Generalized logarithmic Hardy-Littlewood-Sobolev inequality

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This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, which allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

1 Main result and motivation

On $\mathbb{R}^2$, let us define the density of probability $\mu = e^{-V}$ and the external potential $V$ by

$$\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \text{and} \quad V(x) := -\log \mu(x) = 2 \log (1 + |x|^2) + \log \pi \quad \forall \ x \in \mathbb{R}^2.$$ 

We shall denote by $L^1_+(\mathbb{R}^2)$ the set of a.e. nonnegative functions in $L^1(\mathbb{R}^2)$. Our main result is the following generalized logarithmic Hardy-Littlewood-Sobolev inequality.

**Theorem 1.1.** For any $\alpha \geq 0$, we have that

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) \, dx + \alpha \int_{\mathbb{R}^2} V f \, dx + M (1 - \alpha) (1 + \log \pi) \geq \frac{2}{M} (\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy \quad (1)$$

for any function $f \in L^1_+(\mathbb{R}^2)$ with $M = \int_{\mathbb{R}^2} f \, dx > 0$. Moreover, the equality case is achieved by $f_* = M \mu$ and $f_*$ is the unique optimal function for any $\alpha > 0$.

With $\alpha = 0$, the inequality is the classical logarithmic Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) \, dx \geq \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy + M (1 + \log \pi) \geq 0. \quad (2)$$

In that case $f_*$ is an optimal function as well as all functions generated by a translation and a scaling of $f_*$. As long as the parameter $\alpha$ is in the range $0 \leq \alpha < 1$, the coefficient of the right-hand side of (1)
is negative and the inequality is essentially of the same nature as the one with $\alpha = 0$. It can indeed be written as

$$
\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f \, dx + M (1 - \alpha) \left( 1 + \log \pi \right) + \frac{2}{M} (1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| \, dx \, dy \geq 0.
$$

For reasons that will be made clear below, we shall call this range the *attractive range*.

If $\alpha = 1$, the inequality is almost trivial since

$$
\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \int_{\mathbb{R}^2} V f \, dx = \int_{\mathbb{R}^2} f \log \left( \frac{f}{f^*} \right) dx \geq 0 \tag{3}
$$

is a straightforward consequence of Jensen's inequality. Now it is clear that by adding (2) multiplied by $(1 - \alpha)$ and (3) multiplied by $\alpha$, we recover (1) for any $\alpha \in [0, 1]$. As a consequence (1) is a straightforward interpolation between (2) and (3) in the *attractive range*.

Now, let us consider the *repulsive range* $\alpha > 1$. It is clear that the inequality is no more the consequence of a simple interpolation. We can also observe that the coefficient $(\alpha - 1)$ in the right-hand side of (1) is now positive. Since

$$
G(x) = -\frac{1}{2 \pi} \log |x|
$$

is the Green function associated with $-\Delta$ on $\mathbb{R}^2$, so that we can define

$$
(-\Delta)^{-1} f(x) = (G \ast f)(x) = -\frac{1}{2 \pi} \int_{\mathbb{R}^2} \log |x - y| f(y) \, dy,
$$

it is interesting to write (1) as

$$
\int_{\mathbb{R}^2} f \log \left( \frac{f}{M} \right) dx + \alpha \int_{\mathbb{R}^2} V f \, dx + \frac{4 \pi}{M} (\alpha - 1) \int_{\mathbb{R}^2} (-\Delta)^{-1} f \, dx \geq M (\alpha - 1) \left( 1 + \log \pi \right). \tag{4}
$$

If $f$ has a sufficient decay as $|x| \to +\infty$, for instance if $f$ is compactly supported, we know that $(-\Delta)^{-1} f(x) \sim -\frac{M}{2 \pi} \log |x|$ for large values of $|x|$ and as a consequence,

$$
\alpha V + \frac{4 \pi}{M} (\alpha - 1) (-\Delta)^{-1} f \sim -2 (\alpha + 1) \log |x| \to +\infty \quad \text{as} \quad |x| \to +\infty.
$$

In a minimization scheme, this prevents the runaway of the left-hand side in (4). On the other hand, $\int_{\mathbb{R}^2} f \log f \, dx$ prevents any concentration, and this is why it can be heuristically expected that the left-hand side of (4) indeed admits a minimizer.

Inequality (2) was proved in [8] by E. Carlen and M. Loss (also see [2]). An alternative method based on nonlinear flows was given by E. Carlen, J. Carrillo and M. Loss in [7]: see Section 2 for a sketch of their proof. Our proof of Theorem 1.1 relies on an extension of this approach which takes into account the presence of the external potential $V$. A remarkable feature of this approach is that it is insensitive to the sign of $\alpha - 1$.

One of the key motivations for studying (4) arises from entropy methods applied to *drift-diffusion-Poisson* models which, after scaling out all physical parameters, are given by

$$
\frac{\partial f}{\partial t} = \Delta f + \beta \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla \phi) \tag{5}
$$
with a nonlinear coupling given by the Poisson equation

$$- \varepsilon \Delta \phi = f .$$  \hspace{1cm} (6)

Here $V = -\log \mu$ is the external confining potential and we choose it as in the statement of Theorem 1.1, while $\beta \geq 0$ is a coupling parameter with $V$, which measures the strength of the external potential. We shall consider more general potentials at the end of this paper. The coefficient $\varepsilon$ in (6) is either $\varepsilon = -1$, which corresponds to the attractive case, or $\varepsilon = +1$, which corresponds to the repulsive case. In terms of applications, when $\varepsilon = -1$, (6) is the equation for the mean field potential obtained from Newton's law of attraction in gravitation, for applications in astrophysics, or for the Keller-Segel concentration of chemo-attractant in chemotaxis. The case $\varepsilon = +1$ is used for repulsive electrostatic forces in semiconductor physics, electrolytes, plasmas and charged particle models.

In view of entropy methods applied to PDEs (see for instance [15]), it is natural to consider the free energy functional

$$F_\beta[f] := \int_{\mathbb{R}^2} f \log f \, dx + \beta \int_{\mathbb{R}^2} V f \, dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi f \, dx$$  \hspace{1cm} (7)

due to, if $f > 0$ solves (5)-(6) and is smooth enough, with sufficient decay properties at infinity, then

$$\frac{d}{dt} F_\beta[f(t, \cdot)] = - \int_{\mathbb{R}^2} f |\nabla \log f + \beta \nabla V + \nabla \phi|^2 \, dx$$  \hspace{1cm} (8)

so that $F_\beta$ is a Lyapunov functional. Of course, a preliminary question is to establish under which conditions $F_\beta$ is bounded from below. The answer is given by the following result.

**Corollary 1.2.** Let $M > 0$. If $\varepsilon = +1$, the functional $F_\beta$ is bounded from below and admits a minimizer on the set of the functions $f \in L^1_1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f \, dx = M$ if $\beta \geq 1 + M \frac{8}{\pi}$. It is bounded from below if $\varepsilon = -1$, $\beta \geq 1 - M \frac{8}{\pi}$ and $M \leq 8 \pi$. If $\varepsilon = +1$, the minimizer is unique.

As we shall see in Section 3.1, Corollary 1.2 is a simple consequence of Theorem 1.1. In the case of the parabolic-elliptic Keller-Segel model, that is, with $\varepsilon = -1$ and $\beta = 0$, this has been used in [12, 4] to provide a sharp range of existence of the solutions to the evolution problem. In [6], the case $\varepsilon = -1$ with a potential $V$ with quadratic growth at infinity was also considered, in the study of intermediate asymptotics of the parabolic-elliptic Keller-Segel model.

Concerning the drift-diffusion-Poisson model (5)-(6) and considerations on the free energy, in the electrostatic case, we can quote, among many others, [14, 13] and subsequent papers. In the Euclidean space with confining potentials, we shall refer to [10, 11, 3, 1]. However, as far as we know, these papers are primarily devoted to dimensions $d \geq 3$ and the sharp growth condition on $V$ when $d = 2$ has not been studied so far. The goal of this paper is to fill this gap. The specific choice of $V$ has been made to obtain explicit constants and optimal inequalities, but the confining potential plays a role only at infinity if we are interested in the boundedness from below of the free energy. In Section 3.3, we shall give a result for general potentials on $\mathbb{R}^2$: see Theorem 3.4 for a statement.

## 2 Proof of the main result

As an introduction to the key method, we briefly sketch the proof of (2) given by E. Carlen, J. Carrillo and M. Loss in [7]. The main idea is to use the nonlinear diffusion equation

$$\frac{\partial f}{\partial t} = \Delta \sqrt{f}$$
with a nonnegative initial datum \( f_0 \). The equation preserves the mass \( M = \int_{\mathbb{R}^2} f \, dx \) and is such that
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^2} f \log f \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f \, dx \right) = -\frac{8}{M} \left( \int_{\mathbb{R}^2} |\nabla f|^{1/4} |\nabla f|^{3/4} \, dx \right).
\]
According to [9], the Gagliardo-Nirenberg inequality
\[
\| \nabla g \|^2_2 \| g \|^4_4 \geq \pi \| g \|^6_6
\]
applied to \( g = f^{1/4} \) guarantees that the right-hand side is nonpositive. By the general theory of fast diffusion equations (we refer for instance to [17]), we know that the solution behaves for large values of \( t \) like a self-similar solution, the so-called Barenblatt solution, which is given by \( B(t, x) = t^{-2} f_\star (x/t) \).

As a consequence, we find that
\[
\int_{\mathbb{R}^2} f_0 \log f_0 \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_0 \, dx \geq \lim_{t \to +\infty} \int_{\mathbb{R}^2} B \log B \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} B \, dx = \int_{\mathbb{R}^2} f_\star \log f_\star \, dx - \frac{4\pi}{M} \int_{\mathbb{R}^2} f_\star \, dx.
\]
After an elementary computation, we observe that the above inequality is exactly (2) written for \( f = f_0 \).

The point is now to adapt this strategy to the case with an external potential. This justifies why we have to introduce a nonlinear diffusion equation with a drift. As we shall see below, the method is insensitive to \( \alpha \) and applies when \( \alpha > 1 \) exactly as in the case \( \alpha \in (0, 1) \). A natural question is whether solutions are regular enough to perform the computations below and in particular if they have a sufficient decay at infinity to allow all kinds of integrations by parts needed by the method. The answer is twofold. First, we can take an initial datum \( f_0 \) that is as smooth and decaying as \( |x| \to +\infty \) as needed, prove the inequality and argue by density. Second, integrations by parts can be justified by an approximation scheme consisting in a truncation of the problem in larger and larger balls. We refer to [17] for regularity issues and to [15] for the truncation method. In the proof, we will therefore leave these issues aside, as they are purely technical.

**Proof of Theorem 1.1.** By homogeneity, we can assume that \( M = 1 \) without loss of generality and consider the evolution equation
\[
\frac{\partial f}{\partial t} = \Delta \sqrt{f} + 2\sqrt{\pi} \nabla \cdot (xf).
\]
1) Using simple integrations by parts, we compute
\[
\int_{\mathbb{R}^2} (1 + \log f) \Delta \sqrt{f} \, dx = -8 \int_{\mathbb{R}^2} |\nabla f|^{1/4} |\nabla f|^{3/4} \, dx
\]
and
\[
\int_{\mathbb{R}^2} (1 + \log f) \nabla \cdot (xf) \, dx = -\int_{\mathbb{R}^2} \frac{\nabla f \cdot (xf)}{f} \, dx = -\int_{\mathbb{R}^2} x \cdot \nabla f \, dx = 2 \int_{\mathbb{R}^2} f \, dx = 2.
\]
As a consequence, we obtain that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} f \log f \, dx = -8 \int_{\mathbb{R}^2} |\nabla f|^{1/4} |\nabla f|^{3/4} \, dx + 8\pi \int_{\mathbb{R}^2} \mu^{3/2} \, dx
\]
using
\[
\int_{\mathbb{R}^2} \mu^{3/2} \, dx = \frac{1}{2\sqrt{\pi}}.
\]
2) By elementary considerations again, we find that
\[
4\pi \int_{\mathbb{R}^2} f (\Delta)^{-1} \left( \Delta \sqrt{f} \right) dx = -4\pi \int_{\mathbb{R}^2} f^{3/2} dx
\]
and
\[
4\pi \int_{\mathbb{R}^2} \nabla \cdot (x f) (\Delta)^{-1} f dx = -4\pi \int_{\mathbb{R}^2} x f \cdot \nabla (\Delta)^{-1} f dx
\]
\[
= 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \frac{x - y}{|x - y|^2} dxdy
\]
\[
= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) (x - y) \cdot \frac{x - y}{|x - y|^2} dxdy = 1
\]
where, in the last line, we exchanged the variables \(x\) and \(y\) and took the half sum of the two expressions. This proves that
\[
\frac{d}{dt} \left( 4\pi \int_{\mathbb{R}^2} f (\Delta)^{-1} f dx \right) = -8\pi \int_{\mathbb{R}^2} (f^{3/2} - \mu^{3/2}) dx. \tag{11}
\]

3) We observe that
\[
\mu(x) = \frac{1}{\pi (1 + |x|^2)^2} = e^{-V(x)}
\]
solves
\[
\Delta V = -\Delta \log \mu = 8\pi \mu \tag{12}
\]
and, as a consequence,
\[
\int_{\mathbb{R}^2} \nabla \sqrt{f} dx = \int_{\mathbb{R}^2} \Delta \sqrt{f} dx = 8\pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx.
\]
Since
\[
2\sqrt{\pi} \int_{\mathbb{R}^2} \nabla \cdot (x f) dx = -2\sqrt{\pi} \int_{\mathbb{R}^2} f x \cdot \nabla f dx = -8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{|x|^2}{1 + |x|^2} f dx
\]
\[
= -8\sqrt{\pi} + 8\sqrt{\pi} \int_{\mathbb{R}^2} \frac{f}{1 + |x|^2} dx = -8\sqrt{\pi} + 8\pi \int_{\mathbb{R}^2} \mu \sqrt{f} dx,
\]
we conclude that
\[
\frac{d}{dt} \int_{\mathbb{R}^2} f \sqrt{V} dx = 8\pi \int_{\mathbb{R}^2} \left( \mu \sqrt{f} + \sqrt{\mu} f - 2\mu^{3/2} \right) dx. \tag{13}
\]
Let us define
\[
\mathcal{F}[f] := \int_{\mathbb{R}^2} f \log f dx + \alpha \int_{\mathbb{R}^2} V f dx + (1 - \alpha) \left( 1 + \log \pi \right) + 2(1 - \alpha) \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) f(y) \log |x - y| dxdy.
\]
Collecting (10), (11) and (13), we find that
\[
\frac{d}{dt} \mathcal{F}[f(t, \cdot)] = -8 \int_{\mathbb{R}^2} \nabla f^{1/4} dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx - 8\pi \alpha \int_{\mathbb{R}^2} \left( f^{3/2} - \mu \sqrt{f - \sqrt{\mu} f + \mu^{3/2}} \right) dx.
\]
Notice that
\[
\int_{\mathbb{R}^2} \left( f^{3/2} - \mu \sqrt{f - \sqrt{\mu} f + \mu^{3/2}} \right) dx = \int_{\mathbb{R}^2} \varphi \left( \frac{f}{\mu} \right) \mu^{3/2} dx \quad \text{with} \quad \varphi(t) := t^{3/2} - t - \sqrt{t} + 1
\]
and that \( \varphi \) is a strictly convex function on \( \mathbb{R}^+ \) such that \( \varphi(1) = \varphi'(1) = 0 \), so that \( \varphi \) is nonnegative. On the other hand, by (9), we know that

\[
\int_{\mathbb{R}^2} |\nabla f|^{1/4} dx - \pi \int_{\mathbb{R}^2} f^{3/2} dx \geq 0
\]
as in the proof of [7]. Altogether, this proves that \( t \mapsto \mathcal{F}[f(t, \cdot)] \) is monotone nonincreasing. Hence

\[
\mathcal{F}[f_0] \geq \mathcal{F}[f(t, \cdot)] \geq \lim_{t \to +\infty} \mathcal{F}[f(t, \cdot)] = \mathcal{F}[f_*] = 0.
\]
This completes the proof of (1).

3 Consequences

3.1 Proof of Corollary 1.2

To prove the result of Corollary 1.2, we have to establish first that the free energy functional \( \mathcal{F}_\beta \) is bounded from below. Instead of using standard variational methods to prove that a minimizer is achieved, we can rely on the flow associated with (5)-(6).

- **Repulsive case.** Let us consider the free energy functional defined in (7) where \( \phi \) is given by (6) with \( \varepsilon = +1 \), i.e.,

\[
\phi = -\frac{1}{\pi} \log |\cdot| * f.
\]

**Lemma 3.1.** Let \( M > 0 \) and \( \varepsilon = +1 \). Then \( \mathcal{F}_\beta \) is bounded from below on the set of the functions \( f \in L^1(R^2) \) such that \( \int_{\mathbb{R}^2} f dx = M \) if \( \beta \geq 1 + \frac{M}{8\pi} \).

**Proof.** With \( g = \frac{f}{M} \) and \( \alpha = 1 + \frac{M}{8\pi} \), this means that

\[
\frac{1}{M} \mathcal{F}_\beta[f] - \log M = \int_{\mathbb{R}^2} g \log g dx + \beta \int_{\mathbb{R}^2} V g dx - M \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x) g(y) \log |x - y| dx dy
\]

\[
= (\beta - \alpha) \int_{\mathbb{R}^2} V g dx + \int_{\mathbb{R}^2} g \log g dx + \alpha \int_{\mathbb{R}^2} V g dx - 2(\alpha - 1) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} g(x) g(y) \log |x - y| dx dy
\]

\[
\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g dx - (1 - \alpha) (1 + \log \pi)
\]

according to Theorem 1.1; the condition \( \beta \geq \alpha \) is enough to prove that \( \mathcal{F}_\beta[f] \) is bounded from below.

**Proof of Corollary 1.2 with \( \varepsilon = +1 \).** Let us consider a smooth solution of (5)-(6). We refer to [16] for details and to [1] for similar arguments in dimension \( d \geq 3 \). According to (8), \( f \) converges as \( t \to +\infty \) to a solution of

\[
\nabla \log f + \beta \nabla V + \nabla \phi = 0.
\]

Notice that this already proves the existence of a stationary solution. The equation can be solved as

\[
f = M \frac{e^{-\beta V - \phi}}{\int_{\mathbb{R}^2} e^{-\beta V - \phi} dx}
\]
after taking into account the conservation of the mass. With (6), the problem is reduced to solving

\[
-\Delta \psi = M \left( \frac{e^{-\gamma V - \psi}}{\int_{\mathbb{R}^2} e^{-\gamma V - \psi} dx} - \mu \right), \quad \psi = (\beta - \gamma) V + \phi, \quad \gamma = \beta - \frac{M}{8\pi}
\]
using (12). It is a critical point of the functional \( \psi \mapsto \mathcal{J}_{M,Y}[\psi] := \frac{1}{2} \int_{\mathbb{R}^2} |V\psi|^2 \, dx + M \int_{\mathbb{R}^2} \psi \mu \, dx + M \log \left( \int_{\mathbb{R}^2} e^{-\gamma V - \psi} \, dx \right) \). Such a functional is strictly convex as, for instance, in [10, 11]. We conclude that \( \psi \) is up to an additional constant.

\* **Attractive case.** Let us consider the free energy functional (7) \( \mathcal{F}_\beta \) where \( \phi \) is given by (6) with \( \varepsilon = -1 \), i.e., \( \phi = \frac{1}{\varepsilon} \log |x| + f \). Inspired by [12], we have the following estimate.

**Lemma 3.2.** Let \( \varepsilon = -1 \). Then \( \mathcal{F}_\beta \) is bounded from below on the set of the functions \( f \in L^1_\varepsilon(\mathbb{R}^2) \) such that \( \int_{\mathbb{R}^2} f \, dx = M \) if \( M \leq 8\pi \) and \( \beta \geq 1 - \frac{M}{8\pi} \). It is not bounded from below if \( M > 8\pi \).

**Proof.** With \( g = \frac{f}{M} \) and \( \alpha = 1 - \frac{M}{8\pi} \), Theorem 1.1 applied to

\[
\frac{1}{M} \mathcal{F}_\beta[f] - \log M
= (\beta - \alpha) \int_{\mathbb{R}^2} V g \, dx + \int_{\mathbb{R}^2} g \log g \, dx + \alpha \int_{\mathbb{R}^2} V g \, dx + 2(1 - \alpha) \int_{\mathbb{R}^2 \times \mathbb{R}^2} g(x) g(y) \log |x - y| \, dx \, dy
\geq (\beta - \alpha) \int_{\mathbb{R}^2} V g \, dx - (1 - \alpha) (1 + \log \pi)
\]

proves that the free energy is bounded from below if \( M \leq 8\pi \) and \( \beta \geq \alpha \). On the other hand, if \( f_\lambda(x) := \lambda^{-2} f(\lambda^{-1} x) \) and \( M > 8\pi \), then

\[
\mathcal{F}_\beta[f_\lambda] \sim 2 M \left( \frac{M}{8\pi} - 1 \right) \log \lambda \to -\infty \quad \text{as} \quad \lambda \to 0_+,
\]

which proves that \( \mathcal{F}_\beta \) is not bounded from below.

**Proof of Corollary 1.2 with \( \varepsilon = -1 \).** The proof goes as in the case \( \beta = 0 \). We refer to [4] and leave details to the reader.

**Remark 3.3.** Let us notice that \( \mathcal{F}_\beta \) is unbounded from below if \( \beta < 0 \). This follows from the observation that \( \lim_{y \to -\infty} \mathcal{F}_\beta(f_y) = -\infty \) where \( f_y(x) = f(x + y) \) for any admissible \( f \).

### 3.2 Duality

When \( \alpha > 1 \), we can write a first inequality by considering the repulsive case in the proof of Corollary 1.2 and observing that

\[\mathcal{J}_{M,Y}[\psi] \geq \min \mathcal{J}_{M,Y}\]

where \( \psi \in W^{2,1}_{\text{loc}}(\mathbb{R}^2) \) is such that \( \int_{\mathbb{R}^2} (\Delta \psi) \, dx = 0 \) and the minimum is taken on the same set of functions.

When \( \alpha \in [0, 1) \), it is possible to argue by duality as in [5, Section 2]. Since \( f_* \) realizes the equality case in (1), we know that

\[
\int_{\mathbb{R}^2} f_* \log \left( \frac{f_*}{M} \right) \, dx + \alpha \int_{\mathbb{R}^2} V f_* \, dx + M (1 - \alpha) (1 + \log \pi)
= \frac{2}{M} (\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} f_* (x) f_* (y) \log |x - y| \, dx \, dy
\]

and, using the fact that \( f_* \) is a critical point of the difference of the two sides of (1), we also have that

\[
\int_{\mathbb{R}^2} \log \left( \frac{f}{f_*} \right) (f - f_*) \, dx + \alpha \int_{\mathbb{R}^2} V (f - f_*) \, dx
= \frac{4}{M} (\alpha - 1) \int_{\mathbb{R}^2 \times \mathbb{R}^2} (f(x) - f_*(x)) f_*(y) \log |x - y| \, dx \, dy.
\]
By subtracting the first identity to (1) and adding the second identity, we can rephrase (1) as
\[
\mathcal{F}_{(1)}[f] := \int_{\mathbb{R}^2} f \log \left( \frac{f}{\alpha} \right) d\chi \geq \frac{4\pi}{M} (1 - \alpha) \int_{\mathbb{R}^2} \frac{f - f_*}{(f - f_*)^2} d\chi := \mathcal{F}_{(2)}[f].
\]
Let us consider the Legendre transform
\[
\mathcal{F}^*_\{\alpha\}[g] := \sup_f \left( \int_{\mathbb{R}^2} g f d\chi - \mathcal{F}_{\{\alpha\}}[f] \right)
\]
where the supremum is restricted to the set of the functions \( f \in L^1(\mathbb{R}^2) \) such that \( M = \int_{\mathbb{R}^2} f d\chi \). After taking into account the Lagrange multipliers associated with the mass constraint, we obtain that
\[
M \log \left( \int_{\mathbb{R}^2} e^{g - V} d\chi \right) = \mathcal{F}^*_{\{1\}}[g] \leq \frac{M}{16\pi(1 - \alpha)} \int_{\mathbb{R}^2} |\nabla g|^2 d\chi + M \int_{\mathbb{R}^2} e^{-V} d\chi = \mathcal{F}^*_{\{2\}}[g].
\]
We can get rid of \( M \) by homogeneity and recover the standard Euclidean form of the Onofri inequality in the limit case as \( \alpha \to 0^+ \), which is clearly the sharpest one for all possible \( \alpha \in [0, 1) \).

3.3 Extension to general confining potentials with critical asymptotic growth

As a concluding observation, let us consider a general potential \( W \) on \( \mathbb{R}^2 \) such that
\[
W \in C(\mathbb{R}^2) \quad \text{and} \quad \lim_{|x| \to \infty} \frac{W(x)}{V(x)} = \beta
\]
and the associated free energy functional
\[
\mathcal{F}_{\beta, W}[f] := \int_{\mathbb{R}^2} f \log f d\chi + \beta \int_{\mathbb{R}^2} W f d\chi + \frac{1}{2} \int_{\mathbb{R}^2} \phi f d\chi
\]
where \( \phi \) is given in terms of \( f > 0 \) by (6). With previous notations, \( \mathcal{F}_{\beta, V} = \mathcal{F}_{\beta, W} \). Our last result is that the asymptotic behaviour obtained from \( (\mathcal{H}_W) \) is enough to decide whether \( \mathcal{F}_{\beta, W} \) is bounded from below or not. The precise result goes as follows.

**Theorem 3.4.** Under Assumption \( (\mathcal{H}_W) \), \( \mathcal{F}_{\beta, W} \) defined as above is bounded from below if either \( \epsilon = +1 \) and \( \beta > 1 + \frac{M}{8\pi} \), or \( \epsilon = -1 \), \( \beta > 1 - \frac{M}{8\pi} \) and \( M \leq 8\pi \). The result is also true in the limit case if \( (W - \beta V) \in L^\infty(\mathbb{R}^2) \) and either \( \epsilon = +1 \) and \( \beta = 1 + \frac{M}{8\pi} \), or \( \epsilon = -1 \), \( \beta = 1 - \frac{M}{8\pi} \) and \( M \leq 8\pi \).

**Proof.** If \( (W - \beta V) \in L^\infty(\mathbb{R}^2) \), we can write that
\[
\mathcal{F}_{\beta, W}[f] \geq \mathcal{F}_{\beta}[f] - M \|W - \beta V\|_{L^\infty(\mathbb{R}^2)}.
\]
This completes the proof in the limit case. Otherwise, we redo the argument using \( \beta V - (\beta V - W)_+ \) for some \( \beta \in (0, \beta) \) if \( \epsilon = -1 \), and for some \( \beta \in (1 + \frac{M}{8\pi}, \beta) \) if \( \epsilon = +1 \).

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References


