

GEOMETRY OF PHASE SPACE AND SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS IN A BALL

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ABSTRACT. We consider the problem

$$(1) \quad \begin{cases} -\Delta u = u^p + \lambda u & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 \text{ on } \partial B. \end{cases}$$

where B denotes the unit ball in \mathbb{R}^N , $N \geq 3$, $\lambda > 0$ and $p > 1$. Merle and Peletier showed that for $p > \frac{N+2}{N-2}$ there is a unique value $\lambda = \lambda_* > 0$ such that a radial singular solution exists. This value is the only one at which an unbounded sequence of classical solutions of (1) may accumulate. Here we prove that if additionally

$$p < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{or} \quad N \leq 10,$$

then for λ close to λ_* , a large number of classical solutions of (1) exist. In particular infinitely many solutions are present if $\lambda = \lambda_*$. We establish a similar assertion for the problem

$$\begin{cases} -\Delta u = \lambda f(u+1) & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 \text{ on } \partial B. \end{cases}$$

where $f(s) = s^p + s^q$, $1 < q < p$, and p satisfies the same condition as above.

1. INTRODUCTION

Let B denote the unit ball in \mathbb{R}^N , $N \geq 3$ and consider bounded solutions of

$$(1) \quad \begin{cases} -\Delta u = u^p + \lambda u & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 \text{ on } \partial B. \end{cases}$$

According to [17], classical solutions of this problem need to be radially symmetric and decreasing, so that with a straightforward abuse of notations, we may write $u = u(r)$, $r = |x|$, and reduce the existence problem to that of the two-point boundary value problem

$$(2) \quad \begin{cases} u'' + \frac{N-1}{r} u' + u^p + \lambda u = 0, & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0 & \text{and } u(r) > 0 \text{ for } 0 < r < 1. \end{cases}$$

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If the equation $\Delta u + u^p + \lambda u = 0$ is written in a general bounded smooth domain Ω and supplied with zero Dirichlet boundary conditions as in (1), then for any value of the exponent p , no solution exists for $\lambda \geq \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under zero Dirichlet boundary conditions in Ω . This is easily recovered by multiplying the equation by the first eigenfunction and performing an integration by parts. On the other hand, if $\lambda < \lambda_1$, the existence of a positive solution holds for any $p < \frac{N+2}{N-2}$, as it follows from a standard constrained minimization procedure involving compactness of the Sobolev embeddings. Now, by Pohozaev's identity [29], there are no solutions for $\lambda \leq 0$ whenever $p \geq \frac{N+2}{N-2}$ if Ω is star-shaped, and, in particular, if it is a ball.

In [6] Brezis and Nirenberg established that this problem for $p = \frac{N+2}{N-2}$ is solvable for $\underline{\lambda} < \lambda < \lambda_1$, where $\underline{\lambda} = 0$ for $N \geq 4$ and $0 < \underline{\lambda} < \lambda_1$ if $N = 3$. In the case of a ball, they found explicitly that $\underline{\lambda} = \lambda_1/4$. These results can be described in terms of the branch of positive solutions of (1). From standard bifurcation theory, the set of pairs $(\lambda, u(0))$ where u solves (1) is a curve which stems from $\lambda = \lambda_1$ and $u = 0$. In [25, 33, 34, 19, 32, 15], it is shown that, if $p \leq \frac{N+2}{N-2}$, there is at most one solution of (1) for $\lambda < \lambda_1$ (also see [14, 31, 26, 27, 30] for more recent results). Hence in this case the curve goes left, without turning points, and if $p = \frac{N+2}{N-2}$ it blows up as $\lambda \downarrow \underline{\lambda}$.

The situation drastically changes as soon as the critical exponent is crossed. Budd and Norbury in [7] considered (1) for $p > \frac{N+2}{N-2}$, and derived formally qualitative properties of this bifurcation branch. In particular, formal asymptotics and numerical computations suggest that the following takes place: Before reaching $\lambda = 0$, the curve turns right and then oscillates infinitely many times in the form of an exponentially damped sinusoidal along a line $\lambda = \lambda_*$. Merle and Peletier in [23] established rigorously for $N \geq 4$ the existence of this unique $\lambda = \lambda_* > 0$ for which necessarily $\lambda_n \rightarrow \lambda_*$ whenever u_n is an unbounded sequence of classical solutions of (1) for $\lambda = \lambda_n$. Moreover, at $\lambda = \lambda_*$, and only for this value of λ , a unique singular radial solution exists (also see [24]).

In this paper we establish rigorously the validity of the picture suggested by Budd and Norbury [7] in the sense that a large number of solutions of (1) exists for λ close to λ_* . In fact, infinitely many classical solutions exist at $\lambda = \lambda_*$ provided that p is supercritical but not too large.

Theorem 1. *Assume that $N \geq 3$, $p > \frac{N+2}{N-2}$. Suppose moreover that*

$$\text{either } p < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{or } N \leq 10.$$

Then there is a unique number $\lambda_ > 0$ such that, given any integer $k \geq 1$, there exist at least k bounded radial solutions of (1) for any λ sufficiently close to λ_* . In particular, there are infinitely many classical solutions of (1) for $\lambda = \lambda_*$.*

A question that has attracted the interest of various authors is that of the existence of sign-changing solutions when the nonlinearity u^p is replaced by its odd extension $|u|^{p-1}u$, namely

$$(3) \quad \begin{cases} -\Delta u = |u|^{p-1}u + \lambda u & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases}$$

Starting from the n -th radial eigenvalue λ_n of the Laplacian under zero Dirichlet boundary condition, a branch of radial solutions with exactly $n - 1$ zeros bifurcates from $(\lambda_n, 0)$. For the critical exponent $p = \frac{N+2}{N-2}$, the asymptotic behavior of this branch has been analyzed in [9] for $N \geq 7$ and in [1] for $4 \leq N \leq 6$. Roughly speaking, the n -th branch lives in the range $\lambda \in (0, \lambda_n)$ if $N \geq 7$ while it blows-up above $\lambda = 0$ if $N \leq 6$, $n \geq 2$. In the super-critical case, results on classification of singular and classical solutions have been recently found in [5, 13]. Our second result can be visualized by saying that the n -th branch develops at some number $\lambda = \lambda_n^*$ a behavior which is similar to that predicted in Theorem 1 for positive solutions.

Theorem 2. *Assume that $N \geq 3$, and that p satisfies the constraints of Theorem 1. Then there exists a strictly increasing sequence of positive numbers $(\lambda_n^*)_{n \geq 1}$ such that the following holds. Given any integer $k \geq 1$, there exist at least k radial classical solutions of (3) with exactly $n - 1$ zeros for any λ sufficiently close to λ_n^* . In particular, there are infinitely many solutions of (3) for $\lambda = \lambda_n^*$. Besides, a unique singular solution with $n - 1$ zeros, which is positive near 0, exists for this value of λ and there is no other one with the same properties for other values of λ .*

The proof of Theorem 2 is an easy extension of the proof of Theorem 1.

Closely related with (1) is the problem

$$(4) \quad \begin{cases} -\Delta u = \lambda(u+1)^p & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 & \text{on } \partial B. \end{cases}$$

with $\lambda > 0$. This problem has been considered by Joseph and Lundgren in [22]. In this paper, they completely describe the branch (λ, u) of positive radial solutions, which in this case stems from $(0, 0)$. For instance, if p satisfies the constraints of Theorem 1, it is rigorously shown that this branch turns left at some positive value of λ and then oscillates around some number $\lambda = \lambda_*$, at which the same conclusions as the ones of Theorem 1 holds. The main difference between the analysis of Problem (1) and that in [22] of (4) is the fact that in the latter, the associated ODE can be reduced via a suitable change of variables to an autonomous second order ODE. This problem may then be fully analyzed via a phase plane analysis. Recently in [20] the case of p -Laplacian and of k -Hessian operators, where the autonomous reduction is still possible, has been analyzed when the right hand side in (4) is replaced by λe^u . For $p = 2$, this case was also treated in [22] and similar phenomena

were observed for $3 \leq N \leq 9$. Although it looks similar, Problem (1) is considerably more delicate since such a reduction is apparently not possible. In fact it is not clear whether the result in [22] holds true if the power s^p is perturbed by lower order terms. To exemplify the robustness of the method we employ here, we consider the problem

$$(5) \quad \begin{cases} -\Delta u = \lambda f(u+1) & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 \text{ on } \partial B. \end{cases}$$

Theorem 3. *Let $f(s) = s^p + s^q$ in problem (5) where $1 < q < p$. If $p > \frac{N+2}{N-2}$ and*

$$\text{either } p < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{or } N \leq 10,$$

then there is a number $\lambda_ > 0$ such that given any integer $k \geq 1$, there exist at least k classical solutions of (1) for any λ sufficiently close to λ_* .*

The proofs of Theorems 1, 2 and 3 consist in performing a suitable transformation of the problem, the so-called Emden-Fowler transformation [16], also used in [22]. This leads to a second order ODE which is not autonomous. The results follow after a topological analysis of the situation arising in the three-dimensional phase space using tools of the geometric theory of dynamical systems. Although in a slightly different context, phase space analysis has been the issue in several related works on radial solutions to semilinear elliptic equations, see for instance [5, 2, 8, 21]. Also note that a variational analysis can be done in the slightly supercritical case, i.e. for $p - \frac{N+2}{N-2} > 0$, small [10, 12], which also has an ODE interpretation [11].

2. PRELIMINARIES: THE SET-UP

We search for positive radially symmetric solutions $u = u(r)$, $r = |x|$, of (1), namely solutions of the ordinary differential equation (2). Consider the classical Emden-Fowler transformation

$$(6) \quad x(t) = r^{\frac{2}{p-1}} u(r)|_{r=e^t},$$

which recasts Equation (2) into the equivalent problem

$$(7) \quad x'' + \alpha x' + |x|^{p-1} x + e^{2t} \lambda x - \beta x = 0, \quad -\infty < t < 0.$$

where

$$\alpha := (N-2) - \frac{4}{p-1} \quad \text{and} \quad \beta := \frac{2}{p-1} \left(N-2 - \frac{2}{p-1} \right).$$

Straightforward calculations show that finding a positive solution of (2) is equivalent to finding a solution $x(t)$ of (7) such that

$$(8) \quad x(t) > 0 \quad \text{in } (-\infty, 0), \quad \lim_{t \rightarrow -\infty} x(t) = 0 \quad \text{and} \quad x(0) = 0.$$

Now, we introduce the variables $y = x'$ and $z = e^{2t} \lambda$. In these variables, Equation (7) becomes equivalent to the autonomous first order system

$$(9) \quad \begin{cases} x' = y, \\ y' = -\alpha y + \beta x - |x|^{p-1} x - z x, \\ z' = 2z, \end{cases}$$

with the additional constraint $z \geq 0$. At this level, it turns out that finding a solution $x(t)$ of (7)-(8) is equivalent to finding a solution $\varphi(t) = (x(t), y(t), z(t))$ of (9) such that:

$$(10) \quad x(t) > 0 \quad \text{in} \quad (-\infty, 0), \quad \lim_{t \rightarrow -\infty} \varphi(t) = (0, 0, 0) \quad \text{and} \quad x(0) = 0.$$

The rest of this section will be devoted to the phase space analysis of System (9). First we describe the general structure of the orbits and introduce some notations which will be useful in the subsequent analysis. We start with some straightforward facts for the flow in the supercritical case $p > \frac{N+2}{N-2}$:

- (i) The positive z -axis is an orbit for the system.
- (ii) The plane $\{z = 0\}$ is invariant under the flow of (9). It is in correspondence with Equation (2) with $\lambda = 0$ after the Emden-Fowler transformation.
- (iii) System (9) is invariant under $(x, y, z) \mapsto (-x, -y, z)$, so that in what follows, we will concentrate our analysis on $x \geq 0$.
- (iv) Orbits that intersect the plane $\{x = 0\}$ do it transversally. For $y > 0$, they go into the region $x > 0$ while the opposite happens for $y < 0$.
- (v) The two-dimensional foliation Σ given by

$$\Sigma : \left\{ \{(x, y, z) \in \mathbb{R}^3 : z = c\}, \quad c \in \mathbb{R}_+ \right\}$$

is invariant under the flow. Moreover, the planes $\{z = c\}$ move upwards in time, in fact: $z(t) = z_0 e^{\gamma t}$ with γ positive.

- (vi) The system has exactly three singularities,

$$O = (0, 0, 0), \quad P = (\beta^{1/(p-1)}, 0, 0) \quad \text{and} \quad P_- = (-\beta^{1/(p-1)}, 0, 0),$$

which are hyperbolic saddle points. At each of these three points, there are well defined stable and unstable manifolds associated to these singularities.

Linearizing (9) around O we obtain two unstable eigenvalues $2/(p-1)$ and 2 with associated eigenvectors $(1, 2/(p-1), 0)$ and $(0, 0, 1)$, and one stable eigenvalue $2/(p-1) - (N-2)$ with eigenvector $(1, 2/(p-1) - (N-2), 0)$. Thus, from standard invariant manifold theory, see for instance [18], O has a two dimensional unstable manifold $W^u(O)$, constituted by all trajectories approaching O as $t \rightarrow -\infty$, whose tangent plane is spanned by the two unstable eigenvectors. The manifold $W^u(O)$ contains the entire z -axis as

well as the heteroclinic orbit in $\{z = 0\}$ connecting O and P . It is also transversal to the planes $\{z = 0\}$ and $\{x = 0\}$, see Figure 1.

Linearizing around the singularity P , we obtain one unstable eigenvalue, 2, with associated eigenvector

$$\left(1, 2, \frac{(p-1)\beta + 2\alpha + 4}{\beta^{1/(p-1)}}\right)$$

and two stable eigenvalues

$$\frac{1}{2} \left(-\alpha \pm \sqrt{\alpha^2 - 4\beta(p-1)}\right).$$

By a direct computation we see that these eigenvalues are complex if and only if relation

$$(11) \quad p < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{or} \quad N \leq 10$$

holds. Thus, the unstable manifold of P for $z \geq 0$, $W^u(P)$, is one-dimensional, and constituted by a single orbit. Its stable manifold is two-dimensional and lies in the plane $\{z = 0\}$, see Figure 2.

Lemma 1. *The unstable manifold of P , $W^u(P)$ is contained in the closure of the unstable manifold of O , $W^u(O)$.*

We postpone the proof of this fact in the appendix. We shall say that two systems $x' = f(x)$ and $y' = g(y)$ with respective singularities P and Q are C^1 -equivalent around these points if there is a C^1 -diffeomorphism between neighborhoods of these points which transforms trajectories of the first system into trajectories of the other one, preserving orientations. The following fact will be important for our purposes.

Lemma 2. *System (9) is C^1 -equivalent to its linearized system around P , provided that Condition (11) holds.*

We also leave this result for the appendix and go to the proof of Theorem 1.

3. PROOF OF THE THEOREM 1

We look for positive solutions to (2). At the level of the orbit x given by (6), this means that $(x(t), y(t), z(t))$ is attracted by O as $t \rightarrow -\infty$, namely that it lies in $W^u(O)$ and x remain positive for $t < 0$ and $x = 0$ for $t = 0$.

We define λ_* to be the height $z = \lambda_*$ where $W^u(P)$ first intersects the plane $\{x = 0\}$. This intersection is transversal because of the uniqueness of the solution to the initial value problem for the second order Equation (7). Besides, let us observe that a trajectory associated to the one-dimensional manifold $W^u(P)$ is in exact correspondence with the unique positive singular solution to (2), which exists if and only if $\lambda = \lambda_*$, see [23].

Hence our task is to show that under the conditions of Theorem 1 there are infinitely many distinct trajectories lying in the two-dimensional manifold $W^u(O)$ which remain positive in their x -coordinate for $z < \lambda_*$ and are such

that for $z = \lambda_*$ their x -coordinate is 0. The strategy goes as follows: we prove that the curve corresponding to $W^u(O) \cap \{z = \lambda_*\}$ is a spiral curve which winds around the point $W^u(P) \cap \{z = \lambda_*\}$. This fact gives infinitely many intersections of $W^u(O) \cap \{z = \lambda_*\}$ and the plane $\{x = 0\}$.

From Lemma 2, we know that System (9) is C^1 -equivalent in a neighborhood of P to the following linear system:

$$\begin{cases} \bar{x}' = \bar{y} \\ \bar{y}' = a\bar{x} + b\bar{y} + c\bar{z} \\ \bar{z}' = 2\bar{z} \end{cases}$$

where $a = \beta(1-p)$, $b = -\alpha$, $c = -\beta^{1/(p-1)}$. After a suitable linear transformation, we immediately check that this system is also C^1 -equivalent to

$$(12) \quad \begin{cases} \bar{x}' = \bar{y} \\ \bar{y}' = a\bar{x} + b\bar{y} \\ \bar{z}' = 2\bar{z} \end{cases}$$

Let $\Phi : \mathcal{V} \rightarrow \mathcal{U}$ be the diffeomorphism setting the equivalence between (9) and (12), where \mathcal{V} is a neighborhood of P and \mathcal{U} one of $\bar{O} = (0, 0, 0)$. As in the original system (9), the origin in (12) is an attractor focus when (12) is restricted to the plane $\{\bar{z} = 0\}$ and the \bar{z} -axis is its corresponding unstable manifold.

Let us recall that according to Lemma 1 the unstable manifold of O contains the unstable manifold of P in its closure. Hence in the neighborhood \mathcal{V} of P where the equivalence is valid, there is an invariant manifold, namely $W^u(O) \cap \mathcal{V}$. To this manifold, corresponds through the equivalence Φ an invariant manifold \mathcal{M} of System (12) inside \mathcal{U} . Let $\{\psi(t) : t \in \mathbb{R}\}$ be the image through Φ restricted to $\{z = 0\} \cap \mathcal{V}$ of the orbit corresponding to the heteroclinic trajectory which connects O and P . It is easily checked, after making \mathcal{U} and \mathcal{V} smaller if necessary, that the manifold \mathcal{M} must be constituted exactly by the set of points of the form $\{(\psi(t), \bar{z}) : t \in \mathbb{R}, \bar{z} \in \mathbb{R}\}$ which lie inside \mathcal{U} , the neighborhood of O . See Figure 3 for more details.

Let us consider a small z_0 so that the plane $\{z = z_0\}$ intersects \mathcal{V} . Let us consider also the image of $W^u(O) \cap \{z = z_0\}$ through the equivalence Φ . The image of this curve through Φ is a curve which lies on \mathcal{M} . Since \mathcal{M} is given by the set $\{(\psi(t), \bar{z}) : t \in \mathbb{R}, \bar{z} \in \mathbb{R}\} \cap \mathcal{U}$ inside \mathcal{U} , and Φ is a C^1 -diffeomorphism then this curve is a spiral curve which lies inside the image by Φ of the plane $\{z = z_0\}$, which is surface transversal to the flow associated to System (12), and C^1 -close to the plane $\{\bar{z} = \Phi(z_0)\}$ in an appropriately small neighborhood. See Figure 4. This proves that $W^u(O) \cap \{z = z_0\}$ is a spiral curve. Now, by the transition map, we can prove that for all $z_* > 0$, $W^u(O) \cap \{z = z_*\}$ is a spiral curve which winds around the point $W^u(P) \cap \{z = z_*\}$.

It follows from Lemma 4 in the appendix that the curves $W^u(O) \cap \{z = \lambda_*\}$ and $\{z = \lambda_*, x = 0\}$ intersect an infinite number of times and each of these

intersections corresponds to distinct solutions of (2). This explains the oscillating behaviour of the bifurcation diagram: See Figure 5. The continuity of the solutions of the initial value problem in terms of the parameters implies that if λ is close enough to λ_* , a large number of these intersections will persist. This concludes the proof. \square

4. PROOF OF THEOREM 3

Here we consider the equation

$$\begin{cases} -\Delta u = \lambda(u+1)^p + \lambda(u+1)^q & \text{in } B, \\ u > 0 & \text{in } B, \quad u = 0 \quad \text{on } \partial B. \end{cases}$$

We are looking for radially symmetric solutions of this equation, namely solutions of the boundary value problem

$$(13) \quad \begin{cases} u'' + \frac{N-1}{r} u' + \lambda(u+1)^p + \lambda(u+1)^q = 0 & 0 < r < 1, \\ u'(0) = 0, \quad u(1) = 0 & \text{and } u(r) > 0 \quad \text{for } 0 < r < 1. \end{cases}$$

Let us denote $v = u + 1$, so that this problem is equivalent to

$$\begin{cases} v'' + \frac{N-1}{r} v' + \lambda v^p + \lambda v^q = 0, & 0 < r < 1, \\ v'(0) = 0, \quad v(1) = 1 & \text{and } v(r) > 1 \quad \text{for } 0 < r < 1. \end{cases}$$

We absorb the parameter λ by setting $r = s/\sqrt{\lambda}$, $v(r) = w(s)$, so that the problem becomes

$$(14) \quad \begin{cases} w'' + \frac{N-1}{s} w' + w^p + w^q = 0, & 0 < s < 1, \\ w'(0) = 0, \quad w(\sqrt{\lambda}) = 1 & \text{and } w(s) > 1 \quad \text{for } 0 < s < \sqrt{\lambda}. \end{cases}$$

Similarly as before, we introduce for Equation (14) the transformation

$$x(t) = s^{\frac{2}{p-1}} w(s)|_{s=e^t}$$

to obtain the equivalent equation

$$(15) \quad x'' + \alpha x' + x^p + e^{\gamma t} x^q - \beta x = 0 \quad -\infty < t < \nu,$$

where $\nu = \ln(\sqrt{\lambda})$ and α, β, γ are explicit positive constants given by:

$$\alpha = (N-2) - \frac{4}{p-1}, \quad \beta = \frac{2}{p-1} \left(N-2 - \frac{2}{p-1} \right), \quad \gamma = 2 \frac{p-q}{p-1}.$$

Let us observe that conditions on w in (14) become:

$$(16) \quad x(t) > e^{\frac{2t}{p-1}} \quad \text{in } (-\infty, \nu), \quad \lim_{t \rightarrow -\infty} x(t) \rightarrow 0 \quad \text{and} \quad x(\nu) = \lambda^{\frac{1}{p-1}}.$$

Introducing the variables $y = x'$ and $z = e^{\gamma t}$ as in the proof of Theorem 1, we obtain the three-dimensional system

$$(17) \quad \begin{cases} x' = y, \\ y' = -\alpha y + \beta x - x^p - z x^q \\ z' = \gamma z \end{cases}$$

with as before the additional condition $z \geq 0$. A solution $x(t)$ of problem (15)-(16) is in correspondence with a solution $\varphi(t) = (x(t), y(t), z(t))$ of (17) such that:

$$(18) \quad x(t) > e^{\frac{2t}{p-1}} \quad \text{in } (-\infty, \nu), \quad \lim_{t \rightarrow -\infty} \varphi(t) = (0, 0, 0) \quad \text{and} \quad x(\nu) = \lambda^{\frac{1}{p-1}}.$$

In terms of System (17) the condition $x(t) > e^{2t/(p-1)}$ is then equivalent to $x > z^{1/(p-q)}$. In fact, $z = e^{\gamma t}$ and $\gamma = 2 \frac{p-q}{p-1}$. We observe the role of the surface given by $x = z^{1/(p-q)}$ is the same as that of the plane $\{x = 0\}$ in System (9) in the sense of the trajectories which satisfy respectively (18) and (10) must be to the right of these surfaces.

On the other hand, the boundary condition $x(\nu) = \lambda^{1/(p-1)}$ can be written as $x(\nu) = z^{1/(p-q)}(\nu)$. This means that the corresponding solution of System (17) has to intersect the surface $x = z^{1/(p-q)}$ when $t = \nu$.

Let us briefly explain the phase space structure of System (17) which in the end turns out to be analogous to that of System (9). We have two hyperbolic points of (17) which lie on the plane $\{z = 0\}$, $O = (0, 0, 0)$ and $P = (\beta^{1/(p-1)}, 0, 0)$. The system restricted to $z = 0$ is invariant under the associated flow and inside $\{z = 0\}$, there is an orbit which connects the points O and P . This orbit coincides with $W^u(O) \cap \{z = 0\}$. On the other hand, if p satisfies

$$(19) \quad p < \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4} \quad \text{or} \quad N \leq 10,$$

we obtain that P is a spiral attractor in $\{z = 0\}$. Indeed, linearizing around this point, we get the eigenvalues

$$\frac{1}{2} \left(-\alpha \pm \sqrt{\alpha^2 - 4\beta(p-1)} \right)$$

which are complex with non-zero imaginary part if and only relation (19) holds. The point P is an attractor in $\{z = 0\}$, since $\alpha > 0$.

Now, for $z \geq 0$ the unstable manifold of O , $W^u(O)$, is two-dimensional. Linearizing around O , we obtain one negative eigenvalue and the two others are positive: γ and $2/(p-1)$, with respective eigenvectors $(0, 0, 1)$ and $(1, 2/(p-1), 0)$.

Linearizing around P , we have one positive eigenvalue γ , so that the unstable manifold of P , $W^u(P)$, is one-dimensional. Its associated eigenvector

is $(1, \gamma, \frac{(p-1)\beta+2\alpha+\gamma^2}{\beta^{1/(p-1)}})$. The stable manifold of P , $W^s(P)$, is two-dimensional and lies on $z = 0$. Moreover, the heteroclinic orbit which connects O and P is contained in $W^s(P)$. This discussion can be summarized as follows.

Lemma 3. *Consider System (17). The following assertions hold:*

- (i) *The unstable manifold of P , $W^u(P)$, corresponds to the positive singular solution of the Equation (13). This solution is unique since $W^u(P)$ is one-dimensional.*
- (ii) *$W^u(P)$ lies on the boundary of $W^u(O)$.*
- (iii) *Solutions of System (17)-(18) lie inside to $W^u(O)$. Moreover, they must be to the right of the surface $x = z^{1/(p-q)}$ and eventually cross it. The unstable manifold of P crosses this surface at a some first value $z = \lambda_* > 0$.*
- (iv) *The condition $x(\nu) = \lambda^{1/(p-1)}$ in (18), in terms of the $W^u(O)$, means that this manifold intersects the surface $x = z^{1/(p-q)}$.*

Assertions (i) and (ii) have already been discussed for System (9). Assertion (iv) directly follows from Conditions (18). Let us now observe that a solution $u(r)$ of (13) which for some $r_0 > 0$ satisfies $u(r_0) > 0$, also satisfies

$$-(r^{N-1} u'(r))' \geq 2\lambda r^{N-1}$$

in any interval (r_0, r_1) where $u(r) > 0$. It readily follows that there must exist a first $r_1 > r_0$ with $u(r_1) = 0$. The conditions $u = 0$ in terms of System (17) is nothing but the surface $\{z = x^{p-1}\}$. A consequence is that $W^u(O) \cap \{z = x^{p-1}\}$ is a curve in a neighborhood of O , which locally splits $W^u(O)$ into two components. Trajectories which lie below this surface are eventually forced to cross it. This of course also holds for a trajectory representing the unstable manifold of P . The validity of (iii) thus follows. \square

Let us conclude the proof of Theorem 2. The value $z = \lambda_*$ represents exactly the value for which the unique trajectory corresponding to the singular solution crosses the surface defined by the zero boundary value for the solution of (13). The rest of the theorem follows in exactly the same way as in the proof of Theorem 1. \square

5. APPENDIX: SOME TOPOLOGICAL FACTS

This appendix is devoted to the proofs of Lemma 1, Lemma 2, and to the statement (see below) and the proof of Lemma 4 which enters in the proof of Theorems 1 and 3 as follows. We have to prove that the two pairs of curves

$$\begin{array}{ll} W^u(O) \cap \{z = \lambda_*\} & \text{and} \quad \{z = \lambda_*, x = 0\}, \\ W^u(O) \cap \{z = \lambda_*\} & \text{and} \quad \{z = \lambda_*, x = z^{1/(p-q)}\} \end{array}$$

intersect infinitely many times. $W^u(O) \cap \{z = \lambda_*\}$ is a spiral curve which winds around the point given by $W^u(P) \cap \{z = \lambda_*\}$. By definition of λ_* , we also know that this point lies on the line $\{z = \lambda_*, x = 0\}$ or on the curve

$\{z = \lambda_*, x = z^{1/(p-q)}\}$, in case of Theorems 1 and 3 respectively. The result is going to be a consequence of a simple topological lemma.

Let P_* be a point in the plane and consider a spiral curve $\sigma(t)$ around P_* of the following form:

$$\sigma(t) = P_* + r(t) e^{i\theta(t)}, \quad t \in [0, +\infty),$$

where $r > 0$ and θ are continuous functions in $[0, +\infty)$, such that $r(t) \rightarrow 0$ as $t \rightarrow +\infty$ and $\theta(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. We can assume that $0 < r(t) < r(0)$ for $t > 0$, and also that $\theta(t) \leq \theta(0) = 0$ for $t \geq 0$.

Let now $\tau(s)$, $s \in [0, 1]$, be a continuous curve, such that $\tau(0) = P_*$, and assume additionally $\tau(s)$ is to the *right* of P_* for $s \in (0, 1]$. Writing

$$\tau(s) = P_* + \rho(s) e^{i\mu(s)}, \quad s \in (0, 1],$$

where $\rho > 0$ and μ are continuous functions in $(0, 1]$, this means that $\rho(0) = 0$ and there exists an integer $k \in \mathbb{Z}$ such that $-\pi/2 \leq \mu - 2k\pi \leq \pi/2$. Assume further that both σ and τ do not have self-intersections.

Lemma 4. *Let σ and τ be curves as above. Assume that $\rho(s) < r(0)$ for all $s \in (0, 1]$. Then the curves σ and τ intersect an infinite number of times.*

Proof of Lemma 4. In polar coordinates centered around P_* , liftings of σ and τ are respectively given by $\tilde{\sigma}(t) = (r(t), \theta(t))$ and $\tilde{\tau}(s) = (\rho(s), \mu(s))$. Jordan's theorem implies that the curve $\tilde{\sigma}$ separates the strip $(0, r(0)] \times \mathbb{R}$ of the r - θ plane into two components A_- and A_+ to the *left* and to the *right* of the curve $\tilde{\sigma}$ respectively. Also, the curve $\tilde{\tau}$ entirely lies inside this strip. The family of translates $\tilde{\tau}_k = \tilde{\tau} + (0, -2k\pi)$ are also liftings of τ . Given a number n , consider t_n such that $\theta(t) < \theta(t_n)$ and $r(t) < 1/n$ for all $t > t_n$. Then, if k is chosen sufficiently large, the following happens: there are points of $\tilde{\tau}_k$ which lie on A_- , while necessarily $\tilde{\tau}_k(1)$ lies on A^+ . It follows by connectedness that the curve $\tilde{\tau}_k$ intersects $\tilde{\sigma}$. Since the r -coordinate of this point is less than $1/n$, and n is arbitrary, infinitely many intersections of the original curves σ and τ are obtained, as desired. \square

Proof of Lemma 1. For the proof of this result we will make use of the well-known Palis λ -lemma, see for instance [28] or [18]. Its statement in the context we are dealing with is as follows.

Lemma 5. *Consider in \mathbb{R}^n an ODE of the form $x' = f(x)$, where f is of class C^1 , in a neighborhood \mathcal{V} of a hyperbolic singularity P . Let X_t denote its associated flow and consider the (local) stable and unstable manifolds $W^s(P)$, $W^u(P)$, with respective dimensions n_s and n_u . Let D be an n_u dimensional disk which intersects transversally $W^s(P)$ and contains a point Q of $W^u(P)$. Let B^u be any disk inside $W^u(P)$ which contains Q . Let D_t be the connected component of $X_t(D) \cap \mathcal{V}$ which contains $X_t(Q)$. Then given $\varepsilon > 0$, there exists a $t_0 > 0$ such that D_t is C^1 ε -close to B^u for all $t > t_0$. In particular, given any point $Q' \in B^u$, there is a point in D_t which is at a distance less than ε from Q' .*

As we have seen, P is a hyperbolic singularity whose unstable manifold $W^u(P)$ is a one dimensional curve, and $W^s(P)$ two-dimensional. Take a short segment transversal to the plane $\{z = 0\}$ which lies entirely in the two dimensional manifold $W^u(O)$ (take it for instance close and almost parallel to the z -axis). By virtue of the above lemma, the flow will take this segment into a one dimensional segment, still contained in $W^u(O)$, which gets arbitrarily uniformly close to any given finite piece of the curve $W^u(P)$. This proves that $W^u(P)$ lies in the boundary of $W^u(O)$. \square

Proof of Lemma 2. To this end we employ the following result, due to Belickii, [3, 4].

Lemma 6. *Consider an ODE of the form $x' = f(x)$ with $f(x_0) = 0$ and f smooth in a neighborhood of x_0 . Assume that x_0 is a hyperbolic saddle of f , with eigenvalues $\lambda_1, \dots, \lambda_n$. Assume also that none of the relations $Re \lambda_i = Re \lambda_j + Re \lambda_k$ is fulfilled. Then the system is C^1 -equivalent in a neighborhood of x_0 to its linear part, in the sense that there is a C^1 diffeomorphism Φ from a neighborhood of x_0 onto a neighborhood of 0, with $\Phi(x_0) = 0$ which takes trajectories of the system $x' = f(x)$ into trajectories of the linear system $y' = f'(x_0)y$, preserving orientation.*

This result applies immediately to Systems (9) and (17) around P if the unstable eigenvalues are not real, since we have two eigenvalues with the same negative real parts, and a third eigenvalue which is positive. Then none of the relations $Re \lambda_i = Re \lambda_j + Re \lambda_k$ is possible. \square

6. FIGURES

In this section, computations are done in the case: $N = 5$ and $p = 7/3 + 0.3$.

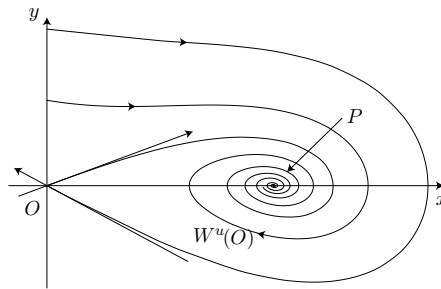


FIGURE 1. The phase space restricted to $\{z = 0\}$ and the heteroclinic trajectory $t \mapsto \psi(t)$ connecting O to P in this plane.

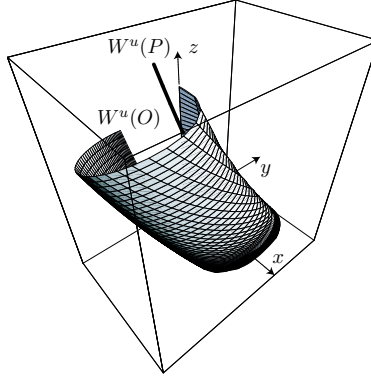


FIGURE 2. The 3-dimensional phase space with the 2-dimensional unstable manifold $W^u(O)$, which contains the axis $\{x = 0, y = 0\}$, and the 1-dimensional unstable manifold $W^u(P)$, which is contained in the adherence of $W^u(O)$.

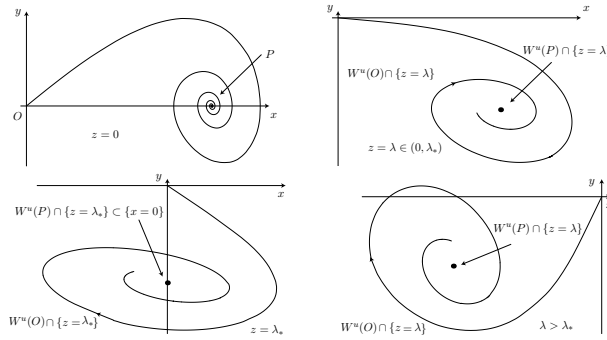


FIGURE 3. Intersection of the unstable manifolds $W^u(O)$ and $W^u(P)$ with planes $\{z = z_*\}$ for $z_* = 0$, $z_* = \lambda \in (0, \lambda_*)$, $z_* = \lambda_*$, $z_* = \lambda > \lambda_*$.

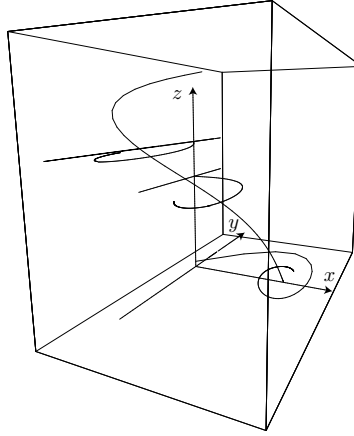


FIGURE 4. Several 3-dimensional sections of the unstable manifold $W^u(O)$ corresponding to the planes $\{z = z_*\}$ for $z_* = \lambda \in (0, \lambda_*)$ and 3-dimensional representation of $W^u(P)$.

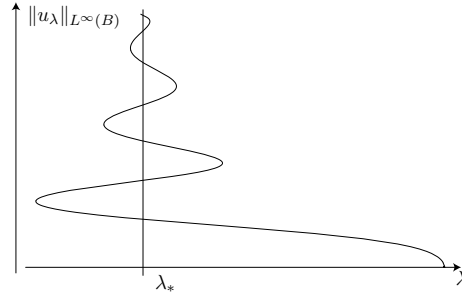


FIGURE 5. The bifurcation diagram in $\mathbb{R}^+ \times L^\infty(B)$.

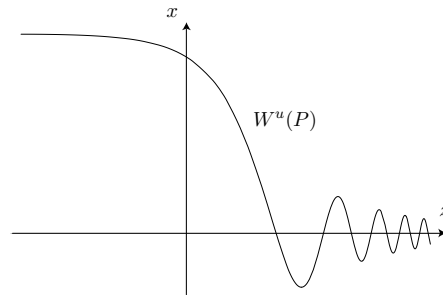


FIGURE 6. The unstable manifold $W^u(P)$ in (z, x) coordinates (z is shown in a logarithmic scale). Finding the first zero determines $z_* = \lambda_*$. Here for $N = 5$ and $p = 7/3 + 0.3$, we find $\lambda_* \approx 12.07\dots$

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