Keller-Lieb-Thirring inequalities
for Schrödinger operators on cylinders

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Abstract

This note is devoted to Keller-Lieb-Thirring spectral estimates for Schrödinger operators on infinite cylinders: the absolute value of the ground state level is bounded by a function of a norm of the potential. Optimal potentials with small norms are shown to depend on a single variable: this is a *symmetry result*. The proof is a perturbation argument based on recent rigidity results for nonlinear elliptic equations on cylinders. Conversely, optimal single variable potentials with large norms must be unstable: this provides a *symmetry breaking result*. The optimal threshold between the two regimes is established in the case of the product of a sphere by a line.

Keywords. Cylinder; Schrödinger operator; eigenvalues; Keller-Lieb-Thirring inequalities; optimal constants; symmetry; symmetry breaking

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Inégalités de Keller-Lieb-Thirring pour des opérateurs de Schrödinger sur des cylindres.

Résumé. Cette note est consacrée à des estimations spectrales de Keller-Lieb-Thirring pour des opérateurs de Schrödinger sur des cylindres infinis : la valeur absolue de l'état fondamental est bornée par une fonction d'une norme du potentiel. Il est montré que les potentiels optimaux de petite norme ne dépendent que d'une seule variable : il s'agit d'un *résultat de symétrie*. La preuve provient d'un argument de perturbation qui repose sur des résultats de rigidité récents pour des équations elliptiques non-linéaires sur des cylindres. A l'inverse, les potentiels optimaux de grande norme qui ne dépendent que d'une seule variable sont instables : cela fournit un *résultat de brisure de symétrie*. La valeur optimale qui sépare les deux régimes est établie dans le cas du produit d'une sphère et d'une droite.

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1. Introduction and main results

Let $(\mathfrak{M}, g)$ be a smooth compact connected Riemannian manifold of dimension $d-1$, without boundary. We denote by Ric the Ricci tensor and by $\lambda_1^{\mathfrak{M}}$ the lowest positive eigenvalue of the Laplace-Beltrami operator $-\Delta_g$ on $\mathfrak{M}$. Let us define the minimum of the Ricci curvature of $\mathfrak{M}$ by $\kappa := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-2}} \mathrm{Ric}(\xi, \xi)$ and consider the infinite cylinder $\mathcal{C} := \mathbb{R} \times \mathfrak{M}$. We shall denote by $x = (s, z)$ the variable on $\mathcal{C}$, so that

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the Laplace-Beltrami operator on $\mathcal{C}$ can be written as $-\left(\partial_s^2 + \Delta_g\right)$. For simplicity, we shall assume that $\mathrm{vol}_g(\mathfrak{M}) = 1$, so that $\|V\|_{\mathrm{L}^q(\mathcal{C})} = \|V\|_{\mathrm{L}^q(\mathbb{R})}$ if V is a potential that depends only on s. The goal of this note is to compare

$$\Lambda(\mu) := \sup\left\{\lambda_1^{\mathcal{C}}[V] : V \in \mathrm{L}^q(\mathcal{C})\,,\ \|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu\right\} \text{ and } \Lambda_{\mathbb{R}}(\mu) := \sup\left\{\lambda_1^{\mathbb{R}}[V] : V \in \mathrm{L}^q(\mathbb{R})\,,\ \|V\|_{\mathrm{L}^q(\mathbb{R})} = \mu\right\}$$

where $-\lambda_1^{\mathcal{C}}[V]$ and $-\lambda_1^{\mathbb{R}}[V]$ denote the lowest eigenvalues of the Schrödinger operators $-\partial_s^2 - \Delta_g - V$ and $-\partial_s^2 - V$ respectively on $\mathcal{C}$ and $\mathbb{R}$. What we aim at is a *symmetry result* that allows us to characterize the regime in which optimal potentials depend only on s. The expression of $\Lambda_{\mathbb{R}}(\mu)$ was found by J.B. Keller in [11] and later rediscovered by E.H. Lieb and W. Thirring in [12]. We refer to this result as the *Keller-Lieb-Thirring inequality*, and to [6] for its use in the context of Caffarelli-Kohn-Nirenberg inequalities. Let us define

$$\mu_1 := q\,(q-1)\left(\tfrac{\sqrt{\pi}\,\Gamma(q)}{\Gamma(q+1/2)}\right)^{1/q}\ .$$

Lemma 1 [11,12] *Assume that $q \in (1, +\infty)$. Then we have*

$$\Lambda_{\mathbb{R}}(\mu) = (q-1)^2\left(\mu/\mu_1\right)^{\beta}\quad \forall\,\mu > 0\,,$$

with $\beta = \frac{2\,q}{2\,q-1}$. As a consequence, if V is a nonnegative real valued potential in $\mathrm{L}^q(\mathbb{R})$, then we have

$$\lambda_1^{\mathbb{R}}[V] \leq \Lambda_{\mathbb{R}}(\|V\|_{\mathrm{L}^q(\mathbb{R})})$$

and equality holds if and only if, up to scalings, translations and multiplications by a positive constant,

$$V(s) = \frac{q\,(q-1)}{(\cosh s)^2} =: V_1(s)\quad \forall\,s \in \mathbb{R}$$

where $\|V_1\|_{\mathrm{L}^q(\mathbb{R})} = \mu_1$, $\lambda_1^{\mathbb{R}}[V_1] = (q-1)^2$. Moreover the function $\varphi(s) = (\cosh s)^{1-q}$ generates the corresponding eigenspace.

The classical Keller inequality in $\mathbb{R}^d$ asserts that for all $\gamma \geq 0$ if $d \geq 3$, $\gamma > 0$ if $d = 2$, and $\gamma > 1/2$ if $d = 1$, the lowest negative eigenvalue, $-\lambda_1^{\mathbb{R}^d}[V]$, of the operator $-\Delta - V$ satisfies

$$\lambda_1^{\mathbb{R}^d}[V]^{\gamma} \leq \mathrm{L}_{\gamma,d}^1\,\|V_+\|_{\mathrm{L}^{\gamma+d/2}(\mathbb{R}^d)}^{\gamma+d/2}\quad \forall\,V \in \mathrm{L}^q(\mathbb{R}^d)$$

where

$$\mathrm{L}_{\gamma,d}^1 = \sup\left\{\left(\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 + \|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\right)^{-(\gamma+d/2)} : \|u\|_{\mathrm{L}^p(\mathbb{R}^d)} = 1\right\}$$

is the best constant in the inequality if $q = p/(p-2)$. See [11,12,4] for details. In the case of infinite cylinders, with $q = \gamma + d/2$ and $\mu = \|V_+\|_{\mathrm{L}^q(\mathbb{R}^d)} \to +\infty$, the inequality in $\mathbb{R}^d$ asymptotically determines the *semi-classical regime* for $\Lambda(\mu)$, but another regime appears for cylinders when $\mu > 0$ is not too large, as in the case of compact manifolds. This is the content of our main result, Theorem 2, for which we need one more definition.

Let us introduce

$$\lambda_\theta := \left(1 + \delta\,\theta\,\frac{d-1}{d-2}\right)\kappa + \delta\,(1-\theta)\,\lambda_1^{\mathfrak{M}}\quad \text{with}\quad \delta = \frac{n-d}{(d-1)\,(n-1)}\,,$$

where the dependence on θ will be discussed at the end of this note, and define

$$\lambda_\star := \lambda_{\theta_\star}\quad \text{where}\quad \theta_\star := \frac{(d-2)\,(n-1)\,\big(3\,n+1-d\,(3\,n+5)\big)}{(d+1)\,\big(d\,(n^2-n-4)-n^2+3\,n+2\big)}\,.$$

Theorem 2 *Let $d \geq 2$ and $q \in (d/2, +\infty)$. The function $\mu \mapsto \Lambda(\mu)$ is convex, positive and such that*

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$$\Lambda(\mu)^{q-d/2} \sim \mathrm{L}^1_{q-\frac{d}{2},\,d}\,\mu^q \quad as \quad \mu \to +\infty\,.$$

Moreover, there exists a positive $\mu_\star$ with

$$\frac{\lambda_\star}{2\,(q-1)}\,\mu_1^\beta \le \mu_\star^\beta \le \frac{\lambda_1^{\mathfrak{M}}}{2\,q-1}\,\mu_1^\beta \tag{1}$$

and $\beta = \frac{2\,q}{2\,q-1}$, such that

$$\Lambda(\mu) = \Lambda_{\mathbb{R}}(\mu) \quad \forall\,\mu \in (0,\mu_\star] \quad and \quad \Lambda(\mu) > \Lambda_{\mathbb{R}}(\mu) \quad \forall\,\mu > \mu_\star\,.$$

As a special case, if $\mathfrak{M} = \mathbb{S}^{d-1}$, inequalities in (1) are in fact equalities.

In other words, we have shown the Keller-Lieb-Thirring inequality on the cylinder $\mathcal{C}$:

$$\lambda_1^{\mathcal{C}}[V] \le \Lambda\big(\|V_+\|_{\mathrm{L}^q(\mathcal{C})}\big) \quad \forall\,V \in \mathrm{L}^q(\mathcal{C}) \tag{KLT}$$

where the function $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ has the properties stated in Theorem 2. If $\|V_+\|_{\mathrm{L}^q(\mathcal{C})} \le \mu_\star$, optimality is achieved, up to scalings, translations and multiplications by a positive constant, by the potential V_1 of Lemma 1. This is based on a rigidity result which, in contrast with results on compact manifolds, involves a non-constant function.

The existence of the function $\mu \mapsto \Lambda(\mu)$ is an easy consequence of a Hölder estimate:

$$\|\partial_s u\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_g u\|^2_{\mathrm{L}^2(\mathcal{C})} - \int_{\mathcal{C}} V\,|u|^2\,dx \ge \|\partial_s u\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_g u\|^2_{\mathrm{L}^2(\mathcal{C})} - \mu\,\|u\|^2_{\mathrm{L}^p(\mathcal{C})}$$

with $\mu = \|V_+\|_{\mathrm{L}^q(\mathcal{C})}$ and $q = p/(p-2)$, and of the Gagliardo-Nirenberg-Sobolev inequality

$$\|\partial_s u\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_g u\|^2_{\mathrm{L}^2(\mathcal{C})} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{C})} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathcal{C})} \quad \forall\,u \in \mathrm{H}^1(\mathcal{C})\,, \tag{GNS}$$

where $\lambda > 0$ is a parameter and $\mu(\lambda)$ is the corresponding optimal constant. The existence of an optimal function for (GNS) can be argued as in [1]. Inverting $\lambda \mapsto \mu(\lambda)$ provides us with $\mu \mapsto \Lambda(\mu)$. See [3] for details and basic properties in a similar case.

The most important point in Theorem 2 is the issue of *symmetry* and *symmetry breaking*. We shall say that there is symmetry if equality in (KLT) is achieved by functions depending only on s, and *symmetry breaking* otherwise. By the method used in [8], there is a continuous curve $p \mapsto \mu_\star(p)$ defined on $(2,2^*)$ such that symmetry holds if $\mu \le \mu_\star$ and symmetry breaking holds if $\mu > \mu_\star$. It is then clear from the definition of Λ and $\Lambda_\star$ that $\Lambda(\mu) \ge \Lambda_{\mathbb{R}}(\mu)$ with equality if and only $\mu \le \mu_\star$. The main issue is henceforth to estimate $\mu_\star$. Our contribution is based on two perturbation methods:

(i) For μ large enough, a non-radial perturbation of an optimal symmetric potential shows symmetry breaking. This is done in the spirit of [1,10]. The computation gives the upper bound on $\mu_\star$ and a detailed proof is given in Section 2.

(ii) For μ not too large, symmetry holds. A sketch of a proof is given in Section 3. The key idea is to consider an optimal potential, symmetric or not, and perturb it adequately to prove that it has to be symmetric. The perturbation depends nonlinearly on the minimizer. The proof is not done at the level of (KLT), but at the level of the dual (GNS) inequality. This gives the lower bound on $\mu_\star$. Details will be given in a forthcoming paper, [7].

Apart from Euclidean spaces, very little is known on estimates like the ones of Theorem 2. A quantitative but non optimal result has been established in [6, Corollary 8]. Some results of symmetry for (KLT) type inequalities have been established for compact manifolds without boundary, see [3,4], and for bounded convex domains in $\mathbb{R}^d$ in relation with the Lin-Ni conjecture, see [9]. To our knowledge, the case of non compact manifolds was open so far, apart from the case of the line which was studied in [5] and the partial results of [6]. Here we give a result which is optimal when $\mathfrak{M}$ is a sphere. Let us finally notice that

various observations connecting the sphere, the Euclidean space and the line have been collected in [5].

2. One-dimensional potentials: proof of Lemma 1 and non-symmetric instability

We start by a short proof of Lemma 1 for the sake of completeness. Notations will be reused in the proof of Lemma 3.

Proof of Lemma 1. By applying Hölder's inequality, we get that

$$\int_{\mathbb{R}} |\partial_s u|^2 \, ds - \int_{\mathbb{R}} V \, |u|^2 \, ds \geq \|\partial_s u\|_{\mathrm{L}^2(\mathbb{R})}^2 - \mu \, \|u\|_{\mathrm{L}^p(\mathbb{R})}^2 \geq -\lambda \, \|u\|_{\mathrm{L}^2(\mathbb{R})}^2 \,,$$

where $\mu = \|V\|_{\mathrm{L}^q(\mathbb{R})}$ and $q = p/(p-2)$. With $V = V_1$, λ is chosen such that $\mu(\lambda) = \mu_1 := \|V_1\|_{\mathrm{L}^q(\mathbb{R})}$ where $\mu(\lambda)$ is the optimal constant in the inequality

$$\|\partial_s u\|_{\mathrm{L}^2(\mathbb{R})}^2 + \lambda \, \|u\|_{\mathrm{L}^2(\mathbb{R})}^2 \geq \mu(\lambda) \, \|u\|_{\mathrm{L}^p(\mathbb{R})}^2 \quad \forall \, u \in \mathrm{H}^1(\mathbb{R}) \,.$$

It is standard (see for instance [5]) that the function $\varphi(s) = (\cosh s)^{-2/(p-2)}$ is optimal for the inequality written with $\mu = \mu_1$ and solves

$$- (p-2)^2 \, \partial_s^2 \, \varphi + 4 \, \varphi - 2 \, p \, \varphi^{p-1} = 0 \,.$$

Altogether, this proves Lemma 1 when $\mu = \mu_1$ and $\lambda_1[V_1] = \frac{4}{(p-2)^2} = (q-1)^2$ because $u = \varphi$ and $V = V_1 = \frac{2 \, p}{(p-2)^2} \, \varphi^{p-2} = q \, (q-1) \, \varphi^{p-2}$ corresponds to the equality case in Hölder's inequality. More details can be found in [2].

If $\mu \neq \mu_1$, we can use scalings. Let $V_\nu(s) = \nu^2 \, V(\nu \, s)$. If $u_1 \neq 0$ solves

$$- \partial_s^2 \, u_1 - V \, u_1 + \lambda_1[V] \, u_1 = 0 \,,$$

then $u_\nu(s) = u_1(\nu \, s)$ is an eigenfunction associated with $\lambda_1[V_\nu] = \nu^2 \, \lambda_1[V]$. A change of variables shows that $\|V_\nu\|_{\mathrm{L}^q(\mathbb{R})} = \nu^{2-1/q} \, \|V\|_{\mathrm{L}^q(\mathbb{R})}$. Optimality is therefore achieved for $V_\nu \in \mathrm{L}^q(\mathbb{R})$ with $\|V_\nu\|_{\mathrm{L}^q(\mathbb{R})} = \mu > 0$ if and only if $\nu^{2-1/q} = \mu/\mu_1$ and $V(s) = V_1(s - s_0)$ for some $s_0 \in \mathbb{R}$, where

$$V_{1,\mu}(s) = \nu^2 \, V_1(\nu \, s) \quad \forall \, s \in \mathbb{R} \quad \text{with} \quad \nu = (\mu/\mu_1)^{\frac{q}{2 \, q - 1}} \,.$$

The corresponding eigenfunction is, up to a multiplication by a constant, $\varphi_\mu(s) = \varphi(\nu \, s)$. The lowest eigenvalue for $V \in \mathrm{L}^q(\mathbb{R})$ such that $\|V\|_{\mathrm{L}^q(\mathbb{R})} = \mu$, which realizes the equality in the Keller-Lieb-Thirring inequality, is $\lambda_1[V_{1,\mu}] = \lambda_1[V_1] \, \nu^2 = \Lambda_{\mathbb{R}}(\mu)$. This completes the proof of Lemma 1. $\square$

Next let us consider a function V of $x = (s, z) \in \mathcal{C}$. Inspired by the results of [1,10] for Caffarelli-Kohn-Nirenberg inequalities, we can prove that $V_{1,\mu}$, considered as a function on $\mathcal{C}$, cannot be optimal for the Keller-Lieb-Thirring inequality on $\mathcal{C}$ if μ is large enough.

Lemma 3 *With the above notations and assumptions, let $V = V(s, z)$ be a nonnegative real valued potential in $\mathrm{L}^q(\mathcal{C})$ for some $q > d/2$ and let $-\lambda_1^{\mathcal{C}}[V]$ be the lowest eigenvalue of the Schrödinger operator $-\partial_s^2 - \Delta_g - V$. If $\Lambda_{\mathbb{R}}(\mu) > 4 \, \lambda_1^{\mathfrak{M}}/(p^2 - 4)$, then*

$$\sup \left\{ \lambda_1^{\mathcal{C}}[V] \, : \, V \in \mathrm{L}^q(\mathcal{C}) \,, \, \|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu \right\} > \Lambda_{\mathbb{R}}(\mu) \,,$$

that is, the above maximization problem cannot be achieved by a potential V depending only on the variable s. The condition $\Lambda_{\mathbb{R}}(\mu) > 4 \, \lambda_1^{\mathfrak{M}}/(p^2 - 4)$ is explicit and equivalent to $\mu^\beta > \mu_1^\beta \, \lambda_1^{\mathfrak{M}}/(2 \, q - 1)$.

Proof. Let φ_μ be as in the proof of Lemma 1. We argue by contradiction and consider

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$$\phi_\varepsilon(s,z) := \varphi_\mu(s) + \varepsilon \left(\varphi_\mu(s)\right)^{p/2} \psi_1(z) \quad \text{and} \quad V_\varepsilon(s,z) := \mu \, \frac{|\phi_\varepsilon(s,z)|^{p-2}}{\|\phi_\varepsilon\|_{\mathrm{L}^p(\mathcal{C})}^{p-2}} \, .$$

Here ψ_1 denotes an eigenfunction of $-\Delta_g$ on $\mathfrak{M}$ such that $\|\psi_1\|_{\mathrm{L}^2(\mathfrak{M})} = 1$ and $-\Delta_g \psi_1 = \lambda_1^{\mathfrak{M}} \psi_1$. Then

$$\|\partial_s \phi_\varepsilon\|_{\mathrm{L}^2(\mathcal{C})}^2 + \|\nabla_g \phi_\varepsilon\|_{\mathrm{L}^2(\mathcal{C})}^2 - \mu \, \|\phi_\varepsilon\|_{\mathrm{L}^p(\mathcal{C})}^2 - \left(\|\partial_s \varphi_\mu\|_{\mathrm{L}^2(\mathbb{R})}^2 - \mu \, \|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^2 \right)$$
$$= \varepsilon^2 \left(\lambda_1^{\mathfrak{M}} \, \|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^p + \|\partial_s \varphi_\mu^{p/2}\|_{\mathrm{L}^2(\mathbb{R})}^2 - (p-1)\, \mu \, \|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^{2-p} \int_{\mathbb{R}} \varphi_\mu^{2\,(p-1)} \, ds \right) + o(\varepsilon^2) \, .$$

Since $\mu \, \|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^{2-p} \, \varphi_\mu^{p-2} = V_{1,\mu}(s) = \nu^2 \, V_1(\nu\, s)$ with $\nu = \left(\mu/\mu_1\right)^\beta$ as in the proof of Lemma 1, since $\chi = \varphi^{p/2}$ solves

$$-\partial_s^2 \chi - (p-1)\, V_1\, \chi = - \left(\frac{p}{p-2}\right)^2 \chi \, ,$$

and since $(p-1)\, q\, (q-1) = \frac{2\, p\, (p-1)}{(p-2)^2}$, we get that

$$\|\partial_s \varphi_\mu^{p/2}\|_{\mathrm{L}^2(\mathbb{R})}^2 - \frac{2\, p\, (p-1)}{(p-2)^2} \, \nu^2 \int_{\mathbb{R}} \varphi_\mu^{2\,(p-1)} \, ds = -\nu^2 \left(\frac{p}{p-2}\right)^2 \|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^p \, .$$

We recall that $\nu^2 = \Lambda_\mathbb{R}(\mu)/(q-1)^2 = \frac{1}{4}\, (p-2)^2\, \Lambda_\mathbb{R}(\mu)$. According to [5, Appendix A], we see that

$$\|\varphi_\mu\|_{\mathrm{L}^p(\mathbb{R})}^p = \frac{4}{p+2} \, \|\varphi_\mu\|_{\mathrm{L}^2(\mathbb{R})}^2 \quad \text{and} \quad \|\phi_\varepsilon\|_{\mathrm{L}^2(\mathcal{C})}^2 = \left(1 + \frac{4\,\varepsilon^2}{p+2}\right) \|\varphi_\mu\|_{\mathrm{L}^2(\mathbb{R})}^2 \, .$$

Hence we finally find that

$$-\lambda_1^{\mathcal{C}}[V_\varepsilon] + \Lambda_\mathbb{R}(\mu) \le \frac{4\,\varepsilon^2}{p+2} \left(\lambda_1^{\mathfrak{M}} - \tfrac{1}{4}\, (p^2 - 4)\, \Lambda_\mathbb{R}(\mu)\right) + o(\varepsilon^2) \, .$$

This shows that $\lambda_1[V_{1,\mu}] - \lambda_1^{\mathcal{C}}[V_\varepsilon] < 0$ is negative for $\varepsilon > 0$, small enough, if $\Lambda_\mathbb{R}(\mu) > \frac{4\,\lambda_1^{\mathfrak{M}}}{p^2-4}$. $\qquad\square$

The condition found by V. Felli and M. Schneider in [10] can be recovered by noticing that $\lambda_1^{\mathbb{S}^{d-1}} = d-1$, when $\mathfrak{M} = \mathbb{S}^{d-1}$. In that case, the above computation are exactly equivalent to the computations for Caffarelli-Kohn-Nirenberg inequalities: see [5] for details.

3. Symmetry: a rigidity result

In this section we get a lower bound on $\mu_\star$ and complete the proof of Theorem 2. Let us define

$$\mathsf{J}[V] := \frac{\|V\|_{\mathrm{L}^q(\mathcal{C})}^q - \|\partial_s V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2 - \|\nabla_g V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2}{\|V^{(q-1)/2}\|_{\mathrm{L}^2(\mathcal{C})}^2} \, .$$

We shall consider a critical point of J and prove that it is symmetric using a well chosen perturbation.

Lemma 4 *With the notations of Section 1 and under the assumptions of Theorem 2, we have*

$$\Lambda(\mu) = \sup \left\{ \mathsf{J}[V] : \|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu \right\} \, .$$

Sketch of the proof of Lemma 4. There exists a nonnegative potential $V \in \mathrm{L}^q \cap C^\infty(\mathcal{C})$ such that $\mathsf{J}[V] = \Lambda(\mu)$. Lemma 4 is based on the equivalence of (KLT) and (GNS), which can also be seen by using $u = V^{(q-1)/2}$ and considering the equality case in Hölder's inequality. See [7] for more details. $\qquad\square$

To any potential $V \geq 0$ we associate the *pressure* function
$$\mathsf{p}_V(r) := r \, V(s)^{-\frac{1}{2}} \quad \forall \, r = e^{-\alpha s} \, .$$
With $\alpha = \frac{1}{q-1} \sqrt{\Lambda_{\mathbb{R}}(\mu)}$, let us define

$$\mathsf{K}[\mathsf{p}] := \frac{2q-1}{2 \, q} \, \alpha^4 \int_{\mathbb{R}^+ \times \mathfrak{M}} \left| \mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_g \mathsf{p}}{\alpha^2 \, (2q-1) \, r^2} \right|^2 \mathsf{p}^{1-2q} \, d\mu$$
$$+ 2 \, \alpha^2 \int_{\mathbb{R}^+ \times \mathfrak{M}} \frac{1}{r^2} \left| \nabla_g \mathsf{p}' - \frac{\nabla_g \mathsf{p}}{r} \right|^2 \mathsf{p}^{1-2q} \, d\mu + \left(\lambda_\star - \frac{2}{q-1} \, \Lambda_{\mathbb{R}}(\mu) \right) \int_{\mathbb{R}^+ \times \mathfrak{M}} \frac{|\nabla_g \mathsf{p}|^2}{r^4} \, \mathsf{p}^{1-2q} \, d\mu \, ,$$

where $d\mu$ is the measure on $\mathbb{R}^+ \times \mathfrak{M}$ with density r^{2q-1}, and $'$ denotes the derivative with respect to r.

Lemma 5 *There exists a positive constant* c *such that, with the above notations, if* V *is a critical point of* J *under the constraint* $\|V\|_{\mathrm{L}^q(\mathcal{C})} = \mu$, *then* $\mathsf{K}[\mathsf{p}_V] = 0$.

The sketch of the proof is as follows. Because V is critical point, we know that
$$0 = \mathsf{J}'[V] \cdot V^{1-q} \left[\partial_s \left(e^{-2(q-1)\alpha s} \, \partial_s \left(e^{(2q-1)\alpha s} \, V^{\frac{2q-1}{2}} \right) \right) + e^{\alpha s} \, \Delta_g V^{\frac{2q-1}{2}} \right] \, .$$

On the other hand, it has been shown in [7] that the r.h.s. is greater or equal than a positive constant times $\mathsf{K}[\mathsf{p}_V]$.

Sketch of the proof of Theorem 2. If $\mu < \mu_\star$, then $\lambda_\star - \frac{2}{q-1} \, \Lambda_{\mathbb{R}}(\mu)$ is positive and we read from the expression of $\mathsf{K}[\mathsf{p}_V]$ that p_V depends only on r or, equivalently, V depends only on s. Up to scalings, translations and multiplications by a positive constant, we get that $V = V_1$ as defined in Lemma 1. $\square$

Note that the constant $\lambda_\star$ is an estimate of the largest constant λ such that
$$\int_{\mathfrak{M}} \left(\tfrac{1}{2} \, \Delta_g(|\nabla_g f|^2) - \nabla_g(\Delta_g f) \cdot \nabla_g f - \tfrac{1}{2q-1} \, (\Delta_g f)^2 - \lambda \, |\nabla_g f|^2 \right) f^{1-2q} \, dv_g \geq 0 \, ,$$

for any positive function $f \in C^3(\mathfrak{M})$. It is estimated by λ_θ with $\theta \in [\theta_\star, 1]$, according to [7]. In the case of the sphere, that is, $\mathfrak{M} = \mathbb{S}^{d-1}$, we have that $\frac{d-1}{d-2} \, \kappa = \lambda_1^{\mathfrak{M}}$ and $\lambda_\theta = \left(1 + \delta \, \frac{d-1}{d-2} \right) \kappa = \left(\frac{d-2}{d-1} + \delta \right) \lambda_1^{\mathfrak{M}}$ is independent of θ. Otherwise, by Lichnerowicz' theorem, we know that $\frac{d-1}{d-2} \, \kappa \leq \lambda_1^{\mathfrak{M}}$. Hence $\theta \mapsto \lambda_\theta$ is a non-increasing function, and since $\theta_\star$ is always negative, we have a simple lower bound for $\lambda_\star$:
$$\lambda_\star \geq \lambda_0 = \kappa + \delta \, \lambda_1^{\mathfrak{M}} \, .$$

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