GENERALIZED SOBOLEV INEQUALITIES AND ASYMPTOTIC BEHAVIOUR IN FAST DIFFUSION AND POROUS MEDIUM PROBLEMS

by

Manuel del Pino [1]

Departamento de Ingeniería Matemática F.C.F.M.

Universidad de Chile

Casilla 170 Correo 3, Santiago, CHILE.

and

Jean Dolbeault [1]

Ceremade, U.M.R. C.N.R.S. no. 7534

Université Paris IX-Dauphine

Place du Maréchal de Lattre de Tassigny

75775 Paris Cédex 16, FRANCE

Abstract

In this paper we prove a new family of inequalities which is intermediate between the classical Sobolev inequalities and the Gross logarithmic Sobolev inequality by the minimization of a well choosen functional and the use of recent uniqueness results for the ground state of the corresponding nonlinear scalar field equation, which allows us to identify the optimal constants . This result is then applied to the equation $u_t = \Delta u^m$ in $I\!R^N$ for $m \in [\frac{N-1}{N}, 1[$ (fast diffusion) and $m > 1$ (porous medium), thus giving an exponential rate of decay for the relative entropy to the stationary solution of a rescaled problem and describing the intermediate asymptotics in the $L^1(I\!R^N)$-norm.

Keywords. Sobolev embeddings – Optimal constants – Minimization – Radial symmetry – Uniqueness – Fast diffusion – Porous medium – Time-dependent rescaling – Relative entropy – Optimal rate of decay – Intermediate asymptotics

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1 Introduction

In this paper we first prove a new family of inequalities which is intermediate between the classical Sobolev inequalities and the Gross logarithmic Sobolev inequality.

Theorem 1.1 *Assume that $N = 2, 3, 4$ and consider $m \in [\frac{N-1}{N}, +\infty[$ ($m \neq \frac{1}{2}, 1$). Then for any $w \in \mathcal{D}^\gamma(I\!\!R^N)$ such that*

$$\int_{I\!\!R^N} |w|^{2\gamma}\, dx = M > 0 \tag{1.1}$$

with $\gamma = \gamma(m) = \frac{1}{2m-1}$, the following identity holds

$$\frac{1}{2}\Big(\frac{2m}{2m-1}\Big)^2 \int_{I\!\!R^N} |\nabla w|^2\, dx + \Big(\frac{1}{1-m} - N\Big) \int_{I\!\!R^N} |w|^{1+\gamma}\, dx + K(m, M) \geq 0 \tag{1.2}$$

where $K(m, M)$ is a negative constant if $m < 1$ and a positive constant if $m > 1$. Here the space $\mathcal{D}^\gamma(I\!\!R^N)$ defined by

$$\mathcal{D}^\gamma(I\!\!R^N) = \{w \in L^{1+\gamma} \cap L^{2\gamma}(I\!\!R^N)\ :\ \nabla w \in L^2(I\!\!R^N)\}\ .$$

The expression of $K(m, M)$, which is optimal for Inequality (1.2) under Constraint (1.1), is explicitely given in Section 2.2, Equation (2.11). Note that Inequality (1.2) also holds in the case $m = \frac{N-1}{N}$ for any $N \geq 3$ and that $K(\frac{N-1}{N}, M)$ is then given in terms of the optimal constant of the Sobolev embedding of $\mathcal{D}^{1,2}(I\!\!R^N) = \{w \in L^{\frac{2N}{N-2}}(I\!\!R^N)\ :\ \nabla w \in L^2(I\!\!R^N)\}$ into $L^{\frac{2N}{N-2}}(I\!\!R^N)$ (see Section 2.4). $\mathcal{D}^\gamma(I\!\!R^N)$ can be defined as the completion of $\mathcal{D}(I\!\!R^N)$ with respect to the norm

$$\|w\| = \|w\|_{L^{1+\gamma}(I\!\!R^N)} + \|w\|_{L^{2\gamma}(I\!\!R^N)} + \|\nabla w\|_{L^2(I\!\!R^N)}\ .$$

$\mathcal{D}^\gamma(I\!\!R^N)$ is a subset of $\mathcal{D}^{1,2}(I\!\!R^N)$ if $N \geq 3$ and $\mathcal{D}^{(\gamma=1)}(I\!\!R^N) = H^1(I\!\!R^N)$. The case $m = 1$ corresponds to the Gross logarithmic Sobolev inequality: for any $w \in \mathcal{D}^{1,2}(I\!\!R^N)$, if $M = \|w\|^2_{L^2(I\!\!R^N)}$,

$$\int_{I\!\!R^N} w^2(x) \log[w^2(x)]\, dx - M\Big[N + \log\Big(\frac{M}{(2\pi)^{N/2}}\Big)\Big] \leq 2 \int_{I\!\!R^N} |\nabla w|^2\, dx$$

or, with $v = w^2$ and $M = \|v\|_{L^1(I\!R^N)}$,

$$\int_{I\!R^N} v \log v \, dx - M\Big[N + \log\Big(\frac{M}{(2\pi)^{N/2}}\Big)\Big] \leq \frac{1}{2}\int_{I\!R^N} \frac{|\nabla v|^2}{v} \, dx \; . \tag{1.3}$$

The main ingredient in our proof comes from two recent papers by L. Erbe and M. Tang [12] for a ball and by P. Pucci and J. Serrin [25] in the whole space, from which the following result can be inferred:

Lemma 1.1 *The radially symmetric solutions (ground states) of*

$$\Delta u - u^\gamma + u^{2\gamma-1} = 0 \; , \quad u > 0 \quad \textit{if} \quad 1 < \gamma < \frac{2N}{N-2} \tag{1.4}$$

and of

$$\Delta u - u^{2\gamma-1} + u^\gamma = 0 \; , \quad u \geq 0 \quad \textit{if} \quad 0 < \gamma < 1 \tag{1.5}$$

such that $\lim_{|x|\to+\infty} u(x) = 0$ *are unique if* $N = 2,\ 3,$ *or* 4.

The sublinear case ($\gamma < 1$) is deduced from [12] while the superlinear case ($\gamma > 1$) comes from [25]. Here we adopt the convention $\frac{2N}{N-2} = +\infty$ if $N = 2$. A more general result (higher dimensions) is given in Appendix A: see Corollaries 4.1 and 4.2. As a consequence of Theorem 1.1, one may prove that for $m = \frac{1+\gamma}{2\gamma} \geq \frac{N-1}{N}$, $m \neq 1$,

$$\begin{aligned} L[v] = \frac{1}{2}\int_{I\!R^N} v(x)\,|x|^2 \, dx \qquad & -\frac{1}{1-m}\int_{I\!R^N} v^m(x)\,dx - K(m,M) \\ & \leq \frac{1}{2}\int_{I\!R^N} v(x)\,\Big|x + \frac{m}{m-1}\nabla(v^{m-1}(x))\Big|^2 \, dx \end{aligned} \tag{1.6}$$

provided $v = w^{2\gamma} \geq 0$ satisfies the additional condition

$$\int_{I\!R^N} v(x)\,|x|^2 \, dx < +\infty \; . \tag{1.7}$$

Inequality (1.6) becomes an equality for

$$v^{m-\frac{1}{2}}(x) = w(x) = w_\infty(x) = \Big(C(m,M) + \frac{1-m}{2m}|x|^2\Big)_+^{-\frac{2m-1}{2(1-m)}} \; ,$$

for some constant which is determined by Condition (1.1). By the way, this also determines the value of $K(m,M)$.

Even if the Condition (1.7) perfectly makes sense with $w = w_\infty$ for any $m > \frac{N-2}{N}$, it is hopeless to expect an inequality similar to (1.6) for an $m \in]\frac{N-2}{N}, \frac{N-1}{N}[$. More precisely, for any $\Lambda \geq 1$

$$\sup_{\substack{w \in \mathcal{D}^\gamma(I\!R^N) \\ \int_{I\!R^N} w|w|^{2\gamma}\, dx = M}} \left(\frac{1}{2}\int_{I\!R^N} |x|^2 \cdot |w|^{2\gamma}\, dx - \frac{1}{1-m}\int_{I\!R^N} |w|^{1+\gamma}\, dx - K(m,M) \right.$$
$$\left. -\frac{\Lambda}{2}\int_{I\!R^N} |x \cdot w^\gamma + \frac{2m}{2m-1}\nabla w|^2\, dx \right) = +\infty \ . \quad (1.8)$$

For more detailed results, the case $N \geq 5$ and a short list of references concerning Sobolev inequalities, we refer to the introduction of Section 2.

In the second part of this paper, we use the results of Theorem 1.1 to study of the asymptotic behavior in time of the solutions of the nonlinear Cauchy problem,

$$u_t = \Delta(u^m), \quad t > 0 \quad x \in I\!R^N \quad (1.9)$$

$$u(0,x) = u_0(x), \quad (1.10)$$

where m is a positive number which we assume different from one, and $u_0 \in L^1(I\!R^N)$ is nonnegative.

The case $m > 1$ arises as a model of slow diffusion of a gas inside a porous container. Unlike the heat equation $m = 1$, this equation exhibits finite speed of propagation in the sense that solutions associated to compactly supported initial data remain compactly supported in space variable at all times (see Aronson [2]). When $0 < m < 1$, the opposite happens. Infinite speed of propagation occurs and solutions may even vanish in finite time. This problem is usually referred to as *the fast diffusion equation.* Let us observe that the elliptic part when written in divergence form becomes

$$m\nabla \cdot (u^{m-1}\nabla u) \ ,$$

and the nonlinear diffusion coefficient u^{m-1} takes small values when u is small if $m > 1$ while the opposite happens if $m < 1$.

These problems are known to be well posed in weak senses and to preserve mass in time whenever $m > (N-2)/N$. Solutions are regular and positive (see [16]), but this is no longer true when m is below this threshold. Discontinuities and finite time vanishing may occur as simple examples show. For $m > 1$ solutions are at least Hölder continuous. Also known is that for compactly supported solutions, their space supports are becoming "ball-like" and of class $C^{1,\alpha}$ for long times (see [6]).

The long-time behavior of solutions to these problems has been the object of a large number of papers. Since mass is preserved, it is natural to ask whether a scaling brings the solution into a certain universal profile as time goes to infinity. This is indeed the case and the role of the limiting profiles is played by an explicit family of self-similar solutions known as the Barenblatt-Prattle solutions [4], characterized by the fact that their initial data is a Dirac mass. These solutions, which come out naturally from the scaling invariance of the equation are given by the explicit formulas

$$U_C(t,x) = \left(\frac{\alpha}{t}\right)^{N\alpha} \cdot v_\infty\Big((\frac{\alpha}{t})^{\alpha} x \Big)$$

where $\alpha = \frac{1}{2-N(1-m)} > 0$ and for $m > \frac{N-2}{N}$, $m \neq 1$,

$$v_\infty(x) = \left(C - \frac{m-1}{2m} |x|^2 \right)_+^{\frac{1}{m-1}}, \tag{1.11}$$

with $(\ )_+$ denoting positive part.

Observe that these solutions have a constant mass in space variable, uniquely determined by the parameter C, and that at time $t = 0$ they become a Dirac measure at the origin. Explicitely, if we set $M = \int_{\mathbb{R}^N} U_C(t,x)dx$ then the value of the constant C is given by

$$C = C(m,M) = \left(\frac{M}{I(m)} \right)^{2(m-1)\alpha}, \tag{1.12}$$

where

$$I(m) = |S^{N-1}| \cdot \int \frac{r^{N-1}}{(1+\frac{1-m}{2m}r^2)^{\frac{1}{1-m}}}\, dr\ ,$$

with the above integral taken on the interval $(0,+\infty)$ if $m<1$ and on the interval $(0,\sqrt{\frac{2m}{m-1}})$ if $m>1$ (see Section 2.1 for more details). Of course the analogues of these functions for $m=1$ are nothing but scalar multiples of the fundamental solution of the heat equation. More generally, the role of self-similar solutions in the asymptotic behavior of solutions is a common pattern to many evolution equations.

Heuristically one thinks that if $u(t,x)$ is a solution of (1.9)-(1.10), then the scaling $u_\lambda(t,x) = \lambda^{N\alpha}u(\lambda^\alpha x, \lambda t)$ which leaves the equation invariant should converge in some sense as $\lambda \to +\infty$ to a solution invariant under the scaling of (1.9)-(1.10) having as initial data the Dirac measure with mass equal to that of u_0, namely to a member of the family of the Barenblatt-Prattle solutions. This would then give account of the asymptotic profile of the original solution u for large times.

This idea was first set and made rigorous by Friedman and Kamin [13] in the context of $u_0 \in L^1(I\!R^N) \cap L^2(I\!R^N)$, both in the cases $m>1$ and $(N-2)/N < m < 1$. These results have been later improved and extended by Vázquez and Kamin (see [18] and [19]). See also [31] for a recent survey and some new results. Thus far it is known that if $u_0 \in L^1(I\!R^N)$ then

$$\lim_{t\to+\infty} \|u(t,.) - U_C(t,.)\|_{L^1(I\!R^N)} = 0 \tag{1.13}$$

and also

$$\lim_{t\to+\infty} t^{N\alpha}\|u(t,.) - U_C(t,.)\|_{L^\infty(I\!R^N)} = 0, \tag{1.14}$$

both in the cases $m>1$ and $(N-2)/N < m < 1$.

Even though these facts have been known for some time, little seems to have been found concerning the *rate* at which the convergence occurs. For the heat equation $m=1$ the following fact is classical.

$$\limsup_{t\to+\infty}\ \sqrt{t}\cdot \|u(t,.) - \frac{\|u_0\|_{L^1(I\!R^N)}}{(2\pi t)^{N/2}}\, e^{-\frac{|x|^2}{2t}}\|_{L^1(I\!R^N)} < +\infty\ . \tag{1.15}$$

In other words the corresponding decay rate of (1.13) when $m = 1$ is of order $O(t^{-1/2})$. Our second main result asserts the validity of an analogous decay estimate for $m \neq 1$.

Theorem 1.2 *Assume that $N = 2, 3, 4$ and $m \geq \frac{N-1}{N}$, $m \neq \frac{1}{2}, 1$. Consider a nonnegative function $u_0 \in L^1(\mathbb{R}^N, (1+|x|^2)dx)$ and assume that u_0^m is bounded in $L^1(\mathbb{R}^N)$. Let $M = \|u_0\|_{L^1(\mathbb{R}^N)}$ and consider the solution of (1.9) with initial data u_0 and the Barenblatt-Prattle solution U_C such that $\|U_C(t,.)\|_{L^1(\mathbb{R}^N)} = M$ for any $t > 0$. Then there exists a constant $C > 0$ depending only on m, M and $L[u_0]$ such that*

(i) if $m \in [\frac{N-1}{N}, 1]$, $t \mapsto t^{-\frac{N(1-m)}{2-N(1-m)}} \|u^m(t,.)\|_{L^1(\mathbb{R}^N)}$ is bounded and

$$\limsup_{t\to+\infty} t^{\frac{1-N(1-m)}{2-N(1-m)}} \|u^m(t,.) - U_C^m(t,.)\|_{L^1(\mathbb{R}^N)} \leq C \; .$$

(ii) if $m \in]1,2]$, $t \mapsto t^{-\frac{N(1-m)}{2-N(1-m)}} \|u(t,.)U_C^{m-1}(t,.)\|_{L^1(\mathbb{R}^N)}$ is bounded and

$$\limsup_{t\to+\infty} t^{\frac{1-N(1-m)}{2-N(1-m)}} \| \; [u(t,.) - U_C(t,.)] \; U_C^{m-1}(t,.) \; \|_{L^1(\mathbb{R}^N)} \leq C \; ,$$

$$\limsup_{t\to+\infty} \Big(t \|u^m(t,.)\|_{L^1(B_t^c)} + \|u(t,x)|x|^2\|_{L^1(B_t^c)} \Big) \leq C$$

where $L^1(B_t^c)$ is the centered ball of radius

$$\sqrt{\frac{2m}{m-1} \, C(m,M)} \cdot [2 - N(1-m)t]^{\frac{1}{2-N(1-m)}} \; .$$

At this point we may notice that if $m < 1$, by interpolation with relation (1.14), we get that for any $p \geq 1$

$$\lim_{t\to+\infty} t^{[1+N(p+m-2)]\frac{\alpha}{p}} \|u(t,.) - U_C(t,.)\|_{L^p(\mathbb{R}^N)} = 0 \; .$$

Similar results hold in the case $m > 1$ (see Section 3 for more detailed results and the case $N \geq 5$). We can also make the following observation. If $m > \frac{N}{N+2}$ (note that $\frac{N}{N+2} < \frac{N-1}{N}$ if $N > 2$), then U_C belongs to $L^1(\mathbb{R}^N, (1+|x|^2)dx)$ and U_C^m belongs to $L^1(\mathbb{R}^N, dx)$, for each fixed $t > 0$: $\int_{\mathbb{R}^N} (U_C(x)|x|^2 + U_C^m(x)) \, dx$ is well defined for $m \in]\frac{N}{N+2}, \frac{N-1}{N}[$ even if Inequality (1.6) does not make sense.

2 Generalized Sobolev inequalities

Assume that $N \geq 2$ and $m > \frac{N-2}{N}$, $m \neq 1$. We define

$$L[v] = \int_{I\!R^N} \left(v(x) \frac{|x|^2}{2} - \frac{1}{1-m} v^m(x) \right) dx - K(m, M) \tag{2.1}$$

where $K(m, M)$ is such that $L[v_\infty] = 0$, with $v_\infty(x) = \left(C + \frac{1-m}{2m} |x|^2 \right)_+^{-\frac{1}{1-m}}$ and $C = C(m, M)$ is given by the condition $\|v_\infty\|_{L^1(I\!R^N)} = M$.

The case $m = 1$ is a special case where the above expressions have to be replaced by their limit as $m \to 1$, $m \neq 1$ (see Section 2.3).

As mentioned in the introduction, a crucial tool in our approach is a uniqueness result for nonnegative radial solutions of

$$\Delta u - u^p + u^q = 0 \tag{2.2}$$

where (with $\gamma = \frac{1}{2m-1}$)

$$1 < p = \gamma < q = 2\gamma - 1 \leq \frac{N+2}{N-2} \quad \text{if} \quad \frac{N-1}{N} \leq m < 1 \Longleftrightarrow 1 < \gamma \leq \frac{N}{N-2} \ ,$$

and (with the convention $\frac{N}{N-2} = +\infty$, $m > \frac{1}{2}$ if $N = 2$)

$$-1 < p = 2\gamma - 1 < q = \gamma < 1 \quad \text{if} \quad m > 1 \Longleftrightarrow 0 < \gamma < 1 \ .$$

According to the results given in Appendix A: Corollaries 4.1 and 4.2), the following conditions are sufficient

(i) (Fast Diffusion case) $m \in]\frac{N-1}{N}, 1[$ and $N = 2, 3, 4$, or $m = \frac{N-1}{N}$ if $N \geq 3$,

$$\text{or} \quad 5 \leq N < 16 \quad \text{and} \quad \gamma \geq \frac{4(N+2)}{5(N-8)} \ , \tag{2.3}$$

(ii) (Porous Medium case) $m > 1$ and $N \leq 16$,

$$\text{or} \quad N > 16 \quad \text{and} \quad \gamma \geq \frac{4(N+2)}{5(N-8)} \ . \tag{2.4}$$

The main result of this section is the

Theorem 2.1 *Assume that* $N \geq 2$*,* $M > 0$ *and* $\gamma = \frac{1}{2m-1}$*. If Conditions (2.3)-(2.4) are satisfied, then*

$$G[w] = \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!R^N} |\nabla w|^2 \, dx + (\frac{1}{1-m} - N) \int_{I\!R^N} |w|^{1+\gamma}(x) \, dx$$

has a unique radial minimizer in $\mathcal{D}^\gamma(I\!R^N)$ *satisfying the constraint*

$$\int_{I\!R^N} |w|^{2\gamma}(x) \, dx = M \ .$$

This minimizer is given by

$$w(x) = w_\infty(x) = \left(C(m,M) + \frac{1-m}{2m}|x|^2\right)_+^{-\frac{2m-1}{2(1-m)}} , \qquad (2.5)$$

$G[w_\infty] = -K(m,M)$ *and* $G[w] = G[w_\infty]$ *if and only if* $w = w_\infty$ *almost everywhere.*

The proof of this result is given in Section 2.5. We shall start with some preliminary computations and scalings which determine the optimal range for m. The case $m = 1$ (heat equation) and $m = \frac{N-1}{N}$, $N \geq 3$ (classical Sobolev embeddings) are treated independently. Before going further, let us give a straightforward corollary which is the case of interest for fast diffusion or porous medium equations. Here $v = w^{2\gamma}$ with the above notations.

Corollary 2.1 *Assume that* $N \geq 2$ *and* $\gamma = \frac{1}{2m-1}$*. If Conditions (2.3)-(2.4) are satisfied, then for any nonnegative function* v *in* $\in L^1(I\!R^N, (1+|x|^2)dx)$ *with* $M = \|v\|_{L^1(I\!R^N)} > 0$*,*

$$\int_{I\!R^N} \left(v(x)\frac{|x|^2}{2} - \frac{1}{1-m}v^m(x)\right) dx - K(m,M) \qquad (2.6)$$
$$\leq \int_{I\!R^N} v(x)|x + \frac{m}{m-1}\nabla(v^{m-1}(x))|^2 \, dx \ .$$

Moreover the inequality is optimal: it becomes an equality if and only if

$$v(x) = v_\infty(x) = w_\infty^{2\gamma}(x) = \left(C(m,M) + \frac{1-m}{2m}|x|^2\right)_+^{-\frac{1}{(1-m)}} \quad \forall\, x \in I\!R^N \ .$$

A very natural question of course is to ask whether the restrictions corresponding to Conditions (2.3)-(2.4) could be removed. With our approach, one has to extend the uniqueness results to the range of parameters which are not covered by the results of Appendix A and this looks difficult. A result can however be obtained with a constant $\Lambda \neq 1$ as shown by the next result

Theorem 2.2 *Assume that* $N \geq 2$*,* $m \geq \frac{N-1}{N}$ *(*$m \neq \frac{1}{2}$*, 1) and* $\gamma = \frac{1}{2m-1}$*. Then for any* $M > 0$*, there exist a constant* $\Lambda = \Lambda(m, M) \geq 1$ *such that for any nonnegative function* v *in* $\in L^1(I\!\!R^N, (1+|x|^2)dx)$ *with* $w = v^{m-\frac{1}{2}} \in \mathcal{D}^\gamma(I\!\!R^N)$ *and* $M = \|v\|_{L^1(I\!\!R^N)}$*,*

$$\begin{aligned} \frac{1}{2}\int_{I\!\!R^N} v(x)\,|x|^2\,dx \quad & -\frac{1}{1-m}\int_{I\!\!R^N} v^m(x)\,dx - K(m,M) \qquad (2.7) \\ & \leq \frac{\Lambda}{2}\int_{I\!\!R^N} |x \cdot w^\gamma + \frac{2m}{2m-1}\nabla w|^2\,dx\ . \end{aligned}$$

A proof of this result, based on a spectral analysis, is given in Appendix C. Note that we may also write Inequality (2.7) as

$$\begin{aligned} \frac{\Lambda}{2}\Big(\frac{2m}{2m-1}\Big)^2 \quad & \int_{I\!\!R^N} |\nabla w|^2\,dx + \frac{1}{2}(\Lambda-1)\int_{I\!\!R^N} w^{2\gamma}(x)\,|x|^2\,dx \\ & +\Big(\frac{1}{1-m} - N\Lambda\Big)\int_{I\!\!R^N} |w|^{1+\gamma}\,dx + K(m,M) \geq 0\ .(2.8) \end{aligned}$$

It would be difficult to give a complete list of references for the Sobolev embeddings or the Gross logarithmic inequalitites. For $m = 1$ one can refer to [1] and references therein. Concerning the optimal Sobolev constant, the minimization methods and the role of radial solutions one may for instance quote [3], [21] and [27]. Further references concerning some special aspects of the problem will be given in the rest of this section.

2.1 Preliminary computations

Assume that $N \geq 2$ and $m > \frac{N-2}{N}$, $m \neq 1$, and consider

$$v_\infty(x) = \Big(C(m,M) + \frac{1-m}{2m}|x|^2\Big)_+^{-\frac{1}{1-m}}$$

where $C(m,M)$ and $M = \|v\|_{L^1(I\!R^N)}$ are related by

$$M \cdot \Big(C(m,M)\Big)^{\frac{1}{1-m}-\frac{N}{2}} = \int \Big(1 + \frac{1-m}{2m}|x|^2\Big)_+^{-\frac{1}{1-m}} \, dx$$

(where the integral is taken over $I\!R^N$ if $m < 1$ and on the ball $B(0, \sqrt{\frac{2m}{m-1}})$ if $m > 1$),

$$C(m,M) = \Big(\frac{M}{I(m)}\Big)^{-\frac{2(1-m)}{2-N(1-m)}}, \quad I(m) = |S^{N-1}| \cdot \int \frac{r^{N-1}}{(1+\frac{1-m}{2m}r^2)^{\frac{1}{1-m}}} \, dr \ ,$$

where the integral defining $I(m)$ has to be taken from 0 to $+\infty$ if $m < 1$ and from 0 to $\sqrt{\frac{2m}{m-1}}$ if $m > 1$.

Similarly, we may define

$$J(m,M) = \int_{I\!R^N} v_\infty^m \, dx = |S^{N-1}| \cdot \int \Big(C(m,M) + \frac{1-m}{2m}|x|^2\Big)_+^{-\frac{m}{1-m}} \, dx \quad (2.9)$$

(where again the integral is taken over $I\!R^N$ if $m < 1$ and on the ball $B(0, \sqrt{\frac{2m}{m-1}})$ if $m > 1$), and a straightforward computation gives

$$J(m,M) = M^{\frac{N(m-1)+2m}{N(m-1)+2}} \cdot J(m,1) \ . \quad (2.10)$$

Proposition 2.1 *On $]0,+\infty[$, $M \mapsto C(m,M)$ is a decreasing function if $m \in]\frac{N-2}{N},1[$ and an increasing function if $m > 1$. If $m > 1$, $M \mapsto J(m,M)$ and $M \mapsto K(m,M)$ are nonnegative convex increasing functions.*

Proof. The convexity of J when $m > 1$ is given by

$$\frac{N(m-1)+2m}{N(m-1)+2} = 1 + \frac{2(m-1)}{N(m-1)+2} > 1 \ .$$

Consider then, with $R(m,M) = \sqrt{\frac{2m}{1-m}C(m,M)}$ if $m > 1$ and $R(m,M) = +\infty$ if $m < 1$,

$$K(m,M) \quad = \frac{1}{2}\int v_\infty(x)|x|^2 \, dx + \frac{1}{m-1}\int v_\infty^m(x) \, dx$$

$$
\begin{aligned}
&= \int v_\infty(x)\Big(\frac{|x|^2}{2} - \frac{1}{1-m}(C(m,M) + \frac{1-m}{2m}|x|^2)\Big)\, dx \\
&= -\frac{MC(m,M)}{1-m} - \frac{1-m}{m}\int v_\infty(x)\frac{|x|^2}{2}\, dx \\
&= \frac{m}{m-1}MC(m,M) - \int v_\infty^m(x)\, dx \\
&= \frac{m}{m-1}MC(m,M) - J(m,M)\ , \qquad (2.11)
\end{aligned}
$$

where the integrals are taken on $B(0, R(m,M))$ and $J(m,M)$ is given by Equations (2.9) and (2.10). This is sufficient to prove the existence for $m > 1$ of a positive constant $c(m,N)$ which does not depend on M such that $K(m,M) = c(m,N) \cdot J(m,M)$. □

Note that $K(m,M)$ is well defined only for $m > \frac{N}{N+2}$ and strictly negative for any $m \in]\frac{N}{N+2}, 1[$. Moreover

$$
\begin{aligned}
0 \quad &= \int_{I\!R^N} v_\infty|x + \frac{m}{m-1}\nabla(v_\infty^{m-1})|^2\, dx \\
&= \int_{I\!R^N} v_\infty|x|^2\, dx + (\frac{m}{m-1})^2\int_{I\!R^N} v_\infty|\nabla(v_\infty^{m-1})|^2\, dx \\
&\qquad +2\int_{I\!R^N} xv_\infty\cdot\frac{m}{m-1}\nabla(v_\infty^{m-1})\, dx \qquad (2.12) \\
&= \int_{I\!R^N} v_\infty|x|^2\, dx + (\frac{2m}{2m-1})^2\int_{I\!R^N} |\nabla(v_\infty^{m-\frac{1}{2}})|^2\, dx - 2N\int_{I\!R^N} v_\infty^m\, dx\ .
\end{aligned}
$$

Remark 2.1 *According to Theorem 2.1, $K(m,M) = -G[w_\infty]$:*

$$
\frac{1}{2}(\frac{2m}{2m-1})^2\int_{I\!R^N}|\nabla(v_\infty^{m-\frac{1}{2}})|^2\, dx + \Big(\frac{1}{1-m} - N\Big)\cdot\int_{I\!R^N} v_\infty^m(x)\, dx + K(m,M) = 0
$$

which together with Equations (2.11) and (2.12) means that

$$
J(m,M) = \int_{I\!R^N} v_\infty^m(x)\, dx\ , \quad K(m,M)\ , \quad \int_{I\!R^N} v_\infty(x)|x|^2\, dx\ ,
$$

$$
\text{and} \quad (\frac{2m}{2m-1})^2\int_{I\!R^N}|\nabla(v_\infty^{m-\frac{1}{2}})|^2\, dx
$$

are proportional (with constants depending only on m and N but not on M) to $MC(m,M) = O(M^{\frac{N(m-1)+2m}{N(m-1)+2}})$.

To conclude with these technical preliminaries, one may consider the Euler-Lagrange equations associated to the optimal results in Inequalities (2.7) and (2.8). Here we adopt the notation $\lambda = \Lambda^{-1}$.

Proposition 2.2 *v_∞ and w_∞ defined by (1.11) and (2.5) are respectively solutions of*

$$-m^2 v^{2m-3}[\Delta v + (m-\frac{3}{2})\frac{|\nabla v|^2}{v}] + \frac{1-\lambda}{2}|x|^2 + (\frac{\lambda}{1-m} - N) m v^{m-1} = \mu \quad (2.13)$$

and

$$-(\frac{2m}{2m-1})^2 \Delta w + \gamma(1-\lambda)|x|^2 w^{2\gamma-1} + (1+\gamma)(\frac{\lambda}{1-m} - N) w^\gamma - 2\gamma\mu w^{2\gamma-1} = 0 \quad (2.14)$$

provided $\mu = \frac{m}{1-m} C(m,M) \cdot \lambda = \frac{m}{1-m}\, \lambda \cdot \left(\frac{M}{I(m)}\right)^{-\frac{2(1-m)}{2-N(1-m)}}$.

The proof is a simple computation (μ will appear as the Lagrange multiplier associated to the constraint $M = \|v\|_{L^1(\mathbb{R}^N)}$). Up to a scaling, Equation (2.14) takes the form (2.2) and the uniqueness of the solutions of Eqiation (2.14) will be used to prove that the minimizer of G in Theorem 2.1 is of the form w_∞. Note that μ is a strictly monotone function of M.

2.2 A scaling argument

In this paragraph, we shall see that simple considerations based on scaling arguments are sufficient to determine the possible ranges of λ and m. Consider

$$\begin{aligned} F_\lambda[v] &= \frac{1}{2}\int_{\mathbb{R}^N} v(x)|x + \frac{m}{m-1}\nabla v^{m-1}(x)|^2\, dx - \lambda L[v] \\ &= \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{\mathbb{R}^N} |\nabla (v^{m-\frac{1}{2}})|^2\, dx + \frac{1-\lambda}{2}\int_{\mathbb{R}^N} v(x)|x|^2\, dx \\ &\quad + (\frac{\lambda}{1-m} - N)\int_{\mathbb{R}^N} v^m(x)\, dx + \lambda K(m,M) \end{aligned}$$

If $v^\tau(x) = \tau^N v(\tau x)$,

$$\begin{aligned} F_\lambda[v^\tau] \quad &= \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!\!R^N} |\nabla(v^{m-\frac{1}{2}})|^2 \, dx \cdot \tau^{2[1-N(1-m)]} \\ &\qquad + \frac{1-\lambda}{2} \int_{I\!\!R^N} v(x)|x|^2 \, dx \cdot \tau^{-2} \\ &\qquad + (\frac{\lambda}{1-m} - N) \int_{I\!\!R^N} v^m(x) \, dx \cdot \tau^{-N(1-m)} \\ &\qquad + \lambda K(m, M) \\ &\sim \frac{1-\lambda}{2} \int_{I\!\!R^N} v(x)|x|^2 \, dx \cdot \tau^{-2} \quad \text{as } \tau \to 0 \end{aligned}$$

provided $m > \frac{N-2}{N}$. As a consequence,

Proposition 2.3 *If $F_\lambda[v]$ is bounded from below uniformly in $v \in X_M = \{v \in L^1(I\!\!R^N) \; : \; v \geq 0, \; |x|^2 v(x) \in L^1(I\!\!R^N), \; v^m \in L^1(I\!\!R^N), \; \nabla v \in L^2(I\!\!R^N) \; , \; M = \|v\|_{L^1(I\!\!R^N)} > 0\}$, then $\lambda \in]-\infty, 1]$.*

Considering now the opposite asymptotics $\tau \to +\infty$ in the case $m < \frac{N-1}{N}$, we have the following

Proposition 2.4 *If $m < \frac{N-1}{N}$, for any $\lambda > 0$,*

$$\inf_{v \in X_M} \frac{1}{2} \int_{I\!\!R^N} v(x)|x + \frac{m}{m-1} \nabla v^{m-1}(x)|^2 \, dx - \lambda L[v] < 0 \; .$$

Proof. As $\tau \to +\infty$, $F_\lambda[v^\tau] \sim \lambda K(m, M) < 0$ if $m < \frac{N-1}{N}$. □

2.3 The Gross logarithmic Sobolev inequality

The case $m = 1$ corresponds to the Gross logarithmic Sobolev inequality. For the completeness of the paper, but also to illustrate the general strategy and anticipating on the results of Section 3, we give here a few results without complete proofs and refer to [15], [28], [29], [30] and [1] for further results and references.

Proposition 2.5 *Assume that $v \in L^1(I\!R^N)$ is a nonnegative function, $M = \|v\|_{L^1(I\!R^N)} > 0$ and $\int_{I\!R^N} v(x)\,|x|^2\,dx < +\infty$. Then*

(i) $L_1[v] = \int_{I\!R^N} v(x)\Big(\frac{|x|^2}{2} + \log(v(x))\Big)\,dx - K(1,M) \geq 0$ provided $K(1,M) = M\log\Big(\frac{M}{(2\pi)^{\frac{N}{2}}}\Big)$. Moreover, the minimum is reached by $v_\infty(x) = M \cdot \frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{\frac{N}{2}}}$, which is the unique minimizer.

(ii) if $\sqrt{v} \in \mathcal{D}^{1,2}(I\!R^N)$, then

$$\frac{1}{8}\|v - v_\infty\|^2_{L^1(I\!R^N)} \leq L_1[v] \leq \frac{1}{2}\int_{I\!R^N} |x\sqrt{v} + 2\nabla(\sqrt{v})|^2\,dx\ . \tag{2.15}$$

(iii) Consider the solution $u \in C^0(I\!R^+, L^1(I\!R^N))$ of the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = -\Delta u + \nabla \cdot (x\ u)$$

with initial data $u(t=0,.) = v$. Then

$$\frac{d}{dt}L_1[u(t,.)] = -\int_{I\!R^N} |x\sqrt{u(t,.)} + 2\nabla(\sqrt{u(t,.)})|^2\,dx$$

and as a consequence,

$$\frac{1}{8}\|u(t,.) - v_\infty\|^2_{L^1(I\!R^N)} \leq L_1[u(t,.)] \leq L_1[v]\cdot e^{-2t}\ .$$

Proof. We just give the general ideas for some proofs of these results. (i) is a consequence of Jensen's inequality applied to $\int_{I\!R^N} s(\frac{f}{g})g\,d\sigma(x)$ with $f = v$, $g = v_\infty$, $s(t) = t\log t$ and $d\sigma(x) = \frac{dx}{v_\infty(x)}$. The left inequality in (ii) is the Csiszár-Kullback inequality applied to $s(t)$ while the right inequality is the classical logarithmic Gross Sobolev inequality. To prove it, a simple method introduced by Toscani is to consider the Fokker-Planck equation and to compute first $\frac{d}{dt}L[u(t,.)]$, and then

$$\begin{aligned}\frac{d}{dt}\Big(&\frac{1}{2}\int_{I\!R^N} |x\sqrt{u(t,.)} + 2\nabla(\sqrt{u(t,.)})|^2\,dx - L[u(t,.)]\Big)\\ &= 4\int_{I\!R^N}\sum_{i,j=1}^N |\partial_{ij}w - \frac{\partial_i w\partial_j w}{w} + \frac{1}{2}w\ \delta_{i,j}|^2\,dx \geq 0\end{aligned}$$

where $w=\sqrt{v}$ is the solution of

$$w_\tau = \Delta w + \frac{|\nabla w|^2}{w} + \frac{N}{2} w + x \cdot \nabla w \ .$$

We may also apply a direct minimisation method like in the cases $m \neq 1$ or simply take the limit $m \to 1$ in Theorem 1.2. The terms involving $\int_{\mathbb{R}^N} v(x)\ |x|^2\ dx$ in the right side of Inequality (2.15) cancel and Inequality (2.15) is equivalent to Inequality (1.3). As a final remark, we can mention that the time-dependent rescaling we shall study in Section 3 for $m=1$: $R(t)=\sqrt{1+2t}$ relates the Fokker-Planck equation to the classical heat equation and gives an easy proof of Estimate (1.15). □

2.4 Classical Sobolev embeddings (case $m = \frac{N-1}{N}$)

Lemma 2.1 *Any function $v \in L^1(\mathbb{R}^N)$ such that $\nabla(v^{m-\frac{1}{2}}) \in L^2(\mathbb{R}^N)$ belongs to $L^{p(m)}(\mathbb{R}^N)$ with $p(m)=(2m-1)\frac{N}{N-2}$. $p(m)$ belongs to $]m,1[$ if and only if $\frac{N}{N+2} < m < \frac{N-1}{N}$. For any $m > \frac{N-1}{N}$, $p(m) > 1$.*

Proof. This corresponds to the critical Sobolev embedding of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ applied to $v^{m-\frac{1}{2}}$. □

Note that the best Sobolev constant (see [21] for instance) is given by

$$\Sigma = \inf_{v \in C_0^\infty(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla v|^2\ dx}{\left(\int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}}\ dx\right)^{\frac{N-2}{N}}} = \left(\frac{N\pi}{N-2}\right)^{\frac{1}{2}} \cdot \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})}\right)^{\frac{2}{N}} .$$

Proposition 2.6 *If $m=\frac{N-1}{N}$, $N \geq 3$, then*

$$\frac{1}{2}\int_{\mathbb{R}^N} v(x)|x + \frac{m}{m-1}\nabla v^{m-1}(x)|^2\ dx \geq L[v] \ . \qquad (2.16)$$

Proof. The Sobolev inequlality is optimal for $v=v_\infty$ and Inequality (2.16) therefore holds (with $\Lambda^{-1}=\lambda=1$ as for $m=1$). Moreover, the constant is optimal: the inequality is strict unless $v = v_\infty$ since

$$\frac{1}{2}\int_{I\!R^N} v(x)|x + \frac{m}{m-1}\nabla v^{m-1}(x)|^2 \, dx - L[v]$$
$$= \frac{1}{2}(\frac{m}{m-1})^2 \int_{I\!R^N} v(x)|\nabla v^{m-1}(x)|^2 \, dx + K(m,M)$$
$$= \frac{1}{2}(\frac{m}{m-1})^2 \Big[\int_{I\!R^N} v(x)|\nabla v^{m-1}(x)|^2 \, dx - \int_{I\!R^N} v_\infty(x)|\nabla v_\infty^{m-1}(x)|^2 \, dx\Big] .$$

Note that for $m = \frac{N-1}{N}$, $p(m) = (m - \frac{1}{2})\frac{2N}{N-2} = 1$ and
$\Sigma^2 = \frac{(N-2)^2}{4M^{\frac{N-2}{N}}} \int_{I\!R^N} v_\infty |\nabla(v_\infty^{m-1}|^2 \, dx \, dx.$ □

2.5 Proof of the Generalized Sobolev inequalities

This Section is devoted to a proof of Theorem 2.1. Arguments concerning the symmetry and the uniqueness of radial solutions have been rejected in Appendices A and B, but are crucial to prove the optimality of $K(m,M)$. Note that in the case $m < 1$ as well as in the case $m > 1$, there is no linear term in the Euler-Lagrange equations, and the result of existence cannot be reduced to the classical framework studied by Berestycki and Lions in [5].

Because of Schwarz' symmetrization method, the minimum of G with the constraint on the $L^{2\gamma}(I\!R^N)$-norm is reached by radially symmetric functions. In the following, we shall prove the existence of one radially symmetric minimizer. Since the solutions of the Euler-Lagrange equations are radially symmetric (see Appendix B: Proposition 5.1) and since the radial problem admits at most one radial solution (see Appendix A: Corollaries 4.1 and 4.2) as soon as Conditions (2.3)-(2.4) are satisfied, the minimizer is nothing else than $w(x) = w_\infty(x)$, which is a solution of the Euler-Lagrange equations (see Remark 2.2 below).

Assume that $\gamma > 1$, $m > \frac{N-1}{N}$ and consider

$$Y_M = \{w \in \mathcal{D}^\gamma(I\!R^N) \ : \ w \geq 0 \ , \ \int_{I\!R^N} w^{2\gamma}(x) \, dx = M\} \ ,$$

$$Y_M(R) = \{w \in Y_M \ : \ \mathrm{supp}(w) \subset B(0,R)\} \ ,$$

$$I_R = \inf_{w\in Y_M(R)} \Big(\frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!R^N} |\nabla w|^2\, dx + (\frac{1}{1-m} - N) \int_{I\!R^N} w^{1+\gamma}(x)\, dx \Big) \, . \tag{2.17}$$

$R \mapsto I_R$ is a decreasing function of R and by density: $\lim_{R\to+\infty} I_R = I_\infty$. If $I_\infty = 0$, there exists a sequence $(w_n)_{n\in I\!N} \subset Y_M$ such that

$$\|w_n\|_{L^{2\gamma}(I\!R^N)} = M \ , \quad \lim_{n\to+\infty} \|\nabla w_n\|_{L^2(I\!R^N)} = 0 \quad \text{and} \quad \|w_n\|_{L^{1+\gamma}(I\!R^N)} = 0 \, .$$

By the Sobolev embedding: $\lim_{n\to+\infty} \|w_n\|_{L^{\frac{2N}{N-2}}(I\!R^N)} = 0$ if $N \geq 3$ (the extension to the case $N = 2$ is evident) a contradiction with Hölder's inequality since $1 + \gamma \leq 2\gamma < \frac{2N}{N-2}$.

I_R is reached by a radially symmetric positive function $w_R \in H^1_0(B(0,R))$ which solves

$$(\frac{2m}{2m-1})^2 \Delta w_R - (\gamma+1)(\frac{1}{1-m} - N) w_R^\gamma + \mu_R\, w_R^{2\gamma-1} = 0$$

where μ_R is the Lagrange multiplier associated to the constraint on the $L^{2\gamma}(I\!R^N)$-norm. The maximum of w_R is attained at $x = 0$: $\Delta w_R(0) \leq 0$, proving that

$$\mu_R \Big(w_R(0)\Big)^{\gamma-1} - (\gamma+1)(\frac{1}{1-m} - N) \geq 0 \, .$$

On the other side,

$$\begin{aligned} \mu_R\, M = \quad & (\frac{2m}{2m-1})^2 \int_{I\!R^N} |\nabla w|^2\, dx + (\gamma+1)(\frac{1}{1-m} - N) \int_{I\!R^N} w^{1+\gamma}(x)\, dx \\ & \leq (\gamma+1) I_R \end{aligned}$$

and thus $w_R(0) \geq \left(\frac{M}{I_R}\, \frac{1-N(1-m)}{(1-m)} \right)^{\frac{1}{\gamma-1}} \to \left(\frac{M}{I_\infty}\, \frac{1-N(1-m)}{(1-m)} \right)^{\frac{1}{\gamma-1}}$ as $R \to +\infty$.

Note also that $\mu_R\, M \geq 2 I_R \to 2 I_\infty > 0$ and that up to the extraction of a subsequence R_n (with $\lim_{n\to+\infty R_n} = +\infty$), we may assume that

$$w_{R_n} \to w_\infty \quad \text{in} \quad C^2_{loc}(I\!R^N) \, ,$$

where w_∞ is a nontrivial (because $w_R(0) \neq 0$) nonnegative solution of

$$(\frac{2m}{2m-1})^2 \Delta w_\infty - (\gamma+1)(\frac{1}{1-m} - N) w_\infty^\gamma + \mu_\infty\, w_\infty^{2\gamma-1} = 0 \, . \tag{2.18}$$

By Hopf's lemma, it is clear that $w_\infty > 0$ in $I\!\!R^N$.

On the other side, any minimizer in Y_M of

$$Q[w] = \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!\!R^N} |\nabla w|^2 \, dx + (\frac{1}{1-m} - N) \int_{I\!\!R^N} w^{1+\gamma}(x) \, dx$$

satisfies

$$\frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!\!R^N} |\nabla w|^2 \, dx = a(m,N) I_\infty$$

and

$$(\frac{1}{1-m} - N) \int_{I\!\!R^N} w^{1+\gamma}(x) \, dx = b(m,N) I_\infty$$

where a and b are two constants independent of M. This is easily seen by a scaling argument: if $w^\lambda(x) = \lambda^{-\frac{N}{2\gamma}} w(\frac{x}{\lambda})$, then: $\frac{d}{d\lambda} Q[w^\lambda]_{|\lambda=1} = 0$, which means

$$\begin{aligned} \frac{1}{2}(\frac{N}{\gamma} - N + 2) \quad & (\frac{2m}{2m-1})^2 \int_{I\!\!R^N} |\nabla w|^2 \, dx \\ & = \frac{N}{2\gamma}(1+\gamma)(\frac{1}{1-m} - N) \int_{I\!\!R^N} w^{1+\gamma}(x) \, dx \, . \end{aligned} \tag{2.19}$$

This together with

$$Q[w] = I_\infty \tag{2.20}$$

provides a unique expression for a and b.

If we consider now the minimizing family w_R,

$$\frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!\!R^N} |\nabla w_R|^2 \, dx - a(m,N) Q[w_R] \to 0$$

and

$$(\frac{1}{1-m} - N) \int_{I\!\!R^N} w^{1+\gamma}(x) \, dx - b(m,N) Q[w_R] \to 0$$

as $R \to +\infty$ (if not, a scaling would again produce a contradiction).

Thus $\mu_\infty = \lim_{R\to+\infty} \mu_R$ is therefore uniquely determined:

$$\mu_\infty = \frac{1}{M} \left[2a(m,N) + (1+\gamma) b(m,N)\right] ,$$

and the convergence of ∇w_R to ∇w_∞ in $L^2(I\!\!R^N)$ and of w_R to w_∞ in $L^{1+\gamma}(I\!\!R^N)$ is strong: as a consequence, $\|w_\infty\|_{L^{2\gamma}(I\!\!R^N)} = M$ and w_∞ belongs to Y_M.

But there is a unique choice of the parameters λ_1 and λ_2 such that $u(x) = \lambda_1 w(\lambda_2 x) > 0$ is a solution of

$$\Delta u - u^\gamma + u^{2\gamma-1} = 0 \ ,$$

which is unique according to Corollary 4.1. w_∞ is therefore nothing else than

$$w_\infty(x) = \left(C(m,M) + \frac{1-m}{2m}|x|^2 \right)^{-\frac{2m-1}{2(1-m)}} .$$

Note that Equations (2.19) and (2.20) together with the condition

$$\int_{I\!\!R^N} w^{2\gamma}(x)\, dx = M$$

and the uniqueness result (Corollary 4.1) prove that any radial minimizer is a solution of Equation (2.18) with μ_∞ fixed as above and is therefore unique.

If $\gamma < 1$, the proof is very similar. Consider again the sets Y_M, $Y_M(R)$ and the infimum I_R. The fact that $I_\infty < 0$ is easily seen by using the function w_∞ as a test function. The first difference comes from the fact that now

$$2\gamma \leq 1+\gamma \leq 2 < \frac{2N}{N-2} \ ,$$

and nothing a priori prevents from some vanishing of mass: passing to the limit, we only know that w_∞ is radial, nonnegative, non-trivial for the same reason as in the fast diffusion case and $\|w_\infty\|_{L^{2\gamma}(I\!\!R^N)} \leq M$. However, the symmetry result given in Appendix B (Proposition 5.1) and the fact that $M \mapsto -K(m,M)$ is concave and decreasing according to Proposition 2.1 proves that the only possible minmizer corresponds to a radially symmetric function v_∞ supported on a single ball in $I\!\!R^N$ such that the constraint on the $L^{2\gamma}(I\!\!R^N)$-norm is satisfied. Then the uniqueness result of Appendix A (Corollary 4.2) applies and the proof goes exactly as in the case $\gamma > 1$. □

Remark 2.2 *We conclude this section by a few remarks concerning the fact that the minimizer is a non-trivial solution of the Euler-Lagrange equations and therefore nothing else than* w_∞. *In the case* $\gamma > 1$, *the limit of the minimizing sequence is a solution of an equation of type (2.2) and therefore by the uniqueness result takes the form* $\left(C + \frac{1-m}{2m}|x|^2\right)^{-\frac{2m-1}{2(1-m)}}$, *but it belongs to* Y_M *which allows us to identify* C *with* $C(m, M)$.

In the case $\gamma < 1$, *the identification of the Lagrange multiplier in terms of the value of the infimum, which then fixes the value of the constant* C, *is crucial. The concavity argument, which is used to prove that the limit satisfies the constraint, is actually deeper. Ignoring the result concerning the uniqueness (up to translations) of the solution (and the fact that the Lagrange multiplier is a strictly monotone function of* M*), the monotonicity and the strict concavity of* $M \mapsto -K(m, M) \leq 0$ *also proves that the minimum cannot be reached by a function supported for instance by a union of* n *disjoint balls, correponding of course to a different value of the Lagrange multiplier, and for which the minimum would be:* $-n\, K(m, \frac{M}{n}) > -K(m, M)$.

3 Long time behaviour of fast diffusion or porous medium equations

Our approach in the proof of these results uses elements which are rather new in the study of nonlinear diffusions of this type but that are already familiar in the field of kinetic equations. In particular our methods are close in spirit to works by Toscani (see [28], [29]) and by Arnold, Markowich, Toscani and Unterreiter (see [1]).

First of all we transform the equation via a change of variables – a time-dependent rescaling – natural for the self-similar structure of the equation, which not only takes care of the dispersion of the profile, but also preserves the mass at all times, and the initial data. The resulting equation is an analogue of the so-called linear Fokker-Planck equation. The original problem is thus translated into the study of the convergence of the rescaled solution to the steady state, uniquely determined by the preservation of mass.

The nonnegative function L which has a unique minimizer turns out to be a convex Lyapunov functional for the rescaled problem. This Lyapunov functional is an analogue of the *relative entropy*, familiar in kinetic equations (see for instance [29]). The observation that this object is indeed decreasing along trajectories in the case of the rescaled porous medium equation ($m > 1$) appears first in Newman and Ralston approach (see [22] and [26]).

An interesting convexity inequality essentially due to Csiszár [8] and Kullback [17] allows us to estimate the difference between the solution and its limit. The essential point in finding the decay rates is given by the results of Section 2.

In this section we first set up the self-similar time-dependent change of variables, introduce the Lyapunov functiona and prove its basic convexity property related to the Cziszar-Kullback inequality. We include a proof of the version we will utilize in Appendix D. The main result of this section is the following theorem, which gives the rate of convergence of the solutions of Equation (1.9).

Theorem 3.1 *Consider a nonnegative function $u_0 \in L^1(\mathbb{R}^N, (1+|x|^2)dx)$ and assume that u_0^m is bounded in $L^1(\mathbb{R}^N)$. Let $M = \|u_0\|_{L^1(\mathbb{R}^N)} > 0$ and consider the solution of Equation (1.9) with initial data u_0. If $u(t,x) = \Big(R(t)\Big)^{-N} \cdot v\Big(\log(R(t)), \frac{x}{R(t)}\Big)$ with $R(t) = \Big(1 + (2 - N(1-m))t\Big)^{\frac{1}{2-N(1-m)}}$, then with the same notations as in Section 2, if $m \geq \frac{N-1}{N}$, $m \neq \frac{1}{2}$, 1,*

$$L[v(\tau,.)] \leq L[u_0] \cdot e^{-2\lambda\tau} \quad \forall\, \tau > 0$$

for some $\lambda \in]0,1]$. Moreover, $\lambda = 1$ if Conditions (2.3)-(2.4) are satisfied.

The proof is a straightforward consequence of Corollary 2.1 and Theorem 2.2. Using the relation $e^{-\lambda\tau} = [R(t)]^{\lambda}$, one proves an algebraic decay in terms of the original time variable t. Combining then this estimate with the

Csiszár-Kullback inequality (see Proposition 3.1), one proves the following result, which contains Theorem 1.2.

Corollary 3.1 *Under the same assumptions as in Theorem 1.2, but without the restriction on the dimension, if* $N \geq 2$ *and if Conditions (2.3)-(2.4) are satisfied, then the same results as in Theorem 1.2 hold. If Conditions (2.3)-(2.4) are not satisfied, then the decay rate is of order* $t^{\frac{N(1-m)-\lambda}{2-N(1-m)}}$ *for* $\|v^m - v_\infty^m\|_{L^1(\mathbb{R}^N)}$ *if* $\frac{N-1}{N} < m < 1$ *and for* $\|(u(t,.) - u_\infty(t,.))u_\infty^{m-1}(t,.)\|_{L^1(\mathbb{R}^N)}$ *if* $1 < m < 2$.

3.1 Time-dependent scalings and the Lyapunov functional

Consider $\tau(t)$ a new time scale and $R(t)$ a lenght scale such that the solution $u(t,x)$ of Equation (1.9) reads as

$$u(t,x) = \Big(R(t)\Big)^{-N} \cdot v\Big(\tau(t), \frac{x}{R(t)}\Big) \tag{3.1}$$

for some nonnegative function v. In this transformation the L^1-norm is preserved: $\|u(t,.)\|_{L^1(\mathbb{R}^N)} = \|v(\tau(t),.)\|_{L^1(\mathbb{R}^N)}$. It is readily checked that $v(\tau,x)$ satisfies the equation

$$v_\tau = \Delta(v^m) + \nabla \cdot (xv) \quad \tau > 0,\ x \in \mathbb{R}^N , \tag{3.2}$$

which for $m = 1$ corresponds to the linear Fokker-Planck equation, provided that $t \mapsto R(t)$ is a solution of the ordinary differential equation

$$\frac{dR}{dt} = R(t)^{N(1-m)-1} . \tag{3.3}$$

and $\tau(t) = \log(R(t))$. If we add the condition $R(0) = 1$ so that the initial data is preserved,

$$v(\tau = 0, x) = u_0(x) ,$$

then

$$R(t) = \Big(1 + (2 - N(1-m))t\Big)^{\frac{1}{2-N(1-m)}} \ , \tag{3.4}$$

provided $m \neq 1$. Observe that $R(t) \to +\infty$ whenever $\frac{N-2}{N} < m$, which is our entire range of interest. The advantage of this change of variables lies on the fact that it eliminates the dispersion taking place in the original function u without introducing any singularity at initial time. This approach has been introduced systematically by J. Dolbeault and G. Rein in [11] for a number of evolution problems in kinetic theory and related models of fluid mechanics or quantum physics.

With the same notations as in Section 1, as $t \to +\infty$, $R(t) \sim (\frac{t}{\alpha})^\alpha$, $u_\infty(t,.) \sim U_C(t,.)$ and the known fact

$$\lim_{t\to+\infty}\Big(\|u(t,.) - u_\infty(t,.)\|_{L^1(I\!R^N)} + t^{N\alpha}\|u(t,.) - u_\infty(t,.)\|_{L^\infty(I\!R^N)}\Big) = 0 \ ,$$

when $(N-2)/N < m < 1$ or $m > 1$ reads in these new scales just as: $v(\tau, x) \to v_\infty(x)$ for $\tau \to +\infty$ both uniformly and in the L^1-sense. Here and henceforth we use the notation, valid both when $m > 1$ and when $\frac{N-2}{N} < m < 1$, $u_\infty(t,x) = \Big(R(t)\Big)^{-N} v_\infty\Big(\log(R(t)), \frac{x}{R(t)}\Big)$,

$$v_\infty(x) = \left(C + \frac{1-m}{2m}|x|^2\right)_+^{-\frac{1}{1-m}} \tag{3.5}$$

with $C = C(m, M)$ given by (1.12) and $M = \int_{I\!R^N} u_0(x)dx$.

We consider now $L[v]$ given by Equation (2.1) which turns out to be a Lyapunov functional for equation (3.1). We shall indeed prove in the next section (Theorem 3.2) that for any solution $v(t,x)$ of Equation (3.2),

$$\frac{d}{dt}L[v(t,.)] = -\int_{I\!R^N} v(t,.)|x + \frac{m}{m-1}\nabla v(t,.)^{m-1}|^2 \, dx \ . \tag{3.6}$$

$L[v(t,.)]$ comes in the evolution problem in natural analogy with similar objects arising in kinetic theory (free energy functional: see [9]) or for the heat equation (relative entropy: see [28], [29], [30], [1]).

Assume that $m \in]\frac{N}{N+2}, 1[$. Let $w = v^{m-\frac{1}{2}}$ be a nonnegative function such that $(1+|x|^2)\, v(x)$ and v^m belong to $L^1(I\!R^N)$. Let us write

$$\bar{L}[v] = \int_{I\!R^N} \Big(\frac{v(x)}{2}(|x|^2 + \frac{2m}{1-m} C) - \frac{1}{1-m} v^m(x) \Big)\, dx$$

The choice $C = C(m, M)$ as in relation (1.12) makes the quantity inside the integral sign to be minimized in v for each fixed x precisely at the value $v = v_\infty(x)$ where $v_\infty(x)$ is given by formula (3.5). Notice that $L[v] \equiv \bar{L}[v] - \bar{L}[v_\infty]$, namely

$$L[v] = \int_{I\!R^N} \Big(v(x)\, \frac{|x|^2}{2} - \frac{1}{1-m} v^m(x) \Big)\, dx - \frac{1+m}{2(1-m)} \int_{I\!R^N} v_\infty^m(x)\, dx \quad (3.7)$$

for $m \neq 1$, $m > \frac{N}{N+2}$ and $L[v_\infty] = 0$. L is strictly convex on its domain of definition and has the function $v_\infty(x)$ as its unique minimizer. This is easily seen using Jensen's inequality in $I\!R^n$ with the measure $d\sigma(x) = \frac{v_\infty^m(x)}{\int_{B(0,R)} v_\infty^m\, dx}\, dx$,

$$\frac{L[v]}{\int_{I\!R^N} v_\infty^m(x)\, dx} = \int_{I\!R^N} s_{FD}\Big(\frac{v^m}{v_\infty^m} \Big)\, d\sigma(x)$$

with $s_{FD}(t) = \frac{mt^{\frac{1}{m}} - t}{1-m} + 1$ and $R = +\infty$ if $m \in]\frac{N}{N+2}, 1[$, and

$$\frac{L[v]}{\int_{B(0,R)} v_\infty^m\, dx} = \int_{B(0,R)} s_{PM}\Big(\frac{v}{v_\infty} \Big)\, d\sigma + \frac{1}{2} \int_{|x|>R} (v \frac{|x|^2}{2} + \frac{v^m}{m-1})\, dx$$

with $s_{PM}(t) = \frac{t^m - mt}{m-1} + 1$ and $R = R(m, M) = \sqrt{\frac{2m}{m-1}\, C(m, M)}$ if $m > 1$. s_{FD} is a strictly convex function if $m > \frac{1}{2}$ (which is obviously the case if $m > \frac{N}{N+2}$, $N \geq 2$) as well as s_{PM}.

An estimate of the difference of v with v_∞ is given by the Csiszár-Kullback inequality:

Proposition 3.1 *Assume that $N \geq 2$. Let v is a nonnegative function such that $(1+|x|^2)\, v(x)$ and $v(x)^m$ belong to $L^1(I\!R^N)$.*

(i) If $\min\{\frac{1}{2}, \frac{N-2}{N}\} < m < 1$*, then there exists a constant* $C > 0$ *which which depends only on* m*,* M *and* $L[v]$ *such that*

$$L[v] \geq C \|v^m - v_\infty^m\|^2_{L^1(I\!\!R^N)} .$$

(ii) If $1 < m < 2$ *and* $R = R(m, M) = \sqrt{\frac{2m}{m-1}\, C(m, M)}$*, then*

$$L[v] \geq \frac{m}{4MC(m, M)} \|(v - v_\infty)v_\infty^{m-1}\|_{L^1(I\!\!R^N)} + \int_{|x|>R} (v\frac{|x|^2}{2} + \frac{v^m}{m-1})dx .$$

Proof. Proposition 3.1 is a direct consequence of Lemma 7.1. For $m < 1$, we take $s(t) = s_{FD}(t)\frac{mt^{\frac{1}{m}} - t}{1-m} + 1$, $K_1 = K_2 = \frac{1}{m}$, $d\mu(x) = dx$ and

$$L[v] = \int_{I\!\!R^N} s\Big(\frac{v^m}{v_\infty^m}\Big)\, v_\infty^m dx .$$

Let us prove that C depends only on m, M and $L[v]$. $\|v^m\|_{L^1(I\!\!R^N)}$ can be bounded as follows: Hölder's inequality applied to $v^m v_\infty^{-m(1-m)} \cdot v_\infty^{m(1-m)}$ gives

$$\int_{I\!\!R^N} v^m\, dx \leq \Big[\int_{I\!\!R^N} v(C(m, M) + \frac{1-m}{2m}|x|^2)\, dx\Big]^m \cdot \Big[\int_{I\!\!R^N} v_\infty^m\, dx\Big]^{1-m} \qquad (3.8)$$

and using the definition of $L[v]$, we get

$$\int_{I\!\!R^N} v\, \frac{|x|^2}{2}\, dx - \frac{1}{1-m}\Big[\int_{I\!\!R^N} v(C(m, M) + \frac{1-m}{2m}|x|^2)dx\Big]^m - K(m, M) \leq L[v],$$

thus obtaining an estimate on $\int_{I\!\!R^N} v\, \frac{|x|^2}{2}\, dx$ and $\|v^m\|_{L^1(I\!\!R^N)}$ which depends only on m, M and $L[v]$.

If $1 < m < 2$, we may write $\frac{|x|^2}{2} = \frac{m}{m-1}\Big(C(m, M) - v_\infty^{m-1}(x)\Big) \geq \frac{m}{m-1}C(m, M)$ for $|x| < R(m, M)$, $\int_{I\!\!R^N} v v_\infty^{m-1}\, dx \geq \frac{m}{m-1}C(m, M)M$ and apply Lemma 7.1 to

$$L[v] = \int_{I\!\!R^N} s\Big(\frac{v}{v_\infty}\Big)\, v_\infty(x)\, d\mu(x) + \int_{B(0,R)^c} \Big(v(x)\, \frac{|x|^2}{2} + \frac{1}{m-1} v^m(x)\Big)\, dx ,$$

$s(t) = s_{PM}(t)\frac{t^m - mt}{m-1} + 1$, $K_1 = K_2 = m$ and $d\mu(x) = v_\infty^{m-1}(x)\, dx$. □

In the next section we shall prove Equation (3.6), thus justifying why we used the denomination "Lyapunov functional" for $L[v]$ by considering the evolution problem and proving that $L[v]$ effectively controls the convergence to v_∞.

3.2 Time evolution and the Lyapunov functional

Theorem 3.2 *Assume that* $m > \frac{N}{N+2}$ *and that* u_0 *is a nonnegative function such that* $(1+|x|^2)u_0 \in L^1(R^N)$. *If* v *and* u *are respectively the solutions of Equations (3.2) and (1.9), then* $\lim_{\tau\to+\infty} L[v(\tau,.)] = 0$ *and if* $m < 1$, *then*

$$\lim_{\tau\to+\infty} \|v^m(\tau,.) - v_\infty^m\|_{L^1(I\!R^N)} = 0 \ ,$$

$$\lim_{t\to+\infty} t^{-N\alpha(1-m)}\|u(t,.)^m - u_\infty(t,.)^m\|_{L^1(I\!R^N)} = 0 \ .$$

In the case $m > 1$, a result similar to the one of Theorem 3.2 for the convergence of v to v_∞ and for the improved decay rate of $u(t,.) - u_\infty$ of course holds using Proposition 3.1.

Proof. Let us assume first that the initial data $u_0(x)$ is smooth and compactly supported in say the ball $B(0,\rho)$ for some $\rho > 0$. Let $v(\tau,x)$ then be the solution of Equation (3.2) such that $v(0,x) = u_0(x)$. In the case $m > 1$, the solution has compact support for any $\tau > 0$ and the computation leading to Equation (3.6) is straightforward. Assume then that $\frac{N}{N+2} < m < 1$. The solution is smooth. We claim that the quantity $L[v(\tau,.)]$ is well defined for $\tau > 0$ and is decreasing. Consider the function

$$w_\rho(x) = (\frac{1-m}{2m})^{-\frac{1}{1-m}} \cdot (|x|^2 - \rho^2)^{-\frac{1}{1-m}} \ . \tag{3.9}$$

It is easily checked that $w_\rho(x)$ is a steady state of (3.2), defined on the region $|x| > \rho$. Since this function takes infinite values on $\partial B(0,\rho)$, the comparison principle implies then that $v(\tau,x) \le w_\rho(x)$ for all $\tau > 0$. Hence $v(x,\tau) = O(|x|^{-\frac{2}{1-m}})$ uniformly in $\tau > 0$. Parabolic estimates then also yield that $\nabla v(x,\tau) = O(|x|^{-\frac{2}{1-m}-1})$. Fix a number $R > 0$ and set $L_0^R[v] = \int_{B(0,R)}(v\frac{|x|^2}{2} - \frac{1}{1-m}v^m)\,dx$. Denoting by $d\sigma(x)$ the measure induced by Lebesgue's measure on $\partial B(,R)$, integrations by parts give

$$\frac{d}{d\tau}\int_{B(0,R)} v\frac{|x|^2}{2} = \int_{B(0,R)} \frac{|x|^2}{2}\nabla\cdot(\nabla v^m + xv)\,dx$$

$$= -\int_{B(0,R)} x\cdot(\nabla v^m + xv)\,dx + \frac{R}{2}\int_{\partial B(0,R)} (\nabla v^m + xv)\cdot x\,d\sigma(x)$$
$$= N\int_{B(0,R)} v^m\,dx - \int_{B(0,R)} |x|^2 v\,dx + O(R^{N+2-\frac{2}{1-m}})$$

A similar integration by parts yields

$$-\frac{1}{1-m}\frac{d}{d\tau}\int_{B(0,R)} v^m\,dx \tag{3.10}$$
$$= \frac{4m^2}{(2m-1)^2}\int_{B(0,R)} |\nabla(v^{m-1/2})|^2 dx - N\int_{B(0,R)} v^m dx + O(R^{N+2-\frac{2}{1-m}})$$

and it follows that

$$\frac{d}{d\tau}L_0^R[v] = -\int_{B(0,R)} v|x - \frac{m}{1-m}\nabla v^{m-1}|^2 dx + O(R^{N+2-\frac{2}{1-m}}) \tag{3.11}$$

where the $O(R^{N+2-\frac{2}{1-m}})$ term is uniform for τ in bounded intervals, and it goes to zero as $R\to+\infty$ since $m > N/(N+2)$. Integrating this last relation between with respect to τ and then letting $R\to+\infty$ we obtain that $L_0(v(\tau,.))$ is well defined and decreasing in τ. The requirement that u_0 is smooth and compactly supported can be removed by a density argument.

We have thus proven that L indeed defines a Lyapunov functional for Equation (3.1). The mass of v is finite and preserved in time, $L[v(\cdot,\tau)]$ is decreasing and therefore uniformly bounded from above in τ, and using Inequality (3.8) if $m<1$, $\int_{I\!R^N} v(\tau,x)\,|x|^2\,dx$ is also uniformly bounded from above in τ (this is straightforward if $m>1$).

The assertion of Theorem 3.2 simply reads

$$\|v^m(\tau,.) - v_\infty^m\|_{L^1(I\!R^N)} \to 0 \quad \text{as } \tau\to+\infty\ .$$

However, we already know that $v(\tau,.)\to v_\infty$ uniformly. To establish the result it suffices to show that $\int_{|x|>R} v^m(\tau,x)dx \to 0$ as $R\to+\infty$, uniformly in τ, which is easily achieved by

$$\int_{|x|>R} v^m(\tau,x)dx \le \left(\int_{|x|>R} v(1+|x|^2)dx\right)^m \left(\int_{|x|>R}(1+|x|^2)^{-\frac{m}{1-m}}dx\right)^{1-m} \tag{3.12}$$

The latter integral is finite for $m > \frac{N}{N+2}$ and goes to 0 as $R \to +\infty$.

v is dominated uniformly in τ by w_ρ and $|x|^{\frac{2}{1-m}-\epsilon} v(\tau,.) \to 0$ uniformly in τ as $|x| \to +\infty$, for any $\epsilon > 0$. Since v approaches uniformly v_∞ and since we are assuming $m > \frac{N}{N+2}$, it follows that $\int |x|^2 v(x,\tau)dx \to \int |x|^2 v_\infty(x)dx$. Hence, from the definition of the Lyapunov functional, $L[v(\tau,.)] \to L[v_\infty] = 0$. The proof of Theorem 3.2 is thus completed. □

The proof of Theorem 1.2 and of its extended version Theorem 3.1 now easily follows. We consider a compactly supported initial data u_0, and keep the notation $v(\tau,x)$ for the solution of Equation (3.1) and $v_\infty(x)$ for the limit. In these terms, the statement becomes an exponential decay estimate:

$$L[v(\cdot,\tau)] \leq C e^{-2\lambda\tau} \; . \tag{3.13}$$

The case $m > 1$ follows from the compuatation (3.11) and the fact that the support of a solution remains compact if it is initially compactly supported. Passing to the limit $R \to +\infty$, we get

$$\frac{d}{d\tau} L[v(\tau,.)] = -\int_{I\!R^N} v|x + \frac{m}{m-1}\nabla v^{m-1}|^2 \, dx \; . \tag{3.14}$$

Using the results of Section 2, we prove (3.13) and can remove by density the assumption of compact support.

If $m < 1$, we may also pass to the limit $R \to +\infty$ in Equation (3.11) and get Identity (3.14) as well and the conclusion holds according to Corollary 2.1 or Theorem 2.2 provided $m > \frac{N-1}{N}$. □

4 Appendix A: uniqueness results

As already mentioned in the introduction, the main ingredient of Section 2 comes from two recent papers by L. Erbe and M. Tang [12] for a ball and its extension to the whole space by P. Pucci and J. Serrin [25], from which Lemma 1.1 can be inferred. The condition (4.1) given in [12] for a ball also applies for the whole space. Lemma 1.1 is deduced from the following theorem (which can be extended to the case of the m-laplacian).

Theorem 4.1 *([12] and [25])* $\Delta u + f(u) = 0$ *admits at most one radial ground state (i.e. a positive solution* $u(x)$ *such that* $\lim_{|x|\to+\infty} u(x) = 0$*) if* $f \in C^0([0,+\infty[\cap C^1(]0,+\infty[$ *,* $f(0) = 0$*,* $f(u)(u-a) < 0$ *for any* $u \in]0,a[\cup]a,+\infty[$ *for some* $a > 0$ *and*

$$\frac{dK}{du} \geq \frac{N-2}{2N} \quad \forall\, u \in]0,a[\cup]a,+\infty[\ , \tag{4.1}$$

where $K(u) = \frac{F(u)}{f(u)}$ *and* $F(u) = \int_0^u f(t)\ dt$*. The same result holds for a positive solution in a ball with zero Dirichlet boundary conditions.*

In the canonical case $f(u) = -u^p + u^q$, $-1 < p < q$, the computation given by Erbe and Tang in [12] applies and Condition (4.1) is equivalent to either

$$(\sigma - q)(p+1)(1+q-p) + (\sigma - p)(q+1)(1+p-q) \leq 0\ , \tag{4.2}$$

or

$$I(p,q) = (p+1)(1+q-p)^2(\sigma - q) - (q+1)(1+p-q)^2(\sigma - p) \geq 0\ , \tag{4.3}$$

where $\sigma = \frac{N+2}{N-2}$ if $N > 2$ and $\sigma = +\infty$ if $N = 2$. Here we take advantage of the relation between p and q to obtain results which improve the ones stated in [25].

Convention. In order to simplify the notations, we adopt the following convention: each time the quantity $N-2$ appears explicitely in the denominator of a constant depending on the dimension, the constant takes the value $+\infty$.

We first apply Theorem 4.1 to the fast diffusion case:

$$\frac{N-1}{N} \leq m < 1 \Longleftrightarrow 1 < \gamma = \frac{1}{2m-1} \leq \frac{N}{N-2}\ ,$$
$$1 < p = \gamma < q = 2\gamma - 1 \leq \sigma\ .$$

Corollary 4.1 *Assume that* $N \geq 2$ *and* $m \in]\frac{N}{N-1}, 1[$. *The positive radially symmetric solution (ground state) of* $\Delta u - u^\gamma + u^{2\gamma-1} = 0$ *such that* $\lim_{|x|\to+\infty} u(x) = 0$ *is unique provided one of the two following conditions is satisfied*

(i) $N = 2$, 3, *or* 4,

(ii) $5 \leq N < 16$ *and* $\gamma \geq \frac{4(N+2)}{5(N-8)}$.

Proof. Condition (4.2) is equivalent to $\frac{1}{2m-1} = \gamma \geq \frac{3N+4}{3N-4}$ and is nonempty if and only if $\frac{3N+4}{3N-4} \leq \frac{N}{N-2} = \gamma_{|(m=\frac{N-1}{N})}$ or $N \leq 4$. It is therefore satisfied provided $N = 2$, 3, or 4 and

$$\frac{N-1}{N} \leq m \leq \frac{3N}{3N+4} < 1 \Longleftrightarrow 1 < \frac{3N+4}{3N-4} \leq \gamma \leq \frac{N}{N-2} \ .$$

Condition (4.3) is equivalent to $I(\gamma, 2\gamma-1) = \gamma(\gamma-1)[(8-\gamma)\sigma - 9\gamma] \geq 0$ or $\gamma \leq \frac{4(N+2)}{5(N-8)}$ and is nonempty provided $\frac{4(N+2)}{5(N-8)} > 1$ which means $N < 16$. For $2 \leq N < 16$, $1 < \gamma \leq \frac{4(N+2)}{5(N-8)}$ which means $\frac{9N}{8(N+2)} \leq m < 1$ (this condition does not imply any restriction for $m \in [\frac{N-1}{N}, 1[$ only if : $\frac{N-1}{N} \geq \frac{9N}{8(N+2)} \Longleftrightarrow (N-4)^2 \leq 0 \Longleftrightarrow N = 4$). □

We may now apply Theorem 4.1 to the porous medium case $m > 1$ $(0 < \gamma < 1)$:

$$-1 < p = 2\gamma - 1 < q = \gamma < 1 \ .$$

Corollary 4.2 *Assume that* $N \geq 2$ *and* $\gamma < 1$. *The positive radially symmetric solution of* $\Delta u - u^{2\gamma-1} + u^\gamma = 0$ *such that* $\lim_{|x|\to+\infty} u(x) = 0$ *is unique provided one of the two following conditions is satisfied*

(i) $N \leq 16$,

(ii) $N > 16$ *and* $\gamma \geq \frac{4(N+2)}{5(N-8)}$.

Proof. Condition (4.2) is equivalent to $2\frac{N+2}{N-2} \le \gamma(\gamma+1)$, which is never satisfied for $\gamma < 1$. Condition (4.3) is equivalent to $I(2\gamma-1,\gamma) = \gamma(\gamma-1)[(\gamma-8)\sigma+9\gamma] \ge 0$ which means $\gamma \le \frac{4(N+2)}{5(N-8)}$ or $m \ge \frac{9N}{8(N+2)}$. Note that it does not give any restriction as long as $\frac{4(N+2)}{5(N-8)} \ge 1$ which corresponds to $N \le 16$. □

5 Appendix B: ground states and symmetry

For the completeness of the paper, we give a sketch of the proof that any nonnegative solution is radially symmetric, which is a classical result. Our arguments are directly inspired from [7]. Only the range of the parameters differs (but the proof given by Cortázar, Elgueta and Felmer still applies).

The notion of "ground state" usually has two meanings in the mathematical literature. One can understand it as the global minimizer of some energy functional, or as a positive (and by extension nonnegative: see [23]) solution of the corresponding Euler-Lagrange equation. Here we prove that such a solution is radially symmetric under some integral condition which is satisfied for minimizers.

Proposition 5.1 *Assume that $N \ge 2$ and consider a nonnegative solution (generalized ground state) of*

$$\Delta u - u^p + u^q = 0 \tag{5.1}$$

in $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N}(u^{p+1}(x)+u^{q+1}(x))\,dx$ is finite. If $-1 < p < q < 1$, then u is supported by a union of disjoint balls on which it is radially symmetric and stricly decreasing along any radius. If $1 < p < q < \frac{N+2}{N-2}$, then u is positive, $\lim_{|x|\to+\infty} u(x) = 0$, u is radially symmetric and stricly decreasing along any radius (up to a translation of the origin).

Proof. Let us define $f(u) = -u^p + u^q$. Since u belongs to $L^{p+1} \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N)$, ∇u is in $L^2(\mathbb{R}^N)$ and by Sobolev's embeddings, u is in $L^{\frac{2N}{N-2}}(\mathbb{R}^N)$. If $s_1 =$

$\frac{N}{N-2} - q$, multiplying Equation (5.1) by u^{s_1} and integrating by parts, we get $u \in L^{2(s_2+q)}(I\!R^N)$ with $s_2 = \frac{s_1+1}{2} \cdot \frac{2N}{N-2} - q$ and then by iteration $u \in L^{2(s_n+q)}(I\!R^N)$ where $s_n = \frac{N}{N-2}\, s_{n-1} + \frac{N}{N-2} - q \to +\infty$ as $n \to +\infty$.

Multiplying now Equation (5.1) by $|\nabla u|^\beta u^\alpha$ for some $\beta \in]0,2[$ and $\alpha > 0$ big enough, we prove that $\nabla u^{\frac{\alpha-1}{\beta+2}}$ belongs $L^{\beta+2}(I\!R^N)$, and iterating again a finite number of times, we prove that ∇u^δ belongs to $L^s(I\!R^N)$ for some $\delta > 0$, $s > N$: u is in $L^\infty(I\!R^N)$ as well as Δu and ∇u is therefore locally bounded in $C^{0,\alpha}(\omega)$ for some $\alpha > 0$ and ω in the support of u. By contradiction, it easily follows that: $\lim_{\rho\to+\infty} \|u\|_{L^\infty(B(0,\rho)^c)} = 0$.

Let us consider first the sublinear case. Assume that u has a non compactly supported connected component K in its support. Without restriction, we may of course assume that $0 \in K$. The radially symmetric function v which is given as the solution of $\Delta v + f(v) = 0$ for $r = |x| > \rho$, with the inital conditions $v(\rho) = \|u\|_{L^\infty(B(0,\rho)^c)}$ and $v'(\rho) = 0$ is such that there exists a $R > \rho$ for which $v(R) = 0$ since $p > -3$, according to the criterion given by Peletier and Serrin in [23]. But for ρ big enough – and u uniformly small enough on $B(0,\rho)^c$ – since f is decreasing on $(0, (\frac{p}{q})^{\frac{1}{q-p}})$, u is dominated by v in $B(0,R) \setminus B(0,\rho)$ according to the Maximum Principle, a contradiction.

The support of u is therefore a union of disjoint compact sets, and with the version of the moving plane method used in [7] or [10] for instance, using the decay of f near 0_+, it is easy to conclude that u is supported by a union of balls on which u is radially symmetric and stricly decreasing along any radius.

The superlinear case is simpler. By Hopf's lemma, any nonnegative solution is positive, of class $C^{1,\alpha}$ and decays to 0 as $|x| \to +\infty$. Then one may apply symmetry results based on the moving plane techniques to get the radial symmetry around some point in $I\!R^N$ (see for instance [14], [20]). $\square$

6 Appendix C: a spectral property

Here we prove Theorem 2.2 (when Conditions (2.3)-(2.4) are not satisfied) by a direct minimization approach, the study of the first and the second variations, and spectral arguments.

Assume that X_M is the set of the nonnegative functions v such that

$$(1+|x|^2)\, v(x) \in L^1(I\!R^N),\ v^m \in L^1(I\!R^N)$$

and $M = \|v\|_{L^1(I\!R^N)} > 0$ and consider

$$\begin{aligned} F_\lambda[v] &= \frac{1}{2}\int_{I\!R^N} v(x)|x + \frac{m}{m-1}\nabla v^{m-1}(x)|^2\, dx - \lambda L[v] \\ &= \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!R^N} |\nabla(v^{m-\frac{1}{2}})|^2\, dx + \frac{1-\lambda}{2}\int_{I\!R^N} v(x)|x|^2\, dx \\ &\qquad + (\frac{\lambda}{1-m} - N)\int_{I\!R^N} v^m(x)\, dx + \lambda K(M) \end{aligned}$$

Lemma 6.1 *For any $\lambda < 1$, if $m > \frac{N-1}{N}$, $m \neq 1$, there exists a minimizer $v_\lambda \in X_M$ of $\inf_{v\in X_M} F_\lambda[v]$.*

Proof. Assume that $\frac{N-1}{N} < m < 1$ and consider a minimizing sequence $(v_n)_{n\in I\!N} \subset X_M$. Using inequality (3.8), we get a bound on the integrals $\int_{I\!R^N} |\nabla v_n^{m-\frac{1}{2}}|^2\, dx$, $\int_{I\!R^N} v|x|^2\, dx$ and $\int_{I\!R^N} v_n^m\, dx$. According to Lemma 2.1, v_n converges weakly in $L^{p(m)}(I\!R^N)$ (with $p(m) = (2m-1)\frac{N}{N-2} > 1$) to some limit v_λ and strongly in $L^{p(m)}_{loc}(I\!R^N)$). By the Dunford-Pettis criterion, since $\int_{I\!R^N} v_n^{p(m)}\, dx$ and $\int_{I\!R^N} v|x|^2\, dx$ are bounded, v_n converges weakly in $L^1(I\!R^N)$ to $v_\lambda \in X_M$. Similarly, as R goes to $+\infty$,

$$\int_{|x|>R} v_n^m\, dx \leq \left(\int_{|x|>R} v_n(C(M) + \frac{1-m}{2m}|x|^2)\, dx\right)^m \cdot \left(\int_{|x|>R} v_\infty^m\, dx\right)^{1-m} \to 0 \tag{6.1}$$

and by the Dunford-Pettis criterion again, we also get $\lim_{n\to+\infty} \int_{I\!R^N} v_n^m\, dx = \int_{I\!R^N} v_\lambda^m\, dx$ (since $m < p(m)$).

Passing to the limit, and by lower semicontinuity of $\int_{I\!R^N} |\nabla v_n^{m-\frac{1}{2}}|^2\, dx$ and $\int_{I\!R^N} v_n|x|^2\, dx$, we obtain: $F[v_\lambda] \leq \liminf_{n\to+\infty} F[v_n]$, which proves the convergence of $|\nabla v_n^{m-\frac{1}{2}}|^2$ and $v_n|x|^2$ respectively strongly in $L^2(I\!R^N)$ and weakly in $L^1(I\!R^N)$.

If $m > 1$, Inequality (3.8) and (6.1) have to be replaced by $\int v^m\ dx \leq \left(\int v\ dx\right)^{\frac{p(m)-m}{p(m)-1}} \cdot \left(\int v^{p(m)}\ dx\right)^{\frac{m-1}{p(m)-1}}$ and the estimate for $|x| > R$ is given by: $\int_{|x|>R} v\ dx \leq \frac{1}{R^2} \int_{I\!R^N} v|x|^2\ dx$. The rest of the proof is the same. $\square$

Lemma 6.2 *With the same notations as in Lemma 6.1, as* $\lambda \to 0$, $v_\lambda \to v_\infty$ *in* $L^1 \cap L^{p(m)}(I\!R^N)$, $\nabla v_\lambda^{m-\frac{1}{2}} \to \nabla v_\infty^{m-\frac{1}{2}}$ *in* $L^2(I\!R^N)$, $v_\lambda^m \to v_\infty^m$ *in* $L^1(I\!R^N)$, *and* $\lim_{\lambda\to 0} \int_{I\!R^N} v_\lambda|x|^2\ dx = \int_{I\!R^N} v_\infty|x|^2\ dx$.

Proof. The same estimates hold for the family $(v_\lambda)_\lambda$ in the limit $\lambda \to 0$ as for the minimizing sequence of Lemma 6.1: the limit is therefore v_∞ which is the unique minimizer in X_M of $\int_{I\!R^N} v|x + \frac{m}{m-1}\nabla v^{m-1}|^2\ dx$. $\square$

We are now facing two possible situations:

- either there exists a $\lambda_0 \in]0,1[$ such that $F_{\lambda_0}[v_{\lambda_0}] \geq 0$ and then $F_\lambda[v_\lambda] \geq 0$ for any $\lambda \leq \lambda_0$,
- or for any $\lambda \in]0,1[$, $F_\lambda[v_\lambda] < 0$. In this case we will pass to the limit $\lambda \to 0$ and get a contradiction by studying for $\lambda \leq 0$ the variations of $F_\lambda[v]$ around $v_\lambda = v_\infty$.

For any $\lambda \leq 0$,

$$
\begin{aligned}
F_\lambda[v] \quad &= \frac{1}{2}\int_{I\!R^N} v(x)|x + \frac{m}{m-1}\nabla v^{m-1}(x)|^2\ dx \\
&\qquad -\lambda\Big(\int_{I\!R^N} v(x)\frac{|x|^2}{2} - \frac{1}{1-m}\int_{I\!R^N} v^m(x)\ dx - K(M)\Big) \\
&= \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{I\!R^N} |\nabla(v^{m-\frac{1}{2}})|^2\ dx + \frac{1-\lambda}{2}\int_{I\!R^N} v(x)|x|^2\ dx \\
&\qquad +(\frac{\lambda}{1-m} - N)\int_{I\!R^N} v^m(x) + \lambda K(M)\ dx \\
&\geq F_\lambda[v_\infty] = 0
\end{aligned}
$$

(the minimum is indeed realized by $v = v_\infty$), the Euler-Lagrange equations associated to the minimization problem are

$$-m^2 v^{2m-3}[\Delta v + (m - \frac{3}{2})\frac{|\nabla v|^2}{v}] + \frac{1-\lambda}{2}|x|^2 + (\frac{\lambda}{1-m} - N) m v^{m-1} = \mu \quad (6.2)$$

where μ is the Lagrange multiplier associated to the constraint $\|v\|_{L^1(\mathbb{R}^N)} = M$. Actually, this equation written for $v = v_\infty$ is a polynomial of degree 2, and the identification of the coefficients gives: $\mu = \frac{m}{1-m} C(m, M) \cdot \lambda$.

To study the second variation, it is easier to consider $w = v^{m-\frac{1}{2}}$ and introduce $G_\lambda[w] = F_\lambda[v] - \mu \int_{\mathbb{R}^N} v(x)\, dx$:

$$\begin{aligned} G_\lambda[w] \quad &= \frac{1}{2}(\frac{2m}{2m-1})^2 \int_{\mathbb{R}^N} |\nabla w|^2 \, dx + \frac{1-\lambda}{2} \int_{\mathbb{R}^N} w^{2\gamma}(x) |x|^2 \, dx \\ &+ (\frac{\lambda}{1-m} - N) \int_{\mathbb{R}^N} w^{1+\gamma}(x)\, dx + \lambda K(M) - \mu \int_{\mathbb{R}^N} w^{2\gamma}(x)\, dx \end{aligned}$$

where $\gamma = \frac{1}{2m-1}$, and to minimize it without constraint. The Euler-Lagrange equations – which are of course identical to Equation (6.2) – are now

$$-(\frac{2m}{2m-1})^2 \Delta w + \gamma(1-\lambda)|x|^2 w^{2\gamma-1} + (1+\gamma)(\frac{\lambda}{1-m} - N) w^\gamma - 2\gamma\mu w^{2\gamma-1} = 0 \quad (6.3)$$

and with $w_\infty = v_\infty^{m-\frac{1}{2}}$,

$$G_\lambda[w_\infty + \epsilon w] - G_\lambda[w_\infty] = \frac{\epsilon^2}{2} (\mathcal{L}^\lambda_{w_\infty} w, w)_{L^2(\mathbb{R}^N)} + o(\epsilon^2) \quad (6.4)$$

with

$$\begin{aligned} \mathcal{L}^\lambda_{w_\infty} \phi = \quad & -(\frac{2m}{2m-1})^2 \Delta \phi + \gamma(2\gamma-1)(1-\lambda) w_\infty^{2(\gamma-1)} |x|^2 \phi \\ & + \gamma(1+\gamma)(\frac{\lambda}{1-m} - N) w_\infty^{\gamma-1} \phi - 2\gamma(2\gamma-1)\mu w_\infty^{2(\gamma-1)} \phi \quad (6.5) \end{aligned}$$

provided the support of w is contained in $B(0,R)$ with $R = R(m,M) = \sqrt{\frac{2m}{m-1} \cdot C(M)}$ whenever $m > 1$.

Because of the positivity of F_λ, for any $\lambda \le 0$, the spectrum of $\mathcal{L}^\lambda_{w_\infty}$ has to be contained in $[0,+\infty[$. Using Equation (6.3), a direct computation shows that $\mathcal{L}^0_{w_\infty}\phi_1 = 0$ for $\phi_1 = w_\infty^\gamma = \left(C(m,M) + \frac{1-m}{2m}|x|^2\right)_+^{-\frac{2}{2(1-m)}}$. Consider now the Rayleigh quotient

$$\lambda_2 = \inf_{<\phi,\phi_1>=0} \frac{(\mathcal{L}_{w_\infty}\phi,\phi)_{L^2(I\!R^N)}}{<\phi,\phi>}$$

where $(.,.)_{L^2(I\!R^N)}$ denotes the standard scalar product in $L^2(I\!R^N)$ and

$$<\phi,\psi>=\int_{B(0,R(m,M))} \frac{\phi(x)\psi(x)}{\left(C(m,M)+\frac{1-m}{2m}|x|^2\right)_+^2}\,dx\ .$$

λ_2 is the second eigenvalue of $\mathcal{L}_{w_\infty}$ for the weighted eigenvalue problem with weight $w_\infty^{\gamma-1} = \left(C(m,M) + \frac{1-m}{2m}|x|^2\right)_+^{-2}$: $\lambda_2 > \lambda_1 > 0$. Note that the constraint $<\phi,\phi_1>=0$ is nothing else than the constraint $\int_{I\!R^N} w^{2\gamma}(x)\,dx = M$.

The end of the proof of Theorem 2.2 goes as follows. Assume by contradiction that for any $\lambda > 0$, if w_λ is a minimizer of G_λ, then $G_\lambda[w_\lambda] < 0$. According to Lemma 6.2, $w_\lambda \to w_\infty$ as $\lambda \to 0_+$, which is in contradiction with the strict positivity of λ_2 and (6.4). □

7 Appendix D: the Csiszár-Kullback inequality

In this appendix, we present a version of the Csiszár-Kullback inequality, which is nothing else than a second order Taylor development of Jensen's inequality. The proof is given for the completeness of the paper and we may refer to see [8], [17] and [1] for more details.

Lemma 7.1 *Assume that* Ω *is a domain in* $I\!R^N$ *and that* s *is a convex nonnegative function on* $I\!R^+$ *such that* $s(1) = 0$ *and* $s'(1) = 0$*. If* μ *is a non-negative measure on* Ω *and if* f *and* g *are nonnegative measurable functions on* Ω *with respect to* μ*, then*

$$\int_\Omega s(\frac{f}{g})g\, d\mu \geq \frac{K}{\max\{\int_\Omega f\, d\mu, \int_\Omega g\, d\mu\}} \cdot \|f-g\|^2_{L^1(\Omega,d\mu)} \tag{7.1}$$

where $K = \frac{1}{4} \cdot \min\{K_1, K_2\}$*,*

$$K_1 = \min_{\eta\in]0,1[} s''(\eta) \quad \textit{and} \quad K_2 = \min_{\substack{\theta\in]0,1[\\ h>0}} s''(1+\theta h)(1+h)\ , \tag{7.2}$$

provided that all the above integrals are finite.

Proof : We may assume without loss of generality that f and g are strictly positive functions. Let us set $h = \frac{f-g}{g}$, so that $\frac{f}{g} = 1 + h$. If ω is any subdomain of Ω and k a positive integrable on ω function, then Cauchy-Schwarz's inequality yields

$$\int_\omega \frac{|f-g|^2}{k}\, d\mu \geq \frac{\left(\int_\omega |f-g|\, d\mu\right)^2}{\int_\omega k\, d\mu}\ . \tag{7.3}$$

The proof of Inequality (7.1) is based on a Taylor's expansion of $s(t)$ around $t = 1$. Since $s(1) = s'(1) = 0$, we have $s(\frac{f}{g}) = s(1+h) = \frac{1}{2}s''(1+\theta h)h^2$ for some function $x \mapsto \theta(x)$ with values in $]0, 1[$. Thus we need to estimate from below the function $\int_\Omega s''(1+\theta h)gh^2\, d\mu$. First, we estimate

$$\int_{f<g} s''(1+\theta h)gh^2\, d\mu = \int_{f<g} s''(1+\theta h)\frac{|f-g|^2}{g}\, d\mu \geq K_1 \int_{f<g} \frac{|f-g|^2}{g}\, d\mu$$

according to (7.2). Using (7.3) with $\omega = \{x \in \Omega\ :\ f(x) < g(x)\}$ and $k = g$, we obtain

$$\int_{f<g} s''(1+\theta h)gh^2\, d\mu \geq K_1 \frac{\left(\int_{f<g} |f-g|\, d\mu\right)^2}{\int_{f<g} g\, d\mu}\ . \tag{7.4}$$

On the other hand, we have

$$\int_{f>g} s''(1+\theta h)gh^2\, d\mu = \int_{f>g} s''(1+\theta h)(1+h)\frac{|f-g|^2}{f}d\mu \geq K_2 \int_{f>g} \frac{|f-g|^2}{f}d\mu$$

using the definition (7.2) of K_2. Now, using again (7.3) with $\omega = \{x \in \Omega : f(x) > g(x)\}$ and $k = f$, we get

$$\int_{f>g} s''(1+\theta h)gh^2\, d\mu \geq K_2 \frac{\left(\int_{f>g} |f-g|\, d\mu\right)^2}{\int_{f>g} f\, d\mu}\ . \tag{7.5}$$

Combining (7.4) and (7.5), we obtain

$$\int_{\Omega} s(\frac{f}{g})p\, d\mu \geq \frac{1}{2}\Big[K_1 \frac{\left(\int_{f<g} |f-g|\, d\mu\right)^2}{\int_{f<g} g\, d\mu} + K_2 \frac{\left(\int_{f>g} |f-g|\, d\mu\right)^2}{\int_{f>g} f\, d\mu}\Big]\ . \tag{7.6}$$

□

Remark 7.1 *If* $M = \int_\Omega f\, d\mu = \int_\Omega g\, d\mu$, *one can improve Inequality (7.1): for nonnegative functions* f *and* g, $\|f-g\|_{L^1(\Omega,d\mu)} = 2\int_{f<g}(f-g)\, d\mu = 2\int_{f>g}(g-f)\, d\mu$. *Inequality (7.6) can indeed be rewritten as* $\int_\Omega s(\frac{f}{g})g\, d\mu \geq \frac{K_1+K_2}{8M} \cdot \|f-g\|^2_{L^1(\Omega,d\mu)}$. *This also holds even if* $s'(1) \neq 0$ *since* $\int_\Omega s'(1)hg\, d\mu = s'(1) \cdot \int_\Omega (f-g)\, d\mu = 0$.

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