Partial differential equations — *A variational proof of Nash's inequality*[*] , by EMERIC BOUIN, JEAN DOLBEAULT AND CHRISTIAN SCHMEISER, January 10, 2019

ABSTRACT. — This paper is intended to give a characterization of the optimality case in Nash's inequality, based on methods of nonlinear analysis for elliptic equations and techniques of the calculus of variations. By embedding the problem into a family of Gagliardo-Nirenberg inequalities, this approach reveals why optimal functions have compact support and also why optimal constants are determined by a simple spectral problem.
KEY WORDS: Nash inequality; interpolation; semi-linear elliptic equations; compactness; compact support; Neumann homogeneous boundary conditions; Laplacian; radial symmetry

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1. INTRODUCTION AND MAIN RESULT

Nash's inequality [17] states that, for any $u\in \mathrm{H}^1(\mathbb{R}^d)$, $d\ge 1$,

$$\|u\|_2^{2+\frac{4}{d}}\le \mathscr{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2\,, \tag{1}$$

where we use the notation $\|v\|_q=\left(\int_{\mathbb{R}^d}|v|^q\,dx\right)^{1/q}$ for any $q\ge 1$. The optimal constant $\mathscr{C}_{\mathrm{Nash}}$ in (1) has been determined by E. Carlen and M. Loss. To state their result, let us introduce ω_d, the volume of the unit ball B_1 in $\mathbb{R}^d$, λ_1, the principal eigenvalue of the Laplacian with homogeneous Neumann boundary conditions, and $x\mapsto\varphi_1(|x|)$ an eigenfunction associated with λ_1, normalized by $\varphi_1(1)=1$.

THEOREM 1.1 ([9]). *Inequality* (1) *holds with optimal constant*

$$\mathscr{C}_{\mathrm{Nash}}=\frac{(d+2)^{1+\frac{2}{d}}}{d\,\lambda_1\,(2\,\omega_d)^{\frac{2}{d}}} \tag{2}$$

Moreover, there is equality in (1) *if and only if, up to translation and scaling of x as well as multiplication of u by a constant,*

$$\overline{u}(x):=\begin{cases}1-\varphi_1(|x|) & \text{for } |x|\le 1\,,\\ 0 & \text{for } |x|>1\,.\end{cases}$$

The *compactness of the support* of the optimizers in (1) can be understood by deriving Nash's inequality as a limiting case of a family of *Gagliardo-Nirenberg inequalities* [14, 18]

$$\|\nabla u\|_2^{\frac{2a}{a+b}}\,\|u\|_p^{\frac{2b}{a+b}}\ge\mathscr{C}_{\mathrm{GN}}(p)\,\|u\|_2^2\,, \tag{3}$$

for all $u\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^d)$, where $1<p<2$, $a=a(p)=d\,(2-p)$, and $b=b(p)=2\,p$. Nash's inequality corresponds to the limit case as $p\to 1$.

The inequality (3) is equivalent to the minimization of $u\mapsto\|\nabla u\|_2^2+\|u\|_p^2$ under the constraint that $\|u\|_2^2$ is a given positive number: any minimizer solves, up to a scaling and a multiplication by a constant, the Euler-Lagrange equation

$$-\Delta u=u-|u|^{p-2}\,u\,. \tag{4}$$

[*] *Dedicated to the memory of Pr. Emilio Gagliardo whose results on functional inequalities have been a constant source of inspiration.*

In our main result, which follows, we use the notation of Theorem 1.1.

THEOREM 1.2. *For any $p \in (1,2)$, equality in (3), written with an optimal constant, is obtained after a possible translation and scaling of x, and multiplication by a constant, by the nonnegative radial solution u_p of (4). The support of u_p is a ball of radius $R_p > 0$, such that $\lim_{p\to 1_+} R_p = R_1 := \sqrt{\lambda_1}$, and u_p converges, as $p \to 1_+$, to $u_1 = \overline{u}(\cdot/R_1)$ in $\mathrm{H}^1 \cap \mathrm{L}^1(\mathbb{R}^d)$.*

Our result is in the spirit of [9]. In their paper, E. Carlen and M. Loss establish the optimality case by direct estimates as chain of inequalities which become equalities in the case of the optimal functions. Our contribution is to establish first why the problem can be reduced to a problem on a ball involving only radial functions, using the theory of nonlinear elliptic PDEs and methods of the calculus of variations. Identifying the optimal case is then an issue of spectral theory.

Let us highlight a little bit why we insist on the compactness of the support of the optimal function. The first application of Nash's inequality by Nash himself in [17] is the computation of the decay rate for parabolic equations. We consider a solution u of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u\,, \tag{5}$$

with initial datum $u_0 \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathbb{R}^d)$. If u_0 is nonnegative, so does $u(t,\cdot)$, and mass is conserved : $\|u(t,\cdot)\|_1 = \|u_0\|_1$, for any $t \ge 0$. For a general initial datum, $\|u(t,\cdot)\|_1 \le \|u_0\|_1$ for any $t \ge 0$. By the estimate

$$\frac{d}{dt}\,\|u(t,\cdot)\|_2^2 = -2\,\|\nabla u(t,\cdot)\|_2^2$$

and Nash's inequality (1), $y(t) := \|u(t,\cdot)\|_2^2$ can be estimated by

$$y' \le -2\,\mathscr{C}_{\mathrm{Nash}}^{-1}\,\|u_0\|_1^{-\frac{4}{d}}\;y^{1+\frac{2}{d}}\,,$$

which, after integration, yields that the solution u of (5) satisfies the estimate

$$\|u(t,\cdot)\|_2 \le \left(\|u_0\|_2^{-\frac{4}{d}} + \frac{4}{d}\,\mathscr{C}_{\mathrm{Nash}}^{-1}\,\|u_0\|_1^{-\frac{4}{d}}\;t\right)^{-\frac{d}{4}}\,, \tag{6}$$

for all $t \ge 0$. This estimate is optimal in the following sense: if we take u_0 to be an optimal function in (1) and differentiate (6) at $t = 0$, it is clear that $\mathscr{C}_{\mathrm{Nash}}$ is the best possible constant in the decay estimate (6).

With this result at hand, it comes a little bit as a surprise that the optimal function has nothing to do with the heat kernel

$$G(t,x) = \frac{1}{(4\pi\,t)^{\frac{d}{2}}}\,\exp\left(-\,\frac{|x|^2}{4\,t}\right)$$

and is even *compactly supported*. This can be explained by noting that (6) is an optimal result for small times, but it can be improved concerning the long time behavior. Estimation of the solution $u(t,\cdot) = G(t,\cdot) * u_0$ of the heat equation by Young's convolution inequality gives

$$\|u(t,\cdot)\|_2 \le \|G(t,\cdot)\|_2 \, \|u_0\|_1 = (8\pi\, t)^{-\frac{d}{4}} \, \|u_0\|_1 \,. \tag{7}$$

The sharpness of this result (take $u_0 = G(\varepsilon,\cdot)$ for an arbitrary small $\varepsilon > 0$) and a comparison with (6) imply

$$8\pi > \frac{4}{d\,\mathscr{C}_{\mathrm{Nash}}}\,. \tag{8}$$

An estimate, which is optimal for both small and large times, is now obviously obtained by taking the minimum of the right hand sides of (6) and of (7). The reader interested in further consideration on Nash's inequality and the heat kernel is invited to refer to [12].

This paper is organized as follows. Theorem 1.1 can be seen as a consequence of Theorem 1.2: see Section 2. Section 3 is devoted to the proof of Theorem 1.2. In Section 4 we adopt a broader perspective and sketch the proof by E. Carlen and M. Loss along with a review of several other methods for proving Nash's inequality.

2. Theorem 1.2 implies Theorem 1.1

The optimal constant $\mathscr{C}_{\mathrm{GN}}(p)$ in (3) is obtained by minimizing the quotient

$$\mathscr{Q}_p[u] := \frac{\|\nabla u\|_2^{\frac{2a}{a+b}} \, \|u\|_p^{\frac{2b}{a+b}}}{\|u\|_2^2}\,.$$

Except when $d = 1$, no explicit expression of $\mathscr{C}_{\mathrm{GN}}(p)$ is available to our knowledge. Theorem 1.2 implies

$$\mathscr{Q}_1[u_1] = \lim_{p\to 1_+} \mathscr{Q}_p[u_p] = \big(\mathscr{C}_{\mathrm{Nash}}\big)^{\frac{d}{d+2}}\,.$$

As a consequence, the support of u_1 is the ball B_{R_1} and on $\partial B_{R_1} \ni x$, $u_1(x) = 0$ and $x\cdot\nabla u_1(x) = 0$. We also deduce from Theorem 1.2 that u_p uniformly converges to u_1 using standard elliptic results of [15] or elementary ODE estimates. Moreover, since $\lim_{p\to 1_+} u_p^{p-1} = 1$, the optimal function u_1 solves the Euler-Lagrange equation

$$-\Delta u_1 = u_1 - 1\,.$$

This means that $v_1 := 1 - u_1(R_1\,\cdot)$ solves

$$\begin{cases} -\Delta v_1 = \lambda_1\, v_1\,, & x \in B_1\,,\\ v_1 = 1\,, \quad x\cdot\nabla v_1 = 0\,, & x\in\partial B_1\,, \end{cases}$$

with $\lambda_1 = R_1^2$. This implies $v_1 = \varphi_1(|\cdot|)$. We recall that λ_1 is defined by

$$\lambda_1 := \inf \int_{B_1} |\nabla\varphi|^2\,dx$$

where the infimum is taken on $\left\{\varphi \in \mathrm{H}^1(B_1) : \int_{B_1} \varphi\,dx = 0 \text{ and } \int_{B_1} |\varphi|^2\,dx = 1\right\}$.

As observed in [5], λ_1 can be computed in terms of the smallest positive zero $z_{d/2}$ of the Bessel function of the first kind $J_{d/2}$ using, for instance, [11, page 492, Chapter VII, Section 8]. Indeed, solving the radial eigenvalue problem

$$\begin{cases} \varphi_1'' + \frac{d-1}{r}\,\varphi_1' + \lambda_1\,\varphi_1 = 0\,, \\ \varphi_1(1) = 1\,, \quad \varphi_1'(1) = 0\,, \quad \varphi_1'(0) = 0, \end{cases}$$

is equivalent to finding the function J such that $\varphi_1 := r^{-\alpha}\,J_\alpha\left(\sqrt{\lambda_1}\,\cdot\right)$ with $\alpha = (d-2)/2$ that solves

$$J_\alpha'' + \frac{1}{r}\,J_\alpha' + \left(1 - \frac{\alpha^2}{r^2}\right) J_\alpha = 0\,.$$

Hence J_α is a Bessel function of the first kind and the boundary condition $\varphi_1'(1) = 0$ is transformed into $\sqrt{\lambda_1}\,J_\alpha'\left(\sqrt{\lambda_1}\right) - \alpha\,J_\alpha\left(\sqrt{\lambda_1}\right) = 0$. With the property

$$z\,J_\alpha'(z) - \alpha\,J_\alpha(z) + J_{\alpha+1}(z) = 0$$

of Bessel functions of the first kind (see, e.g., [27]), we obtain $\lambda_1 = z_{\alpha+1}^2 = z_{d/2}^2$.

Looking at the Euler-Lagrange equation was precisely the way how E. Carlen and M. Loss realized that the optimizers must have compact support, [8]. This information heavily influenced the shape of the proof in [9]. The observations of this section point in the very same direction. For $p \in (1,2)$, the *compact support principle* gives a very simple intuition of why the optimal function for (3) has compact support, and then it is natural to take the limit in the Euler-Lagrange equation (4). As we shall see in the next section, symmetrization can be avoided and replaced by the moving plane method, which also puts the focus on (4).

3. Proof of Theorem 1.2

For readability, we divide the proof into a list of simple statements. Some of these statements are classical and are not fully detailed. Our goal here is to prove that the optimal function u_p is supported in a ball with finite radius and investigate the limit of u_p as $p \to 1_+$.

1. *The non-optimal inequality and the optimal constant.* It is elementary to prove that (1) holds for some positive constant $\mathscr{C}_{\mathrm{Nash}}$ and without loss of generality, we can consider the best possible one. See Section 4 for various proofs of such a statement.

2. *Scalings and Gagliardo-Nirenberg inequalities.* Inequality (3) is equivalent to

$$\|\nabla u\|_2^2+\lambda\,\|u\|_p^2\ge\mathscr{K}_{\rm GN}(p,\lambda)\,\|u\|_2^2\,,\tag{9}$$

for all $u\in\mathrm H^1(\mathbb R^d)\cap\mathrm L^p(\mathbb R^d)$. A scaling argument relates $\mathscr{K}_{\rm GN}(p,\lambda)$ and $\mathscr{C}_{\rm GN}(p)$. Indeed, take any $u\in\mathrm H^1(\mathbb R^d)\cap\mathrm L^p(\mathbb R^d)$, then so does $u_\sigma=u(\sigma\cdot)$ for any $\sigma>0$, and we have

$$\mathscr{K}_{\rm GN}(p,\lambda)\le\frac{\|\nabla u_\sigma\|_2^2+\lambda\,\|u_\sigma\|_p^2}{\|u_\sigma\|_2^2}=\frac{\sigma^2\,\|\nabla u\|_2^2+\lambda\,\|u\|_p^2\,\sigma^{-2\frac ab}}{\|u\|_2^2}\,.$$

where $a=d\,(2-p)$, and $b=2\,p$ as in (3). From this, we may deduce two things. First, choosing $\sigma^2=\lambda^{\frac{b}{a+b}}$ and optimising on u yields

$$\mathscr{K}_{\rm GN}(p,\lambda)=\mathscr{K}_{\rm GN}(p,1)\,\lambda^{\frac{b}{a+b}}\,.$$

Second, an optimization on $\sigma>0$ shows that

$$\mathscr{K}_{\rm GN}(p,\lambda)\le\frac{a+b}{a^{\frac{a}{a+b}}\,b^{\frac{b}{a+b}}}\,\lambda^{\frac{b}{a+b}}\,\mathscr{Q}_p[u]$$

so that optimizing on u yields

$$\mathscr{K}_{\rm GN}(p,\lambda)=\frac{a+b}{a^{\frac{a}{a+b}}\,b^{\frac{b}{a+b}}}\,\lambda^{\frac{b}{a+b}}\,\mathscr{C}_{\rm GN}(p)\,.\tag{10}$$

Throughout this paper we assume that $\mathscr{C}_{\rm GN}(p)$ and $\mathscr{K}_{\rm GN}(p,\lambda)$ are the optimal constants respectively in (3) and (9).

3. *Comparison of the optimal constants.* By taking the limit in $\mathscr{Q}_p[u]$ as $p\to1_+$ for an arbitrary smooth function u and arguing by density, one gets that $p\mapsto\mathscr{C}_{\rm GN}(p)$ is lower semi-continuous and

$$\big(\mathscr{C}_{\rm Nash}\big)^{\frac{d}{d+2}}=\mathscr{C}_{\rm GN}(1)\le\lim_{p\to1_+}\mathscr{C}_{\rm GN}(p)\,.$$

On the other hand, since $\|u\|_p^p\le\|u\|_1^{2-p}\,\|u\|_2^{2\,(p-1)}$ by Hölder's inequality, we find that

$$\mathscr{C}_{\rm GN}(p)\le\big(\mathscr{C}_{\rm Nash}\big)^{\frac{a(p)}{a(p)+b(p)}}\quad\xrightarrow{p\to1_+}\quad\big(\mathscr{C}_{\rm Nash}\big)^{\frac{d}{d+2}}\,,$$

from which we conclude that

$$\big(\mathscr{C}_{\rm Nash}\big)^{\frac{d}{d+2}}=\lim_{p\to1_+}\mathscr{C}_{\rm GN}(p)\,.$$

4. *Nonnegative optimal functions.* We look for optimal functions in (3) by minimizing $\mathscr{Q}_p$. The existence of a minimizer is a classical result in the calculus of variations: see for instance [7]. Without loss of generality, we can consider only nonnegative solutions of (4) because $\mathscr{Q}_p[u]=\mathscr{Q}_p\big[|u|\big]$.

5. *Support and regularity.* According to [10], as a special case of the *compact support principle* (see [20, 21]), nonnegative solutions of (4) have compact support. By convexity of the function $t \mapsto t^{2/p}$, we can consider solutions which have only one connected component in their support. For each p, we can pick one such solution and denote it by u_p. The standard elliptic theory (see for instance [15]) shows that the solution is continuous and smooth in the interior of its support, which is a connected, closed set in $\mathbb{R}^d$.

6. *Symmetry by moving planes.* According to [10], the solution u_p is radially symmetric and supported in a ball of radius R_p. The proof relies on moving plane techniques and applies to nonlinearities such as $u \mapsto u - u^{p-1}$ with $p \in (1,2)$. Up to a translation we can assume that the support is a centered ball. The solution is unique according to [10, Theorem 1.1] and [19, Theorem 3].

7. *Limit as $p \to 1_+$.* This is the point of the proof which requires some care. As $p \to 1_+$, u_p converges to a nonnegative solution u_1 of

$$\begin{cases} -\Delta u_1 = u_1 - 1 & \text{on} \quad B_{R_1}\,, \\ \frac{x}{|x|} \cdot \nabla u_1 = 0\,,\ u_1 = 0 & \text{on} \quad \partial B_{R_1}\,. \end{cases} \tag{11}$$

Let us give some details. For any $p \in (1,2)$, the solution u_p of (4) satisfies the two identities

$$\|\nabla u_p\|_2^2 + \|u_p\|_p^p = \|u_p\|_2^2\,, \quad \frac{d-2}{2\,d}\,\|\nabla u_p\|_2^2 + \frac{1}{p}\,\|u_p\|_p^p = \frac{1}{2}\,\|u_p\|_2^2\,,$$

obtained by testing (4) respectively by u_p and by $x \cdot \nabla u_p$ (Pohozaev's method). The first identity is rewritten as $\|\nabla u_p\|_2^2 + \lambda_p\,\|u_p\|_p^2 = \|u_p\|_2^2$ with $\lambda_p := \|u_p\|_p^{p-2}$. As a consequence, we have that

$$1 = \mathscr{K}_{\rm GN}(p, \lambda_p) = \frac{a+b}{a^{\frac{a}{a+b}}\,b^{\frac{b}{a+b}}}\,\lambda_p^{\frac{b}{a+b}}\,\mathscr{C}_{\rm GN}(p)$$

because of (10). As a consequence we have that $\lim_{p\to 1_+} \lambda_p = \lambda_1$. The two identities and the definition of λ_p provide an explicit expression of $\|\nabla u_p\|_2$, $\|u_p\|_p$ and $\|u_p\|_2$ in terms of λ_p. This proves the strong convergence of u_p to some u_1 in $\mathrm{H}^1 \cap \mathrm{L}^1(\mathbb{R}^d)$ and, as a consequence, this proves that equality in (1) is achieved by u_1 which solves (11).

8. *Convergence of the support.* By adapting the results of [4, 13], it is possible to prove that $\lim_{p\to 1_+} R_p =: R_1 \in (0,\infty)$. Let us give an elementary argument. The dynamical system

$$U' = V\,, \quad V' = 1 - U - \frac{d-1}{r}\,V$$

has smooth solutions such that, if initial data coincide, then $U(r) = u_1(r)$ and $V = u_1'(r)$ on the support of u_1. As a limit of u_p, we know that u_1 is nonnegative. Since the sequence $(r_n)_{n\in\mathbb{N}}$ of the zeros of V, with the convention $r_0 = 0$, is such that the sequence $\big((-1)^n\,U(r_n)\big)_{n\in\mathbb{N}}$ is monotone decaying, u_1 has compact support and $R_1 = r_1$. This completes the proof of Theorem 1.2. □

4. Other proofs of Nash's inequalities

This section is devoted to a brief review of various methods that have been used to derive Nash's inequality and estimate the optimal constant $\mathscr{C}_{\rm Nash}$. The method of E. Carlen and M. Loss is the only one which provides the optimal value of $\mathscr{C}_{\rm Nash}$. A comparison of the various results is summarized in Fig. 1.

4.1. Interpolation with Sobolev's inequality

In dimension $d \geq 3$, it follows from Hölder's inequality that

$$\|u\|_2^2 \leq \|u\|_1^{\frac{4}{d+2}}\,\|u\|_{2^*}^{\frac{2d}{d+2}}$$

with $2^* = \frac{2d}{d-2}$, and from *Sobolev's inequality* that

$$\|u\|_{2^*}^2 \leq \mathsf{S}_d\,\|\nabla u\|_2^2$$

where

$$\mathsf{S}_d = \frac{1}{d\,(d-2)\,\pi}\left(\frac{\Gamma(d)}{\Gamma(d/2)}\right)^{\frac{2}{d}}$$

is the optimal constant in Sobolev's inequality [1, 22, 24], so that $\mathscr{C}_{\rm Nash} \leq \mathsf{S}_d$.

4.2. Nash's inequality and the logarithmic Sobolev inequality are equivalent

In the interpolation strategy, we can replace Sobolev's inequality by the *logarithmic Sobolev inequality*. The equivalence of Nash's inequality and the logarithmic Sobolev inequality is a well known fact, in which optimality of the constants is not preserved: see for instance [2]. In [5], W. Beckner made the following observation: Jensen's inequality applied to the convex function $\sigma(u) := \log(1/u)$ and the probability measure $\|u\|_2^{-2}\,|u|^2\,dx$ shows that

$$\log\left(\frac{\|u\|_2^2}{\|u\|_1}\right) = \sigma\left(\frac{\|u\|_1}{\|u\|_2^2}\right) = \sigma\left(\int_{\mathbb{R}^d}\frac{1}{|u|}\,\frac{|u|^2\,dx}{\|u\|_2^2}\right) \leq \int_{\mathbb{R}^d}\sigma\left(\frac{1}{|u|}\right)\frac{|u|^2\,dx}{\|u\|_2^2} = \int_{\mathbb{R}^d}\log|u|\,\frac{|u|^2\,dx}{\|u\|_2^2}$$

and can be combined with the logarithmic Sobolev inequality in scale invariant form of [23, 28], that is,

$$\int_{\mathbb{R}^d}\log|u|\,\frac{|u|^2\,dx}{\|u\|_2^2} \leq \log\|u\|_2 + \frac{d}{4}\log\left(\frac{2}{\pi\,d\,e}\,\frac{\|\nabla u\|_2^2}{\|u\|_2^2}\right)$$

to prove (1) with the estimate

$$\mathscr{C}_{\rm Nash} \leq \frac{2}{\pi\,d\,e} := \mathscr{C}_1(d)\,.$$

According to [5], this estimate is asymptotically sharp as $d\to\infty$, which can be verified by using classical results on the zeroes of Bessel functions [27]. Notice that *an information-theoretic proof of Nash's inequality* based on Costa's method of entropy powers can be used to directly prove this inequality (see for instance [26], with application to Nash's inequality in [25, Section 4]).

Reciprocally, let us consider Hölder's inequality $\|u\|_q \le \|u\|_p^\alpha \|u\|_s^{1-\alpha}$ with $\alpha = \frac{p}{q}\,\frac{s-q}{s-p}$, $p\le q\le s$, and let us take the logarithm of both sides. Then we obtain

$$\log\left(\frac{\|u\|_q}{\|u\|_p}\right) + (\alpha-1)\,\log\left(\frac{\|u\|_p}{\|u\|_s}\right) \le 0\,.$$

This inequality becomes an equality when $q=p$. We may differentiate it with respect to q at $q=p$ and

$$\int_{\mathbb{R}^d} u^p \log\left(\frac{|u|}{\|u\|_p}\right) dx \le \frac{s}{s-p}\,\|u\|_p \log\left(\frac{\|u\|_s}{\|u\|_s}\right). \tag{12}$$

immediately follows.

The equivalence of Nash's inequality and the logarithmic Sobolev inequality (although with non-optimal constants) is a well known fact that has been exploited in various related problems. See for instance [6, 3].

4.3. The original proof by J. Nash

It is a very simple argument based on Fourier analysis, due originally to E.M. Stein according to J. Nash himself (see [17, page 935]). Let us denote by $\hat u$ the Fourier transform of u defined as

$$\hat u(\xi) = (2\pi)^{-\frac d2}\int_{\mathbb{R}^d} e^{i\,x\,\xi}\,u(x)\,dx\,,$$

so that by Plancherel's formula

$$\|u\|_2^2 = \int_{\mathbb{R}^d} |\hat u(\xi)|^2\,d\xi \le \|\hat u\|_\infty^2 \int_{|\xi|\le R} d\xi + \frac{1}{R^2}\int_{\mathbb{R}^d} |\xi|^2\,|\hat u(\xi)|^2\,d\xi$$
$$\le (2\pi)^{-d}\,\omega_d\,R^d\,\|u\|_1^2 + \frac{1}{R^2}\,\|\nabla u\|_2^2$$

for any $R>0$. By optimizing the right hand side with respect to R, we obtain the estimate

$$\mathscr{C}_{\rm Nash} \le \frac{1}{4\pi}\left(\frac{d+2}{d}\right)^{1+\frac2d}\Gamma\left(\frac d2\right)^{-\frac2d} =: \mathscr{C}_2(d)\,.$$

4.4. The method of E. Carlen and M. Loss

Without loss of generality, we can assume that the function u is nonnegative. If $u^\star$ denotes the spherically non-increasing rearrangement of a function u, then [16]

$$\left\|u^\star\right\|_q = \|u\|_q \quad\text{and}\quad \left\|\nabla u^\star\right\|_2 \le \|\nabla u\|_2\,,$$

so we can consider nonnegative radial non-increasing functions without loss of generality. For any $R>0$, let

$$u_R := u\,\mathbb{1}_{B_R}\,.$$

We observe that

$$u-u_R \le u(R) \le \overline{u}_R := \frac{\|u_R\|_1}{|B_R|}$$

because u is radial non-increasing, so that

$$\|u-u_R\|_2^2 \le \overline{u}_R\,\|u-u_R\|_1 = \frac{\|u_R\|_1}{R^d\,\omega_d}\,\|u-u_R\|_1\,. \tag{13}$$

On the other hand, using

$$\|u_R\|_2^2 = \left\|u_R-\overline{u}_R\right\|_2^2 + \left\|\overline{u}_R\,\mathbb{1}_{B_R}\right\|_2^2\,,$$

we deduce from the Poincaré inequality

$$\int_{B_R}|v|^2\,dx \le \frac{R^2}{\lambda_1}\int_{B_R}|\nabla v|^2\,dx,$$

for all $v\in\mathrm{H}^1(\mathbb{R}^d)$ such that $\int_{B_R}v\,dx=0$ and from the definition of $\overline{u}_R$ that

$$\|u_R\|_2^2 \le \frac{R^2}{\lambda_1}\,\|\nabla u\|_2^2 + \frac{\|u_R\|_1^2}{R^d\,\omega_d}\,, \tag{14}$$

using $\|\nabla u_R\|_2 \le \|\nabla u\|_2$. By definition of u_R, we also know that

$$\|u\|_2^2 = \|u_R\|_2^2 + \|u-u_R\|_2^2\,.$$

After summing (13) and (14), we arrive at

$$\|u\|_2^2 \le \frac{R^2}{\lambda_1}\,\|\nabla u\|_2^2 + \frac{\|u_R\|_1}{R^d\,\omega_d}\left(\|u_R\|_1 + \|u-u_R\|_1\right)$$

and notice that

$$\|u_R\|_1\left(\|u_R\|_1 + \|u-u_R\|_1\right) = \|u_R\|_1\,\|u\|_1 \le \|u\|_1^2\,. \tag{15}$$

Altogether, this means that

$$\|u\|_2^2 \le \frac{R^2}{\lambda_1}\,\|\nabla u\|_2^2 + \frac{\|u\|_1^2}{R^d\,\omega_d}\,.$$

An optimization on $R>0$ determines a unique optimal value $R=R_\star$ and provides the expression of $\mathscr{C}_{\mathrm{Nash}}$. Equality is achieved by functions meeting equalities in all previous inequalities, that is functions such that $u=u_{R_\star}$ and $u_{R_\star}$ is an optimal

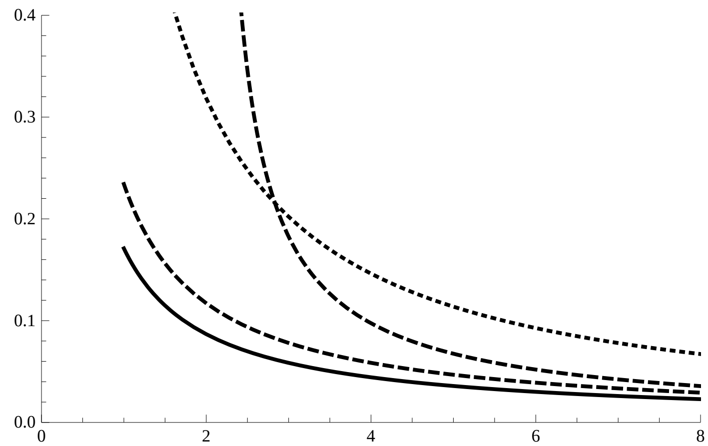

Figure 1: With $d \geq 1$ considered as a real parameter, plots of $d \mapsto S_d$ and $d \mapsto \mathscr{C}_1(d)$ corresponding respectively to an interpolation with Sobolev's inequality (only for $d > 2$) and to an interpolation with the logarithmic Sobolev inequality are shown as dashed curves. The dotted curve is the estimate $d \mapsto \mathscr{C}_2(d)$ of J. Nash in [17]. The optimal value $d \mapsto \mathscr{C}_{\mathrm{Nash}}(d)$ is the plain curve and it is numerically well approximated from below by $d \mapsto 1/(2\pi d)$ deduced from (8).

function for the Poincaré inequality above. This is $u(x) = 1 - \varphi_1(|x|/R_\star)$ on $B_{R_\star}$, extended by $u \equiv 0$ to $\mathbb{R}^d \setminus B_{R_\star}$.

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