{3.15}$$

Now we choose R to be given by a solution of the equation

$$R = \bar r = \sqrt{\frac{g(x,t)\, R}{C\, \psi(R)}}\;, \tag{3.16}$$

which exists, since $\lim_{R\to 0}\sqrt{\frac{g(x,t)\, R}{C\, \psi(R)}} - R > 0$ and $\lim_{R\to\infty}\sqrt{\frac{g(x,t)\, R}{C\, \psi(R)}} - R = -\infty$ for fixed $g(x,t) > 0$. Observe that $\psi(R)/R \ge C_1\, C_2^{3/2}\sqrt{R}$ for any $R > 0$. Using this estimate in (3.16) yields

$$R^{5/4} \le C\,\sqrt{g(x,t)}$$

for a constant $C > 0$. From (3.14) and (3.15) and by using the assumption $g(x,t) < A$ we infer

$$\int_{\Omega_0^{x,t}} f^\varepsilon\; dv \le \sqrt{\frac{g(x,t)\, \Psi(R)}{R}} \le C\sqrt{g(x,t)}\, R^{\frac14} \le C\,\sqrt{g(x,t)}\, g(x,t)^{\frac{1}{10}} \le C\,\sqrt{g(x,t)}\;,$$

where C represents various positive constants.

In the case $(x,t) \in U \setminus U_A$ we estimate $\int_{U\setminus U_A}\left(\int_{\Omega_0^{x,t}} f^\varepsilon \, dv\right) dx\, dt$ by choosing A larger than the bound on ρ given by

$$\left(\int_{\Omega_0^{x,t}} f^\varepsilon \, dv\right)^2 \leq \rho^\varepsilon(x,t)^2 \leq \|\rho^\varepsilon\|_{L^\infty}^2 \leq A \leq g(x,t) \ .$$

Finally by (3.7) we obtain

$$\begin{aligned}\int_U \left(\int_{\Omega_0^{x,t}} f^\varepsilon \, dv\right)^2 dx\ dt &\leq \int_{U_A} Cg\ dx\ dt + \int_{U\setminus U_A} g\ dx\ dt \\ &\leq C \int_U g\ dx\ dt = C\int_U (E_{f^\varepsilon} - E_2)\, f^\varepsilon(x,v,t)\ dx\ dt = O(\varepsilon^2) \ .\end{aligned}$$

Hence (3.12) is bounded uniformly in ε and by combining this with (3.11) and (3.10) we conclude that this also applies to (3.9). The result follows now directly or, for the mixed second moments, by application of the Cauchy-Schwarz inequality.

Before proving rigorously the macroscopic limit in Proposition 10, we will make preparatory statements on second moments of local equilibria, the derivative of which will turn out to represent the diffusivity, and on the strong convergence of the macroscopic density ρ^ε.

Proposition 7 *Under Assumptions* (1.14), (1.15) *and* (1.16), *there exists ρ^0 such that, up to the restriction to a subsequence, $\rho^\varepsilon \to \rho^0$ in L^p_{loc} strongly for all $p \in (1,\infty)$.*

Proof. The proof relies on the Div-Curl Lemma in a similar way as in [21]. By integrating the kinetic equation (1.1) with respect to dv and $v\,dv$ respectively, we obtain the following system

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0 \ , \\ \varepsilon^2\, \partial_t j^\varepsilon + \nabla_x \cdot \displaystyle\int_{\mathbb{R}^3} v \otimes v\, f^\varepsilon\ dv = -j^\varepsilon - \rho^\varepsilon\, \nabla_x V \ . \end{cases} \tag{3.17}$$

Now we split the second moments of f^ε into an equilibrium part and a perturbation,

$$\int_{\mathbb{R}^3} v\otimes v\, f^\varepsilon\ dv = \int_{\mathbb{R}^3} v\otimes v\, G_{f^\varepsilon}\ dv + \int_{\mathbb{R}^3} v \otimes v\,(f^\varepsilon - G_{f^\varepsilon})\ dv = \nu(\rho^\varepsilon)\, I^{3\times 3} + \varepsilon\kappa^\varepsilon \ ,$$

where we used the result of Lemma 1. We use this decomposition to rewrite the system (3.17) as a system of four scalar equations.

$$\begin{cases} \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0 \ , \\ \nabla_x\, \nu(\rho^\varepsilon) = -j^\varepsilon - \rho^\varepsilon\, \nabla_x V - \varepsilon \nabla_x \cdot \kappa^\varepsilon - \varepsilon^2\, \partial_t j^\varepsilon \ . \end{cases} \tag{3.18}$$

We apply the Div-Curl Lemma to

$$U^\varepsilon := (\rho^\varepsilon, j^\varepsilon), \quad V^\varepsilon := (\nu(\rho^\varepsilon), 0, 0, 0) \ .$$

With these definitions and the convention $(\mathrm{curl}\, w)_{ij} = w^i_{x_j} - w^j_{x_i}$ (3.18) becomes

$$\begin{cases} \mathrm{div}_{t,x} U^\varepsilon = 0 \ , \\ (\mathrm{curl}_{t,x} V^\varepsilon)_{1,2\ldots 4} = -j^\varepsilon - \rho^\varepsilon\, \nabla_x V - \varepsilon \nabla_x \cdot \kappa^\varepsilon - \varepsilon^2\, \partial_t j^\varepsilon \ . \end{cases}$$

By assumption (1.16), Proposition 3 and Lemma 6, $\rho^\varepsilon\, \nabla_x V$ and j^ε are bounded in $L^{2,\mathrm{loc}}_{x,t}$ and therefore precompact in $H^{-1,\mathrm{loc}}_{x,t}$. Also by Lemma 6, $\varepsilon\, \kappa^\varepsilon$ and $\varepsilon^2\, j^\varepsilon$ are compact in $L^{2,\mathrm{loc}}_{x,t}$ and their derivatives are therefore compact in $H^{-1,\mathrm{loc}}_{x,t}$.

Due to Proposition 3, the family ρ^ε is weakly * compact in $L^{\infty,\mathrm{loc}}_{x,t}$ and hence weakly compact in $L^{2,\mathrm{loc}}_{x,t}$. The same applies to $\nu(\rho^\varepsilon)$ and $\rho\nu(\rho^\varepsilon)$ as ν is a continuous function. Let $(\varepsilon_i)_{i\in\mathbb{N}}$ be a sequence converging to 0 such that ρ^{ε_i} and $\nu(\rho^{\varepsilon_i})$ weakly converge in $L^{2,\mathrm{loc}}_{x,t}$ and $\rho^{\varepsilon_i}\nu(\rho^{\varepsilon_i})$ converges weak * in $L^{\infty,\mathrm{loc}}_{x,t}$ and consider the corresponding limits for U^{ε_i} and V^{ε_i}:

$$U^{\varepsilon_i} \rightharpoonup (\overline{\rho}, \overline{j}) \quad \text{and} \quad V^{\varepsilon_i} \rightharpoonup (\overline{\nu}, 0, 0, 0) \quad \text{in} \quad L^{2,\mathrm{loc}}_{x,t} \quad \text{as} \quad i \to \infty \ .$$

The Div-Curl Lemma (see [15]) states the following:

Let $U \subseteq \mathbb{R}^m$ be an open, bounded and smooth set and let $(v_k)_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$ be two bounded sequences in $L^2(U,\mathbb{R}^n)$, such that the sequences $(\operatorname{div} v_k)_{k\in\mathbb{N}}$ and $(\operatorname{curl} w_k)_{k\in\mathbb{N}}$ lie in compact subsets of $H^{-1}(U)$ and $H^{-1}(U,\mathbb{R}^{n\times n})$ respectively. Then the sequence of scalar products $(v_k\cdot w_k)_{k\in\mathbb{N}}$ converges to $v\cdot w$ in the sense of distributions, where v and w are the weak limits in $L^2(U,\mathbb{R}^n)$ of $(v_k)_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathbb{N}}$.

Due to the considerations above we obtain from the application of the Div-Curl Lemma that $U^{\varepsilon_i}\cdot V^{\varepsilon_i} = \rho^{\varepsilon_i}\nu(\rho^{\varepsilon_i}) \to \overline{\rho}\,\overline{\nu}$ in the sense of distributions. By uniqueness of the limit in $\mathcal{D}'_{x,t}$ and the weak * limit in $L^{\infty,\mathrm{loc}}_{x,t}$ we get

$$\rho^{\varepsilon_i}\nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\rho}\,\overline{\nu} \quad \text{in} \quad L^{\infty,\mathrm{loc}}_{x,t} \quad \text{as} \quad i\to\infty\,. \tag{3.19}$$

From this identity we shall conclude, in a similar way as in [25], that the convergence of ρ^{ε_i} is strong. In [25] the proof of strong convergence relies on strict convexity. We will instead make use of the strict monotonicity of the function ν, which replaces, in this respect, strict convexity.

We associate the Young measure family $\eta_{x,t}$ to the weak * convergence of ρ^{ε_i} in $L^{\infty,\mathrm{loc}}_{x,t}$ and obtain

$$\begin{cases} \nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\nu} = \displaystyle\int_0^{\rho^{\max}} \nu(\rho)\, d\eta_{x,t}(\rho)\,, \\ \rho^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\rho} = \displaystyle\int_0^{\rho^{\max}} \rho\, d\eta_{x,t}(\rho)\,, \\ \rho^{\varepsilon_i}\,\nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \displaystyle\int_0^{\rho^{\max}} \rho\,\nu(\rho)\, d\eta_{x,t}(\rho) = \overline{\rho}\,\overline{\nu}\,, \end{cases} \tag{3.20}$$

where the final identity is due to (3.19). Now observe that

$$\nu(\rho) = \nu(\overline{\rho}) + \nu'(\tilde{\rho})(\rho-\overline{\rho})$$

for some $\tilde{\rho}\in(0,\rho^{\max})$. From this and (3.20) we conclude

$$0 = \int_0^{\rho^{\max}} \nu(\rho)(\rho-\overline{\rho})\, d\eta_{x,t}(\rho) = \underbrace{\int_0^{\rho^{\max}} \nu(\overline{\rho})(\rho-\overline{\rho})\, d\eta_{x,t}(\rho)}_{=0}$$
$$+ \int_0^{\rho^{\max}} \nu'(\tilde{\rho})(\rho-\overline{\rho})^2\, d\eta_{x,t}(\rho) \ge C\int_0^{\rho^{\max}} (\rho-\overline{\rho})^2\, d\eta_{x,t}(\rho)\,.$$

Here we relied on the second alternative in Lemma 1, which states that $\nu'(\rho)$ assumes its minimum $C>0$ on $[0,\rho^{\max}]$. The measure $\eta_{x,t}$ is therefore the Dirac point mass $\delta(\rho-\overline{\rho})$, which means that the sequence ρ^{ε_i} strongly converges.

In the alternative case, in which $\nu'(\rho)$ is bounded from above on $[0,\rho^{\max}]\ni\rho$, we have to go back to (3.20). Observe that $(\nu^{-1})'(p) = 1/\nu'(\nu^{-1}(p))$ will be bounded from below on the interval $[0,\nu(\rho^{\max})]$ and define $\pi^{\varepsilon_i} := \nu(\rho^{\varepsilon_i})$. By rewriting (3.20) we infer

$$\begin{cases} \pi^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\nu} = \displaystyle\int_0^{\nu(\rho^{\max})} \pi\, d\mu_{x,t}(\pi)\,, \\ \nu^{-1}(\pi^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\rho} = \displaystyle\int_0^{\nu(\rho^{\max})} \nu^{-1}(\pi)\, d\mu_{x,t}(\pi)\,, \\ \nu^{-1}(\pi^{\varepsilon_i})\,\pi^{\varepsilon_i} \overset{*}{\rightharpoonup} \displaystyle\int_0^{\nu(\rho^{\max})} \pi\,\nu^{-1}(\pi)\, d\mu_{x,t}(\pi) = \overline{\rho}\,\overline{\nu}\,, \end{cases}$$

where $\mu_{x,t}$ is the Young measure associated to the weak convergence of $(\pi^{\varepsilon_i})_i$.

In the same way as above for the measure $\eta_{x,t}$, we conclude that $\mu_{x,t}$ is a point mass. Therefore $(\nu(\rho^{\varepsilon_i}))_i$ strongly converges and by continuity of ν^{-1} this also applies to $(\rho^{\varepsilon_i})_i$.

As in Section 1, we denote by G_f the local Gibbs state associated to f: $G_f(x,v,t) = \gamma(|v|^2/2 - \bar{\mu}(\rho_f(x,t)))$.

Lemma 8 *Let f^ε be the solution of* (1.1) *and denote by f^0 its weak limit as $\varepsilon\to 0$. Under the assumptions of Proposition 7, $f^0 = G_{f^0}$.*

Proof. First we show that the differences $f^\varepsilon - G_{f^\varepsilon}$, up to the extraction of a subsequence, converge to zero a.e.: The relation (3.2) implies that a.e. at least one of the two factors in $(\gamma(E_{f^\varepsilon})-f^\varepsilon)(\gamma^{-1}(f^\varepsilon)-E_{f^\varepsilon})$ converges to zero as $\varepsilon \to 0$.

If the first factor converges to zero, then the result holds. Consider therefore the pointwise convergence at a point in phase space where the second factor converges but not the first one. Observe that $\gamma^{-1}(f^\varepsilon) - E_{f^\varepsilon} \to 0$ implies $\gamma^{-1}(f^\varepsilon) - \min(E_{f^\varepsilon}, E_2) \to 0$ because $\gamma^{-1}(g) < E_2$ for all $g > 0$. As the energy profile γ is a diffeomorphism on (E_1, E_2), we obtain

$$-(\gamma^{-1})'(\bar{f}^\varepsilon)(\gamma(\min(E_{f^\varepsilon}, E_2)) - f^\varepsilon) \to 0$$

for some mean value $\bar{f}^\varepsilon$. Remember that γ' is bounded on compact subintervals of (E_1, E_2) and that we evaluate at a point where $|\gamma(E_{f^\varepsilon}) - f^\varepsilon| \geq \delta_1$ for some $\delta_1 > 0$ as $\varepsilon \to 0$. Hence the mean value $\bar{f}^\varepsilon \in (\min(\gamma(E_{f^\varepsilon}), f^\varepsilon), \max(\gamma(E_{f^\varepsilon}), f^\varepsilon))$ stays strictly away from zero, $\bar{f}^\varepsilon > \delta_2$ as $\varepsilon \to 0$ and therefore $-(\gamma^{-1})'(\bar{f}^\varepsilon)$ has a strictly positive lower bound. This yields

$$f^\varepsilon - \gamma(\min(E_{f^\varepsilon}, E_2)) = f^\varepsilon - \gamma(E_{f^\varepsilon}) = f^\varepsilon - G_{f^\varepsilon} \to 0 \,.$$

Summarizing we obtain $f^\varepsilon - G_{f^\varepsilon} \to 0$ a.e. This together with Proposition 7 and the continuity of the mapping $r \to \gamma(|v|^2/2 - \bar{\mu}(r))$ implies the result.

Lemma 9 *Let j^ε be the perturbation of the first moment as defined in* (3.8)*. Then j^ε converges to j^0 in $\mathcal{D}'_{x,t}$, where the limit j^0 is given by*

$$j^0 = -\nabla_x \nu(\rho^0) - \rho^0 \nabla_x V \,.$$

Proof. We define

$$r^\varepsilon := \frac{1}{\varepsilon}\Big(f^\varepsilon - \gamma\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho^\varepsilon)\Big)\Big) \,.$$

After multiplication by ε^{-1}, the left hand side of (1.1) weakly converges to

$$v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = v \cdot \nabla_x G_{f^0} - \nabla_x V \cdot \nabla_v G_{f^0} \,,$$

where we used Lemma 8. As the right hand side of (1.1) can be written as $-\varepsilon\, r^\varepsilon$, we infer that r^ε weakly converges to

$$r^0 = \Big(\nabla_x V \cdot \nabla_v G_{f^0} - v \cdot \nabla_x G_{f^0}\Big) \,.$$

Now observe that

$$\int_{\mathbb{R}^3} v\, r^0 \, dv = -\Big(\rho^0 \,\nabla_x V + \nabla_x \nu(\rho^0)\Big) = j^0 \,.$$

To prove the convergence $j^\varepsilon \to j^0$ in $\mathcal{D}'_{x,t}$, let $\phi \in \mathcal{D}_{x,t}$. For $R > 0$, we choose $\psi_R \in \mathcal{C}_c^\infty(\mathbb{R}^3_v)$ with $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $B_R(0)$ and supp $\psi_R \subset B_{R+1}(0)$ and obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int_{\mathbb{R}^4} \phi\,(j^\varepsilon - j^0)\, dx\, dt &= \lim_{\varepsilon\to 0} \iiint_{\mathbb{R}^7} \phi\, v\,(r^\varepsilon - r^0)\, dv\, dx\, dt \\
&\leq \lim_{R\to\infty} \lim_{\varepsilon\to 0} \Bigg(\Bigg| \iiint_{\mathbb{R}^7} \phi\, \psi_R(v) v (r^\varepsilon - r^0)\, dv\, dx\, dt \Bigg| \\
&\qquad + \frac{3}{R} \iiint_{\mathbb{R}^7} (1 - \psi_R(v)) |\phi|\, |v|^2\, |r^\varepsilon|\, dv\, dx\, dt + \Bigg| \iiint_{\mathbb{R}^7} \phi\,(1 - \psi_R(v))\, v\, r^0\, dv\, dx\, dt \Bigg| \Bigg) \\
&= \lim_{R\to\infty} \Bigg(0 + \frac{3}{R} \sqrt{\mathrm{meas}(\mathrm{supp}\,\phi)}\, \mathcal{M}^2_{\mathrm{supp}\,\phi} + \Bigg| \iiint_{\mathbb{R}^7} \phi\,(1 - \psi_R(v))\, v\, r^0\, dv\, dx\, dt \Bigg| \Bigg) = 0 \,,
\end{aligned}$$

where we used the convergence $r^\varepsilon \to r^0$ in $\mathcal{D}'_{x,v,t}$, the uniform boundedness of κ^ε by Lemma 6 together with an interpolation and the convergence of the last integral.

Proposition 10 *Let f^ε be the solution of* (1.1) *and denote by f^0 its weak limit as $\varepsilon \to 0$. Under the assumptions of Proposition 7, $\rho^0 := \int_{\mathbb{R}^3} f^0\, dv$ satisfies* (1.6), (1.8) *in the weak sense.*

Proof. We multiply the continuity equation in (3.17) by a test function $\phi(x,t) \in \mathcal{D}(\mathbb{R}^4)$, integrate and obtain after integration by parts

$$\int_0^\infty \int_{\mathbb{R}^3} \rho^\varepsilon \partial_t \phi + \nabla_x \phi \cdot j^\varepsilon \, dx \, dt = - \int_{\mathbb{R}^3} \phi(0,x)\, \rho_I(x) \, dx \, .$$

By Proposition 7 and Lemma 9 we can find a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ and pass to the limit as $\varepsilon_j \to 0$ in the above identity. We obtain

$$\int_0^\infty \int_{\mathbb{R}^3} \left[\rho^0\, \partial_t \phi - \rho^0\, \nabla_x V \cdot \nabla_x \phi + \nu(\rho^0) \Delta_x \phi\right] dx \, dt = - \int_{\mathbb{R}^3} \phi(0,x)\, \rho_I(x) \, dx \, ,$$

which is a weak formulation of (1.6), (1.8).

4. Examples

Example 1 *Consider the case of a power law with negative exponent, $\gamma(E) := DE^{-k}$, with $D > 0$ and $k > 5/2$. Then Assumptions 1 and 2 are obviously satisfied with $E_1 = 0$ and $E_2 = \infty$. We easily compute*

$$\bar\mu(\rho) = -\Big(\frac{\rho}{D\beta(k)}\Big)^{\frac{1}{3/2-k}}, \quad \text{where} \quad \beta(k) := 4\pi\sqrt{2}\int_0^\infty \frac{\sqrt{s}}{(s+1)^k} \, ds \, .$$

Substituting this expression in (1.6) yields

$$\partial_t \rho = \nabla \cdot \left(\Theta\, \nabla(\rho^{\frac{k-5/2}{k-3/2}}) + \rho\, \nabla V\right), \quad \text{where} \quad \Theta := \frac{1}{k-5/2}(D\beta(k))^{\frac{1}{k-3/2}} \, .$$

Since $0 < \frac{k-5/2}{k-3/2} < 1$ this is a fast diffusion equation. The confinement condition (1.17) in Assumption 4 is satisfied if, outside of a finite ball, the potential grows faster than a certain power:

$$V(x) \ge C|x|^q, \quad \text{for } |x| > R\,, \quad \text{with } q > \frac{3}{k-5/2} \, .$$

The same threshold was obtained in [9] (HV 6, remark 16, e) in order to guarantee there that the entropy is finite.

Example 2 *For the Maxwellian distribution, $\gamma(E) = \exp(-E)$, we compute*

$$\bar\mu(\rho) = \log \rho - \frac{3}{2}\log(2\pi) \, .$$

and, thus, the linear drift-diffusion equation

$$\partial_t \rho = \nabla \cdot \big(\nabla \rho + \rho\, \nabla V\big) \, .$$

By making use of (3.4) we conclude that the growth assumption on the potential in (1.17) is satisfied if

$$V(x) \ge q \log(|x|), \quad \text{for } |x| > R\,, \quad \text{with } q > 3 \, .$$

Example 3 *Let γ be a cut-off power with positive exponent:*

$$\gamma(E) = (E_2 - E)_+^k \, ,$$

with $D, k > 0$. We obtain

$$\bar\mu(\rho) = \left(\frac{\rho}{D\,\alpha(k)}\right)^{\frac{1}{k+3/2}} - E_2 \, , \quad \text{where} \quad \alpha(k) = 4\pi\sqrt{2}\int_0^1 \sqrt{u}(1-u)^k \, du \, .$$

and, thus, the porous medium type macroscopic equations

$$\partial_t \rho = \nabla \cdot \left(\Theta\, \nabla\left(\rho^{\frac{k+5/2}{k+3/2}}\right) + \rho\, \nabla V\right), \quad \text{where} \quad \Theta := \frac{1}{k+5/2}(D\,\alpha(k))^{\frac{-1}{k+3/2}} \, ,$$

with the exponent satisfying

$$1 < \frac{k+5/2}{k+3/2} < \frac{5}{3} \, .$$

The confinement assumption in (1.17) *is satisfied if*

$$\big(E_2 + \mu^* - V(x)\big)_+ = O(|x|^{-q}) \quad \text{as } |x| \to \infty, \quad \text{with } q > \frac{3}{k+3/2}\ ,$$

where μ^* *is the upper bound for the Fermi energy in assumption* (1.15)*. This assumption is satisfied if* $V \geq E_2 + \mu^*$ *outside of a compact set. In this situation, our results guarantee that the supports of both the kinetic as well as of the macroscopic solutions remain in fixed compact sets for all times. At the macroscopic level, this behaviour of the porous medium equation is well known.*

In the following two examples, we will use the polylogarithmic function

$$\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}\ ,$$

which satisfies, for $\sigma = \pm 1$,

$$\int_0^\infty \frac{k^s\,dk}{\exp(k-\nu)-\sigma} = \sigma \Gamma(s+1)\,\mathrm{Li}_{1+s}(\sigma\exp(\nu)) \quad \text{and} \quad \frac{d}{dz}\,\mathrm{Li}_n(z) = \frac{1}{z}\,\mathrm{Li}_{n-1}(z)\ .$$

Example 4 *Let* γ *be the energy profile of the Fermi-Dirac distribution,* $\gamma(E) = \frac{1}{\exp(E)+\alpha}$*. We obtain*

$$(\bar\mu^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)+1} = -\frac{(2\pi)^{3/2}}{\alpha}\,\mathrm{Li}_{3/2}\big(-\alpha e^\theta\big)\ ,$$

and therefore

$$\bar\mu(\rho) = \log\left(\frac{-\big(\mathrm{Li}_{3/2}^{-1}\big)\big(\frac{-\alpha\rho}{(2\pi)^{3/2}}\big)}{\alpha}\right)\ .$$

As macroscopic equation we obtain

$$\partial_t\rho = \nabla\cdot\Big(D(\rho)\,\nabla\rho + \rho\,\nabla V\Big)\ ,$$

where, by Lemma 1, the diffusivity $D(\rho)$ *is given by*

$$D(\rho) = \nu'(\rho) = \rho\,\bar\mu'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}}\,\frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{-\alpha\rho}{(2\pi)^{3/2}})\big)}\ .$$

Moreover the expansion of $D(\rho)$ *at* $\rho = 0$ *gives*

$$D(\rho) = 1 + \frac{\sqrt{2}}{4}\frac{\alpha\,\rho}{(2\pi)^{3/2}} + \left(\frac{3}{8} - \frac{2\sqrt{3}}{9}\right)\frac{\alpha^2\rho^2}{(2\pi)^3} + O(\rho^3)\ .$$

Example 5 *Let* γ *be the energy profile of the Bose-Einstein distribution,* $\gamma(E) = \frac{1}{\exp(E)-\alpha}$*. It satisfies Assumptions 1-3 with* $E_1 = \log(\alpha)$*,* $E_2 = \infty$*. We obtain*

$$(\bar\mu^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)-1} = \frac{(2\pi)^{3/2}}{\alpha}\,\mathrm{Li}_{3/2}\big(\alpha e^\theta\big)\ ,$$

and therefore

$$\bar\mu(\rho) = \log\left(\frac{\big(\mathrm{Li}_{3/2}^{-1}\big)\big(\frac{\alpha\rho}{(2\pi)^{3/2}}\big)}{\alpha}\right)\ .$$

As macroscopic equation we obtain

$$\partial_t\rho = \nabla\cdot\Big(D(\rho)\,\nabla\rho + \rho\,\nabla V\Big)\ ,$$

where the diffusivity $D(\rho)$ *is given by*

$$D(\rho) = \nu'(\rho) = \rho\bar\mu'(\rho) = \frac{\alpha}{(2\pi)^{3/2}}\,\frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{\alpha\rho}{(2\pi)^{3/2}})\big)}\ .$$

The maximal density $\bar\rho$ *as defined in* (1.13) *is finite in the case of the Bose-Einstein distribution and given by*

$$\overline{\rho} = \frac{(2\pi)^{3/2}\zeta(3/2)}{\alpha} \approx \frac{41.144\ldots}{\alpha}\ .$$

with the Riemann Zeta function given by $\zeta(s) := \mathrm{Li}_s(1) = \sum_{k=1}^\infty \frac{1}{k^s}$*. Observe that* $\lim_{\rho\to\overline{\rho}}\nu'(\rho) = 0$ *and* $\lim_{\rho\to 0}\nu'(\rho) = 1$*.*

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