L^1 and L^∞ intermediate asymptotics for scalar conservation laws

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Abstract

In this paper, using entropy techniques, we study the rate of convergence of nonnegative solutions of a simple scalar conservation law to their asymptotic states in a weighted L^1 norm. After an appropriate rescaling and for a well chosen weight, we obtain an exponential rate of convergence. Written in the original coordinates, this provides intermediate asymptotics estimates in L^1, with an algebraic rate. We also prove a uniform convergence result on the support of the solutions, provided the initial data is compactly supported and has an appropriate behaviour on a neighborhood of the lower end of its support.

Keywords. Scalar conservation laws – asymptotics – entropy – shocks – weighted L^1 norm – self-similar solutions – N-waves – time-dependent rescaling – Rankine-Hugoniot condition – uniform convergence – graph convergence
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1 Introduction and main results

Consider for some $q > 1$ a nonnegative entropy solution of

$$\begin{cases} U_\tau + (U^q)_\xi = 0 \;, \quad \xi \in \mathbb{R} \;, \quad \tau > 0 \;, \\ U(\tau = 0, \cdot) = U_0 \;. \end{cases} \tag{1}$$

To our knowledge, two results are known concerning the convergence of the solution U to an asymptotic state. The first one, by T.-P. Liu & M. Pierre [12] asserts that, for every $p \in [1, +\infty)$,

$$\lim_{\tau \to \infty} \tau^{\frac{1}{q}(1-\frac{1}{p})} \, \|U(\tau) - U_\infty(\tau)\|_p = 0 \;, \tag{2}$$

where U_∞, the so-called self-similar solution, is defined by

$$U_\infty(\tau,\xi) = \begin{cases} \left(\frac{|\xi|}{q\,\tau}\right)^{\frac{1}{q-1}} & 0 \le \xi \le c(\tau) \\ 0 & \text{elsewhere} \end{cases}$$

with $c(\tau) = q\,(\|u_0\|_1/(q-1))^{(q-1)/q}\,\tau^{1/q}$. Here and in the remainder of this paper, we use the notation $\|\cdot\|_q$ to denote the $L^q(\mathbb{R})$ norm. Notice nevertheless that the above result does not give any rate of convergence in L^1. A second and actually much earlier result is due to P. Lax [11] (also see [15, 7]) and says the following. If U is the unique entropy solution to

$$U_\tau + f(U)_\xi = 0\,,\quad U(0,\xi) = U_0(\xi)$$

with $f \in C^2$ near the origin, $f(0) = f'(0) = 0$ and $f'' > 0$, and if $U_0 \ge 0$ is of compact support in the bounded interval (s_-, s_+), then the following estimate holds:

$$\|U(\tau,\cdot) - W_\infty(\tau,\cdot - s_-)\|_1 = O(\tau^{-1/2}) \quad \text{as} \quad \tau \to \infty\,, \tag{3}$$

where $W_\infty(\tau,\xi) = \frac{\xi}{f''(0)}\,\tau^{-1}$ if $0 < \xi < -s_- + s_+ + \sqrt{2\,\|u_0\|_1\,f''(0)}\,\tau^{-1/2}$, and 0 elsewhere. Notice that the function $W_\infty(\cdot,\cdot - s_-)$ in (3) is not a self-similar solution of the Burgers equation:

$$U_\tau + \frac{1}{2}\,f''(0)(U^2)_\xi = 0$$

unless $s_- = 0$. Although sign-changing initial data can be considered [12], for simplicity we will only deal with nonnegative solutions. From now on, we assume that

$$q \in (1,2]$$

without further notice. Our first main result is the following.

Theorem 1 *Let U be a global, piecewise C^1 entropy solution of (1) with a finite number of discontinuities, corresponding to a nonnegative initial data U_0 in $L^1 \cap L^\infty(\mathbb{R})$ which is compactly supported in $(\xi_0, +\infty)$ for some $\xi_0 \in \mathbb{R}$ and such that*

$$\liminf_{\substack{\xi \to \xi_0 \\ \xi > \xi_0}} \frac{U_0(\xi)}{|\xi - \xi_0|^{1/(q-1)}} > 0\,.$$

Then, for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$,

$$\limsup_{\tau \to +\infty} \tau^{\alpha - \epsilon} \int_{\mathbb{R}} \left| U(\tau,\xi) - U_\infty(\tau,\xi - \xi_0) \right| \frac{d\xi}{|\xi - \xi_0|^\alpha} = 0\,. \tag{4}$$

If $\alpha = \frac{q}{q-1}$, then there exists a constant $k > 0$ big enough such that

$$\limsup_{\tau \to +\infty} \frac{\tau^{\frac{q}{q-1}}}{\log \tau} \int_{\mathbb{R}} \left| U(\tau,\xi) - U_\infty(\tau,\xi - \xi_0) \right| |\log \tau + k - \log(\xi - \xi_0)|^{1+\varepsilon} \frac{d\xi}{|\xi - \xi_0|^{\frac{q}{q-1}}} < \infty\,.$$

Sufficient conditions on the initial data for the existence of the solutions considered in Theorem 1 may be found in [16] and references therein. For instance, this regularity holds if U_0 has a finite number of C^1 smooth regions and in each of these regions a finite number of decreasing inflection points.

A straightforward consequence of the above result is the following corollary, which improves (3) as soon as $\frac{1}{2} < \alpha(1 - 1/q) \iff q/(2(q-1)) < \alpha$, although it is not optimal (see [9] and comments below).

Corollary 1 *Under the same assumptions as in Theorem 1, for any $\varepsilon > 0$, there exists a positive constant C_ε such that*

$$\|U(\tau,\cdot) - U_\infty(\tau,\xi-\xi_0)\|_1 \leq C_\varepsilon\, \tau^{-1}(\log\tau)^{1+\varepsilon}\ . \tag{5}$$

The proof of this result, and further decay estimates in weighted L^1 norms will be given in Section 5. At some point, we will need a uniform estimate, which is our second main result. Let us introduce some notations which will be usefull throughout this paper. To a solution U of (1) with initial data U_0, we associate

$$M := \int_{\mathbb{R}} U_0\, d\xi \quad \text{and} \quad c_M := \left(\frac{q\int_{\mathbb{R}} U_0\, d\xi}{q-1}\right)^{(q-1)/q}\ . \tag{6}$$

Theorem 2 *Under the same assumptions as in Theorem 1,*

$$\lim_{\tau\to+\infty}\ \sup_{\xi\in\mathrm{supp}(U(\tau,\cdot))} \tau^{1/q}\ |U(\tau,\xi) - U_\infty(\tau,\cdot-\xi_0)| = 0 \tag{7}$$

and $\rho(\tau) := \max\,[\mathrm{supp}(U(\tau,\cdot))]$ satisfies as $\tau\to+\infty$

$$\lim_{\tau\to+\infty}(1+q\,\tau)^{-1/q}\rho(\tau) = c_M\ ,\quad \rho(\tau)\geq(1+q\,\tau)^{1/q}c_M\,(1+O(\tau^{-1}))\ . \tag{8}$$

The proof of this result is based on elementary estimates which are stated in Section 3. Improving (2) by Hölder interpolation is then straightforward and left to the reader.

The proof of Theorem 1 turns out to be very simple. It is mainly based on the two following tools:

1. A *time-dependent rescaling* which preserves the initial data and replaces the characterization of the intermediate asymptotics by the convergence to a stationary solution.
2. A time-decreasing functional which plays the role of an *entropy,* in the sense that it captures global informations on the evolution of the solution and controls its large time asymptotics.

Rescalings are a usefull tool in the study of the long time behaviour which has been applied to nonlinear parabolic equations [8], hyperbolic conservation laws [12] and more recently to various equations of nonrelativistic mechanics [5]. Recently also, time-dependent rescalings have been widely studied in the context of nonlinear parabolic equations, in connection with entropy methods (see [1, 2, 3, 4] and references therein).

In case of Equation (1), we are going to work directly with a weighted L^1 norm which we shall interpret as an *entropy,* in the above defined sense. The situation is very similar to the *generalized entropy* approach which has been introduced by T.-P. Liu and T. Yang in [13]. Note that this has nothing to do with the notion of *entropy* which is used in hyperbolic problems to select *an entropy solution* to the Cauchy problem [10, 14].

During the completion of this paper, we became aware of a study by Y.-J. Kim, who kindly communicated us a preliminary version of his work [9]. His approach is based on a detailed study of special self-similar solutions. Y.-J. Kim obtains a sharp L^1-norm convergence rate and we do not. Nevertheless, we believe that our method, based on qualitative results and global integral estimates, and our results on the convergence in weighted norms are of interest.

This paper is organized as follows. We first reduce the problem of intermediate asymptotics to the question of the convergence to a stationary solution by using an appropriate time-dependent rescaling. In Section 3, we establish a result of uniform convergence on the support. In Section 4, we derive entropy estimates and establish the convergence in L^1. Further related results are stated in Section 5. The proof of a result of graph convergence which is needed in Section 3 but is more or less standard has been relegated in the Appendix.

2 Time-dependent rescaling

This section is devoted to the proof of Theorem 1. Unless it is specified, we assume for simplicity that $\xi_0 = 0$, which can be achieved by a translation of the initial data.

2.1 Notions of solution, time-dependent rescaling, entropy

As a first step, we consider a time-dependent rescaling which transforms the problem of intermediate asymptotics into the study of the convergence to a stationary solution.

Proposition 2 *Let U be a nonnegative piecewise C^1 entropy solution of (1), whose points of discontinuity are given by the curves $\xi_1(\tau) < \xi_2(\tau) < \cdots < \xi_n(\tau)$. Then the rescaled function*

$$u(t,x) = e^t\, U\left(\frac{1}{q}(e^{qt}-1), e^t x\right) \tag{9}$$

is a piecewise C^1 function, whose points of discontinuity are given by the curves $s_i(t) \equiv e^{-t}\xi_i((e^{qt}-1)/q)$, which satisfy

$$s_i'(t) = \frac{(u_i^+)^q - s_i(t)\, u_i^+ - (u_i^-)^q + s_i(t)\, u_i^-}{u_i^+ - u_i^-} \tag{10}$$

for any $i = 1, 2, \ldots\ n$. Out of the curves $x = s_i(t)$ the function u is a classical solution of

$$u_t = (x\, u - u^q)_x\ , \tag{11}$$

and across these curves it satisfies

$$u_i^- := \lim_{\substack{x\to s_i(t)\\ x<s_i(t)}} u(t,x) > \lim_{\substack{x\to s_i(t)\\ x>s_i(t)}} u(t,x) := u_i^+ . \tag{12}$$

Moreover u and U have the same initial data $U_0 := U(0,\cdot) = u(0,\cdot) =: u_0$. Finally, if $U_0 \in L^1(\mathbb{R})$, then, for all $t > 0$, we have: $||u(t)||_1 = ||U_0||_1$.

Proof. It is well known, *cf.* for instance [14] vol. 1, page 40, that under the hypothesis above, the function U is a classical solution of (1) out of the curves $\xi = \xi_i(\tau)$, $i = 1, \cdots, n$. These curves moreover satisfy the Rankine-Hugoniot condition

$$\xi_i'(\tau) = \frac{(U_i^+)^q - (U_i^-)^q}{U_i^+ - U_i^-}$$

and, across these curves, the function U satisfies:

$$U_i^- := \lim_{\substack{\xi\to\xi_i(\tau)\\ \xi<\xi_i(\tau)}} U(\tau,\xi) > \lim_{\substack{\xi\to\xi_i(\tau)\\ \xi>\xi_i(\tau)}} U(\tau,\xi) := U_i^+ .$$

Now, if we consider u given by $U(\tau,\xi) = R^{-1}\, u(t, R^{-1}\xi)$ with $t(\tau) = \log R(\tau)$, $s_i(t) = \xi_i(\tau)$ and $\tau \mapsto R(\tau)$ given by $R^{q-1} dR/d\tau = 1$, $R(0) = 1$, which means $R(\tau) = (1 + q\,\tau)^{1/q}$, the result follows by straightforward calculus. □

Definition *We shall say in the remainder of this paper that a function u is a* solution *of (11) if and only if it is a piecewise C^1 function whose discontinuity points are given by a finite set of curves $\{s_i(t)\}_{i=1}^n$ satisfying (10), which solves (11) out of these curves and such that (12) holds across them.*

With this definition, if v is a solution of (11) then the function

$$U(\tau,\xi) = (1+q\,\tau)^{-1/q}\, v\left(\frac{1}{q}\log(1+q\,\tau), \frac{\xi}{(1+q\,\tau)^{1/q}}\right)$$

is an entropy solution of (1).

Consider the entropy solution of (1) corresponding to a nonnegative L^1 initial data U_0 such that $M = \|U_0\|_1 > 0$ and the corresponding rescaled solution of (11) with initial data $u_0 = U_0$. For every $c > 0$, let u_∞^c be the stationary solution of (11) defined by

$$u_\infty^c(x) = \begin{cases} x^{1/(q-1)} & 0 \le x \le c\ , \\ \\ 0 & \text{if } x < 0 \text{ or } x > c\ . \end{cases} \tag{13}$$

If $c = c_M := (q\,M/(q-1))^{(q-1)/q}$, we call $u_\infty := u_\infty^{c_M}$. Notice that $\|u_\infty\|_1 = M$. Based on u_∞^c, a relative *entropy* will be defined in Section 4. This entropy is the main tool in the proof of Theorem 1.

2.2 Comparison results

To justify the use of the entropy, some integrations by parts and some intermediate results on the rate of convergence, we are going to prove first some comparison results.

Lemma 3 *Consider two solutions U and V of (1)*

$$U_\tau = -(U^q)_\xi \quad \textit{and} \quad V_\tau = -(V^q)_\xi$$

with nonnegative initial data U_0 and V_0 such that

$$U_0 \le A_0\, V_0 \ \textit{a.e.}$$

for some positive constant A_0. Then

$$U(\tau,\cdot) \le A_0\, V(A_0^{q-1}\tau,\cdot)\ \textit{a.e.} \quad \forall\, \tau \in \mathbb{R}^+\ .$$

Proof. It is based on a scaling argument: $W(\tau,\xi) = A_0\, V(A_0^{q-1}\tau,\xi)$ is a solution of

$$W_\tau = -(W^q)_\xi$$

with initial data $A_0\, V_0$. By the comparison principle for entropy solutions of (1) (*cf.* [14] vol. 1, page 37), for any $\tau > 0$, $U(\tau,\xi) \le W(\tau,\xi)$. □

As a straightforward consequence of Lemma 3, we have the following result for u.

Corollary 4 *Let u be a solution of (11) with a nonnegative initial data u_0 satisfying*

$$u_0 \le A_0\, u_\infty^c\ \textit{a.e.}$$

for some positive constants A_0 and c. Then

$$u(t,x) \le A(t)\, u_\infty^{c(t)}(x)\ \textit{a.e.} \quad \forall\, t \in \mathbb{R}^+$$

with $A(t) = \frac{A_0\, e^{qt/(q-1)}}{\left[1+A_0^{q-1}(e^{qt}-1)\right]^{1/(q-1)}}$ and $c(t) = c\left(\frac{A_0}{A(t)}\right)^{(q-1)/q}$. As a consequence, $u(t,\cdot)$ is supported in $[0, c(t)] \subset [0, c\,(\max(A_0,1))^{(q-1)/q}]$ for any $t \ge 0$ and

$$\|\,(u - x^{1/q})_+\,\|_\infty \le (A(t)-1)_+ \to 0 \quad \textit{as}\ t \to +\infty\ .$$

Proof. With the notations of Lemma 3, if v is a solution of (11) with initial data v_0 such that $u_0 \le A_0\, v_0$ and $v(t,x) = R\,V((R^q-1)/q, R\,x)$ with $R = e^t$, then

$$u(t,x) = R\,U\left(\frac{R^q-1}{q}, R\,x\right) \le A_0\, R\, V\left(A_0^{q-1}\frac{R^q-1}{q}, R\,x\right)$$

by Lemma 3. The right hand side is

$$A_0\, R\, V(\tau,\xi) = \frac{A_0\, R}{(1+q\,\tau)^{1/q}}\, v\left(\frac{1}{q}\log(1+q\,\tau), \frac{\xi}{(1+q\,\tau)^{1/q}}\right)$$

with $\tau = A_0^{q-1}\,(e^{qt}-1)/q$ and $\xi = e^t x$. Conclusion holds with $v_0 = u_\infty^c \equiv v(t,\cdot)$ for any $t \geq 0$. □

We will see in the proof of Proposition 5 (*c.f.* the Appendix) that it is sufficient to know that the initial data is bounded in order to obtain upper estimates like the ones of Lemma 3 and Corollary 4.

3 Uniform estimates

In the rescaled variables, we prove a result of convergence to the asymptotic profile (Theorem 3), which is strictly equivalent to Theorem 2 using the change of variables (9). Recall that the initial data $u_0 = U_0$ is the same before and after rescaling. We need precise estimates close to the shocks. This is the purpose of the following refined *graph convergence* result (see [6] for previous results).

Proposition 5 *Let $\epsilon > 0$, $M = \int u_0\, dx$ and consider a piecewise C^1 nonnegative initial data u_0 with compact support contained in $[0,+\infty)$, such that*

$$\liminf_{\substack{x\to 0\\ x>0}} x^{1/(1-q)} u_0(x) > 0\ .$$

Then there exists a positive T such that, for any $t > T$, if u is a solution of (11),

(i) the support of $u(\cdot,t)$ is an interval $[0,s(t)]$.

(ii) $\inf_{x\in[0,s(t))} x^{1/(1-q)} u(x,t) > 0$.

(iii) there exists a constant $A_0 > 0$ such that

$$u \leq A(t)\, u_\infty^{s(t)} \quad \textit{with } A(t) = \frac{A_0\, e^{q(t-1)/(q-1)}}{\left[1 + A_0^{q-1}(e^{q(t-1)}-1)\right]^{1/(q-1)}}\ .$$

(iv) for any $\epsilon > 0$, there exists a constant κ such that, with the notations (6),

$$u \geq (1-\kappa\, e^{-qt})\, u_\infty^{c_M-\epsilon}\ .$$

The proof of Proposition 5 is given in the Appendix.

Theorem 3 *Let u be an entropy solution of*

$$u_t + (u^q - xu)_x = 0 \tag{11}$$

corresponding to a piecewise C^1 nonnegative initial data u_0 with compact support contained in $[0,+\infty)$ and assume that

$$\liminf_{\substack{x\to 0\\ x>0}} x^{1/(1-q)} u_0(x) > 0\ .$$

Then

$$\lim_{t\to\infty} \sup_{x\in(0,s(t))} |u(t,x) - u_\infty^{s(t)}| = 0\ ,$$

where $[0,s(t)]$ is the support of $u(\cdot,t)$ for $t > 0$ large enough. Moreover, with the notations (6),

$$\lim_{t\to+\infty} s(t) = c_M \quad \textit{and} \quad s(t) \geq c_M - O(e^{-qt}) \quad \textit{as} \quad t \to +\infty\ .$$

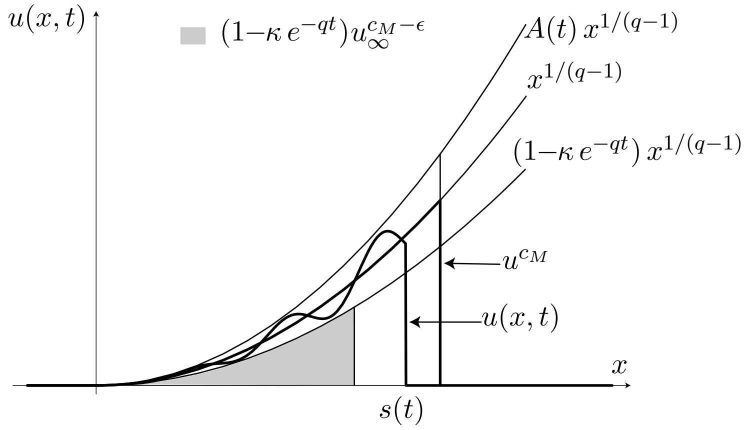

Figure 1: *For $t>0$ large enough, the two cases $s(t)\leq c_M$ and $s(t)>c_M$ are possible.*

The remainder of this section is devoted to the proof of Theorem 3. We first derive a system of ODEs to characterize the position of the last shock. Then we prove the uniform convergence on the support of the solution and finally get an estimate of the behaviour of the upper bound of the support.

Lemma 6 *Consider a solution u of (11) as in Theorem 3. Let $s(t)$ be the upper extremity of the support of u for t karge enough and consider $h(t) := \lim_{\substack{x\to s(t)\\ x<s(t)}} u(x,t)$. Then*

$$\left\{\begin{aligned} \frac{ds}{dt} &= h^{q-1} - s \\ \frac{dh}{dt} &= h\,(1-(u^{q-1})_x) \end{aligned}\right. \tag{14}$$

where by $(u^{q-1})_x$ we denote the quantity $\lim_{\substack{x\to s(t)\\ x<s(t)}} (u^{q-1})_x(x,t)$.

Proof. The equation for s is given by the Rankine-Hugoniot relation. By a direct computation, we obtain

$$\begin{aligned} \tfrac{dh}{dt} &= \tfrac{d}{dt}u(s,t) = u_x(s,t)\,\tfrac{ds}{dt} + u_t(s,t) \\ &= u_x(s,t)\,\tfrac{ds}{dt} + \big(u(x-u^{q-1})\big)_x\Big|_{x=s} \\ &= u_x(s,t)\,(h^{q-1}-s) + \big(u_x(s-h^{q-1})\big) + h\,(1-(u^{q-1})_x) \end{aligned}$$

where $s=s(t)$. □

Lemma 7 *Consider a solution u of (11) as in Theorem 3. Then*

$$(u^{q-1})_x \leq \Big(1-e^{-qt}\Big)^{-1}$$

in the distribution sense.

Proof. If U is an entropy solution of

$$U_\tau + (U^q)_\xi = 0\ ,$$

then it is well known (see for instance [6]) that U satisfies the *entropy inequality*

$$(U^{q-1})_\xi \leq \frac{1}{q\,\tau}$$

in the distribution sense. Using the change of variables (9), the result follows. □

Using the graph convergence result of Proposition 5, we obtain the following result.

Lemma 8 *Let* $u_\infty(x) := x^{1/(q-1)}$ *and consider a solution* u *of (11) as in Theorem 3. Then*

(i) For any $\epsilon > 0$*, there exists* $t_1 > 0$ *such that*

$$s(t) \geq c_M - \epsilon \quad \forall\, t > t_1 \;.$$

(ii) For any $\epsilon > 0$*,* $\delta \in (0,1)$*,* $t_0 > 0$*, there exists* $t_1 > t_0$ *such that*

$$h(t_1) \geq (1-\delta)\, u_\infty(s(t_1)) \;.$$

Proof. First, according to Proposition 5, (iv), $s(t) \geq c_M - \epsilon$ for all $t \geq t_1 = T$. Assume now that (ii) is false: there exists a $\delta \in (0,1)$ and a $t_0 > 0$ such that for any $t > t_0$,

$$\frac{ds}{dt} \leq -\theta\, s(t)$$

with $\theta = 1 - (1-\delta)^{q-1} \in (0,1)$ by (14). This contradicts (i). □

We may now extend the estimate on h to a uniform one.

Lemma 9 *Assume that* $h(t_1) = h_1 > 0$ *for some* $t_1 > 0$*. Then*

$$h(t) \geq h_1\, (1 - e^{-qt_1})^{1/q} \quad \forall\, t > t_1 \;.$$

Proof. Using Lemma 7 and (14), we get successively

$$\frac{dh}{dt} \geq -\frac{h}{e^{qt}-1} \;,$$

$$\frac{d}{dt} \log h \geq -\frac{1}{q}\,\frac{d}{dt}\left(\log\left(1 - e^{-qt}\right)\right)$$

which gives

$$h(t) \geq h_1 \left(\frac{1 - e^{-qt_1}}{1 - e^{-qt}}\right)^{1/q} \quad \forall\, t > t_1 \;.$$

□

This is enough to get a uniform lower bound on u.

Lemma 10 *As* $t \to +\infty$*,* $s(t)$ *converges to* c_M *and for any* $\eta \in (0,1)$*, there exists a* $t_1 \geq 0$ *such that*

$$u(\cdot,t) \geq (1-\eta)\, u_\infty^{s(t)} \quad \forall\, t \geq t_1 \;.$$

Proof. First of all, let us prove that $s(t)$ converges to c_M. Integrating the estimate of Lemma 7 with respect to x, we get

$$u(x,t) \geq \left((h(t))^{q-1} - \frac{s(t)-x}{1-e^{-qt}}\right)_+^{1/(q-1)} =: v(x,t) \quad \forall\, x \in [c_M - \epsilon, s(t)) \;. \tag{15}$$

Integrating u on $(0, s(t))$ and using Proposition 5, (iv), we obtain a lower estimate for the mass:

$$M \geq (1-\eta) \int_0^{c_M-\epsilon} u_\infty\, dx + \int_{c_M-\epsilon}^{s(t)} v\, dx$$

as soon as t is large enough so that $\kappa\, e^{-q\,t} < \eta$. Take $h_0 = h(t_1)$ with the notations of Lemma 8, (ii), and apply Lemma 9:

$$(h(t))^{q-1} \geq (1-\delta)^{q-1}\, s(t_1)\, \left(1 - e^{-qt_1}\right)^{(q-1)/q} =: h_1 \quad \forall\, t > t_1 \;. \tag{16}$$

If there exists a $\zeta > 0$ and a sequence $(t_n)_{n\geq 1}$ with $t_n > t_1$, t_1 given by Lemma 8, (ii), and $\lim_{n\to\infty} t_n = +\infty$, such that

$$s(t_n) > c_M + \zeta \quad \forall\, n \geq 1 \;,$$

then, because of (15),

$$\int_{c_M-\epsilon}^{s(t_n)} v\; dx > \int_0^{\zeta} \left(h_1^{q-1} - \frac{y}{1-e^{-qt_n}} \right)_+^{1/(q-1)} dy \;,$$

which is bounded from below by a positive constant, uniformly in $n \geq 1$. On the other hand, the quantity $M - (1-\eta)\int_0^{c_M-\epsilon} u_\infty\, dx$ is small for ϵ and η sufficiently small, and t large enough, according to Proposition 5, (iii) and (iv), which gives a contradiction. Thus, $s(t) - c_M \leq \zeta$ for any $\zeta > 0$, $t > 0$ large enough, and $s(t) > c_M - \epsilon$ for any $\epsilon > 0$ sufficiently small, $t > 0$ large enough. This proves the first part of Lemma 10.

On $(0, c_M - \epsilon)$, a lower estimate on u is given by Proposition 5, (iv). We may indeed estimate u from below using (15) (the minimum is achieved at $x = c_M - \epsilon$):

$$u(x,t) \geq \left(h(t)^{q-1} - \frac{s(t) - c_M + \epsilon}{1 - e^{-qt}} \right)_+^{1/(q-1)} .$$

Using again (16) and the convergence of $s(t)$ to c_M, we complete the proof of Lemma 10. □

A combination of Proposition 5, (iii) and Lemma 10 proves the uniform convergence of u to $u_\infty^{s(t)}$. To complete the proof of Theorem 3, it remains to estimate $s(t)$. According to Lemma 10, $s(t) \to c_M$. Proposition 5, (iv) implies that

$$u(\cdot,t) \leq u_\infty^{s(t)}\,(1 + O(e^{-qt}))\;,$$

which like in the proof of Lemma 10 can be rephrased into

$$s(t) \geq c_M - O(e^{-qt})$$

(such an estimate makes sense only if $s(t) < c_M$). This ends the proof of Theorem 3 and as a consequence of the change of variables (9), also of Theorem 2. □

4 L^1 intermediate asymptotics

Our main tool in this section is the relative entropy Σ of the solution u of the rescaled equation (11) with respect to the stationary solution u_∞^c given by (13). For any positive constants c and c', this entropy is defined by

$$\Sigma(t) = \int_0^{c'} \mu(x)\,|u(t,x) - u_\infty^c(x)|\; dx = \int_0^c \mu\,|u - u_\infty^c|\; dx + \int_c^{c'} \mu\, u\; dx \qquad (17)$$

for some nonincreasing function μ, which is continuous on $(0,+\infty)$. As we shall see in Section 4.2, the special choice $c' > \sup_{t\in\mathbb{R}^+}\max_{x\in\mathbb{R}}\{\operatorname{supp} u(t,x)\}$ and $c = c_M$ will provide exponential rates of convergence. We finally define

$$f(v) = v - v^q$$

for $v > 0$. Note that the uniform estimate of Theorem 3 will be needed to establish the exponential decay rate in Section 4.2.

Before dealing with the entropy, let us state two elementary properties of f which will be needed to establish rates of decay. Let $q \in (1,+\infty)$ and $f(v) := v - v^q$. Then

(i) For any $v > 0$,

$$|f(v)| \geq (q-1)|v-1| - \frac{1}{2}(q-1)\max(q,2)|v-1|^2 \,. \tag{18}$$

(ii) If $q \in (1,2]$, for any $v > 0$,

$$|f(v)| \leq (q-1)|v-1| + \frac{q}{2}(q-1)|v-1|^2 \,. \tag{19}$$

For simplicity, we will only consider the case $\alpha \in (0, \frac{q}{q-1})$ in the next 3 subsections and explain in subsection 4.4 how to adapt the proof to the case $\alpha = \frac{q}{q-1}$.

4.1 Decay of the entropy

Using the comparison results of Section 2.2, the entropy Σ is now well defined and integrations by parts can be performed in order to compute its time derivative.

Proposition 11 *With the above notations, consider a nonnegative solution u of (11) with initial data u_0, with compact support in $[0,+\infty)$, such that*

$$u_0(x) \leq A_0\, x^{1/(q-1)} \quad \forall\, x \in \mathbb{R}^+$$

for some $A_0 > 0$. Assume that $\lim_{x\to 0,\, x>0} \mu(x)\, u_\infty^q(x) = 0$. Let $c' > 0$ and suppose that the functions $\mu' u_\infty^q$ and $\mu\, u_\infty$ are integrable on $(0,c')$. Then for every fixed $c \in (0,c')$, with Σ defined by (17), for $t \geq 0$ a.e.,

$$\begin{aligned}\frac{d\Sigma}{dt} \leq\; & \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx - \int_c^{c'} \mu'(u_\infty^{c'})^q\, f\left(\frac{u}{u_\infty^{c'}}\right) dx \\ & - \mu(c)\, c^{\frac{q}{q-1}} \left\{ f\left(\frac{u^+(c)}{c^{\frac{1}{q-1}}}\right) + \left| f\left(\frac{u^-(c)}{c^{\frac{1}{q-1}}}\right)\right| \right\} + \mu(c')\,(c')^{\frac{q}{q-1}}\, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right)\end{aligned}$$

where $u^\pm(c) := \lim_{\substack{x\to c\\ \pm(x-c)>0}} u(x)$. If $c = c'$, then

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx \leq 0 \,.$$

Proof. We first notice that

$$\Sigma(t) = \int_0^c \mu\, [u - u_\infty^c]\, \left[1\!\!1_{u>u_\infty^c} - 1\!\!1_{u<u_\infty^c}\right] dx + \int_c^{c'} \mu\, u(t)\, dx \,.$$

We assume for simplicity that $u(t,.)$ has exactly one shock at $x = s(t)$. Let $u^\pm = u^\pm(t) = u_1^\pm$ and $v^\pm = u^\pm(t)/u_\infty^{c'}$, where $u_\infty^{c'}$ stands for $u_\infty^{c'}(s(t))$: $v^- > v^+$ and

$$s'(t) = -(u_\infty^{c'})^{q-1}\, \frac{f(v^+) - f(v^-)}{v^+ - v^-} \,.$$

If $0 < s(t) < c'$, one has to take into account the variation of $s(t)$ and the boundary terms corresponding to the shock. If $c < s(t) < c'$, there is no contribution of the shock, for the following reason. Let us compute

$$\begin{aligned}\frac{d}{dt}\int_c^{c'} \mu\, u\, dx &= \frac{d}{dt}\left(\int_c^{s(t)} \mu\, u\, d + \int_{s(t)}^{c'} \mu\, u\, dx\right) \\ &= \mu(s(t))\, s'(t)\,(u - u^+) + \int_c^{s(t)} \mu\, u_t\, d + \int_{s(t)}^{c'} \mu\, u_t\, dx \,.\end{aligned}$$

Using Equation (11) and integrations by parts, the boundary terms at $x = s(t)$ sum up to

$$\mu(s(t))\, s'(t)\, (u - u^+) + \mu(s(t))\, s'(t)\ \left[(xu - (u^q))_{|x=s(t)^-} - (xu - (u^q))_{|x=s(t)^+}\right] = 0\ ,$$

because of the Rankine-Hugoniot condition (10).

Consider the case $0 < s(t) < c$. Then, with $s = s(t)$,

$$\frac{d\Sigma}{dt} = \int_0^c \mu\, u_t\, \left[1\!\!1_{u>u^c_\infty} - 1\!\!1_{u<u^c_\infty}\right]\, dx + \int_c^{c'} \mu\, u_t\, dx + \left[\mu(s)|u - u^c_\infty(s)| \cdot s'(t)\,\right]^{u=u^-}_{u=u^+}\ ,$$

$$\begin{aligned} \frac{d\Sigma}{dt} &= \int_0^c \mu\ (xu - u^q)_x\ \left[1\!\!1_{u>u^c_\infty} - 1\!\!1_{u<u^c_\infty}\right]\, dx + \int_c^{c'} \mu\,(xu - u^q)_x \\ &\quad + \left[\mu(s)|u - u^c_\infty(s)| \cdot s'(t)\,\right]^{u=u^-}_{u=u^+} \\ &= -\int_0^c \mu\ \left((u^c_\infty)^q \left|f\left(\frac{u}{u^c_\infty}\right)\right|\right)_x dx + \int_c^{c'} \mu\ \left((u^{c'}_\infty)^q f\left(\frac{u}{u^{c'}_\infty}\right)\right)_x dx \\ &\quad + \left[\mu(s)|u - u^c_\infty(s)| \cdot s'(t)\,\right]^{u=u^-}_{u=u^+}\ . \end{aligned}$$

After one integration by parts, we get

$$\begin{aligned} \frac{d\Sigma}{dt} \le& \int_0^c \mu'\,(u^c_\infty)^q \left|f\left(\frac{u}{u^c_\infty}\right)\right| dx + \mu(s)\,(u^c_\infty(s))^q \Psi(v^-, v^+) \\ &- \mu(c)\, c^{\frac{q}{q-1}}\ \left[\ \left|f\left(c^{-\frac{1}{q-1}}\, u^-(t,c)\right)\right| + f\left(c^{-\frac{1}{q-1}}\, u^+(t,c)\right)\right] \\ &- \int_c^{c'} \mu'\,(u^{c'}_\infty)^q\, f\left(\tfrac{u}{u^{c'}_\infty}\right) dx + \mu(c')\,(c')^{q/(q-1)}\, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right) \end{aligned}$$

where

$$\Psi(v^-, v^+) = \left[f(v^+) - f(v^-)\right] \cdot \frac{|v^+ - 1| - |v^- - 1|}{v^+ - v^-} + |f(v^+)| - |f(v^-)|\ .$$

Since $v^+ < v^-$, we have to distinguish three cases:

(i) $1 \le v^+ \le v^-$: $f(v^-) \le f(v^+) \le 0$ and $\Psi(v^-, v^+) = 0$.

(ii) $v^+ < 1 \le v^-$: $f(v^-) \le 0 < f(v^+)$ and by concavity of $f(v) = v - v^q$

$$\begin{aligned} \tfrac{1}{2}\, \Psi(v^-, v^+) = &\ \tfrac{v^- - 1}{v^- - v^+}\, f(v^+) + \tfrac{1 - v^+}{v^- - v^+}\, f(v^-) \\ &\le f\left(\tfrac{v^- - 1}{v^- - v^+}\, v^+ + \tfrac{1 - v^+}{v^- - v^+}\, v^-\right) = f(1) = 0\ . \end{aligned}$$

(iii) $v^+ < v^- \le 1$: $f(v^-) \ge 0$ and $f(v^+) \ge 0$, $\Psi(v^-, v^+) = 0$.

In case $s(t) = c$, a further discussion is needed. Either $s'(t) \neq 0$, and up to a term of zero-measure in t, we can neglect the effect of the shock in the computation of $\frac{d\Sigma}{dt}$ (which is the reason why the inequality in Proposition 11 only holds a.e. in t) or $s'(t) = 0$ and we are left with only the boundary terms, exactly as if there was no shock.

The computations in presence of more than one shock are exactly the same, while the above identity becomes an equality if there is no shock. □

Remark *The term $f(v^+) + |f(v^-)|$ with $v^\pm = c^{-1/(q-1)}\, u^\pm(c)$ which appears in Proposition 11 is nonnegative since $v^+ < v^-$ and $f(v^+) < 0 \Longrightarrow f(v^-) < f(v^+) < 0$.*

Taking $c = c_M$ and letting c' go to $+\infty$, we get the following estimate.

Proposition 12 *Let μ be a nonnegative bounded weight. If u is a solution of (11) corresponding to a nonnegative initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R})$, then*

$$\lim_{t\to+\infty} \int_{C_M}^{+\infty} \mu(x)\, u(t,x)\, dx = 0\ .$$

Proof. For any $t \geq 0$, $u(t,\cdot)$ is nonnegative and uniformly bounded in $L^1(\mathbb{R})$ by $M = \|u_0\|_1$. Since $u(t,\cdot)$ converges to $u_\infty := u_\infty^{C_M}$ in $L^\infty(\mathbb{R})$ as $t \to +\infty$ and $\int_{\mathbb{R}} u(t,\cdot)\, dx = \int_{\mathbb{R}} u_\infty\, dx$, the convergence is also strong in L^1. □

However, we will see in Corollary 14 that, using Proposition 11, we can get a much better estimate.

4.2 Rates of decay

To emphasize the dependence in α, we denote by Σ_α the quantity Σ in case $\mu(x) = |x|^{-\alpha}$. For this special weight, we are going to prove an exponential decay of the entropy, when c and c' are appropriately chosen.

Proposition 13 *Assume that $c \leq c_M$, $c \leq c'$ and $c = c_M$ if $c' > c$. Under the same assumptions as in Proposition 11, if*

$$\mu(x) = x^{-\alpha} \quad \forall\, x > 0$$

for some $\alpha \in (0, \frac{q}{q-1})$, then $\lim_{t\to+\infty} \Sigma_\alpha(t) = 0$ and

$$\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) - \alpha \int_c^{c'} x^{-\alpha}\, u\, dx - r(c') = o\big(\Sigma_\alpha(t)\big) \quad \text{as } t \to +\infty$$

with $r(c') = \mu(c')\,(c')^{q/(q-1)}\, f\left((c')^{-1/(q-1)}\, u^-(c')\right)$.

Proof. By Proposition 11,

$$\frac{d\Sigma_\alpha}{dt} \leq -\alpha \int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx + \alpha \int_c^{c'} x^{-\alpha-1+\frac{q}{q-1}}\, f\left(\frac{u}{u_\infty^{c'}}\right) dx + r(c').$$

On one hand,

$$\int_c^{c'} x^{-\alpha-1+\frac{q}{q-1}}\, f\left(\frac{u}{u_\infty^{c'}}\right) dx = \int_c^{c'} x^{-\alpha}\, u\, dx - \int_c^{c'} x^{-\alpha-1}\, u^q\, dx \leq \int_c^{c'} x^{-\alpha}\, u\, dx\ .$$

On the other hand, since $u/u_\infty^c \to 1$ a.e. on $(0, s(t))$, for $x \in (0, s(t))$, we may write a Taylor expansion of f around 1 with an integral remainder

$$f\left(\frac{u}{u_\infty^c}\right) = (1-q)\left(\frac{u}{u_\infty^c} - 1\right) + q(1-q)\left|\frac{u}{u_\infty^c} - 1\right|^2 \int_0^1 (1-\theta)\left(\theta\frac{u}{u_\infty^c} + 1 - \theta\right)^{q-2} d\theta\ ,$$

from where we get, according to (18),

$$\int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right|\, dx \geq (q-1)\,\Sigma_\alpha - (q-1)\int_0^c x^{-\alpha+\frac{1}{q-1}} \left|\frac{u}{u_\infty^c} - 1\right|^2 dx$$

if $s(t) > c(t)$, and

$$\begin{aligned}
\int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right|\, dx \quad &\geq \quad (q-1)\left(\Sigma_\alpha - \int_{s(t)}^c \mu\, u_\infty^c\, dx\right)\\
&\qquad -(q-1)\int_0^{s(t)} x^{-\alpha+\frac{1}{q-1}} \left|\frac{u}{u_\infty^c} - 1\right|^2 dx
\end{aligned}$$

if $s(t) \leq c$. Thus, with $c(t) := \min(s(t), c)$,

$$\frac{d\Sigma_\alpha}{dt}+(q-1)\,\alpha\,\Sigma_\alpha(t)-r(c') \leq (q-1)\int_0^{c(t)} x^{-\alpha+\frac{1}{q-1}} \left|\frac{u}{u_\infty^c}-1\right|^2 dx+q\,\alpha\int_c^{c'} x^{-\alpha}\,u\,dx+\chi(t)$$

where $\chi(t) \equiv 0$ if $s(t) > c(t)$ and $\chi(t) := \int_{s(t)}^c \mu\, u_\infty^c\, dx$ if $s(t) \leq c(t)$.

By Proposition 5, (iii) and (iv),

$$\left|\frac{u}{u_\infty^{s(t)}}-1\right| \leq O\left(e^{-\frac{q}{q-1}\,t}\right) \quad \forall\, x \in [0, s(t)) \cap [0, c_M-\epsilon]\;.$$

Combining this with the result of Theorem 3, this proves that

$$\left\|\frac{u}{u_\infty^c}-1\right\|_{L^\infty(0,c(t))} \to 0 \quad \text{as} \quad t \to +\infty\;, \tag{20}$$

so that

$$\int_0^c x^{-\alpha+\frac{1}{q-1}}\left(\frac{u}{u_\infty^c}-1\right)^2 dx = \left\|\frac{u}{u_\infty^c}-1\right\|_{L^\infty(0,c(t))}\int_0^c \mu\,|u-u_\infty^c|\,dx$$

is neglectible compared to $\Sigma_\alpha(t)$. □

Note that the result of (20) is slightly stronger than the one of Theorem 3.

Corollary 14 *Under the assumptions of Proposition 13, if* $c = c_M$ *and if*

$$\operatorname{supp} u(t,\cdot) \subset (0, c') \quad \forall\, t \geq 0\;,$$

then for any $\epsilon > 0$*, there exists a positive constant* $C_\alpha(\epsilon)$ *such that*

$$\Sigma_\alpha(t) \leq C_\alpha(\epsilon)\, e^{-[(q-1)\,\alpha-\epsilon]\,t} \quad \forall\, t \geq 0\;.$$

Proof. Take first $c = c' = c_M$. The estimate of Proposition 13 then reduces to

$$\limsup_{t\to+\infty} \frac{d}{dt}\left(e^{[(q-1)\,\alpha-\epsilon/2]t}\int_0^{c_M} \mu\,|u-u_\infty|\,dx\right) < 0$$

for any $\epsilon > 0$ or, in other words, that

$$\int_0^{c_M} \mu\,|u-u_\infty|\,dx = O\left(e^{-[(q-1)\,\alpha-\epsilon/2]\,t}\right)\;.$$

As a consequence,

$$\int_0^{c_M} |u-u_\infty|\,dx \leq c_M^\alpha \int_0^{c_M} \mu\,|u-u_\infty|\,dx \leq C\,e^{-[(q-1)\,\alpha-\epsilon/2]\,t}$$

for some positive constant C. Thus

$$\int_0^{c_M} u\,dx \geq \int_0^{c_M} u_\infty\,dx - \int_0^{c_M} |u-u_\infty|\,dx \geq M - C\,e^{-[(q-1)\,\alpha-\epsilon/2]\,t}\;,$$

which means that

$$\int_{c_M}^{c'} u\,dx = M - \int_0^{c_M} u\,dx \leq C\,e^{-[(q-1)\,\alpha-\epsilon/2]\,t}\;.$$

Now,

$$\int_{c_M}^{c'} x^{-\alpha}\,u\,dx \leq c_M^{-\alpha}\int_{c_M}^{c'} u\,dx \leq c_M^{-\alpha}\,C\,e^{-[(q-1)\,\alpha-\epsilon/2]\,t}\;.$$

Applying Proposition 13 with $c = c_M < \sup \operatorname{supp} u(t,\cdot) < c'$, we get that

$$\frac{d}{dt}\left(e^{[(q-1)\,\alpha-\epsilon/2]\,t}\,\Sigma_\alpha(t)\right)$$

is uniformly bounded, which proves the result. $\square$

Written in terms of the unscaled problem with $t(\tau) = \log R(\tau)$, this means

Corollary 15 *Consider a piecewise C^1 entropy solution U of (1) with a nonnegative initial data U_0 which is compactly supported in $(0,+\infty)$ and such that $\xi^{-1/(q-1)}U_0(\xi)$ is bounded. Then for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$, there exists a positive constant $C_\alpha(\epsilon)$ such that*

$$\int_{\mathbb{R}} |\xi|^{-\alpha}\,|U(\tau,\xi) - V_\infty(\tau,\xi)|\,d\xi \le C_\alpha(\epsilon)\,(1+q\,\tau)^{-\alpha+\epsilon/q} \quad \forall\,\tau \ge 0\,.$$

where $V_\infty(\tau,\xi) = \frac{1}{R}u_\infty(\frac{\xi}{R})$, with $R = R(\tau) = (1+q\,\tau)^{1/q}$.

Proof. We first notice that, by the hypothesis on U_0 and the comparison principle, $\operatorname{supp} U(\tau) \subset (0, c'R(\tau))$ with c' such that $\operatorname{supp} U_0 \subset (0,c')$. This and Corollary 4 readily implies that the function u, defined by (9) satisfies Corollary 14, which gives the conclusion when written in the unscaled variables. $\square$

4.3 End of the proof of Theorem 1

To complete the proof of Theorem 1, we proceed as follows. First, since U_0 is bounded and compactly supported in $(\xi_0,+\infty)$, there exist positive constants A and c' such that, for all $\xi \in \mathbb{R}$, $\operatorname{supp} U_0(\cdot+\xi_0) \subset (0,c')$ and

$$U_0(\xi+\xi_0) \le A\,u_\infty^{c'}(\xi).$$

Then, by Corollary 15 applied to the solution with initial data $U_0(\xi+\xi_0)$, we obtain:

$$\int_0^{c'R(\tau)} |\xi|^{-\alpha}\,|U(\tau,\xi+\xi_0) - V_\infty(\tau,\xi)|\,d\xi \le C_\alpha(\epsilon)\,(1+q\,\tau)^{-\alpha+\epsilon/q} \quad \forall\,\tau \ge 0\,,$$

and, since the supports of $V_\infty(\tau,\cdot)$ and $U(\tau,\cdot+\xi_0)$ are contained in $(0,c'R)$ we deduce,

$$\int_{-\infty}^{+\infty} |\xi|^{-\alpha}\,|U(\tau,\xi+\xi_0) - V_\infty(\tau,\xi)|\,d\xi \le C_\alpha(\epsilon)\,(1+q\,\tau)^{-\alpha+\epsilon/q} \quad \forall\,\tau \ge 0\,,$$

or equivalently, for any $\tau \ge 0$,

$$\int_{-\infty}^{+\infty} |\xi-\xi_0|^{-\alpha}\,|U(\tau,\xi) - V_\infty(\tau,\xi-\xi_0)|\,d\xi \le C_\alpha(\epsilon)\,(1+q\,\tau)^{-\alpha+\epsilon/q}\,.$$

Then, one has to check that it is possible to replace the function V_∞ by the self-similar solution U_∞. But this follows from the fact that, as it is easily checked explicitly,

$$\int_{\xi_0}^{\xi_0+c_M R(\tau)} |\xi-\xi_0|^{-\alpha}\left|U_\infty(\xi-\xi_0,\tau) - \frac{1}{R(\tau)}u_\infty\left(\frac{\xi-\xi_0}{R(\tau)}\right)\right| d\xi \le C_q\,\tau^{-\frac{\alpha}{q}-1}.$$

$\square$

Remark *Unless $\alpha \le 1/(q-1)$, we cannot replace the translated self-similar solution $U_\infty(\tau,\cdot-\xi_0)$ by the self-similar solution $U_\infty(\tau,\cdot)$ itself. This has already been noted in the introduction in the case $q=2$. The problem is due to the fact that as $\tau \to \infty$,*

$$\int_{\xi_0}^{\xi_0+c(\tau)} |\xi-\xi_0|^{-\alpha}|U_\infty(\tau,\xi-\xi_0) - U_\infty(\tau,\xi)|\,d\xi \sim \tau^{-(\alpha+1)/q}$$

and $\tau^{-(\alpha+1)/q} = o(\tau^{-\alpha})$ if $\alpha > 1/(q-1)$. On the contrary, for any $\alpha \in (0,1/(q-1))$, we can replace $U_\infty(\tau,\cdot-\xi_0)$ by $U_\infty(\tau,\cdot)$ in (4) and (5), as it may be explicitely checked.

4.4 The limit case

In order to treat the limit case $\alpha = q/(q-1)$, we modify the weight and consider

$$\mu(x,t) = x^{-\alpha}(at + k - \log x)^{-1-\varepsilon}$$

with $a \geq q-1$ and $k > 0$. For k sufficiently big, this weight is positive for any $t > 0$ since the solution $u(\cdot,t)$ of (11) has a uniformly compact support. Let $s(t) := \max_{x\in\mathbb{R}}\{\operatorname{supp} u(t,x)\}$ as in the previous sections. We choose

$$k > 1 + \log\left(\max_{t\in\mathbb{R}} s(t)\right) . \tag{21}$$

We define Σ as in Section 4 by (17): $\Sigma(t) = \int_0^{c'} \mu(x,t)\,|u(t,x) - u^c_\infty(x)|\,dx$.
1) The conclusion of Proposition 11 holds except that the weight $\mu = \mu(x,t)$ depends on x and t, so there is an additional term:

$$\begin{aligned}\frac{d\Sigma}{dt} \leq & \int_0^c \frac{\partial\mu}{\partial x}\,(u^c_\infty)^q \left| f\left(\frac{u}{u^c_\infty}\right)\right| dx - \int_c^{c'} \frac{\partial\mu}{\partial x}\,(u^c_\infty)^q\, f\left(\frac{u}{u^c_\infty}\right) dx + \int_0^{c'} \frac{\partial\mu}{\partial t}\,|u - u^c_\infty|\,dx \\ & -\mu(c,t)\,c^{\frac{1}{q-1}} \left\{ f\left(\frac{u^+(c)}{c^{\frac{1}{q-1}}}\right) + \left| f\left(\frac{u^-(c)}{c^{\frac{1}{q-1}}}\right)\right| \right\} + \mu(c',t)\,(c')^{\frac{1}{q-1}}\, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right) .\end{aligned}$$

2) Similarly, the final identity in the proof of Proposition 13 has to be replaced by

$$\begin{aligned}\frac{d\Sigma}{dt} + q\,\Sigma(t) \quad \leq \quad & \left\|\frac{u}{u^c_\infty} - 1\right\|_{L^\infty(0,s(t))} \int_0^c x^{-\alpha}\mu\,|u - u^c_\infty|\,dx \\ & + C_\varepsilon\, e^{-qt} + C\int_c^{c'} \mu(x,t)\,u\,dx + r(c')\end{aligned}$$

for some constant $C > 0$. This is a consequence of (21), (18) and (19), as can be checked by following step by step the proof of Proposition 13.

As for Corollary 14, we obtain

$$\Sigma(t) \leq C\,(1+t)\,e^{-qt}$$

for $c = c' = c_M$.

5 Further results

First, let us prove Corollary 1. For every τ large enough and $\xi \in (\xi_0, \xi_0 + c'R(\tau))$, we may estimate $|\xi - \xi_0|^{-\alpha}$ from below by $(c'R(\tau))^{-\alpha}$: using (4), we obtain, as $\tau \to +\infty$,

$$\int_{\mathbb{R}} |U(\tau,\xi) - U_\infty(\tau,\xi-\xi_0)|\,d\xi \leq C_\beta\,\tau^{-\beta} ,$$

for $\beta = \alpha(1 - 1/q) + \varepsilon$. We can then choose α as close to $q/(q-1)$ as we want, which means that we can take β as close to 1 as we wish. Tis ends the proof of Corollary 1.□

Remark *If the initial datum U_0 is supported in $(\xi_0, +\infty)$, but not compactly supported, we do not get Estimate (5). Nevertheless, if we take $c = c' = c_M$ in Propositions 11, 13 and Corollaries 14, 15, this gives*

$$\limsup_{\tau\to+\infty} \tau^{\alpha-\epsilon} \int_{\xi_0}^{\xi_0 + c_M R(\tau)} |\xi - \xi_0|^{-\alpha}\,|U(\tau,\xi) - U_\infty(\tau,\xi-\xi_0)|\,d\xi = 0 .$$

Arguing as in the proof of Corollary 1, we conclude that for every $\beta \in (0,1)$ there is a positive constant C'_β such that

$$\int_{\xi_0}^{\xi_0+C_M R(\tau)} |U(\tau,\xi) - U_\infty(\tau,\xi-\xi_0)|\, d\xi \leq C'_\beta\, \tau^{-\beta}.$$

It is clear that Theorem 1 gives a decay rate in a weighted L^1 space of any nonnegative solution U of (1) such that its initial data satisfies $U_0 \equiv 0$ on $(-\infty,\xi_0)$ for some $\xi_0 \in \mathbb{R}$. If one is interested in decay rates and not in intermediate asymptotics, it is possible to consider other weights corresponding to power laws with negative exponents.

Proposition 16 *If U is a solution of (1) corresponding to a nonnegative continuous by parts initial data U_0 in $L^1 \cap L^\infty$ which is supported in $(\xi_0,+\infty)$ for some $\xi_0 \in \mathbb{R}$, then*

(i) for any $\alpha \in (0,\frac{q}{q-1})$, if U_0 has compact support in $(\xi_0,+\infty)$,

$$\lim_{\tau\to+\infty} \tau^{\frac{\alpha}{q}} \int_{\mathbb{R}} |\xi-\xi_0|^{-\alpha}\, U(\tau,\xi)\, d\xi = \tfrac{q^{1-\alpha}\,(q-1)^{\alpha\left(1-\frac{1}{q}\right)}}{q-\alpha\,(q-1)}\, M^{1-\alpha\left(1-\frac{1}{q}\right)},$$

(ii) for any $\beta > \frac{q}{q-1}$,

$$\limsup_{\tau\to+\infty} \tau^{\frac{1}{q-1}} \int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U(\tau,\xi)\, d\xi < +\infty\ .$$

Proof. According to Theorem 1, the first rate turns out to be sharp since, with the same weight, the difference $U - U_\infty$ has a faster decay. Part (ii) of Theorem 1 is a simple consequence of Hölder's inequality. Assume for simplicity that U has a single shock at $\xi = S(\tau)$. Then

$$\begin{aligned}\tfrac{d}{d\tau}\textstyle\int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U\, d\xi = \int_{\xi_0}^{S(\tau)} |\xi-\xi_0|^{-\beta}\, U_\tau\, d\xi + \int_{S(\tau)}^{+\infty} |\xi-\xi_0|^{-\beta}\, U_\tau\, d\xi \\ +S'(\tau)|S(\tau)-\xi_0|^{-\beta}\,(U^- - U^+)\end{aligned}$$

where $U^\pm = \lim_{\xi-S(\tau)\to 0^\pm} U(\tau,\xi)$. By the Rankine-Hugoniot condition,

$$S'(\tau) = \frac{(U^+)^q - (U^-)^q}{U^+ - U^-}\ ,$$

so that after one integration by parts the boundary terms cancel:

$$\frac{d}{d\tau}\int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U\, d\xi = -\beta \int_{\xi_0}^{+\infty} |\xi-\xi_0|^{-(\beta+1)}\, U^q\, d\xi\ .$$

Hölder's inequality:

$$\int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U(\tau,\xi)\, d\xi \leq \|\, |\xi-\xi_0|^{-\frac{\beta+1}{q}}\, U\|_q \cdot \|\, |\xi-\xi_0|^{\frac{1-\beta(q-1)}{q}}\ 1\!\!1_{\mathrm{supp}(U)}\|_{\frac{q}{q-1}}$$

and the fact that $\inf \mathrm{supp}\,(U) > \xi_0$ then imply the existence of a positive constant C such that

$$\int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U\, d\xi \leq \left[\left(\int_{\mathbb{R}} |\xi-\xi_0|^{-\beta}\, U_0\, d\xi\right)^{1-q} + C\,\tau\right]^{-\frac{1}{q-1}}$$

which concludes the proof of Proposition 16, (ii). $\square$

Appendix. A refined graph convergence result

This Appendix is devoted to a proof of the refined *graph convergence* result stated in Proposition 5. Using very elementary computations, we introduce two families of basic solutions in Section A.1 and then briefly sketch the proof.

A.1 Basic solutions

Consider the equation

$$U_\tau = (U^q)_\xi \ , \quad (\xi, \tau) \in \mathbb{R} \times \mathbb{R}^+ \ . \tag{1}$$

First case: (See Fig. 2) Let

$$U_0 = \kappa_0 \, (\xi - a_0)^{1/(q-1)} \quad \forall \xi \in (a_0, b_0)$$

with $a_0 < b_0$, and $U_0 \equiv 0$ elsewhere. Then

$$U(\xi, \tau) = \kappa(\tau) \, (\xi - a_0)^{1/(q-1)} \quad \forall \xi \in (a_0, b(\tau)) \ ,$$

and $U \equiv 0$ elsewhere, is the unique entropy solution of (1) if

$$\left\{ \begin{array}{l} b(\tau) = a_0 + (b_0 - a_0) \left(1 + \kappa_0^{q-1} q \, \tau\right)^{1/q} \ , \\ \\ \kappa(\tau) = (\kappa_0^{1-q} + q \, \tau)^{1/(1-q)} \ . \end{array} \right.$$

This follows from the equation written for $\xi \in (a_0, b(\tau))$, which is equivalent to

$$\frac{d\kappa}{d\tau} + \frac{q}{q-1} \, \kappa^q = 0$$

and from the Rankine-Hugoniot condition

$$\frac{db}{d\tau} = \kappa^{q-1} \, (b(\tau) - a_0) \ .$$

As a consequence, one can check that the total mass is conserved:

$$\frac{q-1}{q} \, \kappa(\tau) \, (b(\tau) - a_0)^{q/(q-1)} = \frac{q-1}{q} \, \kappa_0 \, (b_0 - a_0)^{q/(q-1)} \ .$$

This solution is known as the *N-wave solution* (see Fig. 2).

Second case: (See Figs. 3, 4) Let

$$\left\{ \begin{array}{ll} U_0(\xi) = \kappa_0 \, (\xi - a_0)^{1/(q-1)} & \forall \xi \in (a_0, b_0) \\ \\ U_0(\xi) = h & \forall \xi \in (b_0, c_0) \end{array} \right.$$

with $a_0 \leq b_0 < c_0$, for some $h \leq \kappa_0 \, (b_0 - a_0)^{1/(q-1)}$, and $U_0 \equiv 0$ elsewhere. Then

$$\left\{ \begin{array}{ll} U(\xi, \tau) = \kappa(\tau) \, (\xi - a_0)^{1/(q-1)} & \forall \xi \in (a_0, b(\tau)) \\ \\ U(\xi, \tau) = h & \forall \xi \in (b(\tau), c(\tau)) \end{array} \right.$$

and $U \equiv 0$ elsewhere, is the unique entropy solution of (1) if

$$\left\{ \begin{array}{l} c(\tau) = c_0 + h^{q-1} \tau \ , \\ \\ \kappa(\tau) = (\kappa_0^{1-q} + q \, \tau)^{1/(1-q)} \ , \end{array} \right.$$

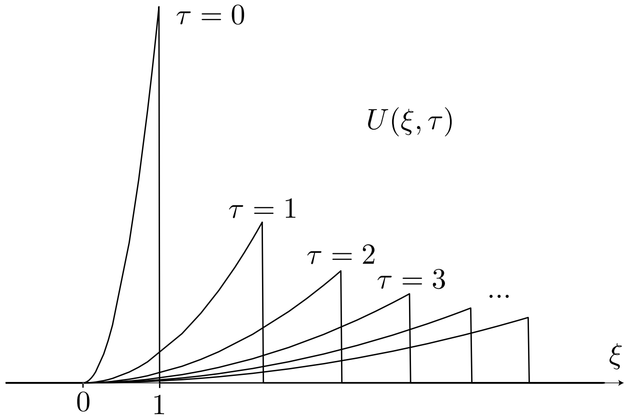

Figure 2: *The N-wave solution of (1) corresponding to* $U_0(\xi) = \frac{q}{q-1}\,\xi^{\frac{1}{q-1}}\,\mathbb{1}_{[0,1]}(\xi)$ *for various* $\tau > 0$*, in case* $q = \frac{3}{2}$.

and if $b = b(\tau)$ is the (unique) solution in $(b_0, c(\tau)$ of

$$\frac{q-1}{q}\,\kappa(\tau)\,b^{\frac{q}{q-1}} - h\,b = \frac{q-1}{q}\,\kappa_0\,b_0^{\frac{q}{q-1}} - h\,b_0 - h^q\tau\ ,$$

as long as $b(\tau) < c(\tau)$. This follows from the equation written for $\xi \in (a_0, b(\tau))$

$$\frac{d\kappa}{d\tau} + \frac{q}{q-1}\,\kappa^q = 0$$

and from the Rankine-Hugoniot conditions written at $\xi = b(\tau)$ and $\xi = c(\tau)$

$$\left\{\begin{array}{l} \frac{db}{d\tau} = \frac{\kappa^q\,(b(\tau)-a_0)^{q/(q-1)} - h^q}{\kappa\,(b(\tau)-a_0)^{1/(q-1)} - h} \\[2ex] \frac{dc}{d\tau} = h^{q-1} \end{array}\right.$$

(See Fig. 3 for a plot in case $h < \kappa_0\,(b_0 - a_0)^{1/(q-1)}$ and Fig. 4 in case $h = \kappa_0\,(b_0 - a_0)^{1/(q-1)}$).

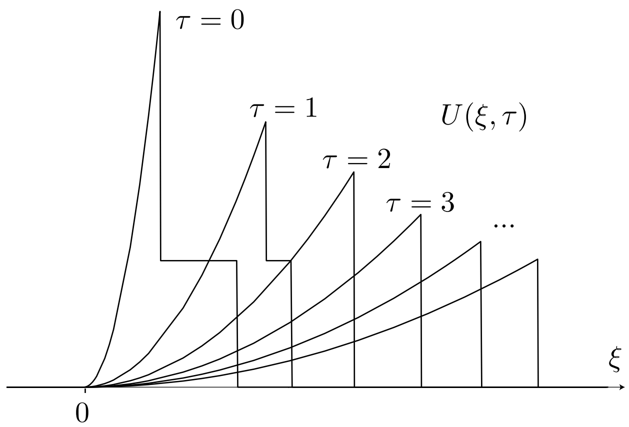

Figure 3: *The solution of (1) corresponding to the second case with* $U_0(\xi) = \kappa_0\,\mathbb{1}_{[a_0,b_0]}(\xi)\,\xi^{\frac{1}{q-1}} + h\,\mathbb{1}_{[b_0,c_0]}(\xi)$ *is plotted here for various* $\tau > 0$*, in case* $q = \frac{3}{2}$, $a_0 = 0$, $b_0 = \frac{1}{2}$, $c_0 = 1$, $h = \frac{1}{2}$ *and* κ_0 *such that* $\int U_0(\xi)\,d\xi = 1$.

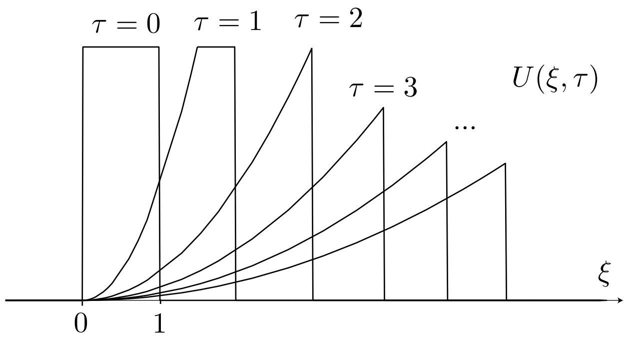

Figure 4: *The solution with $U_0(\xi) = 1\!\!1_{[0,1]}(\xi)$ in case $q = \frac{3}{2}$. This corresponds to the limit situation (in the second case) for which $b_0 = 0$ at $\tau = 0$ and $\kappa(\tau)\,(b(\tau))^{1/(q-1)} = h$ for any $\tau \in (0, \tau_0)$.*

As a consequence, it is immediate to check that the conservation of the total mass is equivalent to the equation for b:

$$\frac{q-1}{q}\,\kappa(\tau)\,(b(\tau)-a(\tau))^{\frac{q}{q-1}} + h^{q-1}(c(\tau)-b(\tau)) = \frac{q-1}{q}\,\kappa_0\,(b_0-a_0)^{\frac{q}{q-1}} + h^{q-1}(c_0-b_0)\ .$$

One recovers the first case in the limit $c_0 = b_0$ and otherwise the solution in the second case evolves according to the first case for $\tau > \tau_0$, where τ_0 is given by

$$\tau_0 := \inf\{\tau > 0\ :\ b(\tau) = c(\tau)\}$$

i.e., as the unique positive solution of

$$\frac{q-1}{q}\,(\kappa_0^{1-q} + q\,\tau)^{\frac{1}{q-1}}(c_0 + h^{q-1}\tau - a_0)^{\frac{q}{q-1}} = \frac{q-1}{q}\,\kappa_0\,(b_0-a_0)^{\frac{q}{q-1}} + h^{q-1}(c_0-b_0)\ .$$

A.2 Proof of Proposition 5

1) Note first that we do not assume that the initial data is continuous. However, using the comparison result stated in Lemma 3, it is clear that only admissible shocks develop for $\tau > 0$ (see Figs. 4, 6). Consider indeed an initial data U_0 which is supported in an interval $[a_0, b_0]$ and discontinuous at $\xi = a_0$. For some $A_0 > 0$ large enough, $U_0 \le A_0\,V_0 := A_0\,1\!\!1_{[a_0,b_0]}$, so that the discontinuity at $\xi = a_0$ disappears for any $\tau > 0$. A similar reasoning allows us to handle the case of a discontinuity which is inside the support and the case of a support made of several intervals.

2) A solution with initial support contained in an interval $[a_0, b_0]$ stays with support contained in $[a_0, b_0]$ as long as no shock develops at $\xi = b_0$ (see Fig. 5). However, by assumption, there exists a positive constant η such that $U_0(\xi) \ge \eta\,(\xi - a_0)_+^{1/(q-1)} =: V_0(\xi)$. Since the solution V of (1) has a shock travelling to $+\infty$, U necessarily also has a shock which develops at $\xi = b_0$ in finite time and then moves to $+\infty$ (see Fig. 6).

3) For the same reason, any solution whose support is initially made of several intervals has a single component support after a finite time. This proves (i).

4) The proof of (ii) easily follows, for instance by contradiction, using an appropriate special solution (second case of Section A.1, see Fig. 7).

5) Applying the same reasoning as in 1) for some interval $[a_0, b_0] \supset \mathrm{supp}(U_0)$, it is clear that the function $A_0\,1\!\!1_{[a_0,b_0]} =: V_0$ is transformed into a function $V = A_1\,U_\infty$ at time

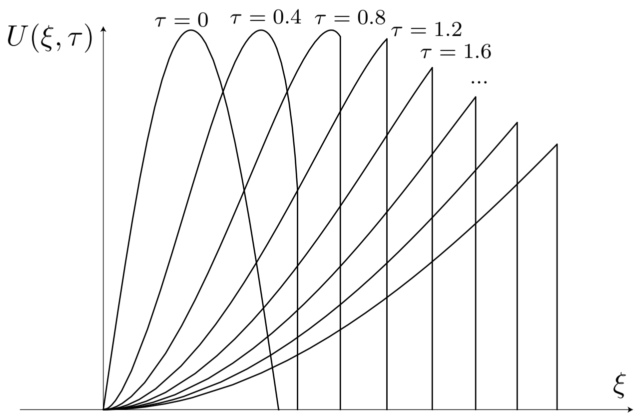

Figure 5: *A typical solution.*

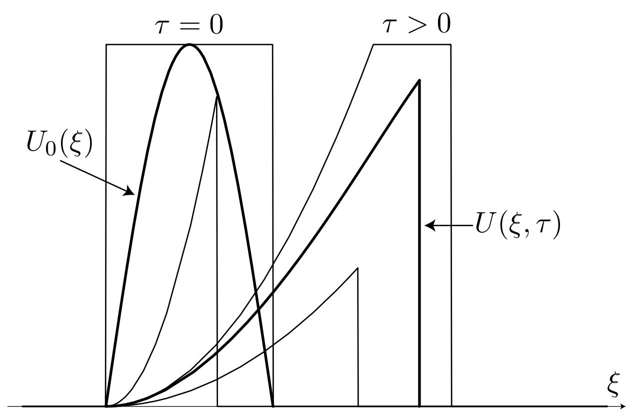

Figure 6: *Upper and lower solutions.*

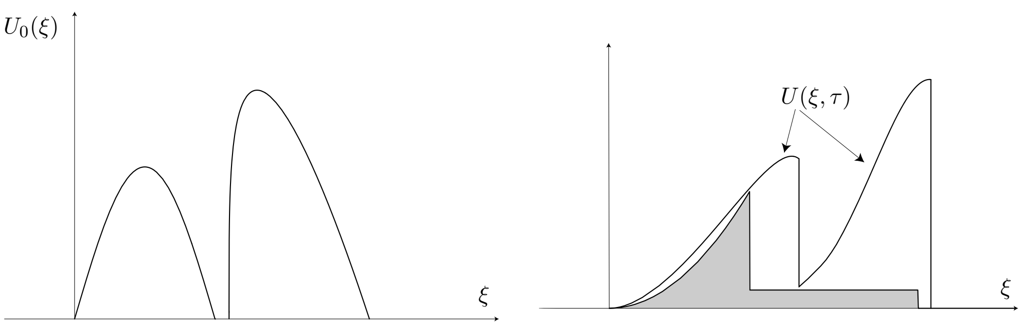

Figure 7: *Left: initial data. Right: for some $\tau > 0$ large enough.*

$\tau = 1$ (for instance), up to some scaling, for A_0 large enough (see Figs. 3, 7). We may then reapply Lemma 3 to get (iii).

6) Using basic functions of the second type of Section A.1 as many times as necessary (see Fig. 7) and the change of variables (9), it is then easy to obtain (iv). □

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