Intermediate asymptotics in L^1 for general nonlinear diffusion equations

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We prove intermediate asymptotics results in L^1 for general nonlinear diffusion equations which behave like power laws at the origin using relative entropy methods and generalized Sobolev inequalities.

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1. Introduction and main result

Relative entropy methods have received a lot of attention in the last few years not only in the context of linear parabolic equations [12,1] but also to handle nonlinear diffusion problems [11,5,7,4,9] and get decay estimates and asymptotic diffusion results. The goal of this letter is to give results on intermediate asymptotics for general nonlinearities. Here, we are not concerned with existence questions (see for instance [4] for a discussion); in all what follows we will assume that the solutions are such that the entropy function and its first derivative are well defined.

Consider a solution $u \in C^0(\mathbb{R}^+, L^1_+(\mathbb{R}^d))$ of

$$u_t = \Delta f(u) \;, \tag{1}$$

corresponding to an initial data $u_{|t=0} = u_0 \geq 0$ and define $M = \|u_0\|_{L^1(\mathbb{R}^d)}$. If the nonlinearity is a power law, *i.e.* if $f(u) = u^m$, the time-dependent rescaling

$$u(t,x) = R^{-d}(t)\, v\left(\tau(t)\,,\frac{x}{R(t)}\right) \tag{2}$$

transforms Equation (1) into a Fokker-Planck type equation, namely

$$v_\tau = \Delta f(v) + \nabla\cdot(xv) \;, \tag{3}$$

provided $\tau(t) = \log(R(t))$, with $v_{|\tau=0} = u_0$ if $R(0) = 1$, and $R(t)$ is a solution of

$$R'(t) = R^{(1-m)d-1} \;. \tag{4}$$

Note that $R(t) \sim t^{\frac{1}{2+(m-1)d}} \to +\infty$ as $t \to +\infty$ if $m > (d-2)/d$.

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The question of finding intermediate asymptotics in L^1 for a solution of Equation (1), *i.e.* a function $u_\infty(t,x)$ depending only on $M = \|u_0\|_{L^1(\mathbb{R}^d)}$ such that

$$\|u^{m\wedge 1}(t,.) - u_\infty^{m\wedge 1}(t,.)\|_{L^1(\mathbb{R}^d,\,d\mu)} = o\left(\|u_\infty^{m\wedge 1}(t,.)\|_{L^1(\mathbb{R}^d,\,d\mu)}\right) = \mathcal{O}(t^{-p(m)}) \qquad (5)$$

where $m\wedge 1$ denotes the minimum of m and 1 and $d\mu = v_\infty^{(m-1)_+}\,dx$, is then transformed into the question of finding the rate of convergence of a solution of (3) to the unique stationary solution with same mass M: $v_\infty(x) = \left[\frac{m-1}{m}\left(\alpha_\infty(M) - \frac{1}{2}|x|^2\right)\right]_+^{1/(m-1)}$ if $m > \frac{d-2}{d}$, $m \neq 1$, with $\alpha_\infty(M)$ such that $\int_{\mathbb{R}^d} v_\infty\,dx = M$ (this solution is known as the Barenblatt-Prattle solution; $(m-1)\cdot\alpha_\infty(M) > 0$) and $v_\infty(x) = \frac{M}{(2\pi)^{d/2}} e^{-|x|^2/2}$ if $m = 1$. To prove such a result, the main tool is the relative entropy $\Sigma[v|v_\infty] = \int_{\mathbb{R}^d} \sigma\left(\frac{v^{m\wedge 1}}{v_\infty^{m\wedge 1}}\right)\, v_\infty^{m\wedge 1}\,d\mu + \int_{\mathbb{R}^d}\frac{1}{2}|x|^2(v - v_\infty)\,dx$, where $\sigma(u) = u\log u$ if $m = 1$ (heat equation), $\sigma(u) = \frac{mu^{1/m}-u}{1-m} + 1$ if $m \in \left(\frac{d-2}{d}, 1\right)$ (fast diffusion equation) and $\sigma(u) = \frac{u^m - mu}{m-1} + 1$ if $m \in (1,+\infty)$ (porous medium equation). In this last case, one has to take into account an additional (nonnegative) term corresponding to the integral of $v\frac{|x|^2}{2} + \frac{v^m}{m-1}$ on $\{x \in \mathbb{R}^d\ :\ v_\infty(x) = 0\}$.

In case $f(u) = u^m$, the generalized Sobolev inequality (see [5,7,11,4]) gives an explicit exponential decay of the relative entropy of a solution of Equation (3): $\Sigma[v(\tau,.)|v_\infty] \leq \Sigma[u_0|v_\infty]\cdot e^{-2\tau}$ provided $m \geq \frac{d-1}{d}$. This is enough to prove that (5) holds with $2p(m) = d(m-1)+1$ using the *Csiszár-Kullback inequality* (see [6,10,2,7]):

Lemma 1.1 *Let ϕ, $\phi_0 \in L^1_+(\mathbb{R}^d, d\mu)$. Then $\int_{\mathbb{R}^d} \sigma\left(\frac{\phi}{\phi_0}\right)\,\phi_0\,d\mu \geq \frac{K}{\mathcal{M}}\|\phi - \phi_0\|^2_{L^1(\mathbb{R}^d,d\mu)}$ with $\mathcal{M} = \max\left\{\|\phi\|_{L^1(\mathbb{R}^d,\,d\mu)}, \|\phi_0\|_{L^1(\mathbb{R}^d,d\mu)}\right\}$, $K = \min\left\{\inf_{t\in[0,1]}\sigma''(t), \inf_{\substack{t\geq 0\\ \theta\in[0,1]}}\sigma''(1+\theta t)(1+t)\right\}$, as soon as σ is a convex function on $\mathbb{R}^+$ such that $0 = \sigma(1) = \min_{\mathbb{R}^+}\sigma$.*

If f is not a power law, the scaling (2) gives an explicitly time-dependent evolution equation for v

$$v_\tau = e^{md\tau}\Delta f\left(e^{-d\tau}v\right) + \nabla\cdot(xv)\,, \qquad (6)$$

and it is reasonable to expect that this equation will give the correct description of the intermediate asymptotics of Equation (1) if R is given by (4), if $f(0) = 0$ and $\lim_{u\to 0^+}\frac{f(u)}{u^m} \in (0,+\infty)$. In order to extend the notion of relative entropy, we will assume throughout this letter that f is strictly increasing in $\mathbb{R}^+$:

$$f'(s) > 0 \text{ in } \mathbb{R}^+\,. \qquad (7)$$

It turns out that relative entropies are a well adapted tool even when the rescaled equation is time-dependent and that the generalized Sobolev inequality (see [4]) can be adapted to (6). To avoid a lengthy statement, we shall simply assume that f is chosen in order that the following *generalized Sobolev inequality* holds: *for any $v \in \mathcal{D}(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d}\left(vh(v) - f(v) + \frac{1}{2}|x|^2 v\right)\,dx - C_d\left(\|v\|_{L^1(\mathbb{R}^d)}\right) \leq \frac{1}{2}\int_{\mathbb{R}^d} v\left|x + \frac{f'(v)}{v}\nabla v\right|^2\,dx, \qquad (8)$$

where h is a primitive of $u \mapsto f'(u)/u$, $C_d(M) = \int_{\mathbb{R}^d}\left(v_\infty h(v_\infty) - f(v_\infty) + \frac{1}{2}|x|^2 v_\infty\right)dx$ with $v_\infty(x) = g\left(\alpha_\infty(M) - \frac{1}{2}|x|^2\right)$, where g is the generalized inverse of h: g is extended *by 0 on $(-\infty,0)$, and by $+\infty$ on $(\sup_{\mathbb{R}^+} h, +\infty)$. Finally, by (7), $\alpha_\infty(M)$ is uniquely determined by the condition $\int_{\mathbb{R}^d} v_\infty(x)\,dx = M = \|v\|_{L^1(\mathbb{R}^d)}$.*

Inequality (8) can, of course, be extended by a density argument to any measurable function for which the integrals are well defined. For sufficient conditions for (8) to hold, we refer to [4]. In particular, in the power law case one has to assume $m \geq \frac{d-1}{d}$ (see [7,4]). If we denote by H the primitive of h satisfying $H(u) = uh(u) - f(u)$, *i.e.* $H(s) = \int_0^s h(\sigma)\,d\sigma$, we may define a relative entropy (which generalizes the one defined in the power law case), with $R = e^\tau$, $v_\infty^R(x) = R^d g\left(R^{-(m-1)d}(\alpha_\infty^R(M) - \frac{1}{2}|x|^2)\right)$ such that $\|v_\infty^R\|_{L^1(\mathbb{R}^d)} = \|v\|_{L^1(\mathbb{R}^d)}$, by

$$\Sigma[\tau, v] = e^{md\tau} \int_{\mathbb{R}^d} \left(H(e^{-d\tau} v) - H(e^{-d\tau} v_\infty^R) + \frac{1}{2}|x|^2 (v - v_\infty^R) \right) dx\,.$$

We shall assume that $f(s) = s^m F(s)$ with $F \in C^0(\mathbb{R}^+) \cap C^1(0,+\infty)$,

$$F>0\,,\ F(0)=1\,,\ F'(s) = \mathcal{O}(s^k),\ k>-1,\ \text{near } s=0\,, \quad s>0\,. \tag{9}$$

Moreover, let us suppose that for all $s > 0$,

$$(m-1)sh(s) - mf(s) \leq 0\,. \tag{10}$$

This last assumption is satisfied for instance if $F' \leq 0$ on $(0,a)$ and $F' \geq 0$ on $(a,+\infty)$ for some $a \in [0,+\infty]$. It covers nonlinearities which are sums of powers and also corresponds to diffusive limits in semiconductors [3,8] or granular media models. Note that if initial data u_0 is a bounded function, the analysis which follows can be greatly simplified and in particular, no assumption on the behavior at infinity of the nonlinearity f will be necessary to prove the main results of this letter. In this case, convergence rates in the norm of L^∞ will be also available, and hence, by interpolation, in $L^p(\mathbb{R}^d)$ for all $p \in [1,+\infty]$.

Theorem 1.2 *Under assumptions (7), (9) and (10), if f is such that inequality (8) holds, then for each solution v of (6), there exist constants $K>0$, $\beta = \min\{2, d(k+1)\} > 0$ such that for all $\tau > 0$,*

$$0 \leq \Sigma[\tau, v] \leq K\, e^{-\beta\tau}\,.$$

We may then prove a result on the intermediate asymptotics of Equation (1) under an adequate assumption on the behavior of f at infinity:

$$\ell_1 := \liminf_{s\to+\infty} f'(s)s^{m-1} > 0 \ \text{ if } m \in (1,2)\ ;\ \ \ell_2 := \liminf_{s\to+\infty} \frac{s\,H''(s)}{|H'(s)|^3} > 0 \text{ if } m \in \left(\frac{d-1}{d}, 1\right). \tag{11}$$

Theorem 1.3 *Under the assumptions of Theorem 1.2, if f satisfies (11), then, with R given by (4), we have as $t \to +\infty$*

$$\|u(t,.) - u_\infty(t,.)\|_{L^1(\mathbb{R}^d,\, u_\infty^{m-1}\,dx)} \leq C\,(R(t))^{-d(m-1)-\frac{\beta}{2}} \qquad \textit{if} \quad 1 < m \leq 2\,,$$

$$\|H(u)(t,.) - H(u_\infty)(t,.)\|_{L^1(\mathbb{R}^d,\, dx)} \leq C\,(R(t))^{-d(m-1)-\frac{\beta}{2}} \qquad \textit{if} \quad (d-1)/d \leq m < 1\,.$$

2. Proofs

We first prove Theorem 1.2 using the scaling properties of the generalized Sobolev inequality, and then, Theorem 1.3.

Proof of Theorem 1.2. Let v be a solution of (6) and consider $S(\tau)=R^{md}\int_{\mathbb{R}^d} H\left(\frac{v}{R^d}\right) dx + \int_{\mathbb{R}^d} \frac{1}{2}|x|^2 v\, dx$ with $R=e^\tau$. A direct computation yields

$$\frac{dS}{d\tau} = dR^{md}\int_{\mathbb{R}^d}\left((m-1)\frac{v}{R^d}h\left(\frac{v}{R^d}\right)-mf\left(\frac{v}{R^d}\right)\right)dx-\int_{\mathbb{R}^d} v\left|x+R^{(m-1)d}f'\left(\frac{v}{R^d}\right)\frac{\nabla v}{v}\right|^2 dx\,. \tag{12}$$

Next, we may use the scaling properties of the generalized Sobolev inequalities:

Lemma 2.1 *If f is a nonlinearity for which (8) holds, then for any $R>0$*

$$\int_{\mathbb{R}^d}\left(R^{md}H\left(\frac{v}{R^d}\right)-R^{md}H\left(\frac{v_\infty^R}{R^d}\right)+\frac{1}{2}|x|^2(v-v_\infty^R)\right)dx \leq \frac{1}{2}\int_{\mathbb{R}^d} v\left|x+R^{(m-1)d}f'\left(\frac{v}{R^d}\right)\frac{\nabla v}{v}\right|^2 dx \tag{13}$$

for any $v\in\mathcal{D}(\mathbb{R}^d)$, where $v_\infty^R(x)=R^d g\left(R^{-(m-1)d}\left(\alpha_\infty^R-\frac{1}{2}|x|^2\right)\right)$ and $\alpha_\infty^R=\alpha_\infty^R(M)$ is such that $\int_{\mathbb{R}^d} v_\infty^R\, dx = M = \|v\|_{L^1(\mathbb{R}^d)}$.

Proof of Lemma 2.1. Let $\tilde f(u)=R^{md}f\left(\frac{u}{R^d}\right)$. Then with standard notations, $\tilde f'(u)=R^{(m-1)d}f'\left(\frac{u}{R^d}\right)$, $\tilde h(u)=R^{(m-1)d}h\left(\frac{u}{R^d}\right)$ and $\tilde H(u)=R^{md}H\left(\frac{u}{R^d}\right)$, and the generalized Sobolev inequality (8) applied to $\tilde f$ gives the result. □

With $S_\infty(\tau):=R^{md}\int_{\mathbb{R}^d} H\left(\frac{v_\infty^R}{R^d}\right)dx+\int_{\mathbb{R}^d}\frac{|x|^2}{2}v_\infty^R\, dx$, (10), (12) and (13) give

$$\frac{d}{d\tau}(S(\tau)-S_\infty(\tau))+2(S(\tau)-S_\infty(\tau))\leq\frac{d}{d\tau}(S_\infty(\tau))\,. \tag{14}$$

Since v_∞^R is a critical point of $\int_{\mathbb{R}^d}\left(R^{md}H(\frac{v}{R^d})+\frac{1}{2}v|x|^2\right)dx$ under the constraint $\|v\|_{L^1(\mathbb{R}^d)} = M$, $\frac{d}{d\tau}S_\infty(\tau)=dR^{md}\int_{\mathbb{R}^d}\left((m-1)\frac{v_\infty^R}{R^d}h\left(\frac{v_\infty^R}{R^d}\right)-mf\left(\frac{v_\infty^R}{R^d}\right)\right)dx=\mathcal{O}(R^{-d(k+1)})$ by our assumptions on F, the r.h.s. in (14) is less than $Ce^{-d(k+1)\tau}$ for some $C>0$ and for τ large enough. Integrating (14), we obtain $|S(\tau)-S_\infty(\tau)|\leq C\,e^{-\max\{2,\,d(k+1)\}\tau}$. □

Proof of Theorem 1.3. We shall distinguish the cases $m<1$ and $m>1$.
Case $1<m\leq 2$. By the definition of the function v_∞^R and the Taylor formula, we have

$$S(\tau)-S_\infty(\tau)=\frac{1}{2}R^{(m-2)d}\int_{\mathbb{R}^d}\frac{f'(w^R)}{w^R}(v-v_\infty^R)^2\, dx\,,$$

with $w^R=\theta v+(1-\theta)v_\infty^R$ for some function θ taking values in the interval $[0,1]$. Let $L_1>0$ be such that for all $s>L_1$, $f'(s)\,s^{m-1}>\ell_1/2$ (see assumption (11)) and define $c_1:=\inf_{\{s\leq L_1\}}f'(s)\,s^{1-m}$, which is a positive number by our assumptions on F. Then, if $A_1:=\{x\in\mathbb{R}^d\,:\,\theta v+(1-\theta)v_\infty^R\leq L_1\,R^d\}$, we find

$$S(\tau)-S_\infty(\tau)\geq\frac{c_1}{2}\int_{A_1}(w^R)^{m-2}\,(v-v_\infty^R)^2\, dx+\frac{\ell_1}{2}\,R^{2(m-1)}\int_{A_1^c}(w^R)^{-m}(v-v_\infty^R)^2 dx\,.$$

But for R sufficiently large, on A_1^c, $w^R \leq v$, and hence, by Hölder's inequality, $\int_{A_1^c} (w^R)^{-m} (v - v_\infty^R)^2 \, dx \geq \left(\int_{A_1^c} |v - v_\infty^R| \, dx\right)^2 (\int_{\mathbb{R}^d} v^m \, dx)^{-1}$. On the other hand, in the set A_1 we can use classical arguments (see [7] for instance) to obtain $\int_{A_1} (w^R)^{m-2}(v - v_\infty^R)^2 dx \geq \left(\int_{A_1} |v - v_\infty^R|(v_\infty^R)^{m-1} dx\right)^2 (\max\{\int_{\mathbb{R}^d} v^m dx, \int_{\mathbb{R}^d} v_\infty^m dx\})^{-1}$. Finally, since the functions v_∞^R are uniformly bounded in $L^\infty(\mathbb{R}^d)$ for R large, we prove the existence of $C > 0$ such that for R large enough

$$\|v - v_\infty^R\|_{L^1\left((v_\infty^R)^{m-1}\, dx\right)} \leq C \max\left\{\int_{\mathbb{R}^d} v^m \, dx, \int_{\mathbb{R}^d} v_\infty^m \, dx\right\}^{1/2} R^{-\beta/2}.$$

Case $(d-1)/d \leq m < 1$**.** Here we choose $h(s)$ to be the primitive of $f'(s)/s$ which tends to 0 as s goes to $+\infty$. We may then rewrite $S(\tau)$ up to a constant as

$$S(\tau) = R^{md} \int_{\mathbb{R}^d} \left(H\left(\frac{v}{R^d}\right) - H'\left(\frac{v_\infty^R}{R^d}\right) \frac{v}{R^d} \right) dx = \int_{\mathbb{R}^d} \left(w - R^{(m-1)d} H'\left(\frac{v_\infty^R}{R^d}\right) \mu^{-1}(w) \right) dx,$$

with $w = \mu(v) := R^{md} H\left(\frac{v}{R^d}\right)$, $v = R^d H^{-1}\left(\frac{w}{R^{md}}\right) = \mu^{-1}(w)$. Using Taylor's formula with respect to w, we obtain

$$S(\tau) - S_\infty(\tau) = \frac{1}{2} \int_{\mathbb{R}^d} R^{-md} \left| H'\left(\frac{v_\infty^R}{R^d}\right) \right| \cdot \frac{H''}{|H'|^3} \left(\frac{\mu^{-1}(w^R)}{R^d}\right) (w - w_\infty^R)^2 \, dx \,,$$

with $w_\infty^R = \mu(v_\infty^R)$, $w^R = \theta w + (1-\theta) w_\infty^R$, for some θ with values in $[0, 1]$. Let us now choose $L_2 > 0$ such that for all $s > L_2$, $H''(s)\, s/|H'(s)|^3 > \ell_2/2$ (see assumption (11)). By our assumptions on F, if we define the set $A_2 := \{x \in \mathbb{R}^d : \mu(L_2 R^d) \leq \theta\mu(v) + (1-\theta)\mu(v_\infty^R) \leq 0\}$, then, on A_2, $0 \geq w^R/R^d \geq R^{(m-1)d} H(L_2) \to 0$, while on $A_2^c = \mathbb{R}^d \setminus A_2$, $v \geq \mu^{-1}(w^R)$ at least for R large enough. Also for R large enough, $R^{(m-1)d}|H'(v_\infty^R/R^d)| \geq \frac{m}{2(1-m)} (v_\infty^R)^{m-1}$. We can therefore find a positive constant C such that for τ large enough

$$\begin{aligned} S(\tau) - S_\infty(\tau) \;\; &\geq \;\; C \int_{A_2} (v_\infty^R)^{m-1} |w^R|^{\frac{1}{m}-2} (w - w_\infty^R)^2 \, dx \\ &\qquad + \frac{\ell_2}{2} R^{2(1-m)d} \int_{A_2^c} \frac{m}{2(1-m)} (v_\infty^R)^{m-1} (w - w_\infty^R)^2 \, |v|^{-1} \, dx \,. \end{aligned}$$

The integral on A_2 can be treated classically like in [7] to obtain

$$\int_{A_2} (v_\infty^R)^{m-1} |w^R|^{\frac{1}{m}-2} (w - w_\infty^R)^2 dx \geq C \left(\int_{A_2} |\mu(v) - \mu(v_\infty^R)| dx \right)^2 (\max\{\|\mu(v)\|_{L^1}, \|\mu(v_\infty^R)\|_{L^1}\})^{-1} ,$$

for some $C > 0$. On the other hand, we notice that $m - 1 < 0$ and that the functions v_∞^R are uniformly bounded in $L^\infty(\mathbb{R}^d)$ for R large enough. Then, in the set A_2^c we use the Cauchy-Schwarz inequality

$$\int_{A_2^c} (v_\infty^R)^{m-1} |v|^{-1} (w - w_\infty^R)^2 \, dx \geq M^{-1} \, \|v_\infty^R\|_{L^1}^{m-1} \left(\int_{A_2^c} |\mu(v) - \mu(v_\infty^R)| \, dx \right)^2 .$$

To finish the proof of Theorem 1.3, we have to make a change of variables to pass from the rescaled function v solution of (6) to the unscaled solution u of (1), and we have also to measure the rate of convergence of v_∞^R to v_∞.

In the case $m > 1$, we need only to control the norm $\|v_\infty - v_\infty^R\|_\infty$ since we have $A_R := \|v - v_\infty\|_{L^1((v_\infty)^{m-1}dx)} \leq CM\|v_\infty^{m-1} - (v_\infty^R)^{m-1}\|_{L^\infty(\mathbb{R}^d)} + \|v - v_\infty^R\|_{L^1\left((v_\infty^R)^{m-1}dx\right)}$, and in the case $m < 1$, the quantity to measure is $B_R := \|\mu(v_\infty) - \mu(v_\infty^R)\|_{L^1(\mathbb{R}^d)}$. In both cases $m > 1$ and $m < 1$, using assumption (9) and the definition of v_∞, v_∞^R, α_∞ and α_∞^R, we can prove that both A_R and B_R are bounded from above by $CR^{-d(k+1)} + C\,|\alpha_\infty(M) - \alpha_\infty^R(M)|$, where $C = C(M, d) > 0$. Actually, the second term is also of the order of $R^{-d(k+1)}$:

$$M = \int_{\mathbb{R}^d} R^d g\left(R^{(1-m)d}\left(\alpha_\infty^R(M) - \frac{|x|^2}{2}\right)\right) dx = \int_{\mathbb{R}^d} \left(\frac{m-1}{m}\left(\alpha_\infty(M) - \frac{|x|^2}{2}\right)_+\right)^{\frac{1}{m-1}} dx\,,$$

and $R^{(1-m)d}\left(\alpha_\infty^R(M) - \frac{|x|^2}{2}\right)$ tends to 0 (resp. $-\infty$) when $R \to +\infty$ and $m > 1$ (resp. $m < 1$). This and assumption (9) easily show that α_∞^R converges to α_∞ as R goes to $+\infty$. Finally, by (9) $h(u) = \frac{m}{m-1}u^{m-1}(1 + \mathcal{O}(|u|^{k+1}))$ as $u \to 0^+$ if $m > 1$ which ends the proof of Theorem 1.3. □

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