The two-dimensional Keller-Segel model after blow-up

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Abstract. In the two-dimensional Keller-Segel model for chemotaxis of biological cells, blow-up of solutions in finite time occurs if the total mass is above a critical value. Blow-up is a concentration event, where point aggregates are created. In this work global existence of generalized solutions is proven, allowing for measure valued densities. This extends the solution concept after blow-up. The existence result is an application of a theory developed by Poupaud, where the cell distribution is characterized by an additional defect measure, which vanishes for smooth cell densities. The global solutions are constructed as limits of solutions of a regularized problem.

A strong formulation is derived under the assumption that the generalized solution consists of a smooth part and a number of smoothly varying point aggregates. Comparison with earlier formal asymptotic results shows that the choice of a solution concept after blow-up is not unique and depends on the type of regularization.

This work is also concerned with local density profiles close to point aggregates. An equation for these profiles is derived by passing to the limit in a rescaled version of the regularized model. Solvability of the profile equation can also be obtained by minimizing a free energy functional.

Key words: Keller-Segel, chemotaxis, aggregation, blow-up, measure valued solutions, defect measure

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1 Introduction

The simplest description of a cell population, which produces a chemical signal and responds to it chemotactically, is the Keller-Segel model

$$\partial_t \varrho + \nabla \cdot (\varrho \nabla S - \nabla \varrho) = 0\,, \tag{1}$$

$$-\Delta S = \varrho\,. \tag{2}$$

The model is written in nondimensionalized form with the cell density $\varrho(t,x)$ and the chemical concentration $S(t,x)$ both depending on time t and position x. The first equation is a convection-diffusion equation, where the drift term $\varrho\nabla S$ describes the chemotactic reaction of the cells, and the second equation is a quasistationary approximation of a reaction-diffusion equation for the chemical concentration. The reaction term ϱ models the production of the chemical by the cells. The model is based on the assumption that characteristic times for the dynamics of the chemical are much shorter than those for the cell dynamics. Also the chemotactic sensitivity, the cell diffusivity, the chemical diffusivity, and the reaction rate (coefficients of $\varrho\nabla S$, $\nabla\varrho$, ΔS, and, respectively, ϱ) have been assumed constant and have been removed by an appropriate scaling.

Since its first formulation [15] and first mathematical investigation [13], this model (as well as variants of it) has received a considerable amount of attention in the mathematical literature. This is caused by the interesting nonlinear effects it shows. In particular, it is well known that in general smooth solutions of initial(-boundary) value problems may only exist for finite time, with L^∞-blow-up of the cell density at the end of the existence interval. This phenomenon strongly depends on the spatial dimension. It does not occur in one-dimensional problems, and it occurs conditionally in higher dimensional situations. For the two-dimensional whole space problem and initial data ϱ_I satisfying

$$\varrho_I \in L^1_+(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\,, \qquad \int_{\mathbb{R}^2} |x|^2 \varrho_I(x)dx < \infty\,,$$

the situation has recently been clarified completely. The qualitative behaviour depends on the total mass

$$M = \int_{\mathbb{R}^2} \varrho_I \, dx\,.$$

For $M < 8\pi$, a global bounded solution of the initial value problem with $\varrho(t=0) = \varrho_I$ exists, which is dispersed for $t \to \infty$. For $M > 8\pi$, blow-up in finite time occurs. First results in this direction have been obtained in [12] and recently completed in [8] and [2]. Actually, even the critical case $M = 8\pi$ is understood now [3]: A global solution exists in this case, which possibly becomes unbounded as $t \to \infty$.

Blow-up scenarios have been investigated in [9], showing that at the blow-up time, mass concentrates in a point. Biologically, this represents aggregation of cells, and the description of the dynamics of these aggregates and of their interaction with the non-aggregated cells is of natural interest.

This led to the study of regularized models which, in some cases, can be seen as more precise descriptions of the actual biological processes. Examples are the inclusion of volume filling effects by a density dependent chemotactic sensitivity [10], [17], [18], [7], the inclusion of a finite sampling radius, which results in a regularization of the chemical concentration [11], and kinetic transport models, whose macroscopic limit is the Keller-Segel model [5]. All these models share the properties that they have global solutions and that they contain the Keller-Segel model as a formal limit.

The asymptotic behaviour of a class of regularized models with density dependent chemotactic sensitivities after blow-up has been investigated in [17], [18]. By formal asymptotics the dynamics of solutions of the regularized problem is studied under the assumption that in the limit the cell density is the sum of a smooth part and of a finite number of point aggregates, mathematically represented as Delta distributions. The result is a partial differential equation for the smooth part coupled to a system of ordinary differential equations for the dynamics of the masses and the positions of the aggregates. Well posedness of this formal limiting problem is proved locally in time. Actually, the system can only be expected to be valid on bounded time intervals between blow-up events and/or collisions of aggregates.

In this work we study the regularized model

$$\partial_t \varrho^\varepsilon + \nabla \cdot (\varrho^\varepsilon \nabla S_\varepsilon[\varrho^\varepsilon] - \nabla \varrho^\varepsilon) = 0\,, \tag{3}$$

where the Poisson equation $\Delta S = -\varrho$ is replaced by the regularized Newtonian potential solution

$$S_\varepsilon[\varrho](x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\varrho(y)dy\,. \tag{4}$$

This regularization is similar to the finite-sampling-radius model [11] mentioned above. We consider the initial value problem

$$\varrho^\varepsilon(t=0) = \varrho_I \in L^1_+(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)\,. \tag{5}$$

Our main result is a rigorous characterization of the limits of solutions as $\varepsilon \to 0$ globally in time and for arbitrary initial mass. We use the framework developed by Poupaud in [16], which he applied to the two-dimensional incompressible Euler equations as well as to the system (3)–(4) without diffusion of cells. The limit of the cell density satisfies a generalized weak formulation of the Keller-Segel model (1)–(2) allowing for measure valued cell densities. Obviously, the main mathematical problem is an appropriate definition of the only nonlinear term, the convective flux $\varrho\nabla S$. Here the fact that the spatial dimension is two, is of essential importance.

The rest of this work is structured as follows: in the following section, a priori estimates for solutions of (3)–(5) are derived and the theory from [16] is outlined. In Section 3, the limit $\varepsilon \to 0$ is carried out and the limiting problem is formulated. A strong formulation is derived under the assumption that the limiting cell density is the sum of a smooth part and a number of point aggregates. It turns out that the strong formulation is similar to the limiting model formally derived by Velázquez [17], [18], except one detail, showing that the limit actually depends on the type of regularization. The subject of Section 4 is the study of local density profiles of the regularized problem approximating point aggregates. After an appropriate rescaling, an equation for these profiles is rigorously derived. Finally, a free energy functional for the regularized problem is introduced and it is shown that solutions of the profile equation can be obtained as minimizers.

2 A priori estimates and diagonal defect measures

Theorem 1 *For every positive ε, the problem (3)–(5) has a global solution satisfying*

$$\|\varrho^\varepsilon(\cdot,t)\|_{L^1(\mathbf{R}^2)} = \|\varrho_I\|_{L^1(\mathbf{R}^2)} := M\,, \tag{6}$$

and

$$\|\varrho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbf{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)\,, \tag{7}$$

with an ε-independent constant c.

Proof: The existence of a local solutions and the mass conservation property (6) are well known results. As a consequence,

$$|\nabla S_\varepsilon[\varrho^\varepsilon](x,t)| \le \frac{1}{2\pi}\int_{\mathbf{R}^2} \frac{\varrho^\varepsilon(y,t)dy}{|x-y|+\varepsilon} \le \frac{M}{2\pi\varepsilon}$$

holds and the second a priori estimate (7) follows from Lemma 1 in [11]. Global existence is an immediate consequence. ∎

Basic and important for what follows is the following representation of the distributional interpretation of the convective flux: For $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\mathbf{R}^2} \varphi\varrho\nabla S_\varepsilon[\varrho]dx = -\frac{1}{4\pi}\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|(|x-y|+\varepsilon)}\varrho(x)\varrho(y)dx\,dy \tag{8}$$

holds, implying the uniform-in-ε estimate

$$\left|\int_{\mathbf{R}^2} \varphi\varrho^\varepsilon\nabla S_\varepsilon[\varrho^\varepsilon]dx\right| \le \frac{M^2}{4\pi}|\varphi|_{1,\infty}\,, \tag{9}$$

where $|\varphi|_{k,\infty} = \max_{k_1+k_2=k}\sup_{\mathbf{R}^2}|\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\varphi|$. The form of the integral kernel in (8) suggests to introduce the family

$$m^\varepsilon(t,x) := \int_{\mathbf{R}^2} \mathcal{K}^\varepsilon(x-y)\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)dy\,, \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x|+\varepsilon)}\,, \tag{10}$$

of matrix valued functions. Following [16] (to which we also refer to for some of the details omitted in the rest of this section), we consider $\varrho^\varepsilon(t,\cdot)$ and $m^\varepsilon(t,\cdot)$ as time dependent measures $\varrho^\varepsilon(t)$ and $m^\varepsilon(t)$.

Lemma 1 *The families $\{\varrho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\varrho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t.*

Proof: The proof is actually contained in the proof of Theorem 3.2 of [16] and repeated here only for completeness. We compute

$$\frac{d}{dt}\int_{\mathbf{R}^2} \varphi\varrho^\varepsilon\,dx = \int_{\mathbf{R}^2} \varrho^\varepsilon(\Delta\varphi + \nabla\varphi\cdot\nabla S_\varepsilon[\varrho^\varepsilon])dx\,,$$

which, by (9), can be estimated by

$$\left|\frac{d}{dt}\int_{\mathbf{R}^2} \varphi\varrho^\varepsilon\,dx\right| \le c\,|\varphi|_{2,\infty}\,.$$

with c independent of ε and t. This implies equicontinuity in $W^{2,\infty}(\mathbb{R}^2)'$:

$$\left|\int_{\mathbf{R}^2}\varphi\varrho^\varepsilon(t,x)\,dx - \int_{\mathbf{R}^2}\varphi\varrho^\varepsilon(s,x)\,dx\right| \le C(\varphi)|t-s|\,.$$

Now let $\varphi \in C_b(\mathbb{R}^2)$. Then for every $\delta > 0$ there exists $\varphi_\delta \in W^{2,\infty}(\mathbb{R}^2)$ such that $\|\varphi - \varphi_\delta\|_{L^\infty(\mathbf{R}^2)} \le \delta$. By the above inequality and by the uniform boundedness of ϱ^ε, we have

$$\left|\int_{\mathbf{R}^2}\varphi\varrho^\varepsilon(t,x)\,dx - \int_{\mathbf{R}^2}\varphi\varrho^\varepsilon(s,x)\,dx\right| \le 2\delta M + C(\varphi_\delta)|t-s|\,,$$

implying, together with $\varrho^\varepsilon(t)(\mathbb{R}^2) = M$, the tight equicontinuity of $\varrho^\varepsilon(t)$.

With a test function $\varphi_R(x) = 1 - \beta(|x|^2/R^2)$ with β nonincreasing, $\beta(r) = 1$ for $0 \le r \le 1/2$, and $\beta(r) = 0$ for $r \ge 1$, the above inequality gives

$$\varrho^\varepsilon(t)(\mathbb{R}^2 \setminus B_R) \le \varrho_I(\mathbb{R}^2 \setminus B_{R/2}) + \frac{c\,t}{R^2}\,,$$

which immediately implies the locally uniform tight boundedness. The result for m^ε is a consequence of the estimate $|m^\varepsilon| \le M\varrho^\varepsilon$. ∎

By the Prokhorov criterium, tight boundedness and equicontinuity of $\varrho^\varepsilon(t)$ provides compactness for $\varrho^\varepsilon(t)$ and $m^\varepsilon(t)$ in the sense that there exist nonnegative bounded time dependent measures $\varrho(t)$ and $m(t)$ such that, restricting to subsequences, $\varrho^\varepsilon(t)$ converges to $\varrho(t)$ tightly and locally uniformly in t, as $\varepsilon \to 0$, and

$$\int_{t_1}^{t_2}\int_{\mathbf{R}^2}\varphi(t,x)m^\varepsilon(t,x)dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbf{R}^2}\varphi(t,x)m(t,x)dx\,dt\,,$$

for all $\varphi \in C_b([t_1,t_2]\times\mathbb{R}^2)$.

Actually, also

$$\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\varphi(x,y)\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\varphi(x,y)\varrho(t,x)\varrho(t,y)\,dx\,dy\,, \tag{11}$$

uniformly in t for all $\varphi \in C_b(\mathbb{R}^2\times\mathbb{R}^2)$ (see [16]). However, by the discontinuity of the limiting kernel in (10), we cannot pass to the limit there, but we have to introduce the defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbf{R}^2}\mathcal{K}(x-y)\varrho(t,x)\varrho(t,y)dy\,,$$

with

$$\mathcal{K}(x) = \lim_{\varepsilon\to 0} \mathcal{K}^\varepsilon(x) = \begin{cases} \frac{x^{\otimes 2}}{|x|^2} & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

The atomic support of the measure $\varrho(t)$ will be denoted by

$$S_{at}(\varrho(t)) := \{a \in \mathbb{R}^2 : \ \varrho(t)(\{a\}) > 0\}\,.$$

It is an at most countable set.

Lemma 2 ([16]) *The defect measure ν is symmetric and nonnegative, and satisfies*

$$\mathrm{tr}(\nu(t,x)) \le \sum_{a\in S_{at}(\varrho(t))} (\varrho(t)(\{a\}))^2 \delta(x-a)\,.$$

Outline of a proof: Symmetry is obvious. For a test function $\varphi \in C_b(\mathbb{R}^2 \times \mathbb{R}^2)$, $(\varphi(x,y)-\varphi(x,x))\mathcal{K}^\varepsilon(x-y)$ converges uniformly to the continuous function $(\varphi(x,y) - \varphi(x,x))\mathcal{K}(x-y)$. Therefore, by (11),

$$\begin{aligned} &\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} (\varphi(x,y) - \varphi(x,x))\mathcal{K}^\varepsilon(x-y)\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)\,dx\,dy \\ &\to \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} (\varphi(x,y) - \varphi(x,x))\mathcal{K}(x-y)\varrho(t,x)\varrho(t,y)\,dx\,dy\,. \end{aligned}$$

By the definitions of m and ν this implies

$$\begin{aligned} &\int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \varphi(x,y)\mathcal{K}^\varepsilon(x-y)\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)\,dx\,dy \\ &\to \int_{\mathbf{R}^2}\int_{\mathbf{R}^2} \varphi(x,y)\mathcal{K}(x-y)\varrho(t,x)\varrho(t,y)\,dx\,dy + \int_{\mathbf{R}^2} \varphi(x,x)\nu(t,x)\,dx\,. \end{aligned}$$

Since $\mathcal{K}^\varepsilon$ is nonnegative, so is the right hand side for a nonnegative test function. Choosing $\varphi(x,y) = \psi(x)\eta(R(x-y)) \ge 0$ with an arbitrary nonnegative ψ and a nonnegative bounded η with compact support and $\eta(0) = 1$, the first term on the right hand side tends to zero for $R \to \infty$, proving nonnegativity of ν. The convergence is indeed a consequence of Lebesgue's theorem of dominated convergence using the fact that $\mathcal{K}(x-y)\eta(R(x-y))$ is bounded and converges to 0 pointwise.

For proving the second statement, note that $\mathrm{tr}(\mathcal{K}^\varepsilon) \le 1$. Combined with the above this gives (again with a nonnegative test function)

$$
\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\varrho(t,x)\varrho(t,y)\,dx\,dy \\
&\ge \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\mathrm{tr}(\mathcal{K}(x-y))\varrho(t,x)\varrho(t,y)\,dx\,dy + \int_{\mathbb{R}^2} \varphi(x,x)\mathrm{tr}(\nu(t,x))\,dx \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)(1-\chi_D(x,y))\varrho(t,x)\varrho(t,y)\,dx\,dy + \int_{\mathbb{R}^2} \varphi(x,x)\mathrm{tr}(\nu(t,x))\,dx\,,
\end{aligned}
$$

where χ_D denotes the characteristic function of the diagonal in $\mathbb{R}^2 \times \mathbb{R}^2$. Since

$$
\chi_D(x,y)\varrho(t,x)\varrho(t,y) = \sum_{a\in S_{at}(\varrho(t))} \varrho(t)(\{a\})^2\,\delta(x-a)\delta(y-a)\,, \tag{12}
$$

the desired result follows. ■

The limit of ϱ^ε is thus characterized by the pair (ϱ,ν) whose properties are collected in the following definition.

Definition 1 *For an interval $I \subset \mathbb{R}$, the set of time dependent measures with diagonal defects is defined by*

$$
\begin{aligned}
\mathcal{DM}^+(I;\mathbb{R}^2) \;=\; &\Big\{(\varrho,\nu):\ \varrho(t)\in\mathcal{M}_1^+(\mathbb{R}^2)\ \forall t\in I,\ \nu\in\mathcal{M}(I\times\mathbb{R}^2)^{2\times2}, \\
&\varrho \text{ is tightly continuous with respect to } t, \\
&\nu \text{ is a nonnegative, symmetric, matrix valued measure,} \\
&\mathrm{tr}(\nu(t,x)) \le \sum_{a\in S_{at}(\varrho(t))} (\varrho(t)(\{a\}))^2\delta(x-a)\Big\}\,,
\end{aligned}
$$

where $\mathcal{M}$ denotes spaces of Radon measures and $\mathcal{M}_1^+$ the subspace of nonnegative bounded measures.

3 Global measure valued solutions and a strong formulation

With the tools presented in the previous section, it is now not hard to pass to the limit in the regularized Keller-Segel model. We again follow along the lines of [16]. Starting from the distributional formulation (8) for the regularized flux, we observe that

$$\frac{(\varphi(x)-\varphi(y))(x-y)}{|x-y|(|x-y|+\varepsilon)} = \mathcal{K}^\varepsilon(x-y)\nabla\varphi(x) + L^\varepsilon(\varphi)(x,y)\,,$$

with

$$L^\varepsilon(\varphi)(x,y) = \frac{(\varphi(x)-\varphi(y)-(x-y)\cdot\nabla\varphi(x))(x-y)}{|x-y|(|x-y|+\varepsilon)}\,,$$

which converges uniformly to the continuous $L^0(\varphi)(x,y)$ for any test function $\varphi \in C_b^1(\mathbb{R}^2)$. For any time interval $(0,T)$, we may therefore pass to the limit in

$$\int_0^T \int_{\mathbf{R}^2} \varphi(t,x)\varrho^\varepsilon(t,x)\nabla S^\varepsilon[\varrho^\varepsilon](t,x)dx\;dt = -\frac{1}{4\pi}\int_0^T\int_{\mathbf{R}^2} m^\varepsilon(t,x)\nabla\varphi(t,x)\;dx\;dt$$
$$-\frac{1}{4\pi}\int_0^T\int_{\mathbf{R}^2}\int_{\mathbf{R}^2}\varrho^\varepsilon(t,x)\varrho^\varepsilon(t,y)L^\varepsilon(\varphi)(t,x,y)\;dx\;dy\;dt\,.$$

As a result, restricting to subsequences, $\varrho^\varepsilon\nabla S_\varepsilon[\varrho^\varepsilon]$ converges to $j[\varrho,\nu]$ in the sense of distributions, with the limiting flux defined by

$$\begin{aligned}&\int_0^T\int_{\mathbf{R}^2}\varphi(t,x)j[\varrho,\nu](t,x)dx\;dt\\ =\;& -\frac{1}{4\pi}\int_0^T\int_{\mathbf{R}^4}(\varphi(t,x)-\varphi(t,y))K(x-y)\varrho(t,x)\varrho(t,y)dx\,dy\,dt\\ &-\frac{1}{4\pi}\int_0^T\int_{\mathbf{R}^2}\nu(t,x)\nabla\varphi(t,x)dx\,dt\,. \end{aligned}\tag{13}$$

for $\varphi\in C_b^1((0,T)\times\mathbb{R}^2)$ with

$$K(x) = \begin{cases}\frac{x}{|x|^2} & \text{for } x\neq 0\,,\\ 0 & \text{for } x = 0\,.\end{cases}\tag{14}$$

This actually completes the proof of our main result.

Theorem 2 *For every $T>0$, as $\varepsilon\to 0$, a subsequence of solutions ϱ^ε of (3)–(5) converges tightly and uniformly in time to a time dependent measure $\varrho(t)$. There exists $\nu(t)$ such that $(\varrho,\nu)\in\mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of*

$$\partial_t\varrho + \nabla\cdot(j[\varrho,\nu]-\nabla\varrho) = 0\,,\tag{15}$$

in the sense that the convective flux $j[\varrho,\nu]$ is given by (13)–(14) and that (15) holds in the sense of distributions. The initial condition $\varrho(t=0)=\varrho_I$ is satisfied in the sense of tight continuity.

Note that, for a ϱ not charging points, $\nu = 0$ and $j[\varrho,0] = \varrho\nabla S_0[\varrho]$, implying that (15) is a generalization of the classical Keller-Segel model.

In order to derive a strong formulation of (15), we decompose the cell density as

$$\varrho = \overline{\varrho} + \hat{\varrho}\,, \quad \text{with } \hat{\varrho}(t,x) = \sum_{n\in N} M_n(t)\delta(x-x_n(t))\,, \quad \delta_n(t,x) = \delta(x-x_n(t))$$

where $N \subset \mathbb{N}$, assuming $\overline{\varrho}$ is smooth and that t varies in a time interval, where the atomic support of ϱ consists of smooth paths $x_n(t)$ carrying smooth weights $M_n(t)$. Then, by $(\varrho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$,

$$\nu(t,x) = \sum_{n\in N} \nu_n(t)\delta_n(t,x)\,,$$

with nonnegative symmetric ν_n satisfying $\mathrm{tr}(\nu_n) \le M_n^2$. The convective flux can be written as

$$j[\varrho,\nu] = \overline{\varrho}\nabla S_0[\overline{\varrho}+\hat{\varrho}] + \sum_n M_n\delta_n\nabla S_0\left[\overline{\varrho} + \sum_{m\neq n} M_m\delta_m\right] + \frac{1}{4\pi}\sum_n \nu_n\nabla\delta_n\,.$$

The equation (15) is then equivalent to

$$\begin{aligned}
\partial_t\overline{\varrho} \;&+\; \nabla\cdot(\overline{\varrho}\nabla S_0[\overline{\varrho}] - \nabla\overline{\varrho}) + \nabla\overline{\varrho}\cdot\nabla S_0[\hat{\varrho}] \\
&+\; \sum_n \delta_n(\dot{M}_n - \overline{\varrho}M_n) \\
&-\; \sum_n M_n\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\varrho} + \sum_{m\neq n} M_m\delta_m\right]\right) \\
&+\; \sum_n \left(\frac{1}{4\pi}\nu_n : \nabla^2\delta_n - M_n\Delta\delta_n\right) = 0\,.
\end{aligned}$$

The terms in the first row are in $L_t^\infty L_x^1$. The other three rows contain zeroth, first and second order derivatives of δ_n. Therefore, all the coefficients (in each row and for every n) have to vanish individually. Starting from the last row this gives

$$\nu_n = 4\pi M_n\,\mathrm{id}\,. \tag{16}$$

As a consequence of $\mathrm{tr}(\nu_n) = 8\pi M_n \le M_n^2$, point masses have to be at least 8π, implying further that there is only a finite number of them.

The other rows give the dynamics of $\overline{\varrho}$, M_n, and x_n:

$$\partial_t \overline{\varrho} + \nabla \cdot (\overline{\varrho} \nabla S_0[\overline{\varrho}] - \nabla \overline{\varrho}) - \frac{1}{2\pi} \nabla \overline{\varrho} \cdot \sum_n M_n \frac{x - x_n}{|x - x_n|^2} = 0 \,, \tag{17}$$

$$\dot{M}_n = \overline{\varrho}(x = x_n) M_n \,, \tag{18}$$

$$\dot{x}_n = \nabla S_0[\overline{\varrho}](x = x_n) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2} \,. \tag{19}$$

Note that the last term in the first equation can be written as a divergence, away from $S_{at}(\varrho)$, where it provides sinks compensating the growth of the point masses:

$$\frac{d}{dt} \left(\int_{\mathbf{R}^2} \overline{\varrho} \, dx + \sum_n M_n \right) = 0 \,.$$

It is interesting to compare the above system with the equations derived in [17] in the formal limit of a different regularization of the Keller-Segel model. The regularization in [17] amounts to replacing the convective term $\varrho \nabla S[\varrho]$ in the Keller-Segel model by $G_\varepsilon(\varrho) \nabla S[\varrho]$ with $G_\varepsilon(\varrho) = \frac{1}{\varepsilon} Q(\varepsilon \varrho)$, where $Q(s)$ is increasing, bounded, and behaves like the identity for $s \to 0$. By means of formal asymptotics, a limiting problem is derived, which is identical with (17)–(19), except for a factor $\Gamma(M_n)$ in front of the right hand side of (19), where Γ can be interpreted as a mean value of the derivative Q' of the profile function. It has the properties $0 < \Gamma(M) < 1$, $\Gamma(8\pi+) = 1$, $\Gamma(\infty) = 0$. In a way, (19) can be seen as a 'weak regularization limit' when $Q' \to 1$.

A local-in-time existence result for initial value problems for (17)–(19) can be found in [19]. In general, one has to expect blow-up events in the smooth part $\overline{\varrho}$ and/or collisions of point aggregates in finite time. At such points in time, a restart is required with either an additional point aggregate after a blow-up event or with a smaller number of point aggregates after a collision. A rigorous theory producing global solutions by such a procedure is missing, however.

4 Long time behaviour

This section is devoted to a brief and formal discussion of the long time behaviour of weak solutions assuming the validity of (16), i.e.

$$\nu(t,x) = 4\pi \,\mathrm{id} \sum_{a\in S_{at}(\varrho(t))} \varrho(t)(\{a\})\delta(x-a)\,, \tag{20}$$

and the existence of the second order moment of the initial density:

$$\int_{\mathbf{R}^2} |x|^2 \varrho_I dx < \infty\,.$$

Solutions with subcritical total mass $M < 8\pi$ are smooth and decay to zero as time tends to infinity [2]. See [1], [3] in case $M = 8\pi$. We shall concentrate on the supercritical case $M > 8\pi$.

Using the product of a time dependent function and a smooth cut-off of $|x|^2$ as a test function in the distributional formulation of (15), and removing the cut-off by a limiting procedure, we obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbf{R}^2} |x|^2\varrho\, dx &= 4M - \frac{1}{2\pi}\int_{\mathbf{R}^4}(1-\chi_D)\varrho\otimes\varrho\, dy\, dx - \frac{1}{2\pi}\int_{\mathbf{R}^2}\mathrm{tr}(\nu)\, dx \\ &= \overline{M}\left(4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi}\right) - \frac{1}{2\pi}\sum_{\substack{a\neq b,\\ a,b\in S_{at}(\varrho(t))}} \varrho(t)(\{a\})\varrho(t)(\{b\})\,, \end{aligned}$$

with $\hat{M} = \sum_{a\in S_{at}(\varrho(t))}\varrho(t)(\{a\})$ and $\overline{M} = M - \hat{M}$. The second equality follows from (12) and (20). A well known identity for smooth solutions of the Keller-Segel model is recovered for $S_{at}(\varrho(t)) = \emptyset$. For $M > 8\pi$, the right hand side is the sum of two nonpositive terms, and the second order moment is a Lyapunov function. Obviously, the dissipation term only vanishes when $\overline{M} = 0$, and when the atomic support of $\varrho(t)$ consists of only one point. Therefore, we expect $\varrho(t,x) \to M\delta(x - X_I)$ as $t\to\infty$. The position X_I of the limiting aggregate is the initial center of mass

$$X_I = \frac{1}{M}\int_{\mathbf{R}^2} x\varrho_I dx\,,$$

since by a computation which goes as the one for the second order moment, the center of mass does not move:

$$\frac{d}{dt}\int_{\mathbf{R}^2} x\varrho dx = 0\,.$$

The result of convergence to one aggregate as $t \to \infty$ has been proven rigorously for the Keller-Segel model without cell diffusion [16]. There the result is valid independently of the value of the total mass, and the proof does not need exact information (as (20)) on the defect measure.

We conclude this section by two examples of how the asymptotic state can be reached. The first one is very simple. Consider a strong solution with $\overline{\varrho} = 0$ and two aggregates: $\varrho(t,x) = M_1\delta(x - x_1(t)) + M_2\delta(x - x_2(t))$. The masses are constant by (18), and the ODEs for the positions can be solved explicitly:

$$x_i(t) = X_I + u_i(t)\frac{x_1(0) - x_2(0)}{|x_1(0) - x_2(0)|}\,,$$

with

$$u_i(t) = (-1)^{i-1}\frac{M_i}{M_1 + M_2}\sqrt{|x_1(0) - x_2(0)|^2 - t(M_1 + M_2)/\pi}\,.$$

Thus, the asymptotic state is reached in finite time

$$t = \frac{\pi|x_1(0) - x_2(0)|^2}{M_1 + M_2}$$

by collision of the aggregates.

As a second example consider another strong solution with one point aggregate and small enough (to be specified below) initial mass

$$\overline{M}(0) = \int_{\mathbf{R}^2} \overline{\varrho}(t = 0)dx$$

of the smooth part. Such a solution may exist only on a finite time interval. The equations (17)–(19) then become

$$\begin{aligned}
&\partial_t\overline{\varrho} + \nabla\cdot(\overline{\varrho}\nabla S_0[\overline{\varrho}] - \nabla\overline{\varrho}) - \frac{M_1}{2\pi}\nabla\overline{\varrho}\cdot\frac{x - x_1}{|x - x_1|^2} = 0\,,\\
&\dot{M}_1 = \overline{\varrho}(x = x_1)M_1\,,\\
&\dot{x}_1 = \nabla S_0[\overline{\varrho}](x = x_1)\,.
\end{aligned}$$

For $q > 1$, a straightforward computation using the Poisson equation $\Delta S_0[\overline{\varrho}] = -\overline{\varrho}$ gives

$$\frac{d}{dt}\int_{\mathbf{R}^2} \overline{\varrho}^q dx = (q-1)\left(\int_{\mathbf{R}^2} \overline{\varrho}^{q+1}dx - \frac{4}{q}\int_{\mathbf{R}^2} |\nabla\overline{\varrho}^{q/2}|^2 dx\right) - \overline{\varrho}(x = x_1)^q M_1\,.$$

With the Gagliardo-Nirenberg inequality

$$\int_{\mathbf{R}^2} |u|^{2(1+1/q)} dx \le C_q \int_{\mathbf{R}^2} |u|^{2/q} dx \int_{\mathbf{R}^2} |\nabla u|^2 dx$$

with $u = \overline{\varrho}^{q/2}$ this implies

$$\frac{d}{dt} \int_{\mathbf{R}^2} \overline{\varrho}^q dx \le (q-1) \left(C_q \overline{M} - \frac{4}{q} \right) \int_{\mathbf{R}^2} |\nabla \overline{\varrho}^{q/2}|^2 dx - \overline{\varrho}(x = x_1)^q M_1 \,.$$

Since $\overline{M}(t)$ is nonincreasing, the right hand side is nonpositive for $\overline{M}(0) < 4/(qC_q)$. For

$$\overline{M}(0) < M_{crit} = \sup_{q>1} \frac{4}{qC_q} \,,$$

the above argument can be made rigorous with an appropriate q, proving global existence and decay to zero of $\overline{\varrho}$. Since the optimal constants C_q are only known for special values of q (see [6]), we cannot compute M_{crit}. The best possible bound for $\overline{M}(0)$ guaranteeing global existence of the strong solution with one aggregate is expected to be 8π, since this presumably prevents a second aggregate to form.

5 Local density profiles

In solutions of the regularized problem, the aggregates of the limiting problem are approximated by local density profiles, concentrating when $\varepsilon \to 0$. In this section, an equation for these profiles is derived by passing to the limit in a rescaled problem. On the other hand, the existence of rotationally symmetric density profiles for supercritical mass is proven by minimization of a free energy functional.

For fixed t and $a \in S_{at}(\varrho(t))$, we introduce the transformations $\varepsilon\xi = x - a$ and $\varepsilon^2 \varrho^\varepsilon = R^\varepsilon$ in (3), leading to

$$\varepsilon^2 \partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0 \,. \tag{21}$$

By (7), R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. As a consequence, the L^∞-weak* limit R of R^ε (restricting to subsequences) satisfies the formal limiting stationary equation

$$\nabla_\xi \cdot (R \nabla_\xi S_1[R] - \nabla_\xi R) = 0 \,. \tag{22}$$

Multiplication by $\ln(R) - S_1[R]$ and integration by parts shows that this quantity is independent of ξ:

$$\ln(R) - S_1[R] = c_a(t)\,.$$

In the weak formulation of (22),

$$\int_{\mathbf{R}^2} (R\Delta\varphi + R\nabla_\xi\varphi \cdot \nabla_\xi S_1[R])d\xi = 0\,,$$

we choose test functions approximating $\varphi(\xi) = (r^2 - |\xi|^2)_+/4$ and let $r \to \infty$, leading to

$$\int_{\mathbf{R}^2} R(\xi)d\xi = \frac{1}{8\pi}\int_{\mathbf{R}^4} \frac{|\xi-\eta|}{|\xi-\eta|+1} R(\xi)R(\eta)d\eta d\xi \le \frac{1}{8\pi}\left(\int_{\mathbf{R}^2} R(\xi)d\xi\right)^2 .$$

This shows that either R vanishes or its mass is not smaller than 8π.

Solutions of (22) carrying a given mass $M > 8\pi$ can also be constructed by minimization of the rescaled free energy functional F_1, where

$$\begin{aligned} F_\varepsilon[\varrho] &:= \int_{\mathbf{R}^2} \left(\varrho\log\varrho - \frac{1}{2}\varrho S_\varepsilon[\varrho]\right) dx \\ &= \int_{\mathbf{R}^2} \varrho\log\varrho\, dx + \frac{1}{4\pi}\int_{\mathbf{R}^4} \log(|x-y|+\varepsilon)\varrho(x)\varrho(y)dy\, dx\,. \end{aligned}$$

A straightforward computation shows the decay of F_ε along solutions of (3):

$$\frac{d}{dt}F_\varepsilon[\varrho^\varepsilon] = -\int_{\mathbf{R}^2} \varrho^\varepsilon |\nabla(\log\varrho^\varepsilon - S_\varepsilon[\varrho^\varepsilon])|^2 dx\,.$$

The free energy has an interesting scaling property. With an arbitrary $a \in \mathbb{R}^2$ and with the transformation $R(\xi) = \varepsilon^2\varrho(a+\varepsilon\xi)$ we have

$$F_\varepsilon[\varrho] = \left(2M - \frac{M^2}{4\pi}\right)\log\frac{1}{\varepsilon} + F_1[R]\,. \tag{23}$$

Lemma 3 *Let $R \in L^1_+(\mathbb{R}^2)$ be a radial function such that $\int_{\mathbf{R}^2}\log(1+|x|)\, R(x)\, dx < \infty$. If $M = \int_{\mathbf{R}^2} R\, dx$, then*

$$\frac{1}{4\pi}\int_{\mathbf{R}^2} \log(1+|x-y|)\, R(y)\, dy \ge \frac{M}{4\pi}\log|x| \quad \forall\, x \in \mathbb{R}^2\,.$$

Proof: Let $R(x) = u(|x|)$ a.e. and observe that

$$\frac{1}{2\pi}\int_{\mathbf{R}^2}\log(1+|x-y|)\,R(y)\,dy \geq \frac{1}{2\pi}\int_{\mathbf{R}^2}\log|x-y|\,R(y)\,dy =: v(|x|)\,,$$

where v is such that

$$\frac{1}{r}(r\,v')' = u\,,\quad \lim_{r\to\infty}\left(\frac{M}{2\pi}\log r - v(r)\right) = 0\,.$$

It is the straightforward to check that

$$v(r) = \log r\int_0^r s\,u(s)\,ds + \int_r^\infty s\,\log s\,u(s)\,ds \geq \log r\int_0^\infty s\,u(s)\,ds = \frac{M}{2\pi}\log r\,.$$

■

Theorem 3 *Defining* $L^1_{+,M} = \{R \in L^1_+(\mathbb{R}^2) :\ \int_{\mathbf{R}^2} R\,d\xi = M\}$ *and*

$$J_M := \inf_{R\in L^1_{+,M}} F_1[R] \geq -\infty\,,$$

then $J_M = -\infty$ *for* $M < 8\pi$*, and* $J_M > -\infty$ *for* $M \geq 8\pi$*. If* $M > 8\pi$*, there exists a minimizer* $R \in L^1_{+,M}$ *with* $F_1[R] = J_M$*, which is rotationally symmetric and nonincreasing as a function of* $|\xi|$*.*

Proof: We first observe that for $M = 8\pi$,

$$F_1[R] \geq F_0[R] \geq M(1+\log\pi+\log M) = 8\pi\log(8/e)$$

by the logarithmic HLS inequality, see [4]. Assume next that $M < 8\pi$. For a fixed $\varrho \in L^1_{+,M}\cap L^\infty(\mathbb{R}^2)$ for which $F_0[\varrho]$ is well defined, set $\varrho^\delta(\xi) = \delta^2\varrho(\delta\xi)$. Then from the scaling property (23),

$$F_1[\varrho^\delta] = \left(\frac{M^2}{4\pi} - 2M\right)\log\frac{1}{\delta} + F_\delta[\varrho] \underset{\delta\to 0}{\sim} 2M\left(\frac{M}{8\pi}-1\right)\log\frac{1}{\delta} + F_0[\varrho] \underset{\delta\to 0}{\longrightarrow} -\infty\,.$$

This proves the statement for $M < 8\pi$.

For the decreasing radial symmetrization R^* of R (see [14]),

$$\int_{\mathbf{R}^2} R\log R\,d\xi = \int_{\mathbf{R}^2} R^*\log R^*\,d\xi$$

holds. On the other hand, we use the Riesz symmetrization inequality (see [14]):

$$\int_{\mathbf{R}^2\times\mathbf{R}^2} R^*(\xi)\, k(|\xi-\eta|)\, R^*(\eta)\; d\xi\, d\eta \geq \int_{\mathbf{R}^2\times\mathbf{R}^2} R(\xi)\, k(|\xi-\eta|)\, R(\eta)\; d\xi\, d\eta\,,$$

which holds for any $R \in L^1_+(\mathbb{R}^2)$ and for nonnegative nonincreasing k such as $k(z) := [\log((1+2r)/(1+z))]_+$. This implies

$$\begin{aligned}&\int_{\mathbf{R}^2\times\mathbf{R}^2} R_r^*(\xi)\, \log(1+|\xi-\eta|)\, R_r^*(\eta)\; d\xi\, d\eta\\ &\leq \int_{\mathbf{R}^2\times\mathbf{R}^2} R_r(\xi)\, \log(1+|\xi-\eta|)\, R_r(\eta)\; d\xi\, d\eta\,,\end{aligned}$$

with $R_r = R\,\chi_{B_r(0)}$. Passing to the limit $r \to \infty$, R_r can be replaced by R, proving a slight generalization of an inequality from [4]. As a consequence, $F_1[R] \geq F_1[R^*]$. Using Lemma 3, we get

$$F_1[R^*] \geq \int_{\mathbf{R}^2} R^* \log R^*\, d\xi + \frac{M}{4\pi}\int_{|\xi|>1} \log|\xi|\, R^*\, d\xi \tag{24}$$

From now on, we consider only radial functions.

Consider then the case $M > 8\pi$. Let $\delta \in [0,1)$,

$$R^\delta_\infty(\xi) := \min\{1, |\xi|^{-M(1-\delta)/(4\pi)}\} \quad \text{and} \quad M^\delta_\infty := \int_{\mathbf{R}^2} R^\delta_\infty\, d\xi\ .$$

By Jensen's inequality

$$\int_{\mathbf{R}^2} R\log\left(\frac{R}{R^\delta_\infty}\right) d\xi = M^\delta_\infty \int_{\mathbf{R}^2} \frac{R}{R^\delta_\infty}\log\left(\frac{R}{R^\delta_\infty}\right)\frac{R^\delta_\infty d\xi}{M^\delta_\infty} \geq M\log\left(\frac{M}{M^\delta_\infty}\right),$$

so that, for the full free energy, we have: $F_1[R] \geq M\log\left(M/M^0_\infty\right)$. It remains to prove the existence of a minimizer when $M > 8\pi$. We choose a $\delta > 0$ small enough such that $M(1-\delta) > 8\pi$, and using (24), we write

$$F_1[R] \geq \int_{\mathbf{R}^2} R\log\left(\frac{R}{R^\delta_\infty}\right) d\xi + \frac{M\delta}{4\pi}\int_{|\xi|>1}\log|\xi|\, R(\xi) d\xi\,,$$

implying

$$F_1[R] \geq M\log\left(\frac{M}{M^\delta_\infty}\right) + \frac{M\delta}{4\pi}\int_{|\xi|>1}\log|\xi|\, R(\xi) d\xi\,,$$

Therefore, for a minimizing sequence $\{R_n\}$, both

$$\int_{\mathbb{R}^2} R_n \log R_n d\xi \quad \text{and} \quad \int_{|\xi|>1} \log|\xi|\, R(\xi) d\xi$$

are bounded and, consequently, $\{R_n\}$ has a weakly convergent subsequence in $L^1(\mathbb{R}^2)$. By lower semicontinuity, we can pass to the limit in $F_1[R_n]$. ∎

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