Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions ⋆

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Abstract

In this paper, we find optimal constants of a special class of Gagliardo-Nirenberg type inequalities which turns out to interpolate between the classical Sobolev inequality and the Gross logarithmic Sobolev inequality. These inequalities provide an optimal decay rate (measured by entropy methods) of the intermediate asymptotics of solutions to nonlinear diffusion equations.

Key words: Gagliardo-Nirenberg inequalities, Logarithmic Sobolev inequality, Optimal constants, Non linear diffusions, Entropy

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1 Introduction and main results

For $d \geq 3$, Sobolev's inequality [40] states the existence of a constant $A > 0$ such that for any function $u \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ with $\nabla u \in L^2(\mathbb{R}^d)$,

$$\|w\|_{\frac{2d}{d-2}} \leq A \, \|\nabla w\|_2 \,. \tag{1}$$

Here and in what follows, we define for $q > 0$

$$\|v\|_q = \left(\int_{\mathbb{R}^d} |v|^q \, dx \right)^{1/q} .$$

The value of the optimal constant is known to be

$$A = \frac{1}{\sqrt{\pi d(d-2)}} \left(\frac{\Gamma(d)}{\Gamma(d/2)} \right)^{1/d}$$

as established by T. Aubin and G. Talenti in [3,41]. This optimal constant is achieved precisely by constant multiples of the functions

$$w_{\sigma,\overline{x}}(x) = \left(\frac{1}{\sigma^2 + |x-\overline{x}|^2} \right)^{\frac{d-2}{2}} ,$$

with $\sigma > 0$, $\overline{x} \in \mathbb{R}^d$. On the other hand, a celebrated logarithmic Sobolev inequality was found in 1975 by Gross [21]. In the case of Lebesgue measure it states that all functions $w \in H^1(\mathbb{R}^d)$, $d \geq 2$ satisfy for any $\sigma > 0$

$$\int_{\mathbb{R}^d} w^2 \log\left(w^2/\|w\|_2^2 \right) \, dx \, + d \left(1 + \log(\sqrt{\pi}\,\sigma) \right) \|w\|_2^2 \leq \sigma^2 \|\nabla w\|_2^2 \,. \tag{2}$$

The extremals of this inequality (which is not stated here in a scaling invariant form) are constant multiples of the Gaussians

$$w(x) = (\pi\sigma^2)^{-d/4} \, e^{-\frac{|x-\overline{x}|^2}{2\sigma^2}} , \tag{3}$$

with $\overline{x} \in \mathbb{R}^d$ [13,42]. In the first part of this work, we will answer the naturally arising question of how these two classical inequalities are related. As we will see, these inequalities correspond to limiting cases of a one-parameter family of optimal Gagliardo-Nirenberg type inequalities [19,35] which we shall describe next.

For $p > 0$, we define

$$\mathcal{D}^p(\mathbb{R}^d) = \{w \in L^{1+p}(\mathbb{R}^d) \ : \ \nabla w \in L^2(\mathbb{R}^d) \text{ and } |w|^{2p} \in L^1(\mathbb{R}^d)\} \,.$$

Our first main result states the validity of the following optimal Gagliardo-Nirenberg inequality.

Theorem 1 *Let $d \geq 2$. If $p > 1$, and $p \leq \frac{d}{d-2}$ for $d \geq 3$, then for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$ the following inequality holds:*

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta} \,, \tag{4}$$

where

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}} ,$$

with

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)} \,, \quad y = \frac{p+1}{p-1} \,.$$

A is optimal, and (4) is reached with equality if and only if w is a constant multiple of one of the functions

$$w_{\sigma,\overline{x}}(x) = \left(\frac{1}{\sigma^2 + |x-\overline{x}|^2}\right)^{\frac{1}{p-1}} ,$$

with $\sigma > 0$ and $\overline{x} \in \mathbb{R}^d$.

An analogous estimate takes place in the case $0 < p < 1$. In fact we have the following result.

Theorem 2 *Let $d \geq 2$ and assume that $0 < p < 1$. Then for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$ the following inequality holds:*

$$\|w\|_{p+1} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{2p}^{1-\theta} \tag{5}$$

where

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y}{2y+d}\right)^{\frac{1-\theta}{2p}} \left(\frac{\Gamma(\frac{d}{2}+1+y)}{\Gamma(1+y)}\right)^{\frac{\theta}{d}} ,$$

with

$$\theta = \frac{d\,(1-p)}{(1+p)\,(d-(d-2)p)} \,, \quad y = \frac{p+1}{1-p} \,.$$

A is optimal, and (5) is reached with equality by the compactly supported functions

$$w_{\sigma,\overline{x}}(x) = \left(\sigma^2 - |x-\overline{x}|^2\right)_+^{\frac{1}{1-p}} ,$$

with $\sigma > 0$ and $\overline{x} \in \mathbb{R}^d$.

The above results are special cases of Gagliardo-Nirenberg inequalities, which are found here in optimal form. Theorem 1 contains the optimal Sobolev inequality when $p = \frac{d}{d-2}$. Moreover, it provides a direct proof of the Gross-Sobolev inequality with an optimal constant as $p \downarrow 1$. In fact, taking the logarithm of both sides of inequality (5) for any $w \in H^1(\mathbb{R}^d)$, we get

$$\frac{1}{\theta} \log \left(\frac{\|w\|_{2p}}{\|w\|_{p+1}} \right) \leq \frac{1}{\theta} \log A + \log \left(\frac{\|\nabla w\|_2}{\|w\|_{p+1}} \right) .$$

Using that $\theta \sim \frac{d}{4}(p-1)$ as $p \downarrow 1$, we get then

$$\frac{2}{d} \int_{\mathbb{R}^d} \left(\frac{w}{\|w\|_2} \right)^2 \log \left(\frac{w}{\|w\|_2} \right) \, dx \leq \lim_{p \downarrow 1} \frac{1}{\theta} \log A + \log \left(\frac{\|\nabla w\|_2}{\|w\|_2} \right) .$$

Since $\lim_{p\downarrow 1} A = 1$, it is enough to compute $\lim_{p\downarrow 1} \frac{A-1}{\theta}$. For that purpose, we choose for A the extremal function: $w_p(x) = (1+\frac{p-1}{2}|x|^2)^{-\frac{1}{p-1}}$, which converges to $e^{-\frac{|x|^2}{2}} = w_1(x)$ as $p \downarrow 1$. Thus

$$\lim_{p\downarrow 1} \frac{A-1}{\theta} = -\log \left(\frac{\|\nabla w_1\|_2}{\|w_1\|_2} \right) + \frac{4}{d} \lim_{p\downarrow 1} \frac{1}{p-1} \left(\frac{\|w_p\|_{2p}}{\|w_p\|_{p+1}} \right) = I + II .$$

Now,

$$II = \frac{2}{d} \int_{\mathbb{R}^d} \left(\frac{w_1}{\|w_1\|_2} \right)^2 \log \left(\frac{w_1}{\|w_1\|_2} \right) \, dx + III - IV$$

where

$$III = \lim_{p\downarrow 1} \frac{1}{p-1} \log \left(\frac{\|w_p\|_{2p}}{\|w_1\|_{2p}} \right) \quad \text{and} \quad IV = \lim_{p\downarrow 1} \frac{1}{p-1} \log \left(\frac{\|w_p\|_{p+1}}{\|w_1\|_{p+1}} \right) .$$

A straightforward computation yields

$$\lim_{p\downarrow 1} \tfrac{1}{p-1} \int_{\mathbb{R}^d} (w_p^{2p} - w_1^{2p}) \, dx = \lim_{p\downarrow 1} \tfrac{1}{p-1} \int_{\mathbb{R}^d} (w_p^{p+1} - w_1^{p+1}) \, dx = \frac{1}{4} \int_{\mathbb{R}^d} e^{-|x|^2} |x|^4 \, dx .$$

It follows that $III - IV = 0$, hence

$$\begin{aligned} \lim_{p\downarrow 1} \tfrac{A-1}{\theta} &= -\log \left(\tfrac{\|\nabla w_1\|_2}{\|w_1\|_2} \right) + \tfrac{2}{d} \textstyle\int_{\mathbb{R}^d} \tfrac{w_1^2}{\|w_1\|_2^2} \log \left(\tfrac{w_1^2}{\|w_1\|_2^2} \right) \, dx \\ &= \tfrac{1}{2} \log \left(\tfrac{2}{\pi d e} \right) , \end{aligned}$$

using the facts $\int_{\mathbb{R}^d} e^{-|x|^2} \, dx = \pi^{\frac{d}{2}}$ and $\int_{\mathbb{R}^d} e^{-|x|^2} |x|^2 \, dx = \frac{d}{2} \pi^{\frac{d}{2}}$. We have then reached the inequality

$$\int_{\mathbb{R}^d} \frac{w^2}{\|w\|_2^2} \log \left(\frac{w^2}{\|w\|_2^2} \right) \, dx \ \leq \ \frac{d}{2} \log \left(\frac{2 \, \|\nabla w\|_2^2}{\pi d e \, \|w\|_2^2} \right) , \tag{6}$$

for any $w \in H^1(\mathbb{R}^N)$. But this inequality is precisely that obtained from (2), when optimizing in $\sigma > 0$. This inequality is the form of the logarithmic Sobolev inequality which is invariant under scaling [45,28]. As a consequence, optimal functions for (6) are any of the Gaussians given by (3) with $\sigma > 0$, $\bar{x} \in \mathbb{R}^d$. We may also notice that, as a subproduct of the above derivation of (6), *this inequality holds with optimal constants*. See Remark 8 for further remarks and references related to (6).

As an application of these optimal inequalities, we will derive some new results for the asymptotic behavior of solutions to the Cauchy problem

$$u_t = \Delta u^m \ , \quad t > 0 \ , \quad x \in \mathbb{R}^d \ , \tag{7}$$

$$u(0,x) = u_0(x) \geq 0 \ , \quad u_0 \in L^1(\mathbb{R}^d) \ . \tag{8}$$

When $m > 0$, $m \neq 1$, this problem has been extensively studied. The case $m > 1$ is the so-called *porous medium equation*. When $0 < m < 1$ it is usually referred to as the *fast diffusion equation*. Both for $m > 1$ and for $0 < m < 1$, this problem is known to be well posed in weak sense. Moreover it preserves *mass* whenever $m > \frac{d-2}{d}$, in the sense that $\int_{\mathbb{R}^d} u(x,t)\, dx$ is constant in $t > 0$. When $\frac{d-2}{d} < m < 1$, solutions are regular and positive for $t > 0$ [22], but this is no longer true when m is below this threshold: for instance, finite time vanishing may occur as simple examples show. For $m > 1$, solutions are at least Hölder continuous.

The qualitative behavior of solutions to these problems has been the subject of a large number of papers. Since mass is preserved, it is natural to ask whether a scaling brings the solution into a certain universal profile as time goes to infinity. This is the case and the role of the limiting profiles is played by an explicit family of self-similar solutions known as the Barenblatt-Prattle solutions [5], characterized by the fact that their initial data is a Dirac mass. These solutions remain invariant under the scaling $u_\lambda(t,x) = \lambda^{d\alpha} u(\lambda^\alpha x, \lambda t)$ with $\alpha = (2 - d(1-m))^{-1} > 0$, which leaves the equation invariant. They are explicitly given by

$$\mathcal{U}(t,x) = t^{-d\alpha} \cdot v_\infty\left(\frac{x}{t^\alpha}\right) \quad \text{with} \quad v_\infty(x) = \left(\sigma^2 - \frac{m-1}{2m}|x|^2\right)_+^{\frac{1}{m-1}} \tag{9}$$

provided $m > \frac{d-2}{d}$, $m \neq 1$. These solutions have a constant mass uniquely determined by the parameter σ.

If σ is chosen so that the mass of $\mathcal{U}$ coincides with that of u_0, it is known that the asymptotic behavior of u itself is well described by $\mathcal{U}$ as $t \to +\infty$. This phenomenon was first rigorously described by A. Friedman and S. Kamin in the context of $u_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, both in the cases $m > 1$ and $(d-2)/d < m < 1$ [18]. These results have been later improved and extended by J.-L. Vázquez and S. Kamin [24,25]. Also see [44] for a recent survey and some new results. Thus far it is well known that if $u_0 \in L^1(\mathbb{R}^d)$ and either $m > 1$ or $(d-2)/d < m < 1$, then

$$\lim_{t\to+\infty} \|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_1 = 0 \ , \quad \lim_{t\to+\infty} t^{d\alpha}\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} = 0 \ . \tag{10}$$

On the other hand, for the heat equation ($m = 1$), the following fact is classical:

$$\limsup_{t\to+\infty} \ \sqrt{t}\cdot \|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^1(\mathbb{R}^d)} < +\infty \ ,$$

with $\mathcal{U}(t,x) = (2\pi t)^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}\, e^{-\frac{|x|^2}{2t}}$. Our next result extends the above asymptotic behavior to the range $\frac{d-1}{d} \leq m < 2$ using an appropriate Lyapunov functional (see Section 4 for more references on the so-called *entropy dissipation techniques*).

Theorem 3 *Assume that the initial datum u_0 is a nonnegative function with*

$$\int_{\mathbb{R}^d} u_0(1+|x|^2)\; dx + \int_{\mathbb{R}^d} u_0^m \; dx \ < \ +\infty \ .$$

If u is the solution of (7)-(8), and $\mathcal{U}$ given by (9) satisfies $\int_{\mathbb{R}^d} \mathcal{U}(t,x)\; dx = \int_{\mathbb{R}^d} u_0\; dx$, then the following facts hold.

(i) *Assume that $\frac{d-1}{d} < m < 1$ if $d \geq 3$, and $\frac{1}{2} < m < 1$ if $d = 2$. Then*

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}}\|u^m(t,\cdot) - \mathcal{U}^m(t,\cdot)\|_{L^1(\mathbb{R}^d)} < +\infty \ .$$

(ii) *Assume that $1 < m < 2$. Then*

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}}\| \ [u(t,\cdot) - \mathcal{U}(t,\cdot)]\ \mathcal{U}^{m-1}(t,\cdot)\ \|_{L^1(\mathbb{R}^d)} < +\infty \ .$$

The main tool in deriving the above result turns out to be the optimal inequalities of Theorems 1 and 2, which are proven in Section 2. We derive some further consequences of independent interest in Section 3, including the key estimate for the proof of Theorem 3, which we carry out in Section 4. Although an exhaustive list of references would have been too long, as much as possible, relevant references will be quoted in the body of this paper.

2 Gagliardo-Nirenberg inequalities

The question of optimal constants has been the subject of many papers. In the case of critical Sobolev injections and scaling invariant inequalities with weights (Hardy-Littlewood-Sobolev and related inequalities), apart from [3,41], one has to cite the remarkable explicit computation by E.H. Lieb [30] and various results based on concentration-compactness methods [31], but the optimality of the constants in Gagliardo-Nirenberg inequalities (see [29] for an estimate) is a long standing question to which we partially answer here. The special case of Nash's inequality [33] has been solved by E. Carlen and M. Loss in [14]. This case, as well as Moser's inequality [32], does not enter in the subclass that we consider here, but it has the striking property that the minimizers are compactly supported, as in Theorem 5. For more details on the connection between Nash's inequality and the logarithmic Sobolev inequality, see [8] and references therein.

In this section, we will establish the validity of Theorems 1 and 2, and derive some consequences that will be useful for later purposes. First, in order to treat the case $p > 1$ of Theorem 1, we will establish Theorem 4 (which is actually equivalent).

Let us consider the functional

$$G(w) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \frac{1}{1+p} \int_{\mathbb{R}^d} |w|^{1+p} \, dx \, .$$

We define the minimization problem

$$I_\infty \equiv \inf_{w \in \mathcal{X}} G(w)$$

over the set $\mathcal{X}$ of all nonnegative functions $w \in \mathcal{D}^p(\mathbb{R}^d)$ that satisfy the constraint

$$\frac{1}{2p} \int_{\mathbb{R}^d} |w|^{2p} \, dx \;=\; J_\infty \, , \tag{11}$$

where for convenience we make the choice:

$$J_\infty := \frac{\pi^{\frac{d}{2}}}{2p} \left(\frac{2p}{d - p(d-2)} \right)^{y+1} \frac{(d-y-1)^d}{p^{d/2}} \; \frac{\Gamma(y+1-\frac{d}{2})}{\Gamma(y+1)}$$

with $y = \frac{p+1}{p-1}$. The following result characterizes the minimizers of I_∞.

Theorem 4 *Assume that* $p > 1$ *and* $p < \frac{d}{d-2}$ *if* $d \geq 3$. *Then* I_∞ *is achieved.*

Moreover, for any minimizer $\overline{w} \in \mathcal{X}$, there exists $\overline{x} \in \mathbb{R}^d$ such that

$$\overline{w}(x) = \left(\frac{a}{b + |x - \overline{x}|^2} \right)^{\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d \ ,$$

where

$$a = 2\, \frac{2p - d\,(p-1)}{(p-1)^2} \quad \text{and} \quad b = \frac{(2p - d\,(p-1))^2}{p(p-1)^2}. \tag{12}$$

Proof. Using Sobolev's and Hölder's inequalities, it is immediately verified that $I_\infty > 0$. For each $R > 0$, we set B_R to be the ball centered at the origin with radius R and $\mathcal{X}_R = \mathcal{X} \cap H^1_0(B_R)$ (here we extend functions of H^1_0 outside of B_R by 0). Let us consider the family of infima

$$I_R = \inf_{w \in \mathcal{X}_R} G(w) \ .$$

I_R is decreasing with R. Besides, by density, $\lim_{R \to +\infty} I_R = I_\infty$. On the other hand, I_R is achieved since $p < \frac{d}{d-2}$ by some nonnegative, radially symmetric function w_R defined on B_R. The minimizer w_R satisfies on B_R the equation

$$-\Delta w_R + w_R^p = \mu_R\, w_R^{2p-1},$$

where μ_R is a Lagrange multiplier. Let us observe that

$$\int_{\mathbb{R}^d} |\nabla w_R|^2\, dx + \int_{\mathbb{R}^d} |w_R|^{1+p}\, dx = \mu_R \int_{\mathbb{R}^d} |w_R|^{2p}\, dx = 2\, p\, \mu_R\, J_\infty \ .$$

Thus

$$\frac{2p}{p+1}\, \mu_R\, J_\infty \le I_R \le p\, \mu_R\, J_\infty \ ,$$

so that μ_R is uniformly controlled from above and from below as $R \to +\infty$, and converges up to the extraction of a subsequence to some limit $\mu_\infty > 0$. Since I_R itself controls the H^1 norm of w_R over each fixed compact subset of B_R, from the equation satisfied by w_R and standard elliptic estimates, we deduce a uniform control over compacts in $C^{2,\alpha}$ norms. Passing to a convenient subsequence of $R \to +\infty$, we may then assume that w_R converges uniformly and in the C^2 sense over compact sets to a radial function w. We may also assume that $w_R \rightharpoonup w$ weakly in $L^{p+1}(\mathbb{R}^d)$ and $\nabla w_R \rightharpoonup \nabla w$ weakly in $L^2(\mathbb{R}^d)$. Besides, since w_R reaches its maximum at the origin, let us also observe from the equation that we get the estimate

$$1 \le \mu_R w_R^{p-1}(0) \ .$$

This relation implies that w_R does not trivialize in the limit. The function w is thus a positive, radially decreasing solution of

$$-\Delta w + w^p = \mu_\infty w^{2p-1},$$

in entire $\mathbb{R}^d$, and $w(|x|) \to 0$ as $|x| \to +\infty$. Now, since the convergence of w_R to w is uniform over compact sets, and w_R is radially decreasing, we may choose a sufficiently large, but fixed number ρ such that on $\rho < |x| < R$, w_R satisfies an inequality of the form

$$-\Delta w + \frac{1}{2} w^p \leq 0 \; .$$

On the other hand, the fact that $p < \frac{d}{d-2}$ yields that the function

$$\zeta(x) = \frac{C}{|x|^{\frac{2}{p-1}}}$$

satisfies for any sufficiently large choice of C,

$$-\Delta \zeta + \frac{1}{2} \zeta^p \geq 0 \; .$$

If we make this choice so that $w_R(\rho) < \zeta(\rho)$ for all large R, then by comparison we obtain that

$$w_R(x) < \frac{C}{|x|^{\frac{2}{p-1}}}, \quad |x| > \rho \; .$$

Now, if we notice that $\frac{2p}{p-1} > d$, then

$$\lim_{M \to +\infty} \sup_{R > M} \int_{M < |x| < R} |w_R|^{2p} \, dx = 0 \; .$$

As a consequence, $w_R \to w$ strongly in $L^{2p}(\mathbb{R}^d)$. Hence $w \in \mathcal{X}$ and since by weak convergence we have $G(w) \leq I_\infty$, the existence of a minimizer is guaranteed.

The Lagrange multiplier μ_∞ is uniquely determined by the system

$$\begin{cases} \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^d} |w|^{1+p} \, dx = I_\infty \\ \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \int_{\mathbb{R}^d} |w|^{1+p} \, dx = 2 \, p \, \mu_\infty \, J_\infty \\ \frac{d-2}{2d} \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \frac{1}{p+1} \int_{\mathbb{R}^d} |w|^{1+p} \, dx = \mu_\infty \, J_\infty \end{cases}$$

which follows respectively from the definition of I_∞, and as a consequence of the equation multiplied by w and $(x \cdot \nabla w)$. The constant μ_∞ therefore depends only on m, p and d.

Finally, let us consider any minimizer w of G over $\mathcal{X}$. It necessarily satisfies the equation

$$-\Delta w + w^p = \mu_\infty w^{2p-1} \; .$$

Ground state solutions of this equation are known to be radial around some point [20]. With no loss of generality, we take it to be the origin. On the other hand, there is a unique choice of a positive paramenter λ such that $\overline{w}(x) = \lambda^{2/(p-1)} w(\lambda x)$ satisfies

$$-\Delta \overline{w} + \overline{w}^p = \overline{w}^{2p-1} \ .$$

Invoking uniqueness results of positive solutions by P. Pucci & J. Serrin [37] and by J. Serrin & M. Tang for quasilinear elliptic equations [39], we deduce that the above equation has only one positive radial ground state. On the other hand, the function

$$\overline{w}(x) = \left(\frac{a}{b + |x|^2} \right)^{\frac{1}{p-1}} ,$$

where the values of a and b are precisely those given by (12), is an explicit solution, hence the unique one. Finally, the fact that $\int_{\mathbb{R}^d} w^{2p} \, dx = J_\infty$ determines exactly what the value of λ is, in fact $\lambda = 1$. This ends to the proof of Theorem 4. $\square$

Next we will state and prove the analogue of Theorem 4 for the case $0 < p < 1$. We consider now the functional

$$\tilde{G}(w) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \frac{1}{2p} \int_{\mathbb{R}^d} |w|^{2p} \, dx \ .$$

We shall denote by

$$\tilde{I}_\infty \equiv \inf_{w \in \tilde{\mathcal{X}}} \tilde{G}(w)$$

the problem of minimizing $\tilde{G}$ over the class $\tilde{\mathcal{X}}$ of all nonnegative functions $w \in \mathcal{D}^p(\mathbb{R}^d)$ that satisfy the constraint

$$\frac{1}{p+1} \int_{\mathbb{R}^d} |w|^{p+1} \, dx \ = \ \tilde{J}_\infty \ ,$$

where $\tilde{J}_\infty$ is now the number

$$\tilde{J}_\infty = \frac{\pi^{\frac{d}{2}}}{p+1} \left(\frac{2p}{d - p(d-2)} \right)^{1-y} \frac{(d+y-1)^d}{p^{d/2}} \ \frac{\Gamma(1+y)}{\Gamma(1+y+\frac{d}{2})}$$

with $y = \frac{p+1}{1-p}$. Then we have the following result

Theorem 5 *Assume that $0 < p < 1$. Then $\tilde{I}_\infty$ is achieved by the radially symmetric function*

$$\overline{w}(x) = a^{-\frac{1}{1-p}} \left(b - |x|^2 \right)_+^{\frac{1}{1-p}}$$

where a and b are given by (12) as in Theorem 4. Moreover, if $p > \frac{1}{2}$, for any minimizer w, there exists $\tilde{x} \in \mathbb{R}^d$ such that $w(x) = \tilde{w}(x - \tilde{x})$, $\forall \, x \in \mathbb{R}^d$.

Proof. The proof goes similarly to that of Theorem 4. We consider the minimization problem on $\tilde{\mathcal{X}}_R = \tilde{\mathcal{X}} \cap H^1_0(B_R)$. By compactness, the minimizer is achieved. Moreover, using decreasing rearrangements, one finds that this minimizer w_R can be chosen radially symmetric and decreasing. It satisfies the equation

$$-\Delta w_R + w_R^{2p-1} = \mu_R \, w^p \ ,$$

within the ball where w_R is strictly positive (we need to be careful with the fact that $2p-1$ may be a negative quantity). Exactly the same analysis as above, yields that μ_R is uniformly controlled and approaches some positive mumber μ_∞. Moser's iteration provides us with a uniform L^∞ bound derived from the H^1 bound. We should observe at this point that the O.D.E. satisfied by w_R easily gives by itself an upper local estimate $C(R_0^2-|x|^2)_+^{1/(1-p)}$ for some $C>0$ in case the support corresponds to $|x| < R_0 \leq R$. If this is the case for some $R_0 > 0$, then the minimizer will be unchanged for any $R > R_0$ and in fact will be the solution of the minimization problem in $\mathbb{R}^d$. On the other hand, a straightforward comparison with barriers of that type [15] actually yields that at some point the minimizer does get compactly supported inside B_R for all R sufficiently large. This minimizer is thus a ground state radial solution of

$$-\Delta w + w^{2p-1} = \mu_\infty \, w^p$$

and for the same reason as in the proof of Theorem 4, μ_∞ is unique. According to the uniqueness results of P. Pucci & J. Serrin and J. Serrin & M. Tang [37,39] again, such a radial minimizer is unique. A scaling argument (with $\overline{w}(x) = \lambda^{1/(p-1)} w(\lambda x)$) similar to the one employed in the proof of Theorem 4 gives that $\mu_\infty = 1$ and w is then nothing but the explicit solution given in the statement of Theorem 5.

In case that $2p-1>0$, it is known that all ground states are compactly supported and radially symmetric on each component of their supports [15]. We obtain then a complete classification of the minimizers as in Theorem 4. When $2p-1<0$, the question arises of whether we do get out of the Euler-Lagrange equation a nice ground state solution, and whether such a solution is symmetric. This does not seem to be known. □

We are now in a position to proceed with the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $w \in \mathcal{D}^p$ satisfy the constraint

$$J[w] := \frac{1}{2p} \int_{\mathbb{R}^d} |w(x)|^{2p} \, dx = J_\infty \ ,$$

with J_∞ given in (11). For $\lambda > 0$, we consider the scaled function $w_\lambda(x) =$

$\lambda^{\frac{d}{2p}} w(\lambda x)$, which still satisfies $J[w_\lambda] = J_\infty$. Then for each $\lambda > 0$,

$$G(w_\lambda) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla w|^2 \, dx \cdot \lambda^{\frac{d}{p}-(d-2)} + \frac{1}{1+p} \int_{\mathbb{R}^d} |w|^{1+p} \, dx \cdot \lambda^{-d \, \frac{p-1}{2p}} \; \geq \; I_\infty \; .$$

Minimizing the left hand side of the above expression in $\lambda > 0$ yields

$$C_* \left[\|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta} \right]^\delta \geq I_\infty \; ,$$

where $C_* = \frac{1}{2} \lambda_*^{\frac{d}{p}-(d-2)} + \frac{1}{p+1} \lambda_*^{-d \, \frac{p-1}{2p}}$, $\lambda_* = \frac{d}{d-p(d-2)} \, \frac{p-1}{p+1}$,

$$\delta = 2p \, \frac{d+2-(d-2)p}{4p-d(p-1)} \quad \text{and} \quad \theta = \frac{d \, (p-1)}{p \, (d+2-p(d-2))} \; .$$

Since $\|w\|_{2p} = 2p \, J_\infty$, we may write

$$\|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta} \geq \left(\frac{I_\infty}{C_*} \right)^{1/\delta} \frac{\|w\|_{2p}}{(2p \, J_\infty)^{1/(2p)}} \; .$$

By homogeneity, the above inequality actually holds for any $w \in \mathcal{D}^p$, with optimal constant $A = (2p \, J_\infty)^{1/(2p)} \left(\frac{C_*}{I_\infty} \right)^{1/\delta}$. □

Remark 6 *The expression of A given in Theorem 1 can be recovered using the invariance under scaling of the inequality. We may indeed write*

$$A = \frac{\|\bar{w}_{a,b}\|_{2p}}{\|\nabla \bar{w}_{a,b}\|_2^\theta \, \|\bar{w}_{a,b}\|_{p+1}^{1-\theta}}$$

for any $\bar{w}_{a,b}(x) = \left(\frac{a}{b+r^2} \right)^{\frac{1}{p-1}}$ with arbitrary positive constants a and b. This fact and a direct computation of this quotient, for instance with $a = b = 1$, yield the expression for A in Theorem 1.

Proof of Theorem 2. It is very similar to the proof of Theorem 1. For any $w \in \mathcal{D}^p$ satisfying the constraint

$$\tilde{J}[w] := \frac{1}{p+1} \int_{\mathbb{R}^d} |w(x)|^{p+1} \, dx = \tilde{J}_\infty$$

and for any $\lambda > 0$, we consider the scaling $w_\lambda(x) = \lambda^{d/(p+1)} w(\lambda x)$, which also satisfies $\tilde{J}[w_\lambda] = \tilde{J}_\infty$. Using now that $\tilde{G}[w_\lambda] \geq \tilde{I}_\infty$, we find, after optimizating on $\lambda > 0$,

$$\tilde{C}_* \left[\|\nabla w\|_2^\theta \, \|w\|_{2p}^{1-\theta} \right]^{\tilde{\delta}} \geq \tilde{I}_\infty \; ,$$

where $\tilde{C}_* = \frac{1}{2}\,\lambda_*^{\frac{2d}{p+1}-(d-2)} + \frac{1}{2p}\,\lambda_*^{-d\,\frac{1-p}{p+1}}$, $\lambda_* = \frac{p-1}{p}\,\frac{d}{d+2-p(d-2)}$,

$$\tilde{\delta} = \frac{(1+p)\,(d-(d-2)p)}{d+1-p(d-1)} \quad \text{and} \quad \theta = \frac{d\,(1-p)}{(1+p)\,(d-(d-2)p)}\;.$$

Since $\|w\|_{1+p} = (p+1)\,\tilde{J}_\infty$, we may write

$$\|\nabla w\|_2^\theta\,\|w\|_{2p}^{1-\theta} \geq \left(\frac{\tilde{I}_\infty}{\tilde{C}_*}\right)^{1/\tilde{\delta}} \frac{\|w\|_{p+1}}{((p+1)\,\tilde{J}_\infty)^{1/(p+1)}}\;.$$

By homogeneity and invariance under scaling, the above inequality is true for any $w \in \mathcal{D}^p$, with optimal constant $A = ((p+1)\,\tilde{J}_\infty)^{1/(p+1)}\left(\frac{\tilde{C}_*}{\tilde{I}_\infty}\right)^{1/\tilde{\delta}}$. □

Remark 7 *Homogeneity and invariance under scaling also yield for A the expression*

$$A = \frac{\|\bar{w}_{a,b}\|_{p+1}}{\|\nabla \bar{w}_{a,b}\|_2^\theta\;\|\bar{w}_{a,b}\|_{2p}^{1-\theta}}$$

for any $\bar{w}_{a,b}(x) = (\frac{b-r^2}{a})_+^{1/(1-p)}$ with a and b arbitrary positive constants. The constant in Theorem 2 then follows by direct computations (with for instance $a=b=1$).

Remark 8 *On the logarithmic Sobolev inequality, we may notice that:*
(i) Finding it as a limit has already been done in [4,9] and several other results show that the logarithmic Sobolev inequality is an endpoint of various families of inequalities: see for instance [6–9]. The point is that we get here the optimal form [45] with optimal constants as the limit of optimal inequalities with optimal constants.
(ii) A proof of (6) based on Theorem 2 and similar to the one given in Theorem 4 can also be established by letting $p \uparrow 1$. It is indeed enough to differentiate the function $p \mapsto A\,\|\nabla w\|_2^\theta \|w\|_{2p}^{1-\theta} - \|w\|_{p+1}$, at $p=1$, where A and θ are considered as functions of p. However one has to assume that w belongs to $\mathcal{D}^p(\mathrm{I\!R}^d)$ for any p in a left neighborhood of 1, and then extend the inequality to $H^1(\mathrm{I\!R}^d) = \mathcal{D}^1(\mathrm{I\!R}^d)$ by a density argument.
(iii) The optimal form (6) of the the logarithmic Sobolev inequality is easily recovered from (2) by applying it to $v(\cdot) = \lambda^{d/2} w(\lambda\cdot)$ and optimizing the corresponding expression with respect to $\lambda > 0$, thus giving $\lambda = (\frac{\pi d}{2\sigma^2})^{1/2}\frac{\|w\|_2}{\|\nabla w\|_2}$.
(iv) The fact that the family of Gaussians (3) are the only minimizers follows by a symmetry argument [20] and by the of result of J. Serrin & M. Tang [39].

3 Some consequences

We may recast the Gagliardo-Nirenberg inequality of Theorem 1 and its extension of Theorem 2 into a single non homogeneous form with still optimal constants. Since the Lagrange multipliers asssociated to the constraints are explicit, this indeed allows to rewrite the minimization problems of Theorems 4 and 5 without constraints (it turns out that both expressions corresponding to $p > 1$ and $p < 1$ can be collected into a single non homogeneous inequality). This form is similar to the standard form of the logarithmic Sobolev inequality (2) compared to the scaled form (6) (also see Remark 11).

Proposition 9 *Let $d \geq 2$, $\sigma > 0$ and $p > 0$ be such that $p \neq 1$, and $p \leq \frac{d}{d-2}$ if $d \geq 3$. Then, for any function $w \in \mathcal{D}^p(\mathbb{R}^d)$, the following inequality holds:*

$$\tfrac{1}{2}\,\sigma^{\frac{d}{p}-d+2}\,\|\nabla w\|_2^2 + \tfrac{\varepsilon}{p+1}\,\sigma^{-d\frac{p-1}{2p}}\,\|w\|_{1+p}^{1+p} - \tfrac{\varepsilon}{2\,p}\,K\,\|w\|_{2p}^{\delta} \geq 0 \tag{13}$$

where ε is the sign of $(p-1)$, $\delta = 2p\,\frac{d+2-p(d-2)}{4p-d(p-1)}$ and $K > 0$ is an optimal constant. For $p > \frac{1}{2}$, $p \neq 1$, optimal functions for Inequality (13) are all given by the family of functions $w_{\sigma,\overline{x}}(x) = \sigma^{-\frac{d}{2p}}\,\overline{w}(\frac{x-\overline{x}}{\sigma})$. For $0 < p \leq \frac{1}{2}$, Inequality (13) is also achieved by the same family of functions. Here $\overline{w}(x) = (\frac{a}{b+\varepsilon|x|^2})_+^{\frac{1}{p-1}}$ with a and b given by (12) (in both cases: $p > 1$ and $p < 1$) and K is explicitely given by (14) (see below).

Proof. Using the scaling $w \mapsto \sigma^{-\frac{d}{2p}} w(\frac{\cdot}{\sigma})$, it is clear that (13) holds for any $\sigma > 0$ if and only if it holds at least for one. For $p > 1$, we take $\sigma = 1$ and (13) is a direct consequence of the proof of Theorem 1, with $K = C_* A^{-\delta}$. The case $p < 1$ is slightly more delicate and we proceed as in the proof of Theorem 5. Let $w_\lambda(x) = \lambda^{\frac{d}{p+1}} w(\lambda x)$. An optimization on $\lambda > 0$ of the quantity

$$\tfrac{1}{2}\,\sigma^{\frac{d}{p}-d+2}\|\nabla w_\lambda\|_2^2 + \frac{K}{2\,p}\,\|w_\lambda\|_{2p}^{\delta} = \tfrac{1}{2}\,\|\nabla w\|_2^2 \cdot \sigma^{\frac{d}{p}-d+2}\lambda^{\frac{2d}{p+1}-d+2} + \frac{K}{2\,p}\,\|w_\lambda\|_{2p}^{\delta} \cdot \lambda^{d\,\frac{p-1}{p+1}\,\frac{\delta}{2p}}$$

shows that it is bounded from below by

$$K^{\frac{1}{2}\,\frac{4p-d(p-1)}{d-p(d-2)}}(C\|\nabla w\|_2^{\theta}\|w_\lambda\|_{2p}^{1-\theta})^{p+1} \cdot \sigma^{-d\,\frac{p-1}{2p}}$$

for some explicit constant $C > 0$, which using Theorem 1 again allows to identify K. □

Remark 10 *The function $\overline{w} = w_{\sigma=1,\bar{x}=0} = \bar{w}_{a,b}$ (with the notations of Remarks 6 and 7, and a, b given by (12)) is a (the unique up to a translation if $p > \frac{1}{2}$) nonnegative radial solution of $-\Delta w + \varepsilon w^p = \varepsilon w^{2p-1}$ (on its support if*

$p \leq \frac{1}{2}$), *which allows us to compute* K *as*

$$K = \frac{1}{2p} \|\overline{w}\|_{2p}^{2p-\delta} = \begin{cases} \frac{1}{2p}\, a^{\frac{4p}{4p-d(p-1)}} b^{-1} \left(\pi^{d/2} \frac{\Gamma(\frac{2p}{p-1} - \frac{d}{2})}{\Gamma(\frac{2p}{p-1})} \right)^{\frac{2(p-1)}{4p-d(p-1)}} & \text{if } p > 1 \\ \frac{1}{2p}\, a^{\frac{4p}{4p-d(p-1)}} b^{-1} \left(\pi^{d/2} \frac{\Gamma(\frac{1+p}{1-p})}{\Gamma(\frac{1+p}{1-p} + \frac{d}{2})} \right)^{\frac{2(p-1)}{4p-d(p-1)}} & \text{if } p > 1 \end{cases} \tag{14}$$

Remark 11 *Inequality (13) is invariant under the scaling* $w \mapsto \mu^{\frac{2}{p-1}} w(\mu \cdot)$, *which makes it clear that minimizers form a one-parameter family (up to any translation in* $\mathbb{R}^d$*). If* $d \geq 3$, *for* $p = \frac{d}{d-2}$, *the dependence in* σ *disappears and (13) is the usual Sobolev inequality (1), with the usual scaling invariance* ($\frac{2}{p-1} = d - 2$). *We observe that in the limit* $p \to 1$, *up to an appropriate scaling, we recover the Gross logarithmic Sobolev inequality in the usual non-homogeneous form (2).*

As noted in [6], the Gaussian weighted forms of the Poincaré inequality and logarithmic Sobolev inequalities may take very simple forms. If we denote by $d\mu$ the measure $(2\pi)^{-d/2} e^{-|x|^2/2}\, dx$, these inequalities are respectively given by

$$\begin{array}{l} \int_{\mathbb{R}^d} |f|^2\, d\mu - \left(\int_{\mathbb{R}^d} |f|^2\, d\mu\right)^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu \\ \text{and} \quad \int_{\mathbb{R}^d} |f|^2 \log\left(\frac{|f|^2}{\int_{\mathbb{R}^d} |f|^2\, d\mu} \right) d\mu \leq 2 \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu\ , \end{array}$$

and a whole family interpolates between both, for $1 \leq p < 2$:

$$\int_{\mathbb{R}^d} |f|^2\, d\mu - \left(\int_{\mathbb{R}^d} |f|^p\, d\mu \right)^{2/p} \leq (2-p) \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu$$

(the logarithmic Sobolev inequality appears as the derivative at $p = 2$). However this family is not optimal (except for $p = 1$ or $p = 2$). Here we will establish a family of optimal inequalities, to the price of weights that are slightly more complicated.

Corollary 12 *Let* $p > 1$ *and consider* $w(x) = \left(\frac{a}{b+|x|^2}\right)^{\frac{1}{p-1}}$ *with* a *and* b *given by (12). Then for any measurable function* f,

$$\frac{K}{p} \left(\int_{\mathbb{R}^d} |f|^{2p} w^{2p} dx \right)^{\frac{\delta}{2p}} - \int_{\mathbb{R}^d} |f|^2\, w^{2p} dx - \int_{\mathbb{R}^d} \left(\frac{2}{p+1} |f|^{p+1} - |f|^2 \right) w^{p+1} dx \leq \int_{\mathbb{R}^d} |\nabla f|^2\, w^2 dx$$

provided all above integrals are well defined. Here K *is an optimal constant, given by (14), and* $\delta = 2p\, \frac{d+2-(d-2)p}{4p-d(p-1)}$.

A similar result holds for $p < 1$.

Proof. It is a straightforward consequence of Inequality (13) with $\sigma = 1$ applied to (fw) and of

$$\int_{\mathbb{R}^d} |\nabla(wf)|^2 \, dx = \int_{\mathbb{R}^d} |\nabla f|^2 \, w^2 dx - \int_{\mathbb{R}^d} f^2 \, w\Delta w \, dx$$

together with $\Delta w = w^p - w^{2p-1}$. □

As another straightforward consequence of Proposition 9, Inequality (13) can be rewritten for $v = w^{2p}$, $m = \frac{p+1}{2p}$ and $\sigma^{-\frac{1}{2p}(4p-d(p-1))} = \frac{d-p(d-2)}{|p^2-1|}$ (for $p < \frac{d}{d-2}$) as follows (this form will be very useful in the next section).

Corollary 13 *Let $d \geq 2$, $m \geq \frac{d-1}{d}$ ($m > \frac{1}{2}$ if $d = 2$), $m \neq 1$ and v be a nonnegative function such that $\nabla v^{m-1/2} \in L^2(\mathbb{R}^d)$, $x \mapsto |x|^2 v(x) \in L^1(\mathbb{R}^d)$ and*

$$\begin{cases} v \in L^1(\mathbb{R}^d) & \textit{if} \quad m > 1 \,, \\ v^m \in L^1(\mathbb{R}^d) & \textit{if} \quad m < 1 \,. \end{cases}$$

Then

$$0 \leq L[v] - L[v_\infty] \leq \frac{1}{2} \int_{\mathbb{R}^d} v|x + \frac{m}{m-1} \nabla(v^{m-1})|^2 \, dx \,, \tag{15}$$

$$\textit{where} \quad L[v] = \int_{\mathbb{R}^d} \left(v\frac{|x|^2}{2} - \frac{1}{1-m} v^m \right) dx$$

and $v_\infty(x) = \left(\sigma^2 + \frac{1-m}{2m}|x|^2\right)_+^{\frac{1}{m-1}}$ with σ defined in order that $M{:=}\|v\|_1{=}\|v_\infty\|_1$. This inequality is optimal and becomes an equality if and only if $v = v_\infty$.

Note that by convexity, v_∞ is the unique minimizer of $L[v]$ under the constraint $\|v\|_1 = M$. The constant σ arising in the expression of v_∞ is explicit:

$$\sigma^{\frac{2-d(1-m)}{1-m}} = \begin{cases} \frac{1}{M} \left(\frac{2m}{1-m}\,\pi\right)^{\frac{d}{2}} \frac{\Gamma(\frac{1}{1-m}-\frac{d}{2})}{\Gamma(\frac{1}{1-m})} & \text{if} \quad m < 1 \,, \\ \frac{1}{M} \left(\frac{2m}{m-1}\,\pi\right)^{\frac{d}{2}} \frac{\Gamma(\frac{m}{m-1})}{\Gamma(\frac{m}{m-1}+\frac{d}{2})} & \text{if} \quad m > 1 \,. \end{cases}$$

4 Long time behaviour of fast diffusion and porous medium equations

In what follows, we denote by $u(x,t)$ the solution of the Cauchy problem (7)-(8). We will also denote henceforth $M = \int_{\mathbb{R}^d} u_0(x)\ dx$. For $m \neq 1$, let us consider the solution of $\dot{R} = R^{(1-m)d-1}$, $R(0) = 1$:

$$R(t) = \Big(1 + (2 - d(1-m))t\Big)^{\frac{1}{2-d(1-m)}}\ , \tag{16}$$

and let $\tau(t) = \log R(t)$. The function $v(x,\tau)$ defined from u by the relation

$$u(t,x) = R(t)^{-d} \cdot v\Big(\tau(t), \frac{x}{R(t)}\Big) \tag{17}$$

satisfies the equation

$$v_\tau = \Delta(v^m) + \nabla \cdot (xv) \quad \tau > 0,\ x \in \mathbb{R}^d\ , \tag{18}$$

which for $m = 1$ corresponds to the linear Fokker-Planck equation. Let us observe that $R(t) \to +\infty$ whenever $\frac{d-2}{d} < m$, which covers our entire range of interest. In (17), the L^1 norm is preserved: $\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|v(\tau(t),\cdot)\|_{L^1(\mathbb{R}^d)}$.

Since $R(0) = 1$ and $\tau(0) = 0$, the initial data is preserved:

$$v(\tau = 0, x) = u_0(x) \quad \forall\, x \in \mathbb{R}^d\ .$$

With the same notations as in Section 1, as $t \to +\infty$, $R(t) \sim t^\alpha$, $u_\infty(t,\cdot) \sim \mathcal{U}(t,\cdot)$ and, according to (10), the known fact $u(t,\cdot) \sim \mathcal{U}(t,\cdot)$ when $\frac{d-2}{d} < m < 1$ or $m > 1$ reads in these new scales just as: $v(\tau,x) \to v_\infty(x)$ for $\tau \to +\infty$, both uniformly and in the L^1 sense, with the notations of Corollary 13.

It turns out that $v \mapsto L[v] = \int_{\mathbb{R}^d}(v\frac{|x|^2}{2} - \frac{1}{1-m}v^m)\ dx$ defines a Lyapunov functional for equation (17) as we shall see below. The proof of Theorem 3 will be a consequence of Propositions 14 and 15 below, and of Corollary 13.

Proposition 14 *Assume that $m > \frac{d}{d+2}$ and that u_0 is a nonnegative function such that $(1 + |x|^2)\, u_0$ and u_0^m belong to $L^1(\mathbb{R}^d)$. Let v be the solutions of equation (18) with initial data u_0. Then, with the above notations,*

$$\frac{d}{d\tau} L[v(\tau,\cdot)] = -\int_{\mathbb{R}^d} v(\tau,\cdot)|x + \frac{m}{m-1}\nabla v(\tau,\cdot)^{m-1}|^2\ dx\ , \tag{19}$$

$$\lim_{\tau \to +\infty} L[v(\tau,\cdot)] = L[v_\infty]\ , \tag{20}$$

and if $\frac{d-1}{d} \leq m < 1$ *for* $d \geq 3$, $\frac{1}{2} < m < 1$ *if* $d = 2$, *or* $m > 1$, *then*

$$0 \leq L[v(\tau,\cdot)] - L[v_\infty] \leq (L[u_0] - L[v_\infty]) \cdot e^{-2\tau} \quad \forall\, \tau > 0\ . \tag{21}$$

Proof. Let us assume first that the initial data $u_0(x)$ is smooth and compactly supported in say the ball $B(0,\rho)$ for some $\rho > 0$. Assume that $\frac{d}{d+2} < m < 1$. The solution is smooth thanks to the results in [22]. Let us consider the function

$$w_\rho(x) = (\frac{1-m}{2m})^{-\frac{1}{1-m}}(|x|^2 - \rho^2)^{-\frac{1}{1-m}}\ .$$

It is easily checked that $w_\rho(x)$ is a steady state of (18), defined on the region $|x| > \rho$. Since this function takes infinite values on $\partial B(0,\rho)$, the comparison principle implies that $v(\tau,x) \leq w_\rho(x)$ for all $\tau > 0$. Hence $v(x,\tau) = O(|x|^{-\frac{2}{1-m}})$ uniformly in $\tau > 0$. Let us fix a number $R > 0$. Integrations by parts then give

$$\begin{aligned}
&\tfrac{d}{d\tau} \textstyle\int_{B(0,R)} v\, \frac{|x|^2}{2}\, dx \\
&\qquad = \textstyle\int_{B(0,R)} \frac{|x|^2}{2} \nabla\cdot(\nabla v^m + xv)\, dx \\
&\qquad = -\textstyle\int_{B(0,R)} x\cdot(\nabla v^m + xv)\, dx + \frac{R}{2}\int_{\partial B(0,R)} (\nabla v^m + xv)\cdot x\, d\sigma(x) \\
&\qquad = d\textstyle\int_{B(0,R)} v^m\, dx - \int_{B(0,R)} |x|^2\, v\, dx + \frac{R}{2}\int_{\partial B(0,R)} (\nabla v^m + xv)\cdot x\, d\sigma\ .
\end{aligned}$$

Integrating with respect to τ, we get

$$\begin{aligned}
&\textstyle\int_{B(0,R)} (v(x,\tau) - u_0(x))\, \frac{|x|^2}{2}\, dx \\
&\qquad = d\textstyle\int_0^\tau \int_{B(0,R)} v^m(x,s)\, dx\, ds + \frac{R}{2}\int_0^\tau \int_{\partial B(0,R)} (\nabla v^m(x,s)\cdot x + v(x,s) R^2)\, d\sigma\ .
\end{aligned}$$

Now, for fixed τ, the rate of decay of $v(x,\tau)$ implies that, as $R \to +\infty$,

$$R^3 \int_0^\tau \int_{\partial B(0,R)} v(x,s)\, d\sigma\, ds = O(R^{d+2-\frac{2}{1-m}})\ .$$

On the other hand, $R^{1-d}\int_{\partial B(0,R)}\int_0^\tau v^m(x,s) d\sigma ds = O(R^{-\frac{2m}{1-m}})$ as $R \to +\infty$, which means that $\int_{\partial B(0,1)}\int_0^\tau v^m(Rz,s)\, d\sigma\, ds = O(R^{-\frac{2m}{1-m}})$. Hence along a sequence $R_n \to +\infty$, we get $\frac{\partial}{\partial R}\int_{\partial B(0,1)}\int_0^\tau v^m(Rz,s)\, d\sigma\, ds_{|R=R_n} = O(R_n^{-\frac{2m}{1-m}-1})$. Equivalently $R_n^{-d}\int_{\partial B(0,R_n)}\int_0^\tau \nabla v^m(x,s)\cdot x\, d\sigma\, ds = O(R_n^{-\frac{2m}{1-m}-1})$, hence

$$R_n \int_0^\tau \int_{\partial B(0,R_n)} \nabla v^m(x,s)\cdot x\, d\sigma\, ds = O(R_n^{d-\frac{2m}{1-m}})\ .$$

The latter term goes to zero as $R_n \to +\infty$ since $m > \frac{d}{d+2}$. We conclude then that

$$\int_{\mathbb{R}^d} (v(x,\tau) - u_0(x))\, \frac{|x|^2}{2}\, dx = d \int_0^\tau \int_{\mathbb{R}^d} v^m(x,s)\, dx\, ds\ .$$

Now, a similar argument leads us to

$$\frac{1}{1-m} \int_{\mathbb{R}^d} (v^m(x,\tau) - v_0^m(x))\, dx = \int_0^\tau \int_{\mathbb{R}^d} \Big(\frac{4m^2}{(2m-1)^2} |\nabla (v^{m-1/2})|^2 - d\, v^m\Big)\, dx\, ds\ .$$

We conclude that $L[v(\tau,\cdot)]$ is well defined and decreasing according to (19).

In the case $m > 1$, the solution has compact support for any $\tau > 0$ and the computation leading to Equation (19) can be carried out directly. Finally, the requirement that u_0 is smooth and compactly supported can be removed by a density argument. The proof of (19) is complete.

If $\frac{d-1}{d} \leq m < 1$ for $d \geq 3$, $\frac{1}{2} < m < 1$ if $d = 2$, or $m > 1$, combining Relation (19) with Estimate (15) of Corollary 13, we get the differential inequality

$$\frac{d}{d\tau} L[v(\tau,\cdot)] \leq -2(L[v(\tau,\cdot)] - L[v_\infty])\ .$$

Since $L[v_\infty]$ minimizes $L[w]$ on $\{w \in L^1_+(\mathbb{R}^d)\ :\ \|w\|_1 = \|u_0\|_1\}$, (21) immediately follows. In that case, (20) is trivial.

Let us establish (20) when $\frac{d}{d+2} < m < \frac{d-1}{d}$. We have proven that L defines a Lyapunov functional for Equation (17). The mass of v is finite and preserved in time, $L[v(\cdot,\tau)]$ is decreasing and therefore uniformly bounded from above in τ. The quantities $\int_{\mathbb{R}^d} v(\tau,x)\,|x|^2\, dx$ and $\int_{\mathbb{R}^d} v^m(\tau,x)\, dx$ are uniformly bounded from above in τ, because of Hölder's inequality applied to $v^m v_\infty^{-m(1-m)} \cdot v_\infty^{m(1-m)}$:

$$\int_\omega v^m\, dx \leq \Big[\int_\omega v(\sigma^2 + \frac{1-m}{2m}|x|^2)\, dx\Big]^m \cdot \Big[\int_\omega v_\infty^m\, dx\Big]^{1-m}\ , \tag{22}$$

for any domain $\omega \subset \mathbb{R}^d$, and because of the definition of $L[v]$:

$$\int_{\mathbb{R}^d} v\, \frac{|x|^2}{2}\, dx - \frac{1}{1-m}\Big[\int_{\mathbb{R}^d} v(\sigma^2 + \frac{1-m}{2m}|x|^2) dx\Big]^m \leq L[v] \tag{23}$$

(with here $\omega = \mathbb{R}^d$), thus giving estimates on $\int_{\mathbb{R}^d} v\, \frac{|x|^2}{2}\, dx$ and $\|v^m\|_{L^1(\mathbb{R}^d)}$ which depend only on m, M and $L[v]$. Next we claim that $\int_{\mathbb{R}^d} v^m\, dx \to \int_{\mathbb{R}^d} v_\infty^m\, dx$ as $\tau \to +\infty$. However, we already know that $v^m(\tau,\cdot) \rightharpoonup v_\infty^m$ in $L^{1/m}(\mathbb{R}^d)$. To establish the result it suffices to show that $\int_{|x|>R} v^m(\tau,x)\, dx \to 0$

as $R \to +\infty$, uniformly in τ, which is easily achieved by applying (22) with $\omega = \{x \in \mathbb{R}^d \;:\; |x| > R\}$. The latter integral is finite for $m > \frac{d}{d+2}$ and goes to 0 as $R \to +\infty$. Using the decay term

$$\int_{\mathbb{R}^d} v|x + \tfrac{m}{m-1}\nabla v^{m-1}|^2 \, dx = \tfrac{4m}{(2m-1)^2} \int_{\mathbb{R}^d} |\nabla v^{m-1/2}|^2 \, dx + \int_{\mathbb{R}^d} v|x|^2 \, dx - 2d \int_{\mathbb{R}^d} v^m \, dx \;,$$

it is clear that at least for a subsequence $\tau_n \to +\infty$, $\int_{\mathbb{R}^d} |x|^2 v(x,\tau_n)\, dx \to \int_{\mathbb{R}^d} |x|^2 v_\infty(x)\, dx$, which proves (20). □

An estimate of the difference between v and v_∞ in terms of L is given by the following result.

Proposition 15 *Assume that $d \geq 2$. Let v is a nonnegative function such that $x \mapsto (1+|x|^2)\, v$ and v^m belong to $L^1(\mathbb{R}^d)$ and consider v_∞ defined as in Corollary 13.*

(i) *If $\frac{d-2}{d} \leq m < 1$, $m > \frac{1}{2}$, then there exists a constant $C > 0$ which depends only on m, $M = \int_{\mathbb{R}^d} v\, dx$ and $L[v]$ such that*

$$C \, \|v^m - v_\infty^m\|^2_{L^1(\mathbb{R}^d)} \leq L[v] - L[v_\infty] \, .$$

(ii) *If $1 < m \leq 2$ and $R = \sqrt{\frac{2m}{m-1}\,\sigma^2}$, then*

$$C \, \|(v - v_\infty) v_\infty^{m-1}\|^2_{L^1(\mathbb{R}^d)} \leq L[v] - L[v_\infty] \, .$$

For the proof of this result, we need a lemma which is a variation of the Csiszár-Kullback inequality. We provide a proof for completeness and refer to [16,26,2] for related results.

Lemma 16 *Assume that Ω is a domain in $\mathbb{R}^d$ and that s is a convex nonnegative function on $\mathbb{R}^+$ such that $s(1) = 0$ and $s'(1) = 0$. If μ is a nonnegative measure on Ω and if f and g are nonnegative measurable functions on Ω with respect to μ, then*

$$\int_\Omega s\Big(\frac{f}{g}\Big)\, g \, d\mu \geq \frac{K}{\max\{\int_\Omega f\, d\mu, \int_\Omega g\, d\mu\}} \cdot \|f - g\|^2_{L^1(\Omega, d\mu)} \tag{24}$$

where $K = \frac{1}{2} \cdot \min\{K_1, K_2\}$,

$$K_1 = \min_{\eta \in]0,1[} s''(\eta) \quad \text{and} \quad K_2 = \min_{\substack{\theta \in]0,1[\\ h > 0}} s''(1+\theta h)(1+h) \;, \tag{25}$$

provided that all the above integrals are finite.

Proof : We may assume without loss of generality that f and g are strictly positive functions. Let us set $h = \frac{f-g}{g}$, so that $\frac{f}{g} = 1 + h$. If ω is any subdomain of Ω and k a positive, integrable on ω, function, then Cauchy-Schwarz's inequality yields

$$\int_\omega \frac{|f-g|^2}{k}\, d\mu \geq \frac{\left(\int_\omega |f-g|\, d\mu\right)^2}{\int_\omega k\, d\mu}\,. \tag{26}$$

The proof of Inequality (24) is based on a Taylor's expansion of $s(t)$ around $t = 1$. Since $s(1) = s'(1) = 0$, we have $s(\frac{f}{g}) = s(1+h) = \frac{1}{2}s''(1+\theta h)h^2$ for some function $x \mapsto \theta(x)$ with values in $]0,1[$. Thus we need to estimate from below the function $\int_\Omega s''(1+\theta h) g h^2\, d\mu$. First, we estimate

$$\int_{f<g} s''(1+\theta h) g h^2\, d\mu = \int_{f<g} s''(1+\theta h) \frac{|f-g|^2}{g}\, d\mu \geq K_1 \int_{f<g} \frac{|f-g|^2}{g}\, d\mu$$

according to the definition (25) of K_1. Using (26) with $\omega = \{x \in \Omega \,:\, f(x) < g(x)\}$ and $k = g$, we obtain

$$\int_{f<g} s''(1+\theta h) g h^2\, d\mu \geq K_1 \frac{\left(\int_{f<g} |f-g|\, d\mu\right)^2}{\int_{f<g} g\, d\mu}\,. \tag{27}$$

On the other hand, we have

$$\int_{f>g} s''(1+\theta h) g h^2\, d\mu = \int_{f>g} s''(1+\theta h)(1+h) \frac{|f-g|^2}{f} d\mu \geq K_2 \int_{f>g} \frac{|f-g|^2}{f} d\mu$$

using the definition (25) of K_2. Now, using again (26) with $\omega = \{x \in \Omega \,:\, f(x) > g(x)\}$ and $k = f$, we get

$$\int_{f>g} s''(1+\theta h) g h^2\, d\mu \geq K_2 \frac{\left(\int_{f>g} |f-g|\, d\mu\right)^2}{\int_{f>g} f\, d\mu}\,. \tag{28}$$

Combining (27) and (28), we obtain (24). □

Proof of Proposition 15. The result is a direct consequence of Lemma 16. For $m < 1$, we take $s(t) = \frac{m t^{\frac{1}{m}} - t}{1-m} + 1$, $K_1 = K_2 = \frac{1}{m}$, $d\mu(x) = dx$ and $L[v] = \int_{\mathbb{R}^d} s\left(\frac{v^m}{v_\infty^m}\right) v_\infty^m dx$. According to (22) and (23), the quantities $\int_{\mathbb{R}^d} v \, \frac{|x|^2}{2}\, dx$ and $\|v^m\|_{L^1(\mathbb{R}^d)}$ depend only on m, M and $L[v]$, which proves the statement on C.

If $1 < m < 2$, we may write $\frac{|x|^2}{2} = \frac{m}{m-1}(\sigma^2 - v_\infty^{m-1}) \le \frac{m}{m-1}\sigma^2$ for $|x| < \sqrt{\frac{2m}{m-1}}\,\sigma$, $\int_{\mathbb{R}^N} v\, v_\infty^{m-1}\, dx \le \frac{m}{m-1}\sigma^2 M$ and apply Lemma 16 to

$$L[v] = \int_{\mathbb{R}^d} s\Big(\frac{v}{v_\infty}\Big)\, v_\infty\, d\mu(x) + \int_{B(0,R)^c} (v\, \frac{|x|^2}{2} + \frac{1}{m-1} v^m)\, dx\ ,$$

with $s(t) = \frac{t^m - mt}{m-1} + 1$, $K_1 = K_2 = m$ and $d\mu(x) = v_\infty^{m-1}(x)\, dx$. □

Proof of Theorem 3. Estimate (21), Proposition 15 and Relation (21) yield that for $m < 1$

$$C\, \|v^m(\cdot,\tau) - v_\infty^m\|^2_{L^1(\mathbb{R}^d)} \le (L[u_0] - L[v_\infty]) \cdot e^{-2\tau} \quad \forall\, \tau > 0\ ,$$

while for $m > 1$

$$C\, \|(v(\cdot,\tau) - v_\infty) v_\infty^{m-1}\|^2_{L^1(\mathbb{R}^d)} \le (L[u_0] - L[v_\infty]) \cdot e^{-2\tau} \quad \forall\, \tau > 0\ .$$

Recalling that in terms of the variable t, $\tau = \tau(t) \sim \log t$, and changing variables into the original definition of v in terms of $u(x,t)$, gives us exactly the relations seeked for in Theorem 3 with $\mathcal{U}$ replaced by $u_\infty(t,x) = R(t)^{-d}\, v_\infty(\log R(t), \frac{x}{R(t)})$ and R given by (16). A straightforward computation shows that $\mathcal{U}$ and u_∞ are asymptotically equivalent and this concludes the proof. □

We should remark that the Lyapunov functional $L[v]$ had already been exhibited by J. Ralston and W.I. Newman in [34,38]. The entropy-entropy dissipation method has been used for the heat equation in [42,43,1] and generalized to nonlinear diffusions in [17,12] (also see [36] by F. Otto on the gradient flow structure of the porous medium equation). More recents developments can be found in [11,23,27,10]. We shall refer to [28] and references therein for earlier works in probability theory and applications to Markov diffusion generators, and to [4] for relations with Sobolev type inequalities.

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