Generalized logarithmic Hardy-Littlewood-Sobolev inequality

Jean Dolbeault[1] **and Xingyu Li**[1]

[1] CEREMADE (CNRS UMR n° 7534), PSL university, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

Correspondence to be sent to: dolbeaul@ceremade.dauphine.fr

This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, which allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

1 Main result and motivation

On $\mathbb{R}^2$, let us define the *density of probability* $\mu = e^{-V}$ and the *external potential* V by

$$\mu(x) := \frac{1}{\pi\left(1+|x|^2\right)^2} \quad \text{and} \quad V(x) := -\log\mu(x) = 2\log\left(1+|x|^2\right) + \log\pi \quad \forall\, x \in \mathbb{R}^2\,.$$

We shall denote by $\mathrm{L}^1_+(\mathbb{R}^2)$ the set of a.e. nonnegative functions in $\mathrm{L}^1(\mathbb{R}^2)$. Our main result is the following *generalized logarithmic Hardy-Littlewood-Sobolev inequality*.

Theorem 1.1. For any $\alpha \ge 0$, we have that

$$\int_{\mathbb{R}^2} f\log\left(\frac{f}{M}\right)dx + \alpha\int_{\mathbb{R}^2} V\,f\,dx + M\,(1-\alpha)\,\big(1+\log\pi\big) \ge \frac{2}{M}\,(\alpha-1)\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\log|x-y|\,dx\,dy \quad (1)$$

for any function $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f\,dx > 0$. Moreover, the equality case is achieved by $f_\star = M\,\mu$ and $f_\star$ is the unique optimal function for any $\alpha > 0$. □

With $\alpha = 0$, the inequality is the classical *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f\log\left(\frac{f}{M}\right)dx + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\log|x-y|\,dx\,dy + M\,\big(1+\log\pi\big) \ge 0\,. \quad (2)$$

In that case $f_\star$ is an optimal function as well as all functions generated by a translation and a scaling of $f_\star$. As long as the parameter α is in the range $0 \le \alpha < 1$, the coefficient of the right-hand side of (1)

December 12, 2019 – File: LogHLS-9.tex
Keywords: logarithmic Hardy-Littlewood-Sobolev inequality; drift-diffusion-Poisson equation; entropy methods; nonlinear parabolic equations
MSC 2010: 26D10; 46E35; 35K55

is negative and the inequality is essentially of the same nature as the one with $\alpha = 0$. It can indeed be written as

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f\, dx + M(1-\alpha)\left(1+\log\pi\right) + \frac{2}{M}(1-\alpha) \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, f(y) \log|x-y|\, dx\, dy \ge 0\,.$$

For reasons that will be made clear below, we shall call this range the *attractive range*.

If $\alpha = 1$, the inequality is almost trivial since

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \int_{\mathbb{R}^2} V f\, dx = \int_{\mathbb{R}^2} f \log\left(\frac{f}{f_\star}\right) dx \ge 0 \tag{3}$$

is a straightforward consequence of Jensen's inequality. Now it is clear that by adding (2) multiplied by $(1-\alpha)$ and (3) multiplied by α, we recover (1) for any $\alpha \in [0,1]$. As a consequence (1) is a straightforward interpolation between (2) and (3) in the *attractive range*.

Now, let us consider the *repulsive range* $\alpha > 1$. It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient $(\alpha-1)$ in the right-hand side of (1) is now positive. Since

$$G(x) = -\frac{1}{2\pi} \log|x|$$

is the Green function associated with $-\Delta$ on $\mathbb{R}^2$, so that we can define

$$(-\Delta)^{-1} f(x) = (G * f)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, f(y)\, dy\,,$$

it is interesting to write (1) as

$$\int_{\mathbb{R}^2} f \log\left(\frac{f}{M}\right) dx + \alpha \int_{\mathbb{R}^2} V f\, dx + \frac{4\pi}{M}(\alpha-1) \int_{\mathbb{R}^2} f\, (-\Delta)^{-1} f\, dx \ge M(\alpha-1)\left(1+\log\pi\right). \tag{4}$$

If f has a sufficient decay as $|x| \to +\infty$, for instance if f is compactly supported, we know that $(-\Delta)^{-1} f(x) \sim -\frac{M}{2\pi} \log|x|$ for large values of $|x|$ and as a consequence,

$$\alpha V + \frac{4\pi}{M}(\alpha-1)(-\Delta)^{-1} f \sim 2(\alpha+1) \log|x| \to +\infty \quad \text{as} \quad |x| \to +\infty\,.$$

In a minimization scheme, this prevents the runaway of the left-hand side in (4). On the other hand, $\int_{\mathbb{R}^2} f \log f\, dx$ prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (4) indeed admits a minimizer.

Inequality (2) was proved in [8] by E. Carlen and M. Loss (also see [2]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [7]: see Section 2 for a sketch of their proof. Our proof of Theorem 1.1 relies on an extension of this approach which takes into account the presence of the external potential V. A remarkable feature of this approach is that it is insensitive to the sign of $\alpha - 1$.

One of the key motivations for studying (4) arises from entropy methods applied to *drift-diffusion-Poisson* models which, after scaling out all physical parameters, are given by

$$\frac{\partial f}{\partial t} = \Delta f + \beta \nabla\cdot(f \nabla V) + \nabla\cdot(f \nabla\phi) \tag{5}$$

with a nonlinear coupling given by the *Poisson* equation

$$-\varepsilon\,\Delta\phi = f\,. \tag{6}$$

Here $V = -\log\mu$ is the external *confining potential* and we choose it as in the statement of Theorem 1.1, while $\beta \geq 0$ is a coupling parameter with V, which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient ε in (6) is either $\varepsilon = -1$, which corresponds to the *attractive* case, or $\varepsilon = +1$, which corresponds to the *repulsive* case. In terms of applications, when $\varepsilon = -1$, (6) is the equation for the mean field potential obtained from Newton's law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case $\varepsilon = +1$ is used for repulsive electrostatic forces in semi-conductor physics, electrolytes, plasmas and charged particle models.

In view of *entropy methods* applied to PDEs (see for instance [15]), it is natural to consider the *free energy functional*

$$\mathscr{F}_\beta[f] := \int_{\mathbb{R}^2} f\log f\,dx + \beta\int_{\mathbb{R}^2} V\,f\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\phi\,f\,dx \tag{7}$$

because, if $f > 0$ solves (5)-(6) and is smooth enough, with sufficient decay properties at infinity, then

$$\frac{d}{dt}\mathscr{F}_\beta[f(t,\cdot)] = -\int_{\mathbb{R}^2} f\,\left|\nabla\log f + \beta\nabla V + \nabla\phi\right|^2 dx \tag{8}$$

so that $\mathscr{F}_\beta$ is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions $\mathscr{F}_\beta$ is bounded from below. The answer is given by the following result.

Corollary 1.2. Let $M > 0$. If $\varepsilon = +1$, the functional $\mathscr{F}_\beta$ is bounded from below and admits a minimizer on the set of the functions $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f\,dx = M$ if $\beta \geq 1 + \frac{M}{8\pi}$. It is bounded from below if $\varepsilon = -1$, $\beta \geq 1 - \frac{M}{8\pi}$ and $M \leq 8\pi$. If $\varepsilon = +1$, the minimizer is unique. □

As we shall see in Section 3.1, Corollary 1.2 is a simple consequence of Theorem 1.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with $\varepsilon = -1$ and $\beta = 0$, this has been used in [12, 4] to provide a sharp range of existence of the solutions to the evolution problem. In [6], the case $\varepsilon = -1$ with a potential V with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the *drift-diffusion-Poisson* model (5)-(6) and considerations on the *free energy*, in the electrostatic case, we can quote, among many others, [14, 13] and subsequent papers. In the Euclidean space with confinig potentials, we shall refer to [10, 11, 3, 1]. However, as far as we know, these papers are primarily devoted to dimensions $d \geq 3$ and the sharp growth condition on V when $d = 2$ has not been studied so far. The goal of this paper is to fill this gap. The specific choice of V has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 3.3, we shall give a result for general potentials on $\mathbb{R}^2$: see Theorem 3.4 for a statement.

2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (2) given by E. Carlen, J. Carrillo and M. Loss in [7]. The main idea is to use the nonlinear diffusion equation

$$\frac{\partial f}{\partial t} = \Delta\sqrt{f}$$

with a nonnegative initial datum f_0. The equation preserves the mass $M = \int_{\mathbb{R}^2} f\,dx$ and is such that

$$\frac{d}{dt}\left(\int_{\mathbb{R}^2} f\log f\,dx - \frac{4\pi}{M}\int_{\mathbb{R}^2} f\left((-\Delta)^{-1} f\right)dx\right) = -\frac{8}{M}\left(\int_{\mathbb{R}^2} \left|\nabla f^{1/4}\right|^2 dx \int_{\mathbb{R}^2} f\,dx - \pi\int_{\mathbb{R}^2} f^{3/2}\,dx\right).$$

According to [9], the Gagliardo-Nirenberg inequality

$$\|\nabla g\|_2^2\,\|g\|_4^4 \ge \pi\,\|g\|_6^6 \tag{9}$$

applied to $g = f^{1/4}$ guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [17]), we know that the solution behaves for large values of t like a self-similar solution, the so-called Barenblatt solution, which is given by $B(t,x) := t^{-2}\,f_\star(x/t)$. As a consequence, we find that

$$\begin{aligned}\int_{\mathbb{R}^2} f_0\log f_0\,dx &- \frac{4\pi}{M}\int_{\mathbb{R}^2} f_0\left((-\Delta)^{-1} f_0\right)dx\\ &\ge \lim_{t\to+\infty}\int_{\mathbb{R}^2} B\log B\,dx - \frac{4\pi}{M}\int_{\mathbb{R}^2} B\left((-\Delta)^{-1}B\right)dx = \int_{\mathbb{R}^2} f_\star\log f_\star\,dx - \frac{4\pi}{M}\int_{\mathbb{R}^2} f_\star\left((-\Delta)^{-1} f_\star\right)dx\end{aligned}$$

After an elementary computation, we observe that the above inequality is exactly (2) written for $f = f_0$.

The point is now to adapt this strategy to the case with an external potential. This justifies why we have to introduce a nonlinear diffusion equation with a drift. As we shall see below, the method is insensitive to α and applies when $\alpha > 1$ exactly as in the case $\alpha \in (0,1)$. A natural question is whether solutions are regular enough to perform the computations below and in particular if they have a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum f_0 that is as smooth and decaying as $|x| \to +\infty$ as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [17] for regularity issues and to [15] for the truncation method. In the proof, we will therefore leave these issues aside, as they are purely technical.

Proof of Theorem 1.1. By homogeneity, we can assume that $M = 1$ without loss of generality and consider the evolution equation

$$\frac{\partial f}{\partial t} = \Delta\sqrt{f} + 2\sqrt{\pi}\,\nabla\cdot(x\,f)\,.$$

1) Using simple integrations by parts, we compute

$$\int_{\mathbb{R}^2} (1+\log f)\,\Delta\sqrt{f}\,dx = -8\int_{\mathbb{R}^2} \left|\nabla f^{1/4}\right|^2 dx$$

and

$$\int_{\mathbb{R}^2} (1+\log f)\,\nabla\cdot(x\,f)\,dx = -\int_{\mathbb{R}^2} \frac{\nabla f}{f}\cdot(x\,f)\,dx = -\int_{\mathbb{R}^2} x\cdot\nabla f\,dx = 2\int_{\mathbb{R}^2} f\,dx = 2\,.$$

As a consequence, we obtain that

$$\frac{d}{dt}\int_{\mathbb{R}^2} f\log f\,dx = -8\int_{\mathbb{R}^2} \left|\nabla f^{1/4}\right|^2 dx + 8\pi\int_{\mathbb{R}^2} \mu^{3/2}\,dx \tag{10}$$

using

$$\int_{\mathbb{R}^2} \mu^{3/2}\,dx = \frac{1}{2\sqrt{\pi}}\,.$$

2) By elementary considerations again, we find that

$$4\pi\int_{\mathbb{R}^2} f\,(-\Delta)^{-1}\left(\Delta\sqrt{f}\right)dx=-4\pi\int_{\mathbb{R}^2} f^{3/2}\,dx$$

and

$$\begin{aligned}4\pi\int_{\mathbb{R}^2}\nabla\cdot(x\,f)\,(-\Delta)^{-1}f\,dx&=-4\pi\int_{\mathbb{R}^2}x\,f\cdot\nabla(-\Delta)^{-1}f\,dx\\&=2\iint_{\mathbb{R}^2\times\mathbb{R}^2}f(x)\,f(y)\,x\cdot\frac{x-y}{|x-y|^2}\,dx\,dy\\&=\iint_{\mathbb{R}^2\times\mathbb{R}^2}f(x)\,f(y)\,(x-y)\cdot\frac{x-y}{|x-y|^2}\,dx\,dy=1\end{aligned}$$

where, in the last line, we exchanged the variables x and y and took the half sum of the two expressions. This proves that

$$\frac{d}{dt}\left(4\pi\int_{\mathbb{R}^2} f\left((-\Delta)^{-1}f\right)dx\right)=-8\pi\int_{\mathbb{R}^2}\left(f^{3/2}-\mu^{3/2}\right)dx. \tag{11}$$

3) We observe that

$$\mu(x)=\frac{1}{\pi\left(1+|x|^2\right)^2}=e^{-V(x)}$$

solves

$$\Delta V=-\Delta\log\mu=8\pi\,\mu \tag{12}$$

and, as a consequence,

$$\int_{\mathbb{R}^2}V\,\Delta\sqrt{f}\,dx=\int_{\mathbb{R}^2}\Delta V\,\sqrt{f}\,dx=8\pi\int_{\mathbb{R}^2}\mu\sqrt{f}\,dx.$$

Since

$$\begin{aligned}2\sqrt{\pi}\int_{\mathbb{R}^2}V\,\nabla\cdot(x\,f)\,dx&=-2\sqrt{\pi}\int_{\mathbb{R}^2}f\,x\cdot\nabla V\,dx=-8\sqrt{\pi}\int_{\mathbb{R}^2}\frac{|x|^2}{1+|x|^2}\,f\,dx\\&=-8\sqrt{\pi}+8\sqrt{\pi}\int_{\mathbb{R}^2}\frac{f}{1+|x|^2}\,dx=-8\sqrt{\pi}+8\pi\int_{\mathbb{R}^2}\sqrt{\mu}\,f\,dx,\end{aligned}$$

we conclude that

$$\frac{d}{dt}\int_{\mathbb{R}^2}f\,V\,dx=8\pi\int_{\mathbb{R}^2}\left(\mu\sqrt{f}+\sqrt{\mu}\,f-2\,\mu^{3/2}\right)dx. \tag{13}$$

Let us define

$$\mathscr{F}[f]:=\int_{\mathbb{R}^2}f\log f\,dx+\alpha\int_{\mathbb{R}^2}V\,f\,dx+(1-\alpha)\left(1+\log\pi\right)+2\,(1-\alpha)\iint_{\mathbb{R}^2\times\mathbb{R}^2}f(x)\,f(y)\,\log|x-y|\,dx\,dy.$$

Collecting (10), (11) and (13), we find that

$$\frac{d}{dt}\mathscr{F}[f(t,\cdot)]=-8\left(\int_{\mathbb{R}^2}\left|\nabla f^{1/4}\right|^2dx-\pi\int_{\mathbb{R}^2}f^{3/2}\,dx\right)-8\pi\,\alpha\int_{\mathbb{R}^2}\left(f^{3/2}-\mu\sqrt{f}-\sqrt{\mu}\,f+\mu^{3/2}\right)dx.$$

Notice that

$$\int_{\mathbb{R}^2}\left(f^{3/2}-\mu\sqrt{f}-\sqrt{\mu}\,f+\mu^{3/2}\right)dx=\int_{\mathbb{R}^2}\varphi\left(\frac{f}{\mu}\right)\mu^{3/2}\,dx\quad\text{with}\quad\varphi(t):=t^{3/2}-t-\sqrt{t}+1$$

and that φ is a strictly convex function on $\mathbb{R}^+$ such that $\varphi(1) = \varphi'(1) = 0$, so that φ is nonnegative. On the other hand, by (9), we know that

$$\int_{\mathbb{R}^2} |\nabla f^{1/4}|^2\,dx - \pi \int_{\mathbb{R}^2} f^{3/2}\,dx \ge 0$$

as in the proof of [7]. Altogether, this proves that $t \mapsto \mathscr{F}[f(t,\cdot)]$ is monotone nonincreasing. Hence

$$\mathscr{F}[f_0] \ge \mathscr{F}[f(t,\cdot)] \ge \lim_{t\to+\infty} \mathscr{F}[f(t,\cdot)] = \mathscr{F}[f_\star] = 0\,.$$

This completes the proof of (1). ■

3 Consequences

3.1 Proof of Corollary 1.2

To prove the result of Corollary 1.2, we have to establish first that the *free energy* functional $\mathscr{F}_\beta$ is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5)-(6).

• ***Repulsive case.*** Let us consider the *free energy* functional defined in (7) where ϕ is given by (6) with $\varepsilon = +1$, *i.e.*, $\phi = -\frac{1}{2\pi} \log|\cdot| * f$.

Lemma 3.1. Let $M > 0$ and $\varepsilon = +1$. Then $\mathscr{F}_\beta$ is bounded from below on the set of the functions $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f\,dx = M$ if $\beta \ge 1 + \frac{M}{8\pi}$. □

Proof. With $g = \frac{f}{M}$ and $\alpha = 1 + \frac{M}{8\pi}$, this means that

$$\begin{aligned}\frac{1}{M}\mathscr{F}_\beta[f] - \log M &= \int_{\mathbb{R}^2} g \log g\,dx + \beta \int_{\mathbb{R}^2} V g\,dx - \frac{M}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\log|x-y|\,dx\,dy \\ &= (\beta-\alpha)\int_{\mathbb{R}^2} V g\,dx + \int_{\mathbb{R}^2} g\log g\,dx + \alpha \int_{\mathbb{R}^2} V g\,dx - 2\,(\alpha-1) \iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\log|x-y|\,dx\,dy \\ &\ge (\beta-\alpha)\int_{\mathbb{R}^2} V g\,dx - (1-\alpha)\,\big(1+\log\pi\big)\end{aligned}$$

according to Theorem 1.1; the condition $\beta \ge \alpha$ is enough to prove that $\mathscr{F}_\beta[f]$ is bounded from below. ■

Proof of Corollary 1.2 with $\varepsilon = +1$. Let us consider a smooth solution of (5)-(6). We refer to [16] for details and to [1] for similar arguments in dimension $d \ge 3$. According to (8), f converges as $t \to +\infty$ to a solution of

$$\nabla \log f + \beta \nabla V + \nabla \phi = 0\,.$$

Notice that this already proves the existence of a stationary solution. The equation can be solved as

$$f = M\,\frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V-\phi}\,dx}$$

after taking into account the conservation of the mass. With (6), the problem is reduced to solving

$$-\Delta\psi = M\left(\frac{e^{-\gamma V - \psi}}{\int_{\mathbb{R}^2} e^{-\gamma V-\psi}\,dx} - \mu\right), \quad \psi = \big(\beta-\gamma\big)\,V + \phi\,, \quad \gamma = \beta - \frac{M}{8\pi}$$

using (12). It is a critical point of the functional $\psi \mapsto \mathscr{J}_{M,\gamma}[\psi] := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla\psi|^2\,dx + M\int_{\mathbb{R}^2} \psi\,\mu\,dx + M\log\big(\int_{\mathbb{R}^2} e^{-\gamma V-\psi}\,dx\big)$. Such a functional is strictly convex as, for instance, in [10, 11]. We conclude that ψ is unique up to an additional constant. ■

• ***Attractive case.*** Let us consider the *free energy* functional (7) $\mathscr{F}_\beta$ where ϕ is given by (6) with $\varepsilon = -1$, *i.e.*, $\phi = \frac{1}{2\pi}\log|\cdot| * f$. Inspired by [12], we have the following estimate.

Lemma 3.2. Let $\varepsilon = -1$. Then $\mathscr{F}_\beta$ is bounded from below on the set of the functions $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f\,dx = M$ if $M \le 8\pi$ and $\beta \ge 1 - \frac{M}{8\pi}$. It is not bounded from below if $M > 8\pi$. □

Proof. With $g = \frac{f}{M}$ and $\alpha = 1 - \frac{M}{8\pi}$, Theorem 1.1 applied to

$$\begin{aligned}
&\frac{1}{M}\mathscr{F}_\beta[f] - \log M\\
&\quad = (\beta-\alpha)\int_{\mathbb{R}^2} V\,g\,dx + \int_{\mathbb{R}^2} g\log g\,dx + \alpha\int_{\mathbb{R}^2} V\,g\,dx + 2\,(1-\alpha)\iint_{\mathbb{R}^2\times\mathbb{R}^2} g(x)\,g(y)\,\log|x-y|\,dx\,dy\\
&\qquad\qquad \ge (\beta-\alpha)\int_{\mathbb{R}^2} V\,g\,dx - (1-\alpha)\,\big(1+\log\pi\big)
\end{aligned}$$

proves that the free energy is bounded from below if $M \le 8\pi$ and $\beta \ge \alpha$. On the other hand, if $f_\lambda(x) := \lambda^{-2} f(\lambda^{-1}x)$ and $M > 8\pi$, then

$$\mathscr{F}_\beta[f_\lambda] \sim 2\,M\left(\frac{M}{8\pi} - 1\right)\log\lambda \to -\infty \quad \text{as} \quad \lambda \to 0_+\,,$$

which proves that $\mathscr{F}_\beta$ is not bounded from below. ■

Proof of Corollary 1.2 with $\varepsilon = -1$. The proof goes as in the case $\beta = 0$. We refer to [4] and leave details to the reader. ■

Remark 3.3. Let us notice that $\mathscr{F}_\beta$ is unbounded from below if $\beta < 0$. This follows from the observation that $\lim_{|y|\to\infty}\mathscr{F}_\beta[f_y] = -\infty$ where $f_y(x) = f(x+y)$ for any admissible f. □

3.2 Duality

When $\alpha > 1$, we can write a first inequality by considering the *repulsive case* in the proof of Corollary 1.2 and observing that

$$\mathscr{J}_{M,\gamma}[\psi] \ge \min \mathscr{J}_{M,\gamma}$$

where $\psi \in \mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^2)$ is such that $\int_{\mathbb{R}^2}(\Delta\psi)\,dx = 0$ and the minimum is taken on the same set of functions.

When $\alpha \in [0,1)$, it is possible to argue by duality as in [5, Section 2]. Since $f_\star$ realizes the equality case in (1), we know that

$$\int_{\mathbb{R}^2} f_\star\log\left(\frac{f_\star}{M}\right)dx + \alpha\int_{\mathbb{R}^2} V\,f_\star\,dx + M\,(1-\alpha)\,\big(1+\log\pi\big) = \frac{2}{M}\,(\alpha-1)\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\star(x)\,f_\star(y)\,\log|x-y|\,dx\,dy$$

and, using the fact that $f_\star$ is a critical point of the difference of the two sides of (1), we also have that

$$\int_{\mathbb{R}^2}\log\left(\frac{f}{f_\star}\right)(f-f_\star)\,dx + \alpha\int_{\mathbb{R}^2} V\,(f-f_\star)\,dx = \frac{4}{M}\,(\alpha-1)\iint_{\mathbb{R}^2\times\mathbb{R}^2}\big(f(x)-f_\star(x)\big)\,f_\star(y)\,\log|x-y|\,dx\,dy\,.$$

By subtracting the first identity to (1) and adding the second identity, we can rephrase (1) as

$$\mathscr{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log\left(\frac{f}{f_\star}\right) dx \ge \frac{4\pi}{M}(1-\alpha)\int_{\mathbb{R}^2}(f-f_\star)(-\Delta)^{-1}(f-f_\star)\,dx := \mathscr{F}_{(2)}[f]\,.$$

Let us consider the Legendre transform

$$\mathscr{F}^*_{(i)}[g] := \sup_f \left(\int_{\mathbb{R}^2} g\,f\,dx - \mathscr{F}_{(i)}[f]\right)$$

where the supremum is restricted to the set of the functions $f \in \mathrm{L}^1_+(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} f\,dx$. After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that

$$M \log\left(\int_{\mathbb{R}^2} e^{g-V}\,dx\right) = \mathscr{F}^*_{(1)}[g] \le \frac{M}{16\pi(1-\alpha)}\int_{\mathbb{R}^2}|\nabla g|^2\,dx + M\int_{\mathbb{R}^2} g\,e^{-V}\,dx = \mathscr{F}^*_{(2)}[g]\,.$$

We can get rid of M by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as $\alpha \to 0_+$, which is clearly the sharpest one for all possible $\alpha \in [0,1)$.

3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential W on $\mathbb{R}^2$ such that

$$W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x|\to+\infty}\frac{W(x)}{V(x)} = \beta \tag{$\mathscr{H}_W$}$$

and the associated *free energy functional*

$$\mathscr{F}_{\beta,W}[f] := \int_{\mathbb{R}^2} f \log f\,dx + \beta\int_{\mathbb{R}^2} W\,f\,dx + \frac{1}{2}\int_{\mathbb{R}^2}\phi\,f\,dx$$

where ϕ is given in terms of $f > 0$ by (6). With previous notations, $\mathscr{F}_\beta = \mathscr{F}_{\beta,V}$. Our last result is that the asymptotic behaviour obtained from ($\mathscr{H}_W$) is enough to decide whether $\mathscr{F}_{\beta,W}$ is bounded from below or not. The precise result goes as follows.

Theorem 3.4. Under Assumption ($\mathscr{H}_W$), $\mathscr{F}_{\beta,W}$ defined as above is bounded from below if either $\varepsilon = +1$ and $\beta > 1 + \frac{M}{8\pi}$, or $\varepsilon = -1$, $\beta > 1 - \frac{M}{8\pi}$ and $M \le 8\pi$. The result is also true in the limit case if $(W - \beta V) \in \mathrm{L}^\infty(\mathbb{R}^2)$ and either $\varepsilon = +1$ and $\beta = 1 + \frac{M}{8\pi}$, or $\varepsilon = -1$, $\beta = 1 - \frac{M}{8\pi}$ and $M \le 8\pi$. □

Proof. If $(W - \beta V) \in \mathrm{L}^\infty(\mathbb{R}^2)$, we can write that

$$\mathscr{F}_{\beta,W}[f] \ge \mathscr{F}_\beta[f] - M\,\|W - \beta V\|_{\mathrm{L}^\infty(\mathbb{R}^2)}\,.$$

This completes the proof in the limit case. Otherwise, we redo the argument using $\tilde\beta V - \big(\tilde\beta V - W\big)_+$ for some $\tilde\beta \in (0,\beta)$ if $\varepsilon = -1$, and for some $\tilde\beta \in \big(1 + \frac{M}{8\pi}, \beta\big)$ if $\varepsilon = +1$. ■

Acknowledgment: This work has been partially supported by the Project EFI (ANR-17-CE40-0030) of the French National Research Agency (ANR). The authors thank L. Jeanjean who pointed them some typographical errors.

References

[1] Arnold, A., Markowich, P., Toscani, G., and Unterreiter, A. (2001). On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations. *Comm. Partial Differential Equations*, 26(1-2):43–100.

[2] Beckner, W. (1993). Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242.

[3] Biler, P. and Dolbeault, J. (2000). Long time behavior of solutions of Nernst-Planck and Debye-Hückel drift-diffusion systems. *Ann. Henri Poincaré*, 1(3):461–472.

[4] Blanchet, A., Dolbeault, J., and Perthame, B. (2006). Two-dimensional Keller-Segel model: optimal critical mass and qualitative properties of the solutions. *Electron. J. Differential Equations*, 44:1–33.

[5] Campos, J. and Dolbeault, J. (2012). A functional framework for the Keller-Segel system: logarithmic Hardy-Littlewood-Sobolev and related spectral gap inequalities. *C. R. Math. Acad. Sci. Paris*, 350(21-22):949–954.

[6] Campos, J. F. and Dolbeault, J. (2014). Asymptotic Estimates for the Parabolic-Elliptic Keller-Segel Model in the Plane. *Comm. Partial Differential Equations*, 39(5):806–841.

[7] Carlen, E. A., Carrillo, J. A., and Loss, M. (2010). Hardy-Littlewood-Sobolev inequalities via fast diffusion flows. *Proc. Natl. Acad. Sci. USA*, 107(46):19696–19701.

[8] Carlen, E. A. and Loss, M. (1992). Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on $\mathbb{S}^n$. *Geom. Funct. Anal.*, 2(1):90–104.

[9] Del Pino, M. and Dolbeault, J. (2002). Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. *J. Math. Pures Appl. (9)*, 81(9):847–875.

[10] Dolbeault, J. (1991). Stationary states in plasma physics: Maxwellian solutions of the Vlasov-Poisson system. *Math. Models Methods Appl. Sci.*, 1(2):183–208.

[11] Dolbeault, J. (1999). Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states). *J. Math. Pures Appl. (9)*, 78(2):121–157.

[12] Dolbeault, J. and Perthame, B. (2004). Optimal critical mass in the two-dimensional Keller-Segel model in $\mathbb{R}^2$. *C. R. Math. Acad. Sci. Paris*, 339(9):611–616.

[13] Dressler, K. (1990). Steady states in plasma physics – the Vlasov-Fokker-Planck equation. *Math. Methods Appl. Sci.*, 12(6):471–487.

[14] Gogny, D. and Lions, P.-L. (1989). Sur les états d'équilibre pour les densités électroniques dans les plasmas. *RAIRO Modél. Math. Anal. Numér.*, 23(1):137–153.

[15] Jüngel, A. (2016). *Entropy methods for diffusive partial differential equations*. SpringerBriefs in Mathematics. Springer, [Cham].

[16] Li, X. Asymptotic behavior of Nernst-Planck equation. Preprint Hal: hal-02310654 and arXiv: 1910.04477, 2019.

[17] Vázquez, J. L. (2006). *Smoothing and decay estimates for nonlinear diffusion equations*, volume 33 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford.