Extremal functions in some interpolation inequalities: Symmetry, symmetry breaking and estimates of the best constants

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This contribution is devoted to a review of some recent results on existence, symmetry and symmetry breaking of optimal functions for Caffarelli-Kohn-Nirenberg (CKN) and weighted logarithmic Hardy (WLH) inequalities. These results have been obtained in a series of papers[1–5] in collaboration with M. del Pino, S. Filippas, M. Loss, G. Tarantello and A. Tertikas and are presented from a new viewpoint.

Keywords: Caffarelli-Kohn-Nirenberg inequality; Gagliardo-Nirenberg inequality; logarithmic Hardy inequality; logarithmic Sobolev inequality; extremal functions; radial symmetry; symmetry breaking; Emden-Fowler transformation; linearization; existence; compactness; optimal constants

1. Two families of interpolation inequalities

Let $d \in \mathbb{N}^*$, $\theta \in [0,1]$, consider the set $\mathcal{D}$ of all smooth functions which are compactly supported in $\mathbb{R}^d \setminus \{0\}$ and define $\vartheta(d,p) := d\,\frac{p-2}{2\,p}$, $a_c := \frac{d-2}{2}$, $\Lambda(a) := (a-a_c)^2$ and $p(a,b) := \frac{2\,d}{d-2+2\,(b-a)}$. We shall also set $2^* := \frac{2\,d}{d-2}$ if $d \geq 3$ and $2^* := \infty$ if $d = 1$ or 2. For any $a < a_c$, we consider the two families of interpolation inequalities:

(CKN) *Caffarelli-Kohn-Nirenberg inequalities*[3,4,6] – Let $b \in (a+1/2, a+1]$ and $\theta \in (1/2,1]$ if $d=1$, $b \in (a,a+1]$ if $d=2$ and $b \in [a,a+1]$ if $d \geq 3$. Assume that $p = p(a,b)$, and $\theta \in [\vartheta(d,p),1]$ if $d \geq 2$. There exists a finite positive constant $\mathsf{C}_{\rm CKN}(\theta,p,a)$ such that, for any $u \in \mathcal{D}$,

$$\||x|^{-b}\,u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathsf{C}_{\rm CKN}(\theta,p,a)\,\||x|^{-a}\,\nabla u\|^{2\,\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\||x|^{-(a+1)}\,u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R}^d)}\,.$$

(WLH) *Weighted logarithmic Hardy inequalities*[3,4] – Let $\gamma \geq d/4$ and $\gamma > 1/2$ if $d=2$. There exists a positive constant $\mathsf{C}_{\rm WLH}(\gamma,a)$ such that, for any $u \in \mathcal{D}$, normalized by $\||x|^{-(a+1)}\,u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$,

$$\int_{\mathbb{R}^d} \frac{|u|^2\,\log\left(|x|^{d-2-2\,a}\,|u|^2\right)}{|x|^{2\,(a+1)}}\,dx \leq 2\,\gamma\,\log\Big[\mathsf{C}_{\rm WLH}(\gamma,a)\,\|\,|x|^{-a}\,\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\Big]\,.$$

(WLH) appears as a limiting case[3,4] of (CKN) with $\theta = \gamma\,(p-2)$ as $p \to 2_+$. By a standard completion argument, these inequalities can be extended to the set

$\mathcal{D}^{1,2}_a(\mathbb{R}^d) := \{u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)\ :\ |x|^{-a}\,\nabla u \in \mathrm{L}^2(\mathbb{R}^d)\ \textit{and}\ |x|^{-(a+1)}\,u \in \mathrm{L}^2(\mathbb{R}^d)\}$. We shall assume that all constants in the inequalities are taken with their optimal values. For brevity, we shall call *extremals* the functions which realize equality in (CKN) or in (WLH).

Let $\mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$ and $\mathsf{C}^*_{\mathrm{WLH}}(\gamma,a)$ denote the optimal constants when admissible functions are restricted to the radial ones. *Radial extremals* are explicit and the values of the constants, $\mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$ and $\mathsf{C}^*_{\mathrm{WLH}}(\gamma,a)$, are known.[3] Moreover, we have

$$\begin{aligned} \mathsf{C}_{\mathrm{CKN}}(\theta,p,a) &\ge \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a_c-1)\,\Lambda(a)^{\frac{p-2}{2p}-\theta}\,,\\ \mathsf{C}_{\mathrm{WLH}}(\gamma,a) &\ge \mathsf{C}^*_{\mathrm{WLH}}(\gamma,a) = \mathsf{C}^*_{\mathrm{WLH}}(\gamma,a_c-1)\,\Lambda(a)^{-1+\frac{1}{4\gamma}}\,. \end{aligned} \tag{1}$$

Radial symmetry for the extremals of (CKN) and (WLH) implies that $\mathsf{C}_{\mathrm{CKN}}(\theta,p,a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$ and $\mathsf{C}_{\mathrm{WLH}}(\gamma,a) = \mathsf{C}^*_{\mathrm{WLH}}(\gamma,a)$, while *symmetry breaking* only means that inequalities in (1) are strict.

2. Existence of extremals

Theorem 2.1. *Equality*[4] *in (CKN) is attained for any* $p \in (2,2^*)$ *and* $\theta \in (\vartheta(p,d),1)$ *or* $\theta = \vartheta(p,d)$, $d \ge 2$ *and* $a \in (a^{\mathrm{CKN}}_\star, a_c)$, *for some* $a^{\mathrm{CKN}}_\star < a_c$. *It is not attained if* $p=2$, *or* $a<0$, $p=2^*$, $\theta=1$ *and* $d\ge 3$, *or* $d=1$ *and* $\theta=\vartheta(p,1)$.

Equality[4] *in (WLH) is attained if* $\gamma \ge 1/4$ *and* $d=1$, *or* $\gamma > 1/2$ *if* $d=2$, *or for* $d\ge 3$ *and either* $\gamma > d/4$ *or* $\gamma = d/4$ *and* $a \in (a^{\mathrm{WLH}}_\star, a_c)$, *where* $a^{\mathrm{WLH}}_\star := a_c - \sqrt{\Lambda^{\mathrm{WLH}}_\star}$ *and* $\Lambda^{\mathrm{WLH}}_\star := (d-1)\,e\,(2^{d+1}\,\pi)^{-1/(d-1)}\,\Gamma(d/2)^{2/(d-1)}$.

Let us give some hints on how to prove such a result. Consider first Gross' logarithmic Sobolev inequality in Weissler's form[7]

$$\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,dx \le \frac{d}{2}\,\log\Big(\mathsf{C}_{\mathrm{LS}}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\Big) \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d) \text{ s.t. } \|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1\,.$$

The function $u(x) = (2\,\pi)^{-d/4}\,\exp(-|x|^2/4)$ is an extremal for such an inequality. By taking $u_n(x) := u(x+n\,\mathsf{e})$ for some $\mathsf{e} \in \mathbb{S}^{d-1}$ and any $n \in \mathbb{N}$ as test functions for (WLH), and letting $n \to +\infty$, we find that $\mathsf{C}_{\mathrm{LS}} \le \mathsf{C}_{\mathrm{WLH}}(d/4,a)$. If equality holds, this is a mechanism of loss of compactness for minimizing sequences. On the opposite, if $\mathsf{C}_{\mathrm{LS}} < \mathsf{C}_{\mathrm{WLH}}(d/4,a)$, which is the case if $a \in (a^{\mathrm{WLH}}_\star, a_c)$ where $a^{\mathrm{WLH}}_\star = a$ is given by the condition $\mathsf{C}_{\mathrm{LS}} = \mathsf{C}^*_{\mathrm{WLH}}(d/4,a)$, we can establish a compactness result which proves that equality is attained in (WLH) in the critical case $\gamma = d/4$.

A similar analysis for (CKN) shows that $\mathsf{C}_{\mathrm{GN}}(p) \le \mathsf{C}_{\mathrm{CKN}}(\theta,p,a)$ in the critical case $\theta = \vartheta(p,d)$, where $\mathsf{C}_{\mathrm{GN}}(p)$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev interpolation inequalities

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)$$

and $p \in (2,2^*)$ if $d=2$ or $p \in (2,2^*]$ if $d \ge 3$. However, extremals are not known explicitly in such inequalities if $d \ge 2$, so we cannot get an explicit interval of existence in terms of a, even if we also know that compactness of minimizing sequences for (CKN) holds when $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\vartheta(p,d),p,a)$. This is the case if $a > a_\star^{\rm CKN}$ where $a = a_\star^{\rm CKN}$ is defined by the condition $\mathsf{C}_{\rm GN}(p) = \mathsf{C}^*_{\rm CKN}(\vartheta(p,d),p,a)$.

It is very convenient to reformulate (CKN) and (WLH) inequalities in cylindrical variables.[8] By means of the Emden-Fowler transformation

$$s = \log|x| \in \mathbb{R}\,,\quad \omega = x/|x| \in \mathbb{S}^{d-1}\,,\quad y = (s,\omega)\,,\quad v(y) = |x|^{a_c - a}\,u(x)\,,$$

(CKN) for u is equivalent to a Gagliardo-Nirenberg-Sobolev inequality on the cylinder $\mathcal{C} := \mathbb{R}\times\mathbb{S}^{d-1}$ for v, namely

$$\|v\|^2_{\mathrm{L}^p(\mathcal{C})} \le \mathsf{C}_{\rm CKN}(\theta,p,a)\,\Big(\,\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^\theta\,\|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}\quad \forall\, v\in \mathrm{H}^1(\mathcal{C})$$

with $\Lambda = \Lambda(a)$. Similarly, with $w(y) = |x|^{a_c - a}\,u(x)$, (WLH) is equivalent to

$$\int_{\mathcal{C}} |w|^2\,\log|w|^2\,dy \le 2\,\gamma\,\log\Big[\mathsf{C}_{\rm WLH}(\gamma,a)\,\Big(\|\nabla w\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\Big)\Big]$$

for any $w \in \mathrm{H}^1(\mathcal{C})$ such that $\|w\|_{\mathrm{L}^2(\mathcal{C})} = 1$. Notice that radial symmetry for u means that v and w depend only on s.

Consider a sequence $(v_n)_n$ of functions in $\mathrm{H}^1(\mathcal{C})$, which minimizes the functional

$$\mathcal{E}^p_{\theta,\Lambda}[v] := \Big(\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\Big)^\theta\,\|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}$$

under the constraint $\|v_n\|_{\mathrm{L}^p(\mathcal{C})} = 1$ for any $n\in\mathbb{N}$. As quickly explained below, if bounded, such a sequence is relatively compact and converges up to translations and the extraction of a subsequence towards a minimizer of $\mathcal{E}^p_{\theta,\Lambda}$.

Assume that $d \ge 3$, let $t := \|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})}/\|v\|^2_{\mathrm{L}^2(\mathcal{C})}$ and $\Lambda = \Lambda(a)$. If v is a minimizer of $\mathcal{E}^p_{\theta,\Lambda}[v]$ such that $\|v\|_{\mathrm{L}^p(\mathcal{C})} = 1$, then we have

$$(t+\Lambda)^\theta = \mathcal{E}^p_{\theta,\Lambda}[v]\,\frac{\|v\|^2_{\mathrm{L}^p(\mathcal{C})}}{\|v\|^2_{\mathrm{L}^2(\mathcal{C})}} = \frac{\|v\|^2_{\mathrm{L}^p(\mathcal{C})}}{\mathsf{C}_{\rm CKN}(\theta,p,a)\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}} \le \frac{\mathsf{S}_d^{\vartheta(d,p)}}{\mathsf{C}_{\rm CKN}(\theta,p,a)}\,\big(t + a_c^2\big)^{\vartheta(d,p)}$$

where $\mathsf{S}_d = \mathsf{C}_{\rm CKN}(1,2^*,0)$ is the optimal Sobolev constant, while we know from (1) that $\lim_{a\to a_c}\mathsf{C}_{\rm CKN}(\theta,p,a) = \infty$ if $d \ge 2$. This provides a bound on t if $\theta > \vartheta(p,d)$. An estimate can be obtained also for v_n, for n large enough, and standard tools of the concentration-compactness method allow to conclude that, up to a subsequence, $(v_n)_n$ converges towards an extremal. A similar approach holds for (CKN) if $d = 2$, or for (WLH).

The above variational approach also provides an existence result of extremals for (CKN) in the critical case $\theta = \vartheta(p,d)$, if $a \in (a_1,a_c)$ where $a_1 := a_c - \sqrt{\Lambda_1}$ and $\Lambda_1 = \min\{(\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\theta}/\,\mathsf{S}_d)^{d/(d-1)}, (a_c^2\,\mathsf{S}_d/\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\theta})^d$.

If symmetry is known, then there are (radially symmetric) extremals.[3] Anticipating on the results of the next section, we can state the following result which arises as a consequence of Schwarz' symmetrization method (see Theorem 3.2, below).

Proposition 2.1. *Let $d \ge 3$. Then (CKN) with $\theta = \vartheta(p,d)$ admits a radial extremal if*[5] *$a \in [a_0,a_c)$ where $a_0 := a_c - \sqrt{\Lambda_0}$ and $\Lambda = \Lambda_0$ is defined by the condition $\Lambda^{(d-1)/d} = \vartheta(p,d)\,\mathsf{C}^*_{\rm CKN}(\theta,p,a_c-1)^{1/\vartheta(d,p)}/\,\mathsf{S}_d$.*

A similar estimate also holds if $\theta > \vartheta(d,p)$, with less explicit computations.[5]

3. Symmetry and symmetry breaking

Define

$$\underline{a}(\theta,p) := a_c - \tfrac{2\sqrt{d-1}}{p+2}\sqrt{\tfrac{2\,p\,\theta}{p-2}-1}\;,\quad \tilde a(\gamma) := a_c - \tfrac{1}{2}\sqrt{(d-1)(4\,\gamma-1)}\;,$$
$$\Lambda_{\rm SB}(\gamma) := \tfrac{1}{8}\,(4\,\gamma-1)\,e\left(\tfrac{\pi^{4\,\gamma-d-1}}{16}\right)^{\frac{1}{4\,\gamma-1}}\left(\tfrac{d}{\gamma}\right)^{\frac{4\,\gamma}{4\,\gamma-1}}\Gamma\left(\tfrac{d}{2}\right)^{\frac{2}{4\,\gamma-1}}\;.$$

Theorem 3.1. *Let $d \ge 2$ and $p \in (2,2^*)$. Symmetry breaking holds in (CKN) if either[3,5] $a < \underline{a}(\theta,p)$ and $\theta \in [\vartheta(p,d),1]$, or[5] $a < a_\star^{\rm CKN}$ and $\theta = \vartheta(p,d)$.*

Assume that $\gamma > 1/2$ if $d = 2$ and $\gamma \ge d/4$ if $d \ge 3$. Symmetry breaking holds in (WLH) if[3,5] $a < \max\{\tilde a(\gamma), a_c - \sqrt{\Lambda_{\rm SB}(\gamma)}\}$.

When $\gamma = d/4$, $d \ge 3$, we observe that $\Lambda_\star^{\rm WLH} = \Lambda_{\rm SB}(d/4) < \Lambda(\tilde a(d/4))$ with the notations of Theorem 2.1 and there is symmetry breaking if $a \in (-\infty, a_\star^{\rm WLH})$, in the sense that $\mathsf{C}_{\rm WLH}(d/4,a) > \mathsf{C}^*_{\rm WLH}(d/4,a)$, although we do not know if extremals for (WLH) exist when $\gamma = d/4$.

Results of symmetry breaking for (CKN) with $a < \underline{a}(\theta,p)$ have been established first[1,8,9] when $\theta = 1$ and later[3] extended to $\theta < 1$. The main idea in case of (CKN) is consider the quadratic form associated to the second variation of $\mathcal{E}^p_{\theta,\Lambda}$ around a minimizer among functions depending on s only and observe that the linear operator $\mathcal{L}^p_{\theta,\Lambda}$ associated to the quadratic form has a negative eigenvalue if $a < \underline{a}$. Results[3] for (WLH), $a < \tilde a(\gamma)$, are based on the same method.

For any $a < a_\star^{\rm CKN}$, we have $\mathsf{C}^*_{\rm CKN}(\vartheta(p,d),p,a) < \mathsf{C}_{\rm GN}(p) \le \mathsf{C}_{\rm CKN}(\vartheta(p,d),p,a)$, which proves symmetry breaking. Using well-chosen test functions, it has been proved[5] that $\underline{a}(\vartheta(p,d),p) < a_\star^{\rm CKN}$ for $p-2 > 0$, small enough, thus also proving symmetry breaking for $a - \underline{a}(\vartheta(p,d),p) > 0$, small, and $\theta - \vartheta(p,d) > 0$, small.

Theorem 3.2. *For all $d \ge 2$, there exists[2,5] a continuous function a^* defined on the set $\{(\theta,p) \in (0,1] \times (2,2^*) \,:\, \theta > \vartheta(p,d)\}$ such that $\lim_{p\to 2_+} a^*(\theta,p) = -\infty$ with the property that (CKN) has only radially symmetric extremals if $(a,p) \in (a^*(\theta,p),a_c) \times (2,2^*)$, and none of the extremals is radially symmetric if $(a,p) \in (-\infty, a^*(\theta,p)) \times (2,2^*)$.*

Similarly, for all $d \ge 2$, there exists[5] a continuous function $a^{**} : (d/4,\infty) \to (-\infty,a_c)$ such that, for any $\gamma > d/4$ and $a \in [a^{**}(\gamma),a_c)$, there is a radially symmetric extremal for (WLH), while for $a < a^{**}(\gamma)$ no extremal is radially symmetric.

Schwarz' symmetrization allows to characterize[5] a subdomain of $(0,a_c) \times (0,1) \ni (a,\theta)$ in which symmetry holds for extremals of (CKN), when $d \ge 3$. If $\theta = \vartheta(p,d)$ and $p > 2$, there are radially symmetric extremals[5] if $a \in [a_0,a_c)$ where a_0 is given in Propositions 2.1.

Symmetry also holds if $a - a_c$ is small enough, for (CKN) as well as for (WLH), or when $p \to 2_+$ in (CKN), for any $d \ge 2$, as a consequence of the existence of the spectral gap of $\mathcal{L}^p_{\theta,\Lambda}$ when $a > \underline{a}(\theta,p)$.

For given θ and p, there is[2,5] a unique $a^* \in (-\infty, a_c)$ for which there is symmetry breaking in $(-\infty, a^*)$ and for which all extremals are radially symmetric when $a \in (a^*, a_c)$. This follows from the observation that, if $v_\sigma(s,\omega) := v(\sigma\, s, \omega)$ for $\sigma > 0$, then $(\mathcal{E}^p_{\theta,\sigma^2\Lambda}[v_\sigma])^{1/\theta} - \sigma^{(2\,\theta-1+2/p)/\theta^2}\,(\mathcal{E}^p_{\theta,\Lambda}[v])^{1/\theta}$ is equal to 0 if v depends only on s, while it has the sign of $\sigma - 1$ otherwise.

From Theorem 3.1, we can infer that radial and non-radial extremals for (CKN) with $\theta > \vartheta(p,d)$ coexist on the threshold, in some cases.

Numerical results illustrating our results on existence and on symmetry / symmetry breaking have been collected in Fig. 1 below in the critical case for (CKN).

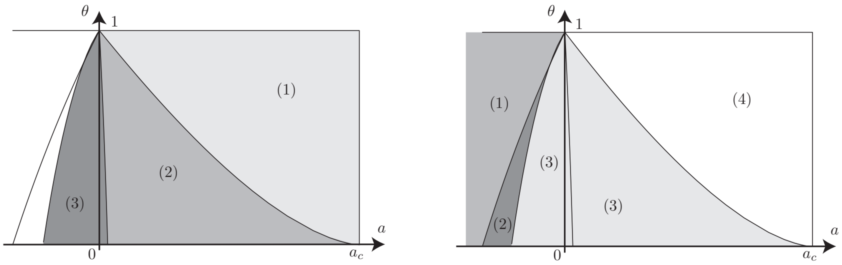

Fig. 1. Critical case for (CKN): $\theta = \vartheta(p,d)$. Here we assume that $d = 5$.
(a) The zones in which existence is known are (1) in which $a \geq a_0$, because extremals are achieved among radial functions, (1)+(2) using the *a priori* estimates: $a > a_1$, and (1)+(2)+(3) by comparison with the Gagliardo-Nirenberg inequality: $a > a_\star^{\rm CKN}$.
(b) The zone of symmetry breaking contains (1) by linearization around radial extremals: $a < \underline{a}(\theta,p)$, and (1)+(2) by comparison with the Gagliardo-Nirenberg inequality: $a < a_\star^{\rm CKN}$; in (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4) symmetry holds by Schwarz' symmetrization: $a_0 \leq a < a_c$.
Numerically, we observe that $\underline{a}$ and $a_\star^{\rm CKN}$ intersect for some $\theta \approx 0.85$.

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