Extremal functions and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

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(joint work with M. Esteban, M. Loss, G. Tarantello, A. Tertikas)

We consider the extremal functions for the interpolation inequalities introduced by Caffarelli, Kohn and Nirenberg in [1], that can be written as

$$(1)\quad \left(\int_{\mathbb{R}^d}\frac{|u|^p}{|x|^{bp}}\,dx\right)^{\frac{2}{p}}\le \mathsf{C}_{\rm CKN}(\theta,a,b)\left(\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d}\frac{|u|^2}{|x|^{2(a+1)}}\,dx\right)^{1-\theta}$$

where u is a smooth function with compact support in $\mathbb{R}^d\setminus\{0\}$ and the parameters are in the range: $b\in(a+1/2,a+1]$ if $d=1$, $b\in(a,a+1]$ if $d=2$ and $b\in[a,a+1]$ if $d\ge 3$, $a\neq(d-2)/2=:a_c$, $p=\frac{2d}{d-2+2(b-a)}$ and $\theta\in[\vartheta(p,d),1]$ with $\vartheta(p,d):=d\,(p-2)/(2\,p)$.

We also consider *weighted logarithmic Hardy* inequalities, introduced in [4], which correspond to the limit $\theta=\gamma\,(p-2)$, $p\to 2_+$ and read as

$$(2)\quad \int_{\mathbb{R}^d}\frac{|u|^2}{|x|^{2(a+1)}}\,\log\left(|x|^{d-2-2a}\,|u|^2\right)dx\le 2\gamma\,\log\left[\mathsf{C}_{\rm GLH}(\gamma,a)\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2a}}\,dx\right]$$

for any smooth function u such that $\||x|^{-(a+1)}|u|\|_{L^2(\mathcal{C})}=1$, with compact support in $\mathbb{R}^d\setminus\{0\}$. The parameters are such that $a<a_c$, $\gamma\ge d/4$ and $\gamma>1/2$ if $d=2$.

Inequalities (1) and (2) can be extended to the larger space $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ obtained by completion with respect to the norm $u\mapsto\int_{\mathbb{R}^d}|x|^{-2a}\,|\nabla u|^2\,dx$. *Extremal* functions are such that the inequalities, written with their optimal constants, become equalities. We shall assume that $\mathsf{C}_{\rm CKN}(\theta,p,a)$ and $\mathsf{C}_{\rm GLH}(\gamma,a)$ are optimal, *i.e.* take their lowest possible value. By a Kelvin transformation (see [7]), the case $a>a_c$ can be reduced to the case $a<a_c$. For simplicity, we shall assume that $a<a_c$.

The case $\theta=1$, $p\in[2,2^*]$ and $d\ge 3$ has been widely discussed in the literature. Existence of extremal functions for (1) has been studied in various papers in case $\theta=1$: see [2] and references therein for details. Radial symmetry of the extremal functions is an important issue, which has been established in a number of cases: see [3, 6, 7, 10, 11, 6]. Extremal functions are then entirely determined and the value of the optimal constants is known. On the other hand, *symmetry breaking*, which means that extremal functions are not radially symmetric, holds for

$$(3)\qquad d\ge 2\;,\quad b<\frac{1}{2}\,(a-a_c)\left[2-\frac{d}{\sqrt{(a-a_c)^2+d-1}}\right],$$

as it has been established in [2, 7, 9]. Moreover, according to [6], a continuous curve $p\mapsto a(p)$ with values in the region $a<0$, $b<a+1$ separates the symmetry breaking region from the region where radial symmetry holds.

The case $\theta<1$ of Inequality (1) has been much less considered. Symmetry breaking has been established in [4] in a region which extends the one found in [7, 9]. If either $d=1$ or $d\ge 2$ but for radial functions, existence of extremal functions for (1) has been proved in [4] for any $\theta>\vartheta(p,d)$. However, the best constant is not achieved if $\theta = \vartheta(p, d)$ and $d = 1$. Existence of extremal functions without symmetry assumption and some results of radial symmetry have also been obtained in [5, 8].

A symmetry breaking result for (2) has been established in [4] when

$$d \geq 2\ , \quad a < -1/2 \quad \text{and} \quad \gamma < \frac{1}{4} + \frac{(a - a_c)^2}{d - 1}\ . \tag{4}$$

It is very convenient to reformulate Inequalities (1) and (2) in cylindrical variables as in [2]. By means of the Emden-Fowler transformation

$$t = \log|x| \in \mathbb{R}\ , \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}\ , \quad y = (t, \omega)\ , \quad v(y) = |x|^{a_c - a}\, u(x)\ ,$$

Inequality (1) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C} := \mathbb{R} \times \mathbb{S}^{d-1}$: for any $v \in H^1(\mathcal{C})$,

$$\left(\int_{\mathcal{C}} |v|^p\, dy \right)^{\frac{2}{p}} \leq \mathsf{C}_{\rm CKN}(\theta, p, a) \left(\int_{\mathcal{C}} |\nabla v|^2\, dy + \Lambda \int_{\mathcal{C}} |v|^2\, dy \right)^{\theta} \left(\int_{\mathcal{C}} |v|^2\, dy \right)^{1-\theta}$$

with $\Lambda := (a_c - a)^2$. Similarly, with $w(y) = |x|^{a_c - a}\, u(x)$, (2) is equivalent to

$$\int_{\mathcal{C}} |w|^2 \log|w|^2\, dy \leq 2\,\gamma\, \log\left[\mathsf{C}_{\rm WLH}(\gamma, a) \Big(\int_{\mathcal{C}} |\nabla w|^2\, dy + \Lambda \Big) \right]$$

for any $w \in H^1(\mathcal{C})$ such that $\|w\|_{L^2(\mathcal{C})} = 1$. We shall denote by $\mathsf{C}^*_{\rm CKN}(\theta, p, a)$ and $\mathsf{C}^*_{\rm WLH}(\gamma, a)$ the optimal constants for (1) and (2) respectively, when the set of functions is restricted to the radially symmetric ones. From [4], we know that

$$\begin{aligned} &\mathsf{C}_{\rm CKN}(\theta, p, a) \geq \mathsf{C}^*_{\rm CKN}(\theta, p, a) = \mathsf{C}^*_{\rm CKN}(\theta, p, a_c - 1)\, \Lambda^{\frac{p-2}{2p} - \theta} \\ &\mathsf{C}_{\rm WLH}(\gamma, a) \geq \mathsf{C}^*_{\rm WLH}(\gamma, a) = \mathsf{C}^*_{\rm WLH}(\gamma, a_c - 1)\, \Lambda^{-1 + 1/(4\gamma)} \end{aligned}$$

where $\Lambda = (a - a_c)^2$. Symmetry breaking means that the above inequalities are strict. Finding extremal functions amounts to minimize the functionals

$$\begin{aligned} &\mathcal{E}[v] := \Big(\|\nabla v\|^2_{L^2(\mathcal{C})} + \Lambda\, \|v\|^2_{L^2(\mathcal{C})} \Big)^{\theta}\, \|v\|^{2(1-\theta)}_{L^2(\mathcal{C})} \big/ \|v\|^2_{L^p(\mathcal{C})}\ , \\ &\mathcal{F}[w] := \frac{\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\, \|w\|^2_{L^2(\mathcal{C})}}{\|w\|^2_{L^2(\mathcal{C})}}\, \exp\left[-\frac{1}{2\gamma} \int_{\mathcal{C}} \frac{|w|^2}{\|w\|^2_{L^2(\mathcal{C})}}\, \log\left(\frac{|w|^2}{\|w\|^2_{L^2(\mathcal{C})}} \right) dy \right]. \end{aligned}$$

Radial symmetry for (1) and (2) means that there are minimizers of $\mathcal{E}$ and $\mathcal{F}$ which depend only on t.

The method of [2, 9, 4] for proving symmetry breaking goes as follows. In case of Inequality (1), consider a symmetric minimizer v_* of $\mathcal{E}$, depending only on t. Up to a scaling and a multiplication by a constant, $v_*(t) = (\cosh t)^{-2/(p-2)}$ solves

$$(p-2)^2\, v'' - 4\, v + 2\, p\, |v|^{p-2}\, v = 0\ .$$

An expansion of $\mathcal{E}[v]$ at order two around v_* involves the operator $\mathcal{L} := -\Delta + \kappa\, {w_*}^{p-2} + \mu$ for some κ and μ which are explicit in terms of θ, p and d. Eigenfunctions are characterized in terms of Legendre's polynomials and spherical harmonic functions. The eigenspace of $\mathcal{L}$ corresponding to the lowest eigenfunction is generated by w_* (after a multiplication by a constant and a scaling). The eigenfunction $\lambda_{1,0}$ associated to the first non trivial spherical harmonic function is not radially symmetric. Condition (3) is determined by requiring that $\lambda_{1,0} < 0$, which implies that $\mathsf{C}_{\rm CKN}(\theta, p, a) > \mathsf{C}^*_{\rm CKN}(\theta, p, a)$. In case of Inequality (2), a similar analysis can be done. The radial minimizer is a Gaussian function in t and the operator $\mathcal{L}$ is the Schrödinger operator with harmonic potential.

Symmetry results in [6, 8] also involves some spectral analysis. By considering sequences $(v_n)_{n\in\mathbb{N}}$ of minimizers of $\mathcal{E}$ appropriately normalized by the condition $\|v_n\|^2_{L^p(\mathcal{C})} = 1$, one proves that $\|\nabla v_n\|^2_{L^2(\mathcal{C})}$ is bounded when either $b = b_n$ converges to $a+1$, or $a = a_n \to 0_-$ if $\theta = 1$, or $a = a_n \to a_{c-}$ if $\theta < 1$. Minimizers being solutions of an elliptic PDE, the convergence to a limit actually holds locally uniformly, which allows to write a linear equation for $D_\omega v_n$, where D_ω denotes an appropriate derivative with respect to ω. By spectral gap considerations, we conclude that $D_\omega v_n \equiv 0$ for n large enough: v_n depends only on t.

Using scaling properties, it has been proved in [6, 8] that there is a curve separating the region of symmetry for (3) from the region of symmetry breaking. The same property holds for (2). However, in both cases, no quantitative estimates are known about the position of the curve in the region $a < 0$. It is an open question to decide whether it coincides with the region defined by (3) and (4) or not.

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