INTERPOLATION INEQUALITIES ON THE SPHERE: RIGIDITY, BRANCHES OF SOLUTIONS, AND SYMMETRY BREAKING

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ABSTRACT. This paper is devoted to three Gagliardo-Nirenberg-Sobolev interpolation inequalities on the sphere. We are interested in branches of optimal functions when a scale parameter varies and investigate whether optimal functions are constant, or not. In the latter case, a symmetry breaking phenomenon occurs and our goal is to decide whether the threshold between symmetry and symmetry breaking is determined by a spectral criterion or not, that is, whether it appears as a perturbation of the constants or not. The first inequality is classical while the two other inequalities are variants which reproduce patterns similar to those observed in Caffarelli-Kohn-Nirenberg inequalities, for weighted inequalities on the Euclidean space. In the simpler setting of the sphere, it is possible to implement a parabolic version of the entropy methods associated to nonlinear diffusion equations, which is so far an open question on weighted Euclidean spaces.

1. INTRODUCTION

In this paper we consider some interpolation inequalities on the sphere which can be written in generic form as

$$F\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)},\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\ge\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\tag{1}$$

where $p>2$ has to be specified. Here the function $F:(\mathbb{R}^+)^2\to\mathbb{R}$ is a two-homogeneous function, *i.e.*,

$$F(h\,\mathsf{a},h\,\mathsf{b})=h^2\,F(\mathsf{a},\mathsf{b})\quad\forall\,(h,\mathsf{a},\mathsf{b})\in(\mathbb{R}^+)^3\,,$$

and we assume that $F(0,1)=1$. We will not give general assumptions of regularity on F as we shall rather focus on three particular cases that illustrate the variety of possible cases. With standard notations, $d\sigma=\left|\mathbb{S}^d\right|^{-1}dv_g$ denotes the uniform probability measure on the unit sphere $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ induced by Lebesgue's measure on $\mathbb{R}^{d+1}$. The space $\mathrm{L}^q(\mathbb{S}^d,d\sigma)$ with $q\in[1,+\infty)$ is the standard Lebesgue space with norm $\|\cdot\|_{\mathrm{L}^q(\mathbb{S}^d)}$ defined by

$$\|u\|_{\mathrm{L}^q(\mathbb{S}^d)}:=\left(\int_{\mathbb{S}^d}|u|^q\,d\sigma\right)^{1/q}.$$

The fact that $d\sigma$ is a probability measure means that $\|u\|_{\mathrm{L}^q(\mathbb{S}^d)}=1$ if $u=1$ a.e. on $\mathbb{S}^d$ and we also have $\|u\|_{\mathrm{L}^{q_1}(\mathbb{S}^d)}\le\|u\|_{\mathrm{L}^{q_2}(\mathbb{S}^d)}$ for any $u\in\mathrm{L}^{q_2}(\mathbb{S}^d,d\sigma)$ as soon as $1\le q_1\le q_2$. The

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space $\mathrm{H}^1(\mathbb{S}^d, d\sigma)$ is obtained by completion of smooth functions on $\mathbb{S}^d$ with respect to the norm $u \mapsto \left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^{1/2}$.

If (1) is an optimal inequality, and if there is some $u_\star \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$ such that equality holds in (1), the question that we address is the *symmetry* versus *symmetry breaking* issue: is $u_\star$ necessarily a constant function (symmetry) or not (symmetry breaking). In this last case, there is a manifold of non-constant optimal functions generated by all possible rotations. We also consider limit cases for which compactness is lost and there is no optimal function (or a sequence of functions approximating the equality case is not compact even after normalization): there is *symmetry breaking* if the optimality case is not achieved by constant functions.

As a side question, let us mention the question of the *rigidity* in related elliptic equations: up to a normalization by a multiplicative constant, is there at most one nonnegative solution to the Euler-Lagrange equation

$$-\partial_{\mathsf{a}} F(\mathsf{a},\mathsf{b})\,\Delta u + \partial_{\mathsf{b}} F(\mathsf{a},\mathsf{b})\,u = \|u\|^{2-p}_{\mathrm{L}^p(\mathbb{S}^d)}\,u^{p-1} \quad \text{with} \quad \mathsf{a} = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{and} \quad \mathsf{b} = \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

or not. Here Δ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$ with the convention that $-\Delta$ is a nonnegative operator. In case of uniqueness and compactness (normalized minimizing sequences for (1) are relatively compact and the limits after extraction of a subsequence solves the equation), then *symmetry* holds and $u = 1$ a.e. is the unique solution, while in the *symmetry breaking* case, there is no rigidity (notice that if u is optimal for (1), then $|u|$ is also optimal).

In order to study the symmetry breaking phenomenon, we shall introduce a *bifurcation* parameter $\lambda > 0$ and families of functions $(F_\lambda)_{\lambda>0}$ depending continuously on λ with $F_\lambda(0,1) = \lambda$. Our goal is to characterize the optimal constant $\mu(\lambda)$ in the inequality

$$F_\lambda\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma) \tag{2}$$

so that *symmetry* means $\mu(\lambda) = \lambda$ while *symmetry breaking* is characterized by $\mu(\lambda) < \lambda$. On $\mathbb{S}^d$, there is an intrinsic length scale, which compares with λ. The three families of *Gagliardo-Nirenberg-Sobolev inequalities* on $\mathbb{S}^d$ studied in this paper are

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu_0(p,\lambda)}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,, \tag{3}$$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu_1(p,\theta,\lambda)}{p-2}\,\|u\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{2(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\,, \tag{4}$$

$$\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\,\|u\|^{2(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)} \ge \left(\frac{\mu_2(p,\theta,\lambda)}{p-2}\right)^\theta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,, \tag{5}$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$, where $\lambda > 0$ is a parameter while $\mu_0(p,\lambda)$ and $\mu_k(\theta,\lambda)$ with $k = 1, 2$ are the optimal constants. Although the case $p \in [1,2)$ makes sense, we shall for simplicity restrict our purpose to $p \in (2,\infty)$ if $d = 1$, 2 and $p \in (2, 2^*]$ if $d \ge 3$, with $2^* = 2\,d/(d-2)$.

By convention, we take $2^* = +\infty$ if $d = 1, 2$. The exponent

$$\theta_\star := d\,\frac{p-2}{2\,p} \tag{6}$$

is the exponent in the Euclidean Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta_\star}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{1-\theta_\star} \ge \mathscr{C}_{\mathrm{GNS}}\,\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^2 \quad \forall\,u \in \mathrm{H}^1(\mathbb{R}^d, dx) \tag{7}$$

and induces the limitation $\theta \ge \theta_\star$ in (5). As we shall see later, the regime for λ small corresponds to symmetry while the asymptotic regime as $\lambda \to +\infty$ is related with optimal functions for (7) on the tangent Euclidean space $\mathbb{R}^d$, which induce symmetry breaking.

Let us give a brief account of the literature. M.-F. Bidault-Véron and L. Véron proved Inequality (3) in [5, Corollary 6.1] as a consequence of a rigidity result. Another proof by W. Beckner in [4] relies on spectral methods, the Funk-Hecke formula and a duality result introduced by E. Lieb in [27]. An earlier version corresponding to the range $p \in (2, 2^\#)$ with $2^\# = (2\,d^2+1)/(d-1)^2$ if $d \ge 2$ and $2^\# = +\infty$ if $d = 1$ was established by D. Bakry and M. Emery in [1, 2], using the *carré du champ* method and the heat flow. We refer to [3] for a general overview of the *carré du champ* method in the context of Markov processes or linear diffusion equations, and to [9, 17] for the extension to nonlinear diffusion equations. Also see Appendix A for further references in the case of the sphere.

Inequalities (4) and (5) are less standard than (3) and we are not aware of specific references. The case $\theta = 1$ in (4) and (5) corresponds to the classical interpolation inequality (3). Inequality (5) appears (with optimal constant) for a special value of θ in [12, Inequality (2.4)] as a consequence of improvements of (3) based on the *carré du champ* method. For a presentation of *entropy methods* and *improved interpolation inequalities*, see [12] and references therein.

An important motivation for (5) arises from pure states in Lieb-Thirring estimates and interpolation inequalities for systems studied in [8]. We refer to [22] for an overview of functional inequalities and branches of solutions in various frameworks, including the quite similar problem of *symmetry breaking* in some of the (CKN) *Caffarelli-Kohn-Nirenberg inequalities*. The sharp threshold for the symmetry range of the parameters in such inequalities is established in [25, 19] in the critical case and in [7, 23] in the subcritical case. A complete parabolic proof based on entropy methods is so far missing and results rely on *rigidity* methods: see [18, 24] for partial results based on flows. In the case of the sphere, Inequalities (3), (4) and (5) have similar properties but by many aspects symmetry *versus* symmetry breaking issues are simpler. For instance, the regularity of the solution of the nonlinear flow on $\mathbb{S}^d$ raises no difficulty: see for instance [28] for some comments in this direction. Branches of optimizers associated with Inequalities (4) and (5) for $\theta < 1$ have qualitative features which are reminiscent of those corresponding to similar (CKN) inequalities (with also some $\theta < 1$) observed in [11, 10, 13]. Beyond numerical observations and a few estimates, these branches for (CKN) are not well understood. Our goal is to state the corresponding properties on the sphere.

This paper is organized as follows. In Section 2, we expose results and strategies of proof for Inequality (3). Sections 3 and 4 are devoted to similar properties for Inequalities (4) and (5) with a special emphasis on symmetry and symmetry breaking. In the Appendix A, for completeness, we provide some computations based on the *carré du champ* method applied with a nonlinear flow in the case of the sphere and further bibliographical references for this technique.

2. Classical Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$

In this preliminary section, we review some known results for Inequality (3) and give some hints for the proofs which will guide us in the study of the two other inequalities.

Theorem 1. *Let $d\ge 1$, $p\in(2,2^*)$ or $p=2^*$ if $d\ge 3$. Inequality (3) holds with $\mu_0(p,\lambda)=\lambda$ if $\lambda\le d$ (symmetry case) and $\mu_0(p,\lambda)<\lambda$ if $\lambda>d$ (symmetry breaking case).*

A standard observation is that $\lambda\mapsto\mu_0(p,\lambda)$ is a concave increasing function as a consequence of

$$\mu_0(p,\lambda)=\inf_{u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}}\frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$$

and that the symmetry breaking point $d=\inf\big\{\lambda>0\,:\,\mu_0(p,\lambda)<\lambda\big\}$ is such that

$$d=\inf_{u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\mathbb{R}}\frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,,$$

where $\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\mathbb{R}$ denotes the set of non-constant functions in $\mathrm{H}^1(\mathbb{S}^d,d\sigma)$.

Let us sketch the proof of Theorem 1.

• ***Symmetry breaking.*** Let us consider the functional

$$\mathscr{F}_\lambda[u]:=\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{\lambda}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right).\tag{8}$$

A Taylor expansion of $\mathscr{F}_\lambda[1+\varepsilon\,\varphi]$ at order two in ε for a function φ such that $\int_{\mathbb{S}^d}\varphi\,d\sigma=0$ shows that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\,\mathscr{F}_\lambda[1+\varepsilon\,\varphi]=\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,.$$

Taking for φ an eigenfunction of the Laplace-Beltrami operator on $\mathbb{S}^d$ associated with its lowest positive eigenvalue, *i.e.*, such that $-\Delta\varphi=d\,\varphi$, we find that

$$\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=(d-\lambda)\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)}<0$$

for any $\lambda>d$, which shows that $\mu_0(p,\lambda)<\lambda$.

• ***Symmetry.*** Let us consider the fast diffusion equation

$$\frac{\partial\rho}{\partial t}=\Delta\rho^m\,,\quad\rho(t=0,\cdot)=|u|^p\tag{9}$$

for some parameter $m \in \big[m_-(d,p), m_+(d,p)\big]$ where

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p+2\pm\sqrt{d\,(p-1)\big(2\,d-(d-2)\,p\big)}\right). \tag{10}$$

A computation based on a nonlinear extension of the *carré du champ method* shows that

$$\frac{d}{dt}\mathscr{F}_\lambda[v(t,\cdot)] \le -2\,\beta^2\,(d-\lambda)\int_{\mathbb{S}^d}\left|\nabla v^{1/\beta}\right|^2 d\sigma$$

where $\beta := 2/\big(2-p\,(1-m)\big)$. References are given in Appendix A and the key computation is detailed in Appendix A.1. Hence

$$\frac{d}{dt}\mathscr{F}_\lambda[v(t,\cdot)] \le 0 \quad \text{and} \quad \lim_{t\to+\infty}\mathscr{F}_\lambda[v(t,\cdot)] = 0$$

if $\lambda \le d$. This proves that $\mathscr{F}_\lambda[u] \ge 0$ for an arbitrary $u \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$. See Lemma 14 and final comments at the end of Section A.2 for further justifications. □

• ***Rigidity.*** As in [5, 6, 17], a remarkable consequence of the proof is the following result.

Corollary 2. *Let $d \ge 1$, $p \in (2,2^*)$ or $p = 2^*$ if $d \ge 3$. If $0 < \lambda \le d$, the equation*

$$-\Delta u + \frac{\lambda}{p-2}\,u = u^{p-1} \tag{11}$$

admits a unique nonnegative solution, which is constant.

The proof is consequence of the *carré du champ method* based on an idea that goes back to [26].

• ***Asymptotics.*** Another piece of information is the asymptotic behavior of the branch $\lambda \mapsto \mu_0(p,\lambda)$. The following result is taken from [16, Proposition 10].

Proposition 3. *Let $d \ge 1$ and $p \in (2,2^*)$. The optimal constant in (3) is such that*

$$\lim_{\lambda\to+\infty}\frac{\mu_0(p,\lambda)}{\lambda^{\theta_\star}} = \frac{\left|\mathbb{S}^d\right|^{-\frac{p-2}{p}}}{\theta_\star^{\theta_\star}\,(1-\theta_\star)^{1-\theta_\star}}\,\mathscr{C}_{\mathrm{GNS}}\,.$$

If $d \ge 3$ and $p = 2^*$, then $\mu_0(p,\lambda) = \min\{\lambda,\, d\}$ is not achieved in $\mathrm{H}^1(\mathbb{S}^d, d\sigma)$ for any $\lambda > d$.

As $\lambda \to +\infty$, the corresponding optimal functions concentrate on points of the sphere. A blow-up analysis shows that the projection on the tangent Euclidean space at a blow-up point can be estimated by (7), and optimal functions for (7) are good test functions, which allows to identify the limit. Notice that concentration is incompatible with symmetry, so that we recover symmetry breaking in the limit as $\lambda \to +\infty$. In the case $d \ge 3$ and $p = 2^*$, loss of compactness occurs and a minimizing sequence behave like concentrating Aubin-Talenti bubbles.

• ***Reparametrization.*** The following observation is elementary. Let us consider the optimal constants for Inequalities (4) and (5) seen as variational problems, namely

$$\mu_1(p,\theta,\lambda) = \inf_{u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}} \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{2(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\,, \tag{12}$$

$$\mu_2(p,\theta,\lambda) = \inf_{u\in \mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}} \left((p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \|u\|^{\frac{2}{\theta}-2}_{\mathrm{L}^2(\mathbb{S}^d)}\,\|u\|^{-\frac{2}{\theta}}_{\mathrm{L}^p(\mathbb{S}^d)}\,. \tag{13}$$

As already noted in Section 1, we know for free that

$$\mu_1(p,1,\lambda) = \mu_2(p,1,\lambda) = \mu_0(p,\lambda)$$

for any $\lambda > 0$. The parameter θ is subject to *admissibility* conditions that will be specified in Sections 3 and 4. Even for $\theta \neq 1$, Inequalities (4) and (5) are related with Inequality (3) as follows.

Proposition 4. *Let $d \ge 1$, $p \in (2,2^*)$ and $\Lambda > 0$, given. If for some admissible θ and either $k=1$ or $k=2$, U is a non-trivial minimizer for $\mu_k(\theta,\Lambda)$, then for an appropriate choice of $\kappa > 0$, the function $u = \kappa\, U$ solves (11) for some $\lambda > 0$. The two parameters κ and λ are explicitly given in terms of $\mu_k(\theta,\Lambda)$, $\|U\|_{\mathrm{L}^2(\mathbb{S}^d)}$ and $\|U\|_{\mathrm{L}^p(\mathbb{S}^d)}$.*

Proof. Let us consider the case $k=1$. The function U solves the Euler-Lagrange equation

$$-\Delta U + \frac{1}{p-2}\left(\Lambda - (1-\theta)\,\mu_1(p,\theta,\Lambda)\,X^{2\theta}\right) U = \frac{\mu_1(p,\theta,\Lambda)\,X^{2\theta-2}}{(p-2)\,\|U\|^{p-2}_{\mathrm{L}^p(\mathbb{S}^d)}}\,U^{p-1}$$

with $X = \|U\|_{\mathrm{L}^p(\mathbb{S}^d)}/\|U\|_{\mathrm{L}^2(\mathbb{S}^d)}$. The proof is complete with the choice

$$\lambda = \frac{\Lambda}{p-2} - (1-\theta)\,\mu_1(p,\theta,\Lambda)\,X^{2\theta} \quad \text{and} \quad \kappa^{2-p} = \frac{\mu_1(p,\theta,\Lambda)\,X^{2\theta-2}}{(p-2)\,\|U\|^{p-2}_{\mathrm{L}^p(\mathbb{S}^d)}}\,.$$

A similar reparametrization holds for $k=2$. □

3. A first family of refined interpolation inequalities on $\mathbb{S}^d$

Let us define

$$\mathcal{Q}^{(1)}_{p,\theta,\lambda}[u] := \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{2(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\,.$$

• ***Admissible parameters.*** We establish the validity range of the parameters in (4).

Lemma 5. *Inequality (4) holds for any $d \ge 1$, $\theta \in [0,1]$, $p \in (2,2^*)$ or $p = 2^*$ if $d \ge 3$, with*

$$\mu_0(p,\lambda) \le \mu_1(p,\theta,\lambda) \le \lambda \quad \forall\,\lambda > 0\,.$$

Moreover, equality holds and $\mu_1(p,\theta,\lambda) = \lambda$ if $\lambda \le d$.

Proof. By Hölder's inequality, $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}$ because $d\sigma$ is a probability measure. Using (12), we learn that

$$\mu_0(p,\lambda) = \inf_{u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}} \mathcal{Q}^{(1)}_{p,1,\lambda}[u] \le \inf_{u\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)\setminus\{0\}} \mathcal{Q}^{(1)}_{p,\theta,\lambda}[u] = \mu_1(p,\theta,\lambda) \le \lambda\,,$$

where the upper bound is obtained by testing $\mathcal{Q}^{(1)}_{p,\theta,\lambda}$ with $u=1$ a.e. If $\lambda\le d$, there is equality because $\mu_0(p,\lambda)=\lambda$ (symmetry case) according to Theorem 1. □

We read from Lemma 5 that (4) is stronger than (3) and this is why we call Inequality (4) a *refined Gagliardo-Nirenberg-Sobolev inequality*. Inequality (4) can also be seen as an *improved* inequality compared to (3) or a *stability result* for (3): see [12] for further considerations on these issues and an example corresponding to a special choice of θ in (4).

As in Section 2, let us consider

$$\mathscr{F}^{(1)}_\lambda[u] := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{\lambda}{p-2}\left(\|u\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{2(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\right).$$

• ***Symmetry breaking.*** As in the case of (3), we obtain an estimate of the symmetry breaking range by Taylor expanding around a constant with a perturbation by a spherical harmonic function.

Lemma 6. *Let $d\ge1$, $\theta\in[0,1]$, $p\in(2,2^*)$ or $p=2^*$ if $d\ge3$. There is* symmetry breaking *in* (4)*, that is, $\mu_1(p,\theta,\lambda)<\lambda$, if $\lambda>d/\theta$.*

Proof. Let us consider $\varphi\in\mathrm{H}^1(\mathbb{S}^d,d\sigma)$ such that $\int_{\mathbb{S}^d}\varphi\,d\sigma=0$, $u_\varepsilon=1+\varepsilon\,\varphi$, and compute

$$\begin{aligned}
&\|\nabla u_\varepsilon\|_2^2 = \varepsilon^2\,\|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + o(\varepsilon^2)\,,\\
&\|u_\varepsilon\|_2^2 = 1+\varepsilon^2\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + o(\varepsilon^2)\,,\\
&\|u_\varepsilon\|_p^2 = 1+\varepsilon^2\,(p-1)\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + o(\varepsilon^2)\,.
\end{aligned}$$

As a consequence, we obtain that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^2}\mathscr{F}^{(1)}_\lambda[u_\varepsilon] = \|\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \lambda\,\theta\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = (d-\lambda\,\theta)\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < 0$$

if we take a non-trivial function φ such that $-\Delta\varphi = d\,\varphi$ and $\lambda>d/\theta$. □

• ***Symmetry.*** Up to a change in the range of λ, the result and its proof are similar to the case studied in Section 2.

Lemma 7. *Let $d\ge1$, $\theta\in[0,1]$, $p\in(2,2^*)$ or $p=2^*$ if $d\ge3$. There is* symmetry *in* (4)*, that is, $\mu_1(p,\theta,\lambda)=\lambda$, if $\lambda\le d/\theta$.*

Proof. If $d=1$, one can consider a periodic interval identified with $\mathbb{S}^1$ and refer to [2] for applying the *carré du champ* method in this elementary setting. Assume that $d\ge2$ and let us consider again the fast diffusion equation (9) with $m\in\big[m_-(d,p),m_+(d,p)\big]$ where $m_\pm(d,p)$ is defined by (10). The same computation as in the proof of Theorem 1 shows that

$$\frac{d}{dt}\mathscr{F}_\lambda^{(1)}[v(t,\cdot)] \le -2\,\beta^2\Big(d+\lambda\big((1-\theta)\,\mathsf{x}-1\big)\Big)\int_{\mathbb{S}^d}\big|\nabla v^{1/\beta}\big|^2\,d\sigma \quad \text{with} \quad \mathsf{x} := \frac{\|v\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}}{\|v\|^{2\theta}_{\mathrm{L}^2(\mathbb{S}^d)}}\,.$$

By a Hölder inequality, we learn that $\mathsf{x} \ge 1$ and are then back to the same arguments as in the case $\theta = 1$, but now under the condition $\lambda \le d/\theta$. □

• ***Asymptotics.*** Heuristically, large values of λ correspond to highly concentrated optimal functions, which behave like the functions obtained by a *stereographic projection* on the tangent Euclidean space at the point of concentration. Let us explain how we define and use the stereographic projection. On $\mathbb{R}^d \ni x$, let $r = |x|$ and $\omega = x/|x|$ denote spherical coordinates. On the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we consider cylindrical coordinates $(\rho\,\omega, z) \in \mathbb{R}^d \times (-1,1)$ with $\rho^2 + z^2 = 1$. The *stereographic projection* $\mathsf{S} : \mathbb{S}^d \setminus \{\mathsf{N}\} \to \mathbb{R}^d$, where $\mathsf{N} \in \mathbb{S}^d$ is the North Pole defined by $z = +1$, is such that $\mathsf{S}(\rho\,\omega, z) = r\,\omega = x$ where

$$z = \frac{r^2-1}{r^2+1} \quad \text{and} \quad \rho = \frac{2\,r}{1+r^2}\,.$$

If v is a function on $\mathbb{R}^d$, let us consider its counterpart $u := \mathsf{S}^{-1}\,v$ on $\mathbb{S}^d$ obtained using the inverse stereographic projection as

$$\big(u\circ\mathsf{S}^{-1}\big)(x) = \mathsf{m}(r)^{d-2}\,v(x) \quad \forall\,x \in \mathbb{R}^d \quad \text{with} \quad \mathsf{m}(r) = \sqrt{(1+r^2)/2}\,.$$

For any $q \ge 1$, with $\delta(q) := 2\,d - q\,(d-2)$, we obtain

$$\int_{\mathbb{S}^d}|u|^q\,d\sigma = \big|\mathbb{S}^d\big|^{-1}\int_{\mathbb{R}^d}\frac{|v|^q}{\mathsf{m}(r)^{\delta(q)}}\,dx\,,$$
$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\sigma + \frac14\,d\,(d-2)\int_{\mathbb{S}^d}|u|^2\,d\sigma = \big|\mathbb{S}^d\big|^{-1}\int_{\mathbb{R}^d}|\nabla v|^2\,dx\,.$$

Notice that $\delta(2) = 4$. We refer to [12, Theorem 2.1] for a statement concerning the inequality obtained from (3) and a special case of (4) by the stereographic projection.

As another observation, we shall use the optimal constant $\mathscr{K}_{p,d}$ in the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^d$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathscr{K}_{p,d}\,\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{R}^d, dx)\,. \tag{14}$$

An optimization under scaling (apply the inequality to $u_h(x) = h^{d/p}\,(h\,x)$ and optimize its left-hand side with respect to $h > 0$) shows that this inequality is equivalent to (7) and, with $\theta_\star$ as in (6),

$$\theta_\star^{\theta_\star}\,(1-\theta_\star)^{1-\theta_\star}\,\mathscr{K}_{p,d} = \mathscr{C}_{\mathrm{GNS}}\,.$$

After these preliminaries, we can state the following result, which extends the results for (3) of [16, Proposition 9 and 10] to Inequality (4).

Lemma 8. *Let $d \ge 1$, $\theta \in [0,1]$, $p \in (2,2^*)$ or $p = 2^*$ if $d \ge 3$. There is an explicit constant $\mu_1^\infty(p,\theta) > 0$ such that*

$$\mu_1(p,\theta,\lambda) = \mu_1^\infty(p,\theta)\,\lambda^\gamma\,\big(1+o(1)\big) \quad \textit{as} \quad \lambda \to +\infty$$

where $\gamma = 1 - d\,(p-2)\,\theta/2p = 1 - \theta\,\theta_\star$.

An explicit expression of μ_1^∞ can be deduced from the proof.

Proof. We do a *blow-up analysis* based on a change of variables depending on $\lambda \to +\infty$. Let $u_\lambda := \mathsf{S}^{-1}\,v_\lambda$ where

$$v_\lambda(x) := v\left(2\sqrt{\tfrac{(\lambda-\lambda_\star)}{p-2}}\,x\right) \quad \forall\,x \in \mathbb{R}^d\,.$$

We take $\lambda_\star := \frac14\,d\,(d-2)\,(p-2)$ and v as an optimal function for (14). With $r = |x|$, we compute

$$\begin{aligned}
\mathcal{Q}^{(1)}_{p,\theta,\lambda}[u_\lambda] &= \frac{(p-2)\left(\|\nabla u_\lambda\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac14\,d\,(d-2)\,\|u_\lambda\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + (\lambda-\lambda_\star)\,\|u_\lambda\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_\lambda\|^{2\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u_\lambda\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\\
&= \frac{(p-2)\,\|\nabla v_\lambda\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + (\lambda-\lambda_\star)\,\|\mathsf{m}(r)^{-2}\,v_\lambda\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{|\mathbb{S}^d|^{\theta\,(1-2/p)}\,\|\mathsf{m}(r)^{-\delta(p)/p}\,v_\lambda\|^{2\theta}_{\mathrm{L}^p(\mathbb{R}^d)}\,\|\mathsf{m}(r)^{-2}\,v_\lambda\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R}^d)}}\\
&= \frac{(\lambda-\lambda_\star)\left(\frac{4\,(\lambda-\lambda_\star)}{p-2}\right)^{-d\,\frac{p-2}{2p}\,\theta}\left(4\,\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|\mathsf{m}(\varepsilon\,r)^{-2}\,v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right)}{|\mathbb{S}^d|^{\theta\,(1-2/p)}\,\|\mathsf{m}(\varepsilon\,r)^{-\delta(p)/p}\,v\|^{2\theta}_{\mathrm{L}^p(\mathbb{R}^d)}\,\|\mathsf{m}(\varepsilon\,r)^{-2}\,v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{R}^d)}}\,.
\end{aligned}$$

with $\varepsilon = \sqrt{\frac{p-2}{4\,(\lambda-\lambda_\star)}}$. By taking the limit as $\lambda \to +\infty$ and using Lebesgue's theorem of dominated convergence with $\mathsf{m}(\varepsilon\,r) \to 1/\sqrt2$ a.e., and by using the fact that the quotient is proportional to the Gagliardo-Nirenberg-Sobolev inequality up to a numerical constant, we conclude that

$$\mathcal{Q}^{(1)}_{p,\theta,\lambda}[u_\lambda] = (\lambda-\lambda_\star)^{1-\theta\theta_\star}\,\mu_1^\infty(p,\theta)\big(1+o(1)\big)$$

as $\lambda \to +\infty$, for a constant $\mu_1^\infty(p,\theta)$ that can be computed explicitly, up to tedious but elementary considerations. This proves that

$$\lim_{\lambda\to+\infty} \lambda^{-\gamma}\,\mu_1(p,\theta,\lambda) \le \mu_1^\infty(p,\theta).$$

We have now to prove the reverse inequality. As in [16, Proposition 10], we argue by contradiction. Let $(\lambda_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be such that $u_n \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$ with $\|u_n\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$ for any $n \in \mathbb{N}$, $\lim_{n\to+\infty}\lambda_n = +\infty$,

$$\mathcal{Q}^{(1)}_{p,\theta,\lambda_n}[u_n] = \mu_1(p,\theta,\lambda_n) \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n^{-\gamma}\,\mu_1(p,\theta,\lambda_n) \le \mu_1^\infty(p,\theta) - \eta \tag{15}$$

for some $\eta > 0$. We learn from the expression of $\mathcal{Q}^{(1)}_{p,\theta,\lambda}$ that $\lambda_n\,\|u_n\|^{2\theta}_{\mathrm{L}^2(\mathbb{S}^d)} \le \mu_1^\infty(p,\theta)\,\lambda_n^\gamma$ as $n \to +\infty$ so that $\lim_{n\to+\infty}\|u_n\|_{\mathrm{L}^2(\mathbb{S}^d)} = 0$. Concentration occurs: there exists an eventually finite sequence $(y_i)_{i\in\mathbb{N}}$ of points in $\mathbb{S}^d$, sequences of positive numbers $(\zeta_i)_{i\in\mathbb{N}}$ and $(r_{i,n})_{i,n\in\mathbb{N}}$ and functions $u_{i,n} \in \mathrm{H}^1(\mathbb{S}^d, d\sigma)$ with

$$u_{i,n} = u_n \quad \text{on} \quad \mathbb{S}^d \cap B(y_i, r_{i,n}) \quad \text{and} \quad \operatorname{supp} u_{i,n} \subset \mathbb{S}^d \cap B(y_i, 2\,r_{i,n})$$

such that, as $n\to+\infty$,

$$\sum_i \|\nabla u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \sim \|\nabla u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad\text{and}\quad \sum_i \|u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \sim \|u_n\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,,$$

with, for all i,

$$\lim_{n\to+\infty} r_{i,n}=0\,,\quad \Sigma_{i\in\mathbb{N}}\zeta_i=1 \quad\text{and}\quad \lim_{n\to+\infty}\int_{\mathbb{S}^d\cap B(y_i,r_{i,n})}|u_{i,n}|^p\,d\sigma=\zeta_i\,.$$

With a *blow up* argument applied to $(u_{i,n})_{n\in\mathbb{N}}$, we prove for all i that

$$\lim_{n\to+\infty}\lambda_n^{-\gamma}\left((p-2)\,\|\nabla u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\lambda_n\,\|u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\|u_{i,n}\|^{-2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\ge\mu_1^\infty(p,\theta)\,\zeta_i^{2\theta/p}\,.$$

We may notice that

$$\frac{(p-2)\,\|\nabla u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\lambda_n\,\|u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_{i,n}\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\ge\frac{(p-2)\,\|\nabla u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\lambda_n\,\|u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_n\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\,.$$

Let us choose an integer N such that

$$\Sigma_{i=1}^N\zeta_i^{2\theta/p}\ge\left(\Sigma_{i=1}^N\zeta_i\right)^{2\theta/p}>1-\frac{\eta}{2}\left(\mu_1^\infty(p,\theta)\right)^{-1}.$$

For n large enough, by writing

$$\lambda_n^{-\gamma}\sum_{i=1}^N\frac{(p-2)\,\|\nabla u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\lambda_n\,\|u_{i,n}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u_{i,n}\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}\ge\mu_1^\infty(p,\theta)\sum_{i=1}^N\zeta_i^{2\theta/p}\left(1+o(1)\right)\ge\mu_1^\infty(p,\theta)-\frac{\eta}{2}\,,$$

we obtain a contradiction with (15). This concludes the proof. □

We collect our results on inequality (4) with optimal constant $\mu_1(p,\theta,\lambda)$ as follows.

Theorem 9. *Let $d\ge1$, $\theta\in[0,1]$, $p\in(2,2^*)$ or $p=2^*$ if $d\ge3$. Inequality (3) holds with $\mu_0(p,\lambda)=\lambda$ if $\lambda\le d/\theta$ (symmetry case) and $\mu_0(p,\lambda)<\lambda$ if $\lambda>d/\theta$ (symmetry breaking case). The function $\lambda\mapsto\mu_1(p,\theta,\lambda)$ is monotone increasing, concave and such that for some explicit $\mu_1^\infty(p,\theta)>0$, we have $\mu_1(p,\theta,\lambda)\sim\mu_1^\infty(p,\theta)\,\lambda^\gamma$ as $\lambda\to+\infty$ with $\gamma=1-2\theta\,\theta_\star$.*

4. A SECOND FAMILY OF REFINED INTERPOLATION INEQUALITIES ON $\mathbb{S}^d$

In this section, we consider (5). Let us define

$$\mathcal{Q}^{(2)}_{p,\theta,\lambda}[u]:=\frac{\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\,.$$

• ***Admissible parameters.*** We establish the validity range of the parameters in (5). We recall that $\theta_\star=d\,(p-2)/(2\,p)$ according to the definition (6). Let us define

$$q:=\frac{2\,p\,\theta}{2-p\,(1-\theta)}\tag{16}$$

so that $q=2^\star$ if $\theta=\theta_\star$ and $q=p$ if $\theta=1$.

Lemma 10. *Inequality* (5) *holds for any d, θ and p such that*

$$d\ge 1\,,\quad \theta\in[\theta_\star,1]\cap(1-2/p,1]\,,\quad p\in(2,2^*)\quad or\quad p=2^*\quad if\quad d\ge 3 \tag{17}$$

with an optimal constant $\mu_2(p,\theta,\lambda)$ such that

$$\mu_0\big(q,\lambda\,\tfrac{q-2}{p-2}\big)\le\mu_2(p,\theta,\lambda)\le\lambda\quad\forall\,\lambda>0$$

where q is given in terms of p and θ by (16). *Moreover, $\mu_2(p,\theta,\lambda)=\lambda$ if $\lambda\le d\,\big(1-(1-\theta)\,\frac p2\big)$.*

Proof. By Hölder's inequality, $\|u\|_{\mathrm L^2(\mathbb S^d)}\le\|u\|_{\mathrm L^p(\mathbb S^d)}$ because $d\sigma$ is a probability measure. Using (12), we learn that

$$\mu_0\big(q,\tilde\lambda\big)=\inf_{u\in\mathrm H^1(\mathbb S^d,d\sigma)\setminus\{0\}}\mathcal Q^{(1)}_{q,1,\tilde\lambda}[u]\le\inf_{u\in\mathrm H^1(\mathbb S^d,d\sigma)\setminus\{0\}}\Big(\mathcal Q^{(2)}_{p,\theta,\lambda}[u]\Big)^{1/\theta}\le\lambda$$

with

$$\tilde\lambda:=\lambda\,\frac{q-2}{p-2}\,.$$

The upper bound is obtained by testing $\mathcal Q^{(2)}_{p,\theta,\lambda}$ with $u=1$ a.e. If $\tilde\lambda\le d$, there is equality because $\mu_0\big(q,\tilde\lambda\big)=\tilde\lambda$ (symmetry case) according to Theorem 1, which gives us a *symmetry* case. Notice that $1-(1-\theta)\,\frac p2$ is positive if $\theta>1-2/p$. □

As above, let us consider

$$\mathscr F^{(2)}_\lambda[u]:=\left(\|\nabla u\|^2_{\mathrm L^2(\mathbb S^d)}+\frac{\lambda}{p-2}\,\|u\|^2_{\mathrm L^2(\mathbb S^d)}\right)^\theta\,\|u\|^{2\,(1-\theta)}_{\mathrm L^2(\mathbb S^d)}-\left(\frac{\lambda}{p-2}\right)^\theta\,\|u\|^2_{\mathrm L^p(\mathbb S^d)}\,.$$

• ***Symmetry breaking.*** Again symmetry breaking is obtained by perturbation.

Lemma 11. *Assume that d, θ and p satisfy* (17). *There is* symmetry breaking *in* (5), *that is, $\mu_2(p,\theta,\lambda)<\lambda$, if $\lambda>\theta\,d$.*

Proof. With the same computations as in the proof of Lemma 11 and a non-trivial spherical harmonic function φ, we obtain that

$$\lim_{\varepsilon\to0}\frac1{\varepsilon^2}\mathscr F^{(2)}_\lambda[u_\varepsilon]=\left(\frac{\lambda}{p-2}\right)^\theta\left(\theta\,\|\nabla\varphi\|^2_{\mathrm L^2(\mathbb S^d)}-\lambda\,\|\varphi\|^2_{\mathrm L^2(\mathbb S^d)}\right)=(d\,\theta-\lambda)\,\|\varphi\|^2_{\mathrm L^2(\mathbb S^d)}<0\,,$$

which concludes the proof. □

• ***Symmetry.*** We apply the same methods as in Sections 2 and 3.

Lemma 12. *Assume that d, θ and p satisfy* (17). *If $\theta\le1/2$, there is* symmetry *in* (5), i.e., *$\mu_2(p,\theta,\lambda)=\lambda$, if $\lambda\le\theta\,d$.*

It is important to notice that the compatibility of the condition $\theta\le1/2$ with (17) induces restrictions not only on θ but also on p if $d\ge2$: $p<2$ if $d=2$ and $p\le2\,d/(d-1)$ if $d\ge3$.

Proof. Let us consider an optimal function $u\in\mathrm H^1(\mathbb S^d,d\sigma)$ for (5) such that $\|u\|_{\mathrm L^p(\mathbb S^d)}=1$ and let $\mathsf a:=\|\nabla u\|^2_{\mathrm L^2(\mathbb S^d)}$ and $\mathsf b:=\|u\|^2_{\mathrm L^2(\mathbb S^d)}$. We assume that u is not a constant so that $\mathsf b<1$

by Hölder's inequality. By optimality, we have that

$$(p-2)\,\mathsf{a}+\lambda\,\mathsf{b}=\mu_2(p,\theta,\lambda)\,\mathsf{b}^{\frac{1}{\theta}-1}<\lambda\,\mathsf{b}^{\frac{1}{\theta}-1}\,.$$

Let us consider the same flow as in Lemma 12. At $t=0$, we know that $\frac{d}{dt}\mathscr{F}^{(2)}_{\lambda}[v(t,\cdot)]=0$ because u is a critical point, and the computation shows that this amounts to

$$\frac{\theta\,(\lambda-d)}{(p-2)\,\mathsf{a}+\lambda\,\mathsf{b}}+\frac{1-\theta}{\mathsf{b}}=0\,.$$

By Lemma 10 symmetry eventually holds only if $\lambda\le\theta\,d<d$. Let us assume that our non symmetric minimizer corresponds to the range $\lambda\le\theta\,d$. Hence $\lambda-d<0$ and

$$\lambda-\theta\,d>\theta\,(\lambda-d)+(1-\theta)\,\lambda\,\mathsf{b}^{\frac{1}{\theta}-2}>0$$

if $\theta<1/2$ because $\mathsf{b}^{\frac{1}{\theta}-2}<1$. This is obviously a contradiction. We conclude that there is no non-symmetric optimizer if $\theta<1/2$ and $\lambda\le\theta\,d$.

So far, this proof is formal as we assume that a minimizer exists. By standard compactness results, this is always true, except for some limiting values of θ or p. However, one can always approximate an inequality like (5) by another inequality in the same family, with slightly different exponents and pass to the limit. This concludes the proof. □

Let us conclude by a few remarks. The issue of *symmetry* versus *symmetry breaking* is far from fully answered in case of inequality (5), which is not unexpected in view of similar difficulties in corresponding *Caffarelli-Kohn-Nirenberg inequalities*. This is clearly an open and interesting direction of research. A good point is that the *carré du champ* method at least provides some partial answers, which are not available in similar inequalities on the Euclidean space with weights. There are many other questions which are pending and will be the subject of future work: asymptotics as $\lambda\to+\infty$ for (5), reparametrization and numerical computation of the branches, expansion of the branches near the bifurcation point, energy estimates in relation with symmetry breaking in the asymptotic regime corresponding to $p-2>0$, small, *etc.*

APPENDIX A. NONLINEAR CARRÉ DU CHAMP METHOD ON THE SPHERE

For completeness, we give a summary of the *carré du champ* method in integral form applied to the sphere with entropy methods and a nonlinear diffusion equation of fast diffusion or porous medium type. More details can be found in a series of earlier papers on manifolds (see [9, 17], [20, Section 5.2] or [23]) with similar computations done in the perspective of uniqueness for positive solutions of elliptic equations (rigidity) that can be found in [26, 5]. In the specific case of the sphere, it is possible to reduce the problem to a computation based on the ultraspherical operator (see [14, 15, 21, 12, 24]). The method in the $d=1$ case is slightly different (see for instance [20, Section 5.3]) and results do not require the use of nonlinear diffusion equations as it is known for a long time from [1, 2], so this will not be discussed here. There are exceptional values of the exponent ($p=6$ if $d=3$, or $p=3$ if $d=4$) which require a specific treatment that will not be detailed here, but the result can easily be recovered by considering the limit as $p\to 6$ if $d=3$ and $p\to 3$ if $d=4$ in the inequalities: see for instance [21]. Here we adopt the presentation of [17] and go along the lines of the proof of [23, Lemma 4.3]. We adopt simplified notations and write Id instead of using the standard metric, or simply write the Laplace-Beltrami operator as $-\Delta$. We split the method in two steps, with a first algebraic part closely related with the computations of [5] and entropy methods applied to a parabolic equation in order to establish the inequalities. Because of the regularizing effects of the parabolic equation, there is no regularity issue in considering positive solutions and solutions will be considered as smooth without further notice. An approximation scheme has of course to be done for initial data in $H^1(\mathbb{S}^d,d\sigma)$: see for instance [24] for details in the case of the ultraspherical operator.

A.1. **A purely algebraic computation.** Let $\mathsf{H}u$ denote the *Hessian* of a C^2 function u on $\mathbb{S}^d$ and define the *trace free Hessian* by

$$\mathsf{L}u:=\mathsf{H}u-\frac{1}{d-1}\,(\Delta u)\,\mathrm{Id}\,.$$

We also consider the following trace free tensor

$$\mathsf{M}u:=\frac{\nabla u\otimes\nabla u}{u}-\frac{1}{d-1}\,\frac{|\nabla u|^2}{u}\,\mathrm{Id}\,,$$

where

$$\nabla u\otimes\nabla u:=\big(\partial_i u\,\partial_j u\big)_{ij}\quad\text{and}\quad\|\nabla u\otimes\nabla u\|^2=|\nabla u|^4=\sum_{ij}\big(\partial_i u\big)^2\big(\partial_j u\big)^2\,.$$

Using $\mathsf{L}:\mathrm{Id}=\mathsf{M}:\mathrm{Id}=0$ and

$$\|\mathsf{L}u\|^2=\|\mathsf{H}u\|^2-\frac{1}{d}\,(\Delta u)^2\,,$$

$$\|\mathsf{M}u\|^2=\left\|\frac{\nabla u\otimes\nabla u}{u}\right\|^2-\frac{1}{d}\,\frac{|\nabla u|^4}{u^2}=\frac{d-1}{d}\,\frac{|\nabla u|^4}{u^2}\,,$$

we deduce from

$$\int_{\mathbb{S}^d}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma=\int_{\mathbb{S}^d}\frac{|\nabla u|^4}{u^2}\,d\sigma-2\int_{\mathbb{S}^d}\mathsf{H}u:\frac{\nabla u\otimes\nabla u}{u}\,d\sigma$$
$$=\frac{d-1}{d-2}\int_{\mathbb{S}^d}\|\mathsf{M}u\|^2\,d\sigma-2\int_{\mathbb{S}^d}\mathsf{L}u:\frac{\nabla u\otimes\nabla u}{u}\,d\sigma-\frac{2}{d}\int_{\mathbb{S}^d}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma$$

that

$$\int_{\mathbb{S}^d}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma=\frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathsf{M}u\|^2\,d\sigma-2\int_{\mathbb{S}^d}\mathsf{L}u:\frac{\nabla u\otimes\nabla u}{u}\,d\sigma\right).$$

This provides us with a first identity,

$$\int_{\mathbb{S}^d}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma=\frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathsf{M}u\|^2\,d\sigma-2\int_{\mathbb{S}^d}\mathsf{L}u:\mathsf{M}u\,d\sigma\right).\tag{18}$$

The Bochner-Lichnerowicz-Weitzenböck formula on $\mathbb{S}^d$ takes the simple form

$$\tfrac12\,\Delta\,(|\nabla u|^2)=\|\mathsf{H}u\|^2+\nabla(\Delta u)\cdot\nabla u+(d-1)\,|\nabla u|^2$$

where the last term, *i.e.*, $\mathrm{Ric}(\nabla u,\nabla u)=(d-1)\,|\nabla u|^2$, accounts for the Ricci curvature tensor contracted with $\nabla u\otimes\nabla u$. An integration of this formula on $\mathbb{S}^d$ shows a second identity,

$$\int_{\mathbb{S}^d}(\Delta u)^2\,d\sigma=\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathsf{L}u\|^2\,d\sigma+d\int_{\mathbb{S}^d}|\nabla u|^2\,d\sigma\,. \tag{19}$$

Hence for some parameters β and κ to be fixed later, the functional

$$\begin{aligned}\mathsf{k}[u]&:=\int_{\mathbb{S}^d}\left(\Delta u+\kappa\,\frac{|\nabla u|^2}{u}\right)\left(\Delta u+(\beta-1)\,\frac{|\nabla u|^2}{u}\right)d\sigma\\&=\int_{\mathbb{S}^d}(\Delta u)^2\,d\sigma+(\kappa+\beta-1)\int_{\mathbb{S}^d}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma+\kappa\,(\beta-1)\int_{\mathbb{S}^d}\frac{|\nabla u|^4}{u^2}\,d\sigma\end{aligned} \tag{20}$$

can be rewritten using (18) and (19) as

$$\mathsf{k}[u]=\int_{\mathbb{S}^d}\mathsf{q}[u]\,d\sigma+d\int_{\mathbb{S}^d}|\nabla u|^2\,d\sigma$$

where

$$\mathsf{q}[u]:=\frac{d}{d-1}\left(\mathsf{a}\,\|\mathsf{L}u\|^2-2\,\mathsf{b}\,\mathsf{L}u:\mathsf{M}u+\mathsf{c}\,\|\mathsf{M}u\|^2\right)$$

and

$$\mathsf{a}=1\,,\quad\mathsf{b}=(\kappa+\beta-1)\,\frac{d-1}{d+2}\,,\quad\mathsf{c}=(\kappa+\beta-1)\,\frac{d}{d+2}+\kappa\,(\beta-1)\,. \tag{21}$$

Let us consider the choice of the parameters

$$\kappa=\beta\,(p-2)+1\quad\text{and}\quad m=1+\frac{2}{p}\left(\frac{1}{\beta}-1\right) \tag{22}$$

and recall that by definition (10), we have

$$m_\pm(d,p):=\frac{1}{(d+2)\,p}\left(d\,p+2\pm\sqrt{d\,(p-1)\big(2\,d-(d-2)\,p\big)}\right).$$

Lemma 13. *Let $d\ge2$, $p\ge1$ if $d=2$, and $p\in[1,2^*]$ if $d\ge3$. If $m_-(d,p)\le m\le m_+(d,p)$, then q is a positive quadratic form if κ and β are given in terms m and p by (22). As a consequence, we have*

$$\mathsf{k}[u]\ge d\int_{\mathbb{S}^d}|\nabla u|^2\,d\sigma\quad\forall\,u\in C^2(\mathbb{S}^d)\,.$$

Proof. It follows from the reduced discriminant condition $\mathsf{b}^2-\mathsf{a}\,\mathsf{c}<0$ by tedious but elementary computations. In the cases $p=6$ if $d=3$ and $p=3$ if $d=4$, computations have to be done directly with m, without using the parameter β (see below). □

In dimension $d=1$, we have a similar result with $m=\beta=1$ using the Poincaré inequality on the circle $\mathbb{S}^1$.

A.2. **Entropy methods and nonlinear flows.** Let us introduce the parabolic evolution setting corresponding to the nonlinear diffusion equation

$$\frac{\partial w}{\partial t}=w^{2-2\beta}\left(\Delta w+\kappa\,\frac{|\nabla w|^2}{w}\right). \tag{23}$$

The function

$$\rho(t,x) = \big(w(m\,t,x)\big)^{\beta\,p} \quad \forall\,(t,x)\in\mathbb{R}^+\times\mathbb{S}^d$$

with κ and β as in (22) solves the standard fast-diffusion equation (9). From this observation, it is clear that

$$\frac{d}{dt}\int_{\mathbb{S}^d} w^{\beta\,p}\,d\sigma = \frac{d}{dt}\int_{\mathbb{S}^d}\rho\,d\sigma = 0\,.$$

Lemma 14. *Let $d\ge 2$, $p\ge 1$ if $d=2$, $m>0$ and $p\in[1,2^*]$ if $d\ge 3$. With κ and β as in (22), any smooth positive solution of (23) is such that*

$$\frac{d}{dt}\int_{\mathbb{S}^d} w^{2\beta}\,d\sigma = 2\,\beta^2\,(p-2)\int_{\mathbb{S}^d}|\nabla w|^2\,d\sigma$$

and, with k defined by (20),

$$\frac{d}{dt}\int_{\mathbb{S}^d}\big|\nabla w^\beta\big|^2\,d\sigma = -\,2\,\beta^2\,\mathsf{k}[w]\,.$$

As a consequence, if $m_-(d,p)\le m\le m_+(d,p)$, then we have

$$\frac{d}{dt}\int_{\mathbb{S}^d}\big|\nabla w^\beta\big|^2\,d\sigma \le -\,2\,\beta^2\,d\int_{\mathbb{S}^d}|\nabla w|^2\,d\sigma$$

Proof. These results are easily proved using a few integrations by parts and Lemma 13. □

A standard consequence is, for instance, the fact that

$$\frac{d}{dt}\mathscr{F}_\lambda\big[w^\beta\big] \le -\,2\,\beta^2\,(d-\lambda)\int_{\mathbb{S}^d}|\nabla w|^2\,d\sigma$$

if w solves (23) and $\mathscr{F}_\lambda$ is defined by (8). Hence if $\lambda\le d$ and (23) is supplemented with the initial datum $w(0,\cdot)=u^{1/\beta}$, then

$$\mathscr{F}_\lambda[u] \ge \mathscr{F}_\lambda\big[w^\beta(t,\cdot)\big] = \lim_{s\to+\infty}\mathscr{F}_\lambda\big[w^\beta(s,\cdot)\big] = 0 \quad \forall\,t\ge 0\,.$$

The limit as $t\to+\infty$ can be identified by analyzing $\mathsf{k}[w]=0$ and proving that this means that w is then a constant if $p<2^*$. In the case $p=2^*$, identifying w as a constant can be done only by considering w as a limit as $t\to+\infty$ of a solution of (23).

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References

[1] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math., Springer, Berlin, 1985, pp. 177–206.

[2] D. Bakry and M. Émery, *Inégalités de Sobolev pour un semi-groupe symétrique*, C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), pp. 411–413.

[3] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and geometry of Markov diffusion operators*, vol. 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Cham, 2014.

[4] W. BECKNER, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2), 138 (1993), pp. 213–242.

[5] M.-F. BIDAUT-VÉRON AND L. VÉRON, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math., 106 (1991), pp. 489–539.

[6] ———, *Erratum: "Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations" [Invent. Math.* **106** *(1991), no. 3, 489–539; MR1134481 (93a:35045)]*, Invent. Math., 112 (1993), p. 445.

[7] M. BONFORTE, J. DOLBEAULT, M. MURATORI, AND B. NAZARET, *Weighted fast diffusion equations (Part I): Sharp asymptotic rates without symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities*, Kinetic and Related Models, 10 (2017), pp. 33–59.

[8] D. BONHEURE, J. DOLBEAULT, AND H. TAVARES, *Some systems arising from Lieb-Thirring kinetic energy inequalities*. Work in progress.

[9] J. DEMANGE, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611.

[10] J. DOLBEAULT AND M. J. ESTEBAN, *A scenario for symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities*, Journal of Numerical Mathematics, 20 (2013), pp. 233—249.

[11] ———, *Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations*, Nonlinearity, 27 (2014), p. 435.

[12] ———, *Improved interpolation inequalities and stability*, Advanced Nonlinear Studies, 20 (2020), pp. 277–291.

[13] J. DOLBEAULT, M. J. ESTEBAN, S. FILIPPAS, AND A. TERTIKAS, *Rigidity results with applications to best constants and symmetry of Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities*, Calculus of Variations and Partial Differential Equations, 54 (2015), pp. 2465–2481.

[14] J. DOLBEAULT, M. J. ESTEBAN, M. KOWALCZYK, AND M. LOSS, *Sharp interpolation inequalities on the sphere: New methods and consequences*, Chinese Annals of Mathematics, Series B, 34 (2013), pp. 99–112.

[15] ———, *Improved interpolation inequalities on the sphere*, Discrete and Continuous Dynamical Systems Series S (DCDS-S), 7 (2014), pp. 695–724.

[16] J. DOLBEAULT, M. J. ESTEBAN, AND A. LAPTEV, *Spectral estimates on the sphere*, Analysis & PDE, 7 (2014), pp. 435–460.

[17] J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Nonlinear flows and rigidity results on compact manifolds*, Journal of Functional Analysis, 267 (2014), pp. 1338 – 1363.

[18] J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*, Journal of elliptic and parabolic equations, 2 (2016), pp. 267–295.

[19] ———, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, Inventiones Mathematicae, 206 (2016), pp. 397–440.

[20] ———, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, Invent. Math., 206 (2016), pp. 397–440.

[21] ———, *Interpolation inequalities on the sphere: linear vs. nonlinear flows (inégalités d'interpolation sur la sphère : flots non-linéaires vs. flots linéaires)*, Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2017), pp. 351–379.

[22] ———, *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs.*, Proc. Int. Cong. of Math. 2018, Rio de Janeiro, 3 (2018), pp. 2279–2304.

[23] J. DOLBEAULT, M. J. ESTEBAN, M. LOSS, AND M. MURATORI, *Symmetry for extremal functions in subcritical Caffarelli-Kohn-Nirenberg inequalities*, Comptes Rendus Mathématique. Académie des Sciences. Paris, 355 (2017), pp. 133–154.

[24] J. DOLBEAULT AND A. ZHANG, *Parabolic methods for ultraspherical interpolation inequalities*, Discrete and Continuous Dynamical Systems, (2022).

[25] V. FELLI AND M. SCHNEIDER, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, Journal of Differential Equations, 191 (2003), pp. 121–142.

[26] B. GIDAS AND J. SPRUCK, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math., 34 (1981), pp. 525–598.

[27] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2), 118 (1983), pp. 349–374.

[28] C. VILLANI, *Optimal transport*, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009. Old and new.

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