TIME-DEPENDENT RESCALINGS AND DISPERSION FOR THE BOLTZMANN EQUATION

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Summary: Using the notion of time-dependent rescalings introduced in [DR], we prove explicit dispersion results. The main tool is a Lyapunov functional which is given by the energy after rescaling and the entropy dissipation. The relation with asymptotically self-similar solutions is investigated. The method applies to the solutions of the Boltzmann, Landau and BGK equations. In the case of the Vlasov-Fokker-Planck equation, the difference with the self-similar solutions has a faster decay, which is estimated by a classical method for parabolic equations and interpolation estimates.

I. Introduction

Consider in $I\!R^3$ the Boltzmann equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_B(f,f) \tag{B}$$

with

$$Q_B(f,f) = \int\int_{I\!R^3 \times S^2} B(v-v_*,\omega)\Big(f'f'_* - ff_*\Big)\, dv_* d\omega \ .$$

Here f', f'_*, f and f_* are the standard notations for respectively $f(t,x,v')$, $f(t,x,v'_*)$, $f(t,x,v)$ and $f(t,x,v_*)$ where (v',v'_*) and (v,v_*) are the incoming and outcoming velocities corresponding to a deflection given by the angle $\omega \in S^2$:

$$\begin{cases} v' = v - [(v-v_*)\cdot\omega]\omega \ , \\ v'_* = v_* + [(v-v_*)\cdot\omega]\omega \ , \end{cases}$$

We assume that the cross-section takes the form

$$B(v-v_*,\omega) = |v-v_*|^\gamma \ b(\frac{v-v_*}{|v-v_*|}\cdot\omega) \ . \tag{PL}$$

γ is determined by the 2-body power law of interaction as follows: if the 2-body force is proportional to $\frac{1}{r^s}$ with $s \geq 2$, then

$$\gamma = \frac{s-5}{s-1} \ .$$

$s = 2$ is the case of the Coulomb force term, which is a limit case for the Boltzmann equation.

The corresponding Landau equation in $I\!R^3$ is

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q_L(f,f) \tag{L}$$

with

$$Q_L(f,f) = \sum_{i,j=1}^{3} \frac{\partial}{\partial v_i} \int_{I\!R^3} |v - v_*|^{\gamma+2} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2}_{|z=v-v_*} \right) \left(f_* \frac{\partial f}{\partial v_j} - f \frac{\partial f_*}{\partial v_{*j}} \right) dv_* \ .$$

The BGK equation is a simplified version of the Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{T_0}(M_{\rho,u,\theta} - f) \tag{BGK}$$

where T_0 is a positive constant and ρ, u, θ are defined in terms of the moments of f:

$$\Big(\rho(t,x), \rho(t,x)u(t,x), 3\rho(t,x)\theta(t,x)\Big) = \left(\int_{I\!R^3} f(t,x,v)\ dv\ ,\ \int_{I\!R^3} f(t,x,v)v\ dv\ ,\ \int_{I\!R^3} f(t,x,v)|v-u|^2\ dv \right) .$$

As a caricature(*) of the Landau equation, we may also consider the (linear) Vlasov-Fokker-Planck equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \beta \mathrm{div}_v(vf) + \sigma \Delta_v f\ , \tag{VFP}$$

where β and σ are two nonnegative constants (friction and diffusion coefficients respectively). The corresponding Green function is explicitely known: using its invariances, we will prove in the last section that the difference with the solution to the Cauchy problem has a faster decay.

We will not give detailed references on the existence theory of the Boltzmann equation, and will refer to [L1] for a more extended list of references, and also to [CIP]. Let us simply mention [IS], [G] and [BPT] for the existence of classical solutions, and [DPL1], [DPL2] and [L1] for the existence of renormalized solutions (we will work in that framework). Concerning the Landau equation, we may quote the results by P.-L. Lions (see [L2]) and C. Villani (see [V1]) and for the BGK model, [P1], [P3] and [M].

For the notions of dispersion (see [MP] for an application to the Boltzmann equation), we will refer to [IR], [P1], [P2], [CDP] and [DR]. The time-dependent rescaling method has been developped in the context of kinetic equation in the last of these references.

One has to mention here that the notion of dispersion has been used a long time ago by R. Illner and M. Shindbrot (see [IS]) to obtain an existence result for the Boltzmann equation. Since they were building solutions in L^∞, they were getting the decay one might expect for the free transport equation, actually that $\rho(t,.) = \int f(t,.,v)\ dv$ decays like $\frac{1}{t^3}$. Here we deal with weak solutions (actually even renormalized solutions for the Boltzmann and the Landau equations) for which only the entropy and some moments are known to be bounded. This gives rise to a much slower decay. How to fill the gap between these situations is still open.

Transformations of kinetic equations have been extensively used to build special solutions. The notations τ, η we shall use thereafter are the classical notations of the celebrated Nikolskii transform, and we will refer

(*) If $s = 5$, the homogeneous Landau equation can be written (see [V2]) after a rescaling and a convenient renormalization as

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \sum_{i,j=1}^{3} \Big((|v|^2 \delta_{ij} - v_i v_j) *_v f \Big) \frac{\partial^2 f}{\partial v_i \partial v_j} - 3f\ ,$$

and the equation is reduced to the Vlasov-Fokker-Planck equation if f is radially symmetric in $|v|$ (isotropic case). However, this reduction is valid only for a correctly normalized distribution function. This explains why the scaling properties of the (linear) Vlasov-Fokker-Planck equation differ from the scaling properties of the Landau equation.

to [Co] or [BCS] for more references in this direction. However, our purpose is a little bit different, since we are intereted in building integral quantities (that could even be exhibited without any justifications concerning the transformations of the equations) to obtain dispersion results and we will therefore not insist on that point.

II. The Boltzmann equation

As in [DR], we may consider the time-dependent rescaling given by

$$\dot{\tau} = \frac{1}{A^2(t)} ,$$

$$\xi(t,x) = \frac{x}{R(t)} ,$$

$$\eta(t,x,v) = \frac{A^2(t)}{R(t)}(v - \frac{\dot{R}(t)}{R(t)}x) .$$

Here $\dot{}$ denotes the derivative with respect to t. A natural requirement is the preservation of the L^1-norm: if F is the rescaled distribution function given by

$$f(t,x,v) = \left(\frac{A(t)}{R(t)}\right)^6 F(\tau,\xi,\eta) ,$$

it has to satisfy the rescaled Boltzmann equation (without cut-off):

$$\frac{\partial F}{\partial \tau} + \eta \cdot \nabla_\xi F + 2A^2(t)\left(\frac{\dot{A}(t)}{A(t)} - \frac{\dot{R}(t)}{R(t)}\right)\mathrm{div}_\eta(\eta F) - \frac{\ddot{R}(t)A^4(t)}{R(t)}\xi \cdot \nabla_\eta F = A^{2(1-\gamma)}(t)R^{\gamma-3}(t)Q_B(F,F) .$$

With the ansatz

$$A = R , \quad \ddot{R} = \frac{1}{R^3} ,$$

the rescaled distribution function is a solution of

$$\frac{\partial F}{\partial \tau} + \eta \cdot \nabla_\xi F - \xi \cdot \nabla_\eta F = \epsilon(\tau) Q_B(F,F) , \qquad (RB)$$

with

$$\epsilon(\tau(t)) = \frac{1}{R^{1+\gamma}(t)} .$$

Assuming further that $R(0) = 1$, $\dot{R}(0) = 0$ and $\tau(0) = 0$, which means that f and F satisfy the same initial conditions, the change of variables means

$$R(t) = \sqrt{1+t^2} , \quad \dot{R}(t) = \frac{t}{\sqrt{1+t^2}} ,$$

$$\tau(t) = \mathrm{Arctg}(t) ,$$

$$\xi(t,x) = \frac{x}{\sqrt{1+t^2}} ,$$

$$\eta(t,x,v) = \sqrt{1+t^2}(v - \frac{t}{1+t^2}x) ,$$

$$f(t,x,v) = F(\tau(t),\xi(t,x),\eta(t,x,v))\ .$$

Exactly as for the usual Boltzmann equation, the following *a priori* estimates hold, at least at a formal level:

$$\frac{d}{d\tau}\int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\ d\xi d\eta = 0\ , \qquad (Mass)$$

$$\frac{d}{d\tau}\int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\ (\frac{1}{2}|\eta|^2+\frac{1}{2}|\xi|^2)\ d\xi d\eta = 0\ , \qquad (Energy)$$

and Boltzmann's H theorem is now

$$\frac{d}{d\tau}\int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\log\Big(F(\tau,\xi,\eta)\Big)\ d\xi d\eta = -\epsilon(\tau)\int\int\int\int_{(I\!R^3)^3\times S^2} B(\eta-\eta_*,\omega)\Big(F'F'_*-FF_*\Big) \cdot\log\Big(\frac{F'F'_*}{FF_*}\Big)\ d\xi d\eta d\eta_* d\omega\ .$$

Remark 1.

$$\begin{aligned} L(t) &= \frac{1}{2}\int\int_{I\!R^3\times I\!R^3} F(\tau(t),\xi,\eta)\ (|\eta|^2+|\xi|^2)\ d\xi d\eta \\ &= \frac{1}{2}(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{t}{1+t^2}x|^2\ dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2\ dxdv \end{aligned}$$

plays the same role as the Lyapunov functional in [DR] but is actually (at least formally) a conserved quantity. It can be seen as the limit as $\alpha\to 0_+$ of Lyapunov functionals L_α given by

$$L_\alpha(t) = \frac{1}{2}\int\int_{I\!R^3\times I\!R^3} F_\alpha(\tau(t),\xi,\eta)\ (|\eta|^2+|\xi|^2)\ d\xi d\eta$$

where F_α is given by the same formula as F except that the new ansatz for R and A is now

$$A = R^{1-\alpha}(t)\ ,\quad \ddot R = R^{4\alpha-3}\ .$$

In that case,

$$\frac{d}{d\tau}\int\int_{I\!R^3\times I\!R^3} F_\alpha(\tau,\xi,\eta)\ (|\eta|^2+|\xi|^2)\ d\xi d\eta = -4\alpha R^{1-2\alpha}(t)\dot R(t)\int\int_{I\!R^3\times I\!R^3} F_\alpha(\tau,\xi,\eta)\ |\eta|^2\ d\xi d\eta\ ,$$

$$\frac{d}{dt}L_\alpha(t) = -2\alpha\dot R(t)R(t)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{\dot R(t)}{R(t)}x|^2\ dxdv\ .$$

We shall use the identity (with $\alpha=1$) in Theorem 1 to prove the dispersion result. However, this essentially says nothing but:
1) $\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x-vt|^2\ dxdv$ is preserved.
2) $\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2\ dxdv$ grows at most like t^2 (as $t\to+\infty$) which is clear because of 1) and the conservation of the energy.

In the framework of the renormalized solutions of R.J. DiPerna and P.-L. Lions (see [DPL1], [DPL1]), these estimates are transformed into *a priori* estimates under the following assumptions on the collision kernel:

(H1) The Boltzmann collision kernel satifies the "weak angular cut-off": $B\in L^1_{loc}(I\!R^3\times S^2)$ which means

$$b\in L^1(S^2)\quad\text{and } \gamma>-3 \Longleftrightarrow s>2$$

in the case of a power law 2-body interaction. Note that in this case the cross-section derived from the corresponding force, which goes like $\frac{1}{r^s}$ gives a singularity of the order of $(\frac{v-v_*}{|v-v_*|}\cdot\omega)^{-\alpha}$ with $\alpha = \frac{s+1}{s-1}$ and that this singularity here is removed (but the term $b(\frac{v-v_*}{|v-v_*|}\cdot\omega)$ is invariant under the time-dependent rescaling).

(H2) The Boltzmann collision kernel satifies the "Mild Growth Condition":

$$\lim_{|z|\to+\infty} \frac{1}{1+|z|^2}\int_{|z-v|<R}(\int_{S^2} B(v,\omega)\, d\omega)\, dv = 0 \quad \forall\, R>0\,,$$

which means for the case of a power law 2-body interaction

$$\gamma < 2 \Longleftrightarrow s > 1\,.$$

(H3) Almost everywhere in $(v,\omega)\in I\!R^3\times S^2$, B is positive.

We first give a dispersion result for the Boltzmann equation which improves the one given in [P1]. This result does not depend on the special form (PL) we assumed in Section 1.

Theorem 1. *Assume that the collision kernel satifies assumptions (H1), (H2) and (H3), and consider a renormalized solution $f\in C^0(I\!R^+, L^1(I\!R^3\times I\!R^3))$ corresponding to an initial datum $f_0\in L^1(I\!R^3\times I\!R^3)$ such that $f_0(|x|^2+|v|^2+|\log f_0|)$ is bounded in $L^1(I\!R^3\times I\!R^3)$, and such that for any $t>0$, the estimate*

$$\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v|^2\, dxdv + \int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x-vt|^2\, dxdv \le \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(|x|^2+|v|^2)\, dxdv\,,$$

and the entropy estimate

$$\begin{aligned}&\int\int_{I\!R^3\times I\!R^3} f(t,x,v)\log f(t,x,v)\, dxdv\\ &\qquad + \int_0^t(\int\int\int\int_{(I\!R^3)^3\times S^2} B(v-v*,\omega)\Big(f'f'_*-ff_*\Big)\cdot\log\Big(\frac{f'f'_*}{ff_*}\Big)\, dxdvdv_*d\omega)\, ds\\ &\qquad\le \int\int_{I\!R^3\times I\!R^3} f_0(x,v)\log f_0(x,v)\, dxdv\,,\end{aligned}$$

hold. Then

(i) for any $t>0$,

$$\begin{aligned}(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{t}{1+t^2}x|^2\, dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2\, dxdv&\\ \le \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(|x|^2+|v|^2)\, dxdv\,,&\end{aligned}\tag{L}$$

(ii) there exists a positive constant $C=C(f_0)$ which depends only on f_0 such that for any $r>0$,

$$m(r,t) := \int_{|x|<r}(\int_{I\!R^3} f(t,x,v)\, dv)dx \le \frac{C(f_0)}{\log\Big(\frac{\sqrt{1+t^2}}{r}\Big)}\,.\tag{D}$$

Proof : The simplest method to prove (i) is to notice that

$$\begin{aligned}(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{t}{1+t^2}x|^2\, dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2\, dxdv\\ = \int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v|^2\, dxdv + \int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x-vt|^2\, dxdv\,.\end{aligned}$$

One may also use the method of P.-L. Lions for the Vlasov-Boltzmann equation (see [L1], III) to prove directly (using approximating solutions and passing to the limit) that any solution $F \in C^0([0,\frac{\pi}{2}[, L^1(I\!R^3 \times I\!R^3))$ of (RB) which is a limit of classical solutions of regularized problems has also to satisfy

$$\int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\,(\frac{1}{2}|\eta|^2+\frac{1}{2}|\xi|^2)\,d\xi d\eta \le \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(\frac{1}{2}|v|^2+\frac{1}{2}|x|^2)\,dxdv$$

for any $\tau \in [0,\frac{\pi}{2}[$, which is equivalent.

We now use (i) to obtain a dispersion relation via an interpolation which is in a sense the limit case as $p \to 1$ of an interpolation between moments and an L^p-norm. The result is obtained using several times Jensen's inequality: *if f and g are two nonnegative $L^1(\Omega)$ solutions such that $f(|\log f|+|\log g|)$ belongs to $L^1(\Omega)$, Jensen's inequality applied to $t\mapsto t\log t = s(t)$ with the measure $d\mu(y)=\frac{g(y)dy}{\int_\Omega g(y)\,dy}$ gives*

$$\int_\Omega f\log(\frac{f}{g})\,dy = \int_\Omega g(y)\,dy\cdot\int_\Omega s(\frac{f}{g})\,d\mu(y) \ge \int_\Omega g(y)\,dy\cdot s\Big(\int_\Omega \frac{f}{g}\,d\mu(y)\Big) = \int_\Omega f(y)\,dy\log\Big(\frac{\int_\Omega f(y)\,dy}{\int_\Omega g(y)\,dy}\Big)\,. \tag{J}$$

Applying first this inequality to $g=e^{-(1+t^2)|v-\frac{t}{1+t^2}x|^2}$ with $y=v$, $\Omega=I\!R^3$, and then integrating with respect to x, we get

$$\begin{aligned}\int_{I\!R^3}\rho(t,x)\log\rho(t,x)\,dx \le & \int\int_{I\!R^3\times I\!R^3} f(t,x,v)\log f(t,x,v)\,dxdv \\ & + (1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{t}{1+t^2}x|^2\,dxdv - \frac{3}{2}M\log\Big(\frac{1+t^2}{\pi}\Big)\end{aligned} \tag{T1a}$$

where

$$M = m(\infty,t) = \int\int_{I\!R^3\times I\!R^3} f(t,x,v)\,dxdv = \int_{I\!R^3}\rho(t,x)\,dx\,.$$

Applying now (J) to $\Omega = B(0,R)$, $g\equiv 1$, $y=x$, we find

$$m(r,t)\log m(r,t) \le m(r,t)\log(\frac{4\pi}{3}r^3) + \int_{|x|<r}\rho(t,x)\log\rho(t,x)\,dx\,. \tag{T1b}$$

But

$$\int_{|x|<r}\rho(t,x)\log\rho(t,x)\,dx = \int_{I\!R^3}\rho(t,x)\log\rho(t,x)\,dx - \int_{|x|>r}\rho(t,x)\log\rho(t,x)\,dx\,. \tag{T1c}$$

Applying again (J) to $\Omega = I\!R^3\setminus B(0,r)$, $y=x$, $g=e^{-\frac{|x|^2}{1+t^2}}$, we obtain

$$\begin{aligned}-\int_{|x|>r}\rho(t,x)\log\rho(t,x)\,dx \le & \int_{|x|>r}\rho(t,x)\frac{|x|^2}{1+t^2}\,dx \\ & - (M-m(r,t))\log(M-m(r,t)) + \frac{3}{2}(M-m(r,t))\log[\pi(1+t^2)]\,.\end{aligned} \tag{T1d}$$

Combining (T1a), (T1b), (T1c) and (T1d), we find

$$\begin{aligned}3m(r,t)\log\Big(\frac{\sqrt{1+t^2}}{r}\Big) \le & \int\int_{I\!R^3\times I\!R^3} f(t,x,v)\Big(\log f(t,x,v) + (1+t^2)|v-\frac{t}{1+t^2}x|^2 + \frac{|x|^2}{1+t^2}\Big)\,dxdv \\ & - m(r,t)\log m(r,t) - (M-m(r,t))\log(M-m(r,t)) \\ & + 3M\log(\pi) - m(r,t)\log(\frac{4}{3\sqrt{\pi}})\,.\end{aligned}$$

$t\mapsto -t\log t$ is bounded from above: combining the decay of $t\mapsto \int\int_{I\!R^3\times I\!R^3} f(t,x,v)\log f(t,x,v)\,dxdv$ which is given by the "H-Theorem", and (L1), we get the dispersion estimate (D). ◻

However, the entropy production term for the rescaled distribution function provides further informations. Essentially, it controls the weak L^1-convergence to the Maxwellian functions and the dynamical stability in the sense of Y. Guo, G. Rein (see for instance [G1], [G2], [GR]) but is not sufficient to give the strong L^1-convergence to the Maxwellian functions as will be shown on a large class of counter-examples. These three properties are the subject of the next three propositions.

Proposition 1. Weak L^1-convergence. *Under the same assumptions as in Theorem 1, for any sequence τ_n such that $\lim_{n\to+\infty}\tau_n=\frac{\pi}{2}$, up to the extraction of a subsequence, there exists a function $D\in L^1(I\!R^3)$ and two functions $\xi\mapsto\theta(\xi)$ and $\xi\mapsto U(\xi)$ such that $F(\tau_n,.,.)$ converges weakly in $L^1(I\!R^3\times I\!R^3)$ as $n\to+\infty$ to the local Maxwellian function*

$$\mathcal{M}_M(\xi,\eta)=D(\xi)\cdot\frac{e^{-\frac{1}{2\theta}|\eta-U(\xi)|^2}}{(2\pi\theta)^{3/2}} \qquad (M)$$

with $M=||D||_{L^1(I\!R^3)}=||f_0||_{L^1(I\!R^3\times I\!R^3)}$.

Notation : In the rest of this section, we denote by $e[f]$ the entropy production rate:

$$e[f]=\int\int\int\int_{(I\!R^3)^3\times S^2}B(v-v*,\omega)\Big(f'f'_*-ff_*\Big)\cdot\log\left(\frac{f'f'_*}{ff_*}\right)\,d\xi,d\eta,\eta_*d\omega\ .$$

Proof : The key estimate is the decay of the entropy. Since $\int_0^\tau\epsilon(\tau)\,d\tau$ diverges, there exists an (increasing) sequence $(\tau_n)_{n\in I\!N}$ with $\lim_{n\to+\infty}\tau_n=\frac{\pi}{2}$ such that $\lim_{n\to+\infty}e[F(\tau_n,.,.)]=0$. But

$$\tau\mapsto\int\int_{I\!R^3\times I\!R^3}F(\tau,\xi,\eta)\,\left(\frac{1}{2}|\eta|^2+\frac{1}{2}|\xi|^2+\log\big(F(\tau,\xi,\eta)\big)\right)\,d\xi d\eta$$

is decreasing and bounded by its initial value

$$\int\int_{I\!R^3\times I\!R^3}f_0(\xi,\eta)\,\left(\frac{1}{2}|\eta|^2+\frac{1}{2}|\xi|^2+\log\big(f_0(\xi,\eta)\big)\right)\,d\xi d\eta\ .$$

Using the Dunford-Pettis criterion, we get the existence of an increasing subsequence of $(\tau_n)_{n\in I\!N}$ (we still denote it by $(\tau_n)_{n\in I\!N}$) and of a function $\mathcal{M}_M(\xi,\eta)\in L^1(I\!R^3\times I\!R^3)$ such that

$$F(\tau_n,.,.)\to\mathcal{M}_M$$

weakly in $L^1(I\!R^3\times I\!R^3)$. The usual (see [L1]) convexity argument then proves:

$$e[\mathcal{M}_M]=0\ .$$

According to [L1] for instance (see also [A], [CIP], [C]), $\mathcal{M}_M$ is a Maxwellian function: there exist three functions D, θ and U respectively in $L^1(I\!R^3)$, $L^\infty_{loc}(I\!R^3)$ and $L^\infty_{loc}(I\!R^3)$ such that Equation (M) holds. □

The dynamical stability is a global property of the solution to the Cauchy problem, which is preserved by the evolution. Since we deal with weak solutions, one cannot expect to prove more than such a result in general. The nice property here is that the free energy controls the L^1-norm of the difference with the stationary solution. This situation is simpler than several situations that have been studied earlier (see [G1], [G2], [GR], [R] for instance).

Proposition 2. Dynamical stability.
(i) For any $\theta>0$, there exists a unique minimum in

$$X=\{F\in L^1(I\!R^3\times I\!R^3)\ :\ F\geq 0\ a.e.\ ,||F||_{L^1(I\!R^3\times I\!R^3)=M}\}$$

of

$$\mathcal{F}_{M,\theta}[F] = \int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\,\left(\frac{1}{2}|\eta|^2 + \frac{1}{2}|\xi|^2 + \theta\log\big(F(\tau,\xi,\eta)\big)\right)\,d\xi d\eta$$

which is reached by the following minimizer

$$G_{M,\theta}(\xi,\eta) = \frac{M}{(2\pi\theta)^3} e^{-\frac{\theta}{2}(|\eta|^2|\xi|^2)}\ .$$

In the following, we note

$$m_{M,\theta} = \mathcal{F}_{M,\theta}[G_{M,\theta}]\ .$$

(ii) For any $\theta > 0$, if F is a rescaled solution of the Boltzmann equation like in Theorem 1,

$$\tau \mapsto \inf_{\theta>0} \mathcal{F}_{M,\theta}[F(\tau,.,.)] - \mathcal{F}_{M,\theta}[m_{M,\theta}]$$

is decreasing, and for any $\tau \in [0,\frac{\pi}{2}]$,

$$\frac{1}{2}||F(\tau,.,.) - m_{M,\theta}||^2_{L^1(I\!R^3\times I\!R^3)} \le \mathcal{F}_{M,\theta}[F(\tau,.,.)] - \mathcal{F}_{M,\theta}[m_{M,\theta}] \le \mathcal{F}_{M,\theta}[f_0(,.,.)] - \mathcal{F}_{M,\theta}[m_{M,\theta}]\ .$$

Proof : The proof is a straightforward consequence of the Csiszár-Kullback inequality (see [Cs], [K], [AMTU]) and the fact that $\tau \mapsto \mathcal{F}_{M,\theta}[F(\tau,.,.)]$ is decreasing. □

To conclude this section, we exhibit examples of stationary or time-periodic solutions of the rescaled Boltzmann equation that do not have a free energy as low as the one of the minimizer.

Proposition 3. Counter-examples to the strong convergence: stationary and time-periodic solutions.
(i) Stationary solutions. For any $\theta > 0$ and any $\omega \in I\!R^3$ with $|\omega| < 1$,

$$H_{M,\theta,\omega}(\xi,\eta) = \frac{M}{(2\pi\theta)^3} e^{-\frac{1}{2\theta}(|\xi|^2+|\eta|^2-2\omega\cdot\xi\wedge\eta)}$$

is a solution whose free energy is

$$\mathcal{F}_{M,\theta}[H_{M,\theta,\omega}] = \mathcal{F}_{M,\theta}[m_{M,\theta}] + 4M\theta\frac{|\omega|^2}{|\omega|^2-1}\ .$$

(ii) Time-periodic solutions. Given any solution of

$$\ddot{\xi}_0 + \xi_0 = 0$$

($E_0 = \frac{1}{2}|\xi_0|^2 + \frac{1}{2}|\dot{\xi}_0|^2$ is a constant of the motion),

$$J_{\xi_0}(\xi,\eta) = \frac{M}{(2\pi\theta)^3} e^{-\frac{1}{2\theta}(|\xi-\xi_0|^2+|\eta-\dot{\xi}_0|^2-2\omega\cdot\xi\wedge\eta)}$$

is a time-periodic solution of period 2π whose free energy is

$$\mathcal{F}_{M,\theta}[J_{\xi_0}] = \mathcal{F}_{M,\theta}[m_{M,\theta}] + E_0\ .$$

(See Appendix B for the most general type of time-periodic solutions.)

Proof :
Stationary solutions. A class of stationary solutions is given by functions which are Maxwellian in the velocities and depend only of the integrals of the motion according to Newton's law:

$$\ddot{\xi} = -\xi \ ,$$

Since the force term derives from the harmonic potential $\frac{1}{2}|\xi|^2$, which is radially symmetric, there are two integrals of the motion:

$$|\xi|^2 + |\dot{\xi}|^2 \quad \text{and} \quad \xi \wedge \dot{\xi} \ ,$$

which give rise to the stationary solutions of Proposition 3. We may then write:

$$|\xi|^2 + |\eta|^2 - 2\omega \cdot \xi \wedge \eta = |\eta - \omega \wedge \xi|^2 + |\xi|^2 - |\omega \wedge \xi|^2$$

and the computation of $\mathcal{F}_{M,\theta}[H_{M,\theta,\omega}]$ immediately follows.
Time-periodic solutions. A straightforward computation shows that

$$\frac{\partial J_{\xi_0}}{\partial t} + \eta \cdot \nabla_\xi J_{\xi_0} - \xi \cdot \nabla_\eta J_{\xi_0} = \frac{1}{\theta}(\xi_0 \dot{\xi}_0 + \dot{\xi}_0 \ddot{\xi}_0) J_{\xi_0} = 0 \ .$$

□

Remark 2. The fact that there is no other stationary solution than the ones that depend of the integrals of the motion is usually called Jeans' theorem (see [Do1] for a general approach of these ideas). That there are no other stationary solutions than the ones considered above can be studied directly following Desvillettes' ordinary differential equations approach (see [De]): this method is detailed in Appendix B.

Remark 3. One may also study directly the dynamical stability of the solutions $H_{M,\theta,\omega}$ with the modified free energy

$$\tilde{\mathcal{F}}_{M,\theta,\omega}[F] = \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} F(\tau,\xi,\eta) \left(\frac{1}{2}|\eta - \omega \wedge \xi|^2 + \frac{1}{2}(|\xi|^2 - |\omega \wedge \xi|^2) + \theta \log\Big(F(\tau,\xi,\eta) \Big) \right) d\xi d\eta \ .$$

To obtain further results, θ and ω should be choosen consistently with the initial data, assuming for instance that

$$\int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(\xi,\eta) \ \xi \wedge \eta \ d\xi d\eta = \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(\xi,\eta) \ \xi \wedge (\omega \wedge \xi) \ d\xi d\eta$$

and

$$\int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(\xi,\eta) \ |\eta - \omega \wedge \xi|^2 \ d\xi d\eta = 3\theta \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(\xi,\eta) \ d\xi d\eta \ .$$

However, it is not clear if these quantities, which are conserved for classical solutions, are still preserved during the evolution for renormalized solutions.

Remark 4. If $t \mapsto R(t)$ is a solution of

$$\ddot{R} = \frac{1}{R^3} \quad \text{with } R(0) = 1 \ , \quad \dot{R}(0) = 0 \ ,$$

$t \mapsto R_\lambda(t) = \sqrt{\lambda} R(\frac{t}{\lambda})$ is a solution of the same equation with $R_\lambda(0) = \sqrt{\lambda}$ and $\dot{R}_\lambda(0) = 0$. The generic solution is therefore (for any $R_0 > 0$)

$$R(t) = R_0 \sqrt{1 + (\frac{t}{R_0^2})^2} \ , \quad t \in \mathbb{R} \ .$$

To the family of the global Maxwellian stationary solutions $H_{M,\theta,\omega}(\xi,\eta)$ corresponds a whole family of solutions of the Boltzmann equation

$$f^{R_0}_{M,\theta,\omega}(t,x,v) = \frac{M}{(2\pi\theta)^3}\exp -\frac{1}{2\theta}\left(\frac{|x|^2}{R_0^2+\frac{t^2}{R_0^2}} + (R_0^2+\frac{t^2}{R_0^2})\left|v-\frac{t}{R_0^4+t^2}x\right|^2 - 2\omega\, x\wedge v\right) .$$

The fact that there is no limit as $R_0 \to 0_+$ reflects why (see [DR]) self-similar solutions do not appear explicitely in the Boltzmann equation.

An even more general class of special solutions is given by the local Maxwellian that are solutions of the free transport equation:

$$f(t,x,v) = g\Big(\text{Arctg}(\frac{t}{R_0}), \frac{x}{R(t)}, R(t)(v-\frac{\dot R(t)}{R(t)}x)\Big)$$

where g is given in Proposition 6 (g is a local Maxwellian which is a solution of the transport equation in an external harmonic potential: see Appendix B). However, these solutions are classical. The decay is then of course of order $O(t^{-3})$ like for the solutions of R. Illner and M. Shindbrot (see [IS]), in an appropriate norm.

Note that the above remarks apply as well to the Landau or BGK equations (see Section III and IV).

III. The Landau equation

The same scaling as for the Boltzmann equation holds for the Landau equation. If

$$\dot\tau = \frac{1}{A^2(t)}\ ,\quad \xi(t,x) = \frac{x}{R(t)} \quad \text{and} \quad \eta(t,x,v) = \frac{A^2(t)}{R(t)}(v-\frac{\dot R(t)}{R(t)}x)\ .$$

the rescaled distribution function F such that

$$f(t,x,v) = \left(\frac{A(t)}{R(t)}\right)^6 F(\tau,\xi,\eta)$$

has to satisfy the rescaled Landau equation:

$$\frac{\partial F}{\partial\tau} + \eta\cdot\nabla_\xi F + 2A^2(t)\Big(\frac{\dot A(t)}{A(t)} - \frac{\dot R(t)}{R(t)}\Big)\text{div}_\eta(\eta F) - \frac{\ddot R(t)A^4(t)}{R(t)}\xi\cdot\nabla_\eta F = A^{2(1-\gamma)}(t)R^{\gamma-3}(t)Q_L(F,F)\ .$$

With the same ansatz as in Section II,

$$A = R\ ,\quad \ddot R = \frac{1}{R^3}\ ,\quad R(0)=1\ ,\quad \dot R(0) = 0 \quad\text{and}\quad \tau(0)=0\ ,$$

F becomes a solution of

$$\frac{\partial F}{\partial\tau} + \eta\cdot\nabla_\xi F - \xi\cdot\nabla_\eta F = \epsilon(\tau)Q_L(F,F)\ , \qquad (RL)$$

with

$$\epsilon(\tau(t)) = \frac{1}{R^{1+\gamma}(t)}\ ,\quad R(t)=\sqrt{1+t^2}\ ,\quad \dot R(t) = \frac{t}{\sqrt{1+t^2}} \quad\text{and}\quad \tau(t) = \text{Arctg}(t)\ .$$

The estimates are the same except the H Theorem which now is

$$\frac{d}{d\tau}\int\int_{I\!R^3\times I\!R^3} F(\tau,\xi,\eta)\log\Big(F(\tau,\xi,\eta)\Big)\ d\xi d\eta = -\epsilon(\tau)e[F](\tau)\ ,$$

$$e[F](\tau) = \int\int_{(I\!R^3)^3} |\eta_* - \eta|^{\gamma+2} (\nabla F - \nabla_* F)^t \, \Pi(\eta_* - \eta)(\nabla F - \nabla_* F) d\xi d\eta_* d\eta \ ,$$

$\Pi(\eta_* - \eta)$ beeing the orthogonal projection on $(\eta_* - \eta)^\perp$.

As a consequence, assuming that there exists a renormalized solution (see [L2], [V1]), we may state a result analogous to the one we obtained for the Boltzmann equation (the proof is exactly the same).

Theorem 2. *Assume that the collision kernel satifies assumptions (H1), (H2) and (H3), and that f is a renormalized solution of the Landau equation with the same a priori estimates as in Theorem 1. Then*

(i) for any $t > 0$,

$$(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2 \, dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2 \, dxdv$$
$$\leq \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(|x|^2 + |v|^2) \, dxdv \ ,$$

(ii) there exists a positive constant $C = C(f_0)$ which depends only on f_0 such that for any $r > 0$,

$$m(r,t) := \int_{|x|>r} (\int_{I\!R^3} f(t,x,v) \, dv) dx \leq \frac{C(f_0)}{\log\left(\frac{\sqrt{1+t^2}}{r}\right)} \ .$$

Note that the weak L^1-convergence, the dynamical stability and the counter-examples hold exactly as for the Boltzmann equation.

IV. The BGK and Vlasov-Fokker-Planck equations

Since the quantities preserved by the BGK equation are the same as the ones of the Boltzmannn equation or the Landau equation, the same type of results holds. Dispersion estimates for a modified BGK equation that satisfies a maximum principle have been obtained by B. Perthame in [P1]. Here we simply use the entropy estimate to control the dispersion.

Theorem 3. *Assume that f is a solution of the BGK equation with the same a priori estimates as in [P1]. Then*

(i) for any $t > 0$,

$$(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2 \, dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2 \, dxdv$$
$$\leq \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(|x|^2 + |v|^2) \, dxdv \ ,$$

(ii) there exists a positive constant $C = C(f_0)$ which depends only on f_0 such that for any $r > 0$,

$$m(r,t) := \int_{|x|>r} (\int_{I\!R^3} f(t,x,v) \, dv) dx \leq \frac{C(f_0)}{\log\left(\frac{\sqrt{1+t^2}}{r}\right)} \ .$$

The same techniques also applies to the Vlasov-Fokker-Planck equation except that the two terms (friction and diffusion) of the Fokker-Planck term do not have the same scaling properties, which makes the analysis of the dispersion somehow a little bit tricky. We use here the same approach as in [DR] for the Vlasov-Poisson-Fokker-Planck system.

If

$$f(t,x,v) = F(\tau(t), \frac{x}{R(t)}, R(t)(v - \frac{\dot{R}(t)}{R(t)}x)) \ , \quad \tau(t) = \mathrm{Arctg}(t) \quad \text{and} \quad R(t) = \sqrt{1+t^2} \ ,$$

then F is a solution of

$$\frac{\partial F}{\partial \tau} + \eta \cdot \nabla_\xi F - \xi \cdot \nabla_\eta F = \beta R^2(t) \mathrm{div}_\eta(\eta F) + \sigma R^4(t) \Delta_\eta F \ . \qquad (RVFP)$$

The free energy is as usual (see [BD], [D], [DR])

$$\begin{aligned}\mathcal{F}(\tau) = \int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(\tau,\xi,\eta) \left(\frac{1}{2}|\eta|^2 + \frac{1}{2}|\xi|^2 + \frac{\sigma}{\beta} R^2(t) \log\big(F(\tau,\xi,\eta)\big)\right) d\xi d\eta \\ - \frac{\sigma}{\beta} R^2(t) ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \\ + 3\frac{\sigma}{\beta} R^2(t) ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log\left(2\pi \frac{\sigma}{\beta} \cdot R^2(t)\right) \geq 0 \ ,\end{aligned}$$

with of course $\tau = \tau(t) = \mathrm{Arctg}(t)$. Since $t \mapsto R(t)$ is increasing, it is more convenient to consider

$$L(\tau) = \frac{1}{R^2(t)} \mathcal{F}(\tau) \ ,$$

$$\begin{aligned}L(\tau) = \frac{1}{R^2(t)} E(\tau) + \frac{\sigma}{\beta} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(\tau,\xi,\eta) \ \log\big(F(\tau,\xi,\eta)\big) \ d\xi d\eta \\ - \frac{\sigma}{\beta} ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \\ + 3\frac{\sigma}{\beta} ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log\left(2\pi \frac{\sigma}{\beta} \cdot R^2(t)\right) \ ,\end{aligned}$$

with

$$E(\tau) = \frac{1}{2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(\tau,\xi,\eta) \left(|\eta|^2 + |\xi|^2\right) d\xi d\eta \ .$$

Then

$$\begin{aligned}\frac{dL}{d\tau}(\tau) = -2\frac{\dot{R}(t)}{R^3(t)} (\frac{d\tau}{dt})^{-1} E(\tau) + 6\frac{\dot{R}(t)}{R(t)} (\frac{d\tau}{dt})^{-1} \frac{\sigma}{\beta} ||F||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \\ - \beta \int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(\tau,\xi,\eta) \ |\eta + \frac{\sigma}{\beta} R^2(t) \frac{\partial_\eta F}{F}|^2 \ d\xi d\eta \ .\end{aligned}$$

Going back to the original variables, this means that

$$\begin{aligned}\frac{d}{dt}\Bigg[\frac{1}{2}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2 \ dxdv + \frac{1}{2(1+t^2)^2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|x|^2 \ dxdv \\ + \frac{\sigma}{\beta} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v) \log f(t,x,v) \ dxdv \\ - \frac{\sigma}{\beta} ||f||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log ||f||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} + 3\frac{\sigma}{\beta} ||f||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \log\left(2\pi\frac{\sigma}{\beta} \cdot (1+t^2)\right)\Bigg] \\ = -\frac{t}{1+t^2}\left[\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2 \ dxdv + \frac{1}{(1+t^2)^2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|x|^2 \ dxdv\right] \\ + \frac{6t}{1+t^2} \frac{\sigma}{\beta} ||f||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} - \frac{\beta}{1+t^2} \int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v) \ |v - \frac{t}{1+t^2}x + \frac{\sigma}{\beta}\sqrt{1+t^2} \ \frac{\partial_v f}{f}|^2 \ dxdv \ .\end{aligned}$$

With this computation and using the estimates given in Proposition 5 (Appendix A), we are in position to state the following results on the long time behaviour of f.

Theorem 4 . *Consider in $C^0([0,+\infty[, L^1(I\!R^3 \times I\!R^3))$ the solution of the Vlasov-Fokker-Planck equation corresponding to a nonnegative initial data f_0 in $L^\infty(I\!R^3 \times I\!R^3)$ such that*

$$(x,v) \mapsto f_0(x,v)\ (|x|^2 + |v|^2 + |\log f_0(x,v)|)$$

belongs to $L^1(I\!R^3 \times I\!R^3)$. Then

$$t \mapsto \frac{1}{\log(1+t^2)} \int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2\ dxdv$$

is uniformly bounded on $I\!R^+$, f belongs to $C^0([0,+\infty[, L^1 \cap L^\infty(I\!R^3\times I\!R^3))$ and for any $(p,q) \in [1,+\infty]\times[1,\frac{5}{3}]$,

$$||f(t,.,.)||_{L^p(I\!R^3\times I\!R^3)} = o\Big(t^{-\frac{3}{2}(1-\frac{1}{p})}\Big) \quad \text{and} \quad ||\int_{I\!R^3} f(t,.,v)\ dv||_{L^q(I\!R^3)} = o\Big([\log(1+t^2)]^{\frac{3(q-1)}{2q}} t^{-\frac{3}{2}(q-1)}\Big)\ ,$$

as $t \to +\infty$.

Proof : The decay of $||f(t,.,.)||_{L^p(I\!R^3\times I\!R^3)}(t)$ is given by the results of Appendix A and interpolation inequalities based on Hölder's inequality.

Using that, on one side

$$L(\tau(t)) \leq L(0) + 3\log(1+t^2)\ \frac{\sigma}{\beta}||f_0||_{L^1(I\!R^3\times I\!R^3)}\ ,$$

and on the other side, by the mean of Jensen's inequality,

$$L(t) - \frac{1}{4}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2\ dxdv \geq -\frac{3\sigma}{2\beta}||f_0||_{L^1(I\!R^3\times I\!R^3)} \log 2\ ,$$

one proves that $\frac{1}{\log(1+t^2)}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2\ dxdv$ is bounded.

The decay of $||\int_{I\!R^3} f(t,.,v)\ dv||_{L^q(I\!R^3)}(t)$ is then a consequence of the following interpolation inequality (see [Do3] for more details): there exists a constant $C(p) > 0$ such that

$$||\int_{I\!R^3} f(t,.,v)\ dv||_{L^q(I\!R^3)} \leq C(p)||f(t,.,.)\ dv||^{\theta}_{L^p(I\!R^3\times I\!R^3)} \cdot \Big(\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v - \frac{t}{1+t^2}x|^2\ dxdv\Big)^{1-\theta}$$

with $q = \frac{5p-3}{3p-1}$ and $\theta = \frac{2p}{5p-3}$. □

Remark 5.
1) Note that the decay given above is mainly a consequence of the estimate given by the Green function, not by a decay of the moment.
2) More precise estimates on the decay might be obtained using the expression of the Green function.
3) Using again interpolations results like in [Do3], one could also give decay estimates on moments in v of order strictly less than 2.
4) The decay estimates on the moment in $(v - \frac{t}{1+t^2}x)$ might be improved using one of the following logarithmic Sobolev inequalities: for any nonnegative function g such that $\sqrt{g}$ belongs to $H^1(I\!R^d)$ ($d \geq 1$), for any $a > 0$,

$$\int_{I\!R^d} g(y)(\log g(y) + \frac{|y|^2}{2a})\ dy + ||g||_{L^1(I\!R^d)} \log\Big(\frac{(2\pi a)^{d/2}}{||g||_{L^1(I\!R^d)}}\Big) \leq \frac{a}{2}\int_{I\!R^d} g(y)\ |\frac{\nabla g}{g} + \frac{y}{a}|^2\ dy\ ,$$

$$\int_{I\!R^d} g(y)\log g(y)\ dy + d\left[1 + ||g||_{L^1(I\!R^d)} \log\Big(\frac{\sqrt{2\pi a}}{||g||_{L^1(I\!R^d)}}\Big)\right] \leq \frac{a}{2}\int_{I\!R^d} \frac{|\nabla g|^2}{g}\ dy\ .$$

V. Conclusion

This paper has been focused on the dispersion properties of kinetic equations with collision terms. However the rescaled equation does not provide much information because most of the solutions are defined only in a very weak sense. The available information is too poor to prove that the rescaled solution converges to the stationary solution of the rescaled equation. This is very similar to the situation that occurs for kinetic equations without collision terms, for the Vlasov-Poisson system for instance (see [DR]) in the sense that the results one can establish are simply based on the decay of the Lyapunov functional, which can be computed directly.

Here, the method provides dispersion results because the collision kernels are local in x and (except for the Vlasov-Fokker-Planck equation) and preserve the local energy (the momentum of order 2 in v is conserved).

As a consequence, the rate of dispersion is determined more by the quantities one uses for the interpolation than by the structure of the equation itself. For instance, in the case of the Boltzmann equation, the solution found by R. Illner and M. Shindbrot (see [IS]) are such that $||\rho(t,.)||_{L^\infty(I\!R^3)}$ has a decay of order $O(t^{-3})$ while one proves only a decay of order $O(\frac{1}{\log(t)})$ for the renormalized solution of R. DiPerna and P.-L. Lions, simply because the interpolation uses the estimate $\int\int_{I\!R^3\times I\!R^3} f(t,x,v)\ \log(f(t,x,v))\ dxdv$ instead of an L^∞ (with weight) norm of f.

Similarly, for non-collisional kinetic equations, one may expect that the dispersion rate one can obtain with interpolation methods will depend in a crucial way whether the solution is strong, weak, or renormalized. A very simple illustration of this fact is given by the Vlasov-Poisson system: while the decay is of order $O(t^{-3})$ for classical solutions (see [BaDe]), the decay turns out to be weaker for strong solutions (see [P1], [IR]), actually of order $O(t^{-3/5})$. It is still weaker if we consider the renormalized solutions defined by R. J. DiPerna and P.-L. Lions in [DPL3] as shown by the following result.

Proposition 4. *Consider a renormalized solution of the Vlasov-Poisson system*

$$\begin{cases} \dfrac{\partial f}{\partial t} + v\cdot\nabla_x f - \nabla_x U\cdot\nabla_v f = 0 \\ -\Delta U = \displaystyle\int_{I\!R^3} f(t,x,v)\ dv \end{cases}$$

such that

$$\left(\int\int_{I\!R^3\times I\!R^3} f(t,x,v)\ (|v|^2+|x-vt|^2+\log(f(t,x,v))\ dxdv + \int_{I\!R^3}|\nabla_x U(t,x)|^2\ dx\right)_{|t=0} < +\infty\ .$$

Assume moreover that $f(t=0,.,.)=f_0$ has a spherical symmetry (i.e. depends only on t, $|x|$, $|v|$ and $(x\cdot v)$, and that at $t=0$, $f(t=0,.,.)$ is supported in the set $\{(x,v)\in I\!R^3\times I\!R^3\ :\ (x\cdot v)>0\}$. Then f is spherically symmetric for any $t>0$,

$$\sup_{t>0}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)\ (|v|^2+|x-vt|^2+\log(f(t,x,v))\ dxdv + \int_{I\!R^3}|\nabla_x U(t,x)|^2\ dx < +\infty$$

and

(i) for any $t>0$,

$$(1+t^2)\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|v-\frac{t}{1+t^2}x|^2\ dxdv + \frac{1}{1+t^2}\int\int_{I\!R^3\times I\!R^3} f(t,x,v)|x|^2\ dxdv$$
$$\leq \int\int_{I\!R^3\times I\!R^3} f_0(x,v)(|x|^2+|v|^2)\ dxdv\ ,$$

(ii) there exists a positive constant $C = C(f_0)$ which depends only on f_0 such that for any $r > 0$,

$$m(r,t) := \int_{|x|<r} \Big(\int_{I\!R^3} f(t,x,v)\ dv \Big) dx \le \frac{C(f_0)}{\log\Big(\frac{\sqrt{1+t^2}}{r}\Big)} \ .$$

The proof of (ii) is essentially the same as before except that we have to prove first that under the symmetry assumptions made on f_0, the solution has a Lyapunov functional which essentially has the same decay as the solution of the free transport equation. We may note that for a strong solution, one would get a decay of order $O(t^{-6/5})$.

Of course, for a spherically symmetric solution, more results on the regularity are available and it is possible to consider a stronger form of the solution than simply renormalized solutions. However, we will here only use the *a priori* estimates that are required to define a renormalized solution (for the interpolation method). The symmetry assumptions are done only to prove the decay of the Lyapunov functional. Without these assumptions, the usual Lyapunov functional, or even its slightly improved form (see [DR]) involving a moment of order 2 in x, is not sufficient to apply our interpolation method.

Proof : First, we may notice that for a spherically symmetric solution, the potential U is radial and decaying: $x \cdot \nabla_x U(t,x) \le 0$ for $(t,x) \in I\!R^+ \times I\!R^3$ a.e. The argument is then very simple. Assume first that the solution is a classical one (assertion (i) will be satisfied as soon as it is verified uniformly for classical solutions). One may see that for any characteristic starting in the support of f_0 at $t = 0$, for any $t > 0$,

$$\frac{d}{dt}\Big(\frac{1}{2}|v|^2\Big) = (x \cdot v) \quad \text{and} \quad \frac{d}{dt}(x \cdot v) = |v|^2 - (x \cdot \nabla_x U) \ge 0 \ ,$$

$$\Big(\frac{1}{2}|v|^2\Big)_{|t=0} \ge 0 \quad \text{and} \quad (x \cdot v)_{|t=0} \ge 0 \ .$$

The kinetic energy is therefore increasing, and because of the conservation of energy,

$$\frac{d}{dt} \int_{I\!R^3} |\nabla_x U(t,x)|^2\ dx \le 0 \ .$$

Exactly as in [DR], instead of making a complete change of variables in t, x and v like in section II, one may simply consider the equation for F defined by

$$f(t,x,v) = F\Big(t,x,\eta = v - \frac{\dot R}{R} x\Big) \ .$$

F is a solution of

$$\frac{\partial F}{\partial t} + v \cdot \nabla_x F - \nabla_x U \cdot \nabla_v F - \frac{\ddot R}{R} x \cdot \nabla_v F + \frac{\dot R}{R} \nabla_x \cdot (xF) - \frac{\dot R}{R} \nabla_v \cdot (vF) = 0 \ .$$

For any given function $t \mapsto B(t)$ in C^1, a computation provides the identity

$$\begin{aligned}
&\frac{d}{dt}\Big[B(t) \cdot \Big(\int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |\eta|^2\ dx d\eta + \frac{\ddot R}{R} \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |x|^2\ dx d\eta + \int_{I\!R^3} |\nabla_x U(t,x)|^2\ dx \Big)\Big] \\
&= B\Big(\frac{\dot B}{B} - 2\frac{\dot R}{R}\Big) \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |\eta|^2\ dx d\eta \frac{B\ddot R}{R}\Big(\frac{\dot B}{B} + \frac{(\frac{d^3 R}{dt^3})}{\ddot R} + \frac{\dot R}{R}\Big) \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |x|^2\ dx d\eta \\
&\quad + \dot B \int_{I\!R^3} |\nabla_x U(t,x)|^2\ dx + \frac{B\dot R}{R} \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ (x \cdot \nabla_x U(t,x))\ dx \\
&\le B\Big(\frac{\dot B}{B} - 2\frac{\dot R}{R}\Big) \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |\eta|^2\ dx d\eta \\
&\quad + \frac{B\ddot R}{R}\Big(\frac{\dot B}{B} + \frac{(\frac{d^3 R}{dt^3})}{\ddot R} + \frac{\dot R}{R}\Big) \int\!\!\int_{I\!R^3 \times I\!R^3} F(t,x,\eta)\ |x|^2\ dx d\eta + \dot B \int_{I\!R^3} |\nabla_x U(t,x)|^2\ dx
\end{aligned}$$

as soon as $\dot R > 0$. Assuming now simply that $R(t)=\sqrt{1+t^2}$ and $B(t)=R^2(t)=1+t^2$, we get

$$\frac{d}{dt}\left[B(t)\cdot\Big(\int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(t,x,\eta)\,|\eta|^2\,dxd\eta+\frac{\ddot R}{R}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} F(t,x,\eta)\,|x|^2\,dxd\eta\Big)\right]\le \frac{d}{dt}\int_{\mathbb{R}^3}|\nabla_x U(t,x)|^2\,dx\le 0$$

which proves (i) when one replaces η and $R(t)$ by their values. □

Appendix A: Green's function of the Vlasov-Fokker-Planck operator

In order to give a complete description of the asymptotic behaviour of the Vlasov-Fokker-Planck equation, we state in this Appendix the decay properties that arise from the symmetries of the Green's function. The proof exactly follows the method one may apply to the heat equation.

Proposition 5.

(i) The Green function $G\in C^1(\mathbb{R}^+;\mathcal{D}(\mathbb{R}^{4N}))$ of the Vlasov-Fokker-Planck operator, i.e. the solution of

$$\begin{cases}\partial_t G+v\cdot\partial_x G-\beta\partial_v\cdot(vG)-\sigma\Delta_v G=0\\ G(t=0,x,v,y,w)=\delta(x-y,v-w)\end{cases}$$

is

$$G(t,x,v,y,w)=e^{N\beta t}F(t,x+\frac{1-e^{\beta t}}{\beta}-y,e^{\beta t}v-w)\ ,\quad \text{with } F(t,x,v)=\frac{e^{-\frac{\lambda(t)|v|^2-2\mu(t)(x\cdot v)+\nu(t)|x|^2}{4\sigma(\lambda(t)\nu(t)-\mu^2(t))}}}{(4\pi\sigma)^N(\lambda(t)\nu(t)-\mu^2(t))^{N/2}}\ ,$$

or

$$G(t,x,v,y,w)=J(t)\cdot H\Big(M(t)\big((x,v)-U(t)(y,w)\big)\Big)$$

with

$$H(x,v)=(4\pi)^{-N}\cdot e^{-\frac14(|x|^2+|v|^2)}\ ,$$

$$J(t)=\big(a(t)d(t)-b(t)c(t)\big)^N=\frac{e^{N\beta t}}{\big(\sigma^2(\lambda(t)\nu(t)-\mu^2(t))\big)^{N/2}}\ ,$$

$$U(t)(y,w)=\Big(y-\frac{1}{\beta}(e^{-\beta t}-1)w,e^{-\beta t}w\Big)$$

and $\quad M(t)(x,v)=\big(a(t)x+b(t)v,c(t)x+d(t)v\big)$,

$$a(t)=\frac{1}{\sqrt{\sigma\lambda(t)}}\ ,\qquad b(t)=\frac{1}{\sqrt{\sigma\lambda(t)}}\cdot\frac{1-e^{\beta t}}{\beta}\ ,$$

$$c(t)=-\frac{\mu(t)}{\lambda(t)\sqrt{\sigma(\nu(t)-\frac{\mu^2(t)}{\lambda(t)})}}\ ,\quad d(t)=\frac{e^{\beta t}-\frac{\mu(t)}{\lambda(t)}\cdot\frac{1-e^{\beta t}}{\beta}}{\sqrt{\sigma(\nu(t)-\frac{\mu^2(t)}{\lambda(t)})}}\ ,$$

$$\lambda(t)=\frac{\frac32-2\,e^{\beta t}+\frac{e^{2\beta t}}{2}}{\beta^3}+\frac{t}{\beta^2}\ ,\quad \mu(t)=\frac{-1+2\,e^{\beta t}-e^{2\beta t}}{2\,\beta^2}\quad\text{and}\quad \nu(t)=\frac{-1+e^{2\beta t}}{2\,\beta}\ .$$

(ii) f is a solution of the linear Vlasov-Fokker-Planck equation

$$\partial_t f+v\cdot\partial_x f-\beta\partial_v\cdot(vf)-\sigma\Delta_v f=g$$

corresponding to the initial condition f_0 if and only if

$$f(t,x,v) = f_0 * G(t,.) + \int_0^t g(s,.) * G(t-s,.)\,ds$$

where

$$\big(h * G(t,.)\big)(t,x,v) = \int\!\!\int_{\mathbb{R}^N\times\mathbb{R}^N} h(y,w)G(t,x,v,y,w)\,dydw\;.$$

(iii) The fundamental solution is invariant under the following transformation: for any $\alpha > 0$,

$$G(t,x,v,y,w) = k(\alpha,t)\cdot G(\alpha t, T(\alpha,t)(x,v), U^{-1}(\alpha t)\mathrm{o}T(\alpha,t)\mathrm{o}U(t)(y,w))$$
$$\forall\ (t,x,v,y,w) \in]0,+\infty[\times\mathbb{R}^{4N}$$

with

$$k(\alpha,t) = \frac{J(t)}{J(\alpha t)} \quad \text{and} \quad T(\alpha,t) = M^{-1}(\alpha t)\ \mathrm{o}\ M(t)\;.$$

As a consequence, if $g \equiv 0$ and if f_0 belongs to $L^1(\mathbb{R}^N\times\mathbb{R}^N)$, then

$$\lim_{t\to+\infty} t^{N/2}\cdot ||f(t,x,v) - ||f_0||_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\cdot G(t,x,v,0,0)||_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N;\,dxdv)} = 0\;,$$

$$\lim_{t\to+\infty} ||f(t,x,v) - ||f_0||_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\cdot G(t,x,v,0,0)||_{L^1(\mathbb{R}^N\times\mathbb{R}^N;\,dxdv)} = 0\;.$$

Proof : (i) is a classical computation which can be found in [Bo2] or [H]. Let us give a sketch of the proof: with the change of variables

$$G(t,x,v,y,w) = e^{N\beta t}F(t, x + \frac{1-e^{\beta t}}{\beta}v - y, e^{\beta t}v - y)\;,$$

F has to be the fundamental solution of

$$\partial_t F(t,x,v) = \sigma\Big(\frac{d\lambda}{dt}(t)\cdot\frac{\partial^2 F}{\partial x^2}(t,x,v) + 2\frac{d\mu}{dt}(t)\cdot\frac{\partial^2 F}{\partial x\partial v}(t,x,v) + \frac{d\nu}{dt}(t)\cdot\frac{\partial^2 F}{\partial v^2}(t,x,v)\Big)$$

A Fourier transform shows that

$$\hat F(t,\hat x,\hat v) = \int\!\!\int_{\mathbb{R}^N\times\mathbb{R}^N} F(t,x,v)e^{i(x\cdot\hat x + v\cdot\hat v)}\,dxdv$$

is a solution of

$$\partial_t \hat F(t,\hat x,\hat v) = -\sigma\Big(\frac{d\lambda}{dt}(t)\cdot|\hat x|^2 + 2\frac{d\mu}{dt}(t)\cdot(\hat x\cdot\hat v) + \frac{d\nu}{dt}(t)\cdot|\hat v|^2\Big)\hat F(t,\hat x,\hat v)$$

with the normalization condition $\hat F(t,0,0) = 1$ for any $t > 0$.

$$\log\Big(\frac{\hat F(t,\hat x,\hat v}{\hat F(t,0,0)})\Big) = -\sigma\Big(\lambda(t)|\hat x|^2 + 2\mu(t)\cdot(\hat x\cdot\hat v) + \nu(t)|\hat v|^2\Big) = -\sigma\Big(\lambda(t)\big|\hat x + \frac{\mu(t)}{\lambda(t)}\hat v\big|^2 + \big(\nu(t) - \frac{\mu^2(t)}{\lambda(t)}\big)\big|\hat v\big|^2\Big)$$

An inverse Fourier transform now gives

$$
\begin{aligned}
&F(t,x,v)\\
&= \int\int_{I\!R^N\times I\!R^N} e^{-i(x\cdot\hat{x}+v\cdot\hat{v})}\hat{F}(t,\hat{x},\hat{v})\ \frac{d\hat{x}d\hat{v}}{(2\pi)^{2N}}\\
&= \frac{e^{-\frac{|x|^2}{4\sigma\lambda(t)}}}{(4\pi\sigma\lambda(t))^{N/2}}\cdot\int_{I\!R^N} e^{-iv\cdot\hat{v}}\cdot e^{i\frac{\mu(t)}{\lambda(t)}\cdot(x\cdot\hat{v})}\cdot e^{-\frac{\sigma}{\lambda(t)}(\lambda(t)\nu(t)-\mu^2(t))|\hat{v}|^2}\ \frac{d\hat{v}}{(2\pi)^N}\\
&= \frac{e^{-\frac{|x|^2}{4\sigma\lambda(t)}}}{(4\pi\sigma\lambda(t))^{N/2}}\cdot\int_{I\!R^N} e^{-iv\cdot\hat{v}}\cdot e^{-\frac{\sigma}{\lambda(t)}(\lambda(t)\nu(t)-\mu^2(t))|\hat{v}-\frac{i\mu(t)}{2\sigma(\lambda(t)\nu(t)-\mu^2(t))}x|^2}\cdot e^{\frac{\mu^2(t)}{4\sigma\lambda(t)(\lambda(t)\nu(t)-\mu^2(t))}|x|^2}\ \frac{d\hat{v}}{(2\pi)^N}\\
&= \frac{e^{-\frac{1}{4\sigma\lambda(t)}(1+\frac{\mu^2(t)}{\lambda(t)\nu(t)-\ mu^2(t)})|x|^2}}{(4\pi\sigma\lambda(t))^{N/2}}\cdot\frac{e^{-\frac{\lambda(t)}{4\sigma(\lambda(t)\nu(t)-\mu^2(t))}|v|^2}}{\left(4\pi\sigma\cdot\frac{\lambda(t)\nu(t)-\mu^2(t)}{\lambda(t)}\right)^{N/2}}\cdot e^{-\frac{\mu(t)}{2\sigma(\lambda(t)\nu(t)-\mu^2(t))}\cdot(x\cdot v)}\\
&= \frac{e^{-\frac{\lambda(t)|v|^2-2\mu(t)(x\cdot v)+\nu(t)|x|^2}{4\sigma(\lambda(t)\nu(t)-\mu^2(t))}}}{(4\pi\sigma)^N(\lambda(t)\nu(t)-\mu^2(t))^{N/2}}\ ,
\end{aligned}
$$

which provides the result for G, since

$$
\frac{\lambda(t)\cdot|v|^2-2\mu(t)\cdot(x\cdot v)+\nu(t)\cdot|x|^2}{\sigma(\lambda(t)\nu(t)-\mu^2(t))} = \left|\sqrt{\frac{\lambda(t)}{\sigma(\lambda(t)\nu(t)-\mu^2(t))}}\left(v-\frac{\mu(t)}{\lambda(t)}x\right)\right|^2 + \left|\frac{1}{\sqrt{\sigma\lambda(t)}}\ x\right|^2 .
$$

(ii) The integral form of the equation is classically given by the following computation: if G is the Green function corresponding to

$$\frac{\partial G}{\partial t} = LG\ ,\quad G(t=0,.) = \delta\ ,$$

a solution of

$$\frac{\partial f}{\partial t} = Lf + g\ ,\quad f(t=0,.) = f_0$$

is then formally given by

$$f = G * f_0 + \int_0^t g(s,.) * G(t-s,.)\ ds$$

(apply $(\frac{\partial}{\partial t} - L)$ to $[G*f_0+\int_0^t g(s,.)*G(t-s,.)\ ds]$ and integrate by parts). Since the solution of the equation is unique, (ii) is proved.

(iii) The invariance of G is easily derived from the following identities:

$$k(\alpha,t)J(\alpha t)\cdot H(M(\alpha t)\ \mathrm{o}\ T(\alpha,t)(x,v)) = J(t)H(M(t)(x,v))\ ,$$

$$G(t,x,v,y,w) = G(t,(x,v)-U(t)(y,w),0,0)\ .$$

In the following we denote by G_0 the fundamental solution:

$$G_0(t,x,v) = G(t,x,v,0,0)\ .$$

Like for the heat equation we may now compute

$$
\begin{aligned}
k(t,1)f(t,T(t,1)(x,v)) &= \int\int_{I\!R^N\times I\!R^N} f_0(y,w)\ k(t,1)\ G_0\Big(t,T(t,1)[(x,v)-T^{-1}(t,1)\mathrm{o}U(t)(y,w)]\Big)\ dydw\\
&= \int\int_{I\!R^N\times I\!R^N} f_0(y,w)\ G_0\Big(1,(x,v)-T^{-1}(t,1)\mathrm{o}U(t)(y,w)\Big)\ dydw
\end{aligned}
$$

where

$$T^{-1}(t,1) = M^{-1}(1) \text{ o } M(t) \sim \frac{\sqrt{\frac{\beta}{\sigma}}}{a(1)d(1)-b(1)c(1)} \begin{pmatrix} d(1) & -b(1) \\ -c(1) & a(1) \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2}\beta e^{-\beta t} & -\sqrt{2} \\ \sqrt{\frac{\beta}{t}} & \frac{1}{\sqrt{\beta t}} \end{pmatrix} ,$$

$$T^{-1}(t,1)\text{o}U(t) \sim \frac{\sqrt{\frac{\beta}{\sigma}}}{a(1)d(1)-b(1)c(1)} \begin{pmatrix} -\sqrt{\frac{\beta}{t}} \cdot b(1) & -\sqrt{2}\, d(1) \\ \sqrt{\frac{\beta}{t}} a(1) & \sqrt{2}\, c(1) \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{\beta} \\ 0 & 0 \end{pmatrix} \quad \text{as } t \to +\infty .$$

This proves that

$$\lim_{t \to +\infty} k(t,1) f(t, T(t,1)(x,v)) = \int\int_{I\!R^N \times I\!R^N} f_0(y,w)\; dydw G_0(1,x,v) .$$

To conclude the proof, we have to notice that

$$\begin{aligned} &\lim_{t\to+\infty} \sup_{(x,v)\in I\!R^N\times I\!R^N} k(t,1) \cdot \Big| f(t,x,v) - ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G_0(t,x,v) \Big| \\ &\quad = \lim_{t\to+\infty} \sup_{(x,v)\in I\!R^N\times I\!R^N} k(t,1) \cdot \Big| f\big(t, T(t,1)(x,v)\big) - ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G_0\big(t, T(t,1)(x,v)\big) \Big| \\ &\qquad \leq \lim_{t\to+\infty} \sup_{(x,v)\in I\!R^N\times I\!R^N} \int\int_{I\!R^N\times I\!R^N} f_0(y,w)\; k(t,1) \cdot \Big| G_0\Big(t, T(t,1)[(x,v) - T^{-1}(t,1)\text{o}U(t)(y,w)]\Big) \\ &\qquad\qquad - G_0(t, T(t,1)[(x,v)) \Big|\; dydw \\ &\qquad = 0 \end{aligned}$$

because of Lebesgue's theorem of dominated convergence. The result holds for the L^∞-estimate since $k(t,1) \sim \frac{J(1)\sigma^N}{(2\beta)^{N/2}} \cdot t^{N/2}$ as $t \to +\infty$. The result for the L^1-norm is obtained in the same way, but

$$\begin{aligned} &\int\int_{I\!R^N\times I\!R^N} \Big| f(t,x,v) - ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G_0(t,x,v) \Big|\; dxdv \\ &\qquad = k(t,1) \cdot \int\int_{I\!R^N\times I\!R^N} \Big| f\big(t, T(t,1)(x,v)\big) - ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G_0(t,x,v) \Big|\; dxdv , \end{aligned}$$

which explains why there is no explicit w.r.t. t uniform convergence. □

Remark 6. The above computation also shows that $||\; ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G(t,x,v,0,0)||_{L^1(I\!R^N\times I\!R^N;\, dxdv)}$ is a constant (which is consistent with the fact that the L^1-norm of f is conserved) and that

$$||f(t,x,v)||_{L^\infty(I\!R^N\times I\!R^N;\, dxdv)} \sim ||f_0||_{L^1(I\!R^N\times I\!R^N)} \cdot G(t,x,v,0,0)||_{L^\infty(I\!R^N\times I\!R^N;\, dxdv)} = O(t^{-N/2}) \quad \text{as } t \to +\infty .$$

From a heuristical point of view, the equivalence results as $t \to +\infty$ are easily recovered if one notices that f is a solution of the Vlasov-Fokker-Planck equation corresponding to an initial data f_0 if and only if

$$f^\alpha(t,x,v) = k(\alpha,t) f(\alpha t, T(\alpha,t)(x,v))$$

is a solution of the Vlasov-Fokker-Planck equation corresponding to an initial data

$$f_0^\alpha(x,v) = k(\alpha,t) f_0\Big(U^{-1}(t)\text{o}T^{-1}(\alpha,t)\text{o}U(\alpha t)(x,v)\Big)$$

(the proof relies on a simple change of variables). But for a fixed t,

$$f_0^\alpha \to ||f_0||_{L^1(I\!R^N\times I\!R^N)}\cdot\delta \quad \text{as} \quad \alpha\to+\infty$$

(in the sense of the distributions), which proves the assertion :

$$f(t,x,v)=f^t(1,x,v)\sim ||f_0||_{L^1(I\!R^N\times I\!R^N)}\cdot G(t,x,v,0,0) \quad \text{as} \quad t\to+\infty\ .$$

Appendix B: Positive Maxwellian solutions of the Vlasov equation in an harmonic potential

In this last part of the paper, we give the general expression of nonnegative local Maxwellian solutions of the transport equation in an harmonic potential by solving the corresponding equations for the coefficients. These solutions form a bigger class than the periodic solutions given in Proposition 3 (Section II) and might even be pseudo-periodic.

Most probably, the main interest of such solutions is that they straightforwardly give special solutions of the Boltzmann, Landau or BGK equations in the whole space simply by using the time dependant rescaling. These solutions (for the unscaled equations) are of course local Maxwellians.

Proposition 6. *Any nonnegative local Maxwellian solution*

$$g(t,x,v)=r(t,x)\exp\{-\sigma(t,x)|v-u(t,x)|^2\} \tag{B1}$$

of the transport equation in an harmonic potential:

$$\frac{\partial g}{\partial t}+v\cdot\nabla_x g-x\cdot\nabla_v g=0 \tag{B2}$$

such that the coefficients $r(t,x)$, $u(t,x)$ *and* $\sigma(t,x)$ *are Lipschitz and such that* $(t,x,v)\mapsto g(t,x,v)\ (1+|x|^2+|v|^2+|\log g(t,x,v)|)$ *belongs to* $L^\infty(I\!R,L^1(I\!R^3\times I\!R^3))$ *is given by*

$$r(t,x)=M(\frac{\sigma_0}{\pi})^{\frac{3}{2}}(1-\frac{|\omega_0|^2}{\sigma_0^2})\ e^{-\sigma_0|x-x_0(t)|^2+\frac{1}{\sigma_0}|\omega_0\wedge(x-x_0(t))|^2}\ ,$$

$$\sigma(t,x)=\sigma_0\ ,$$

$$u(t,x)=u_\perp^1\cos(\frac{|\omega_0|}{\sigma_0}t+\varphi_1)e_\perp^1+u_\perp^2\cos(\frac{|\omega_0|}{\sigma_0}t+\varphi_2)e_\perp^2+u_{\omega_0}\cos(t+\varphi_{\omega_0})\frac{\omega_0}{|\omega_0|}\ ,$$

where $(\sigma_0,\omega_0)\in]0,+\infty[\times I\!R^3$ *are constants such that*

$$|\omega_0|<\sigma_0\ ,$$

$(e_\perp^1,e_\perp^2,\frac{\omega_0}{|\omega_0|})$ *is a given orthonormal basis in* $I\!R^3$, $\Big((u_\perp^1,u_\perp^2,u_{\omega_0}),(\varphi_1,\varphi_2,\varphi_{\omega_0})\Big)\in I\!R^3\times[0,2\pi[^3$, $M=||g(t,.,.)||_{L^1(I\!R^3\times I\!R^3)}$ *and* $x_0(t)$ *is given by*

$$x_0(t)=\frac{\sigma_0^2}{\sigma_0^2-|\omega_0|^2}\left[\frac{1}{|\omega_0|^2}\omega_0\wedge\big(\omega_0\wedge u_0'(t)\big)-\frac{1}{\sigma_0}\omega_0\wedge u_0(t)\right]-\frac{u_0'(t)\cdot\omega_0}{|\omega_0|^2}\omega_0\ .$$

Proof : If g is a Maxwellian solution of Equation (B2) of the form given in Equation (B1), its coefficients have to satisfy to the following system

$$\frac{\partial r}{\partial t} - |v-u|^2 r \frac{\partial \sigma}{\partial t} + 2r\sigma \frac{\partial u}{\partial t} \cdot (v-u) + v \cdot \nabla_x r - |v-u|^2 r v \cdot \nabla_x \sigma$$

$$+2r\sigma \nabla_x u(v, v-u) + 2r\sigma x \cdot (v-u) = 0$$

or, assuming that the coefficients of the polynomial in $v-u$ cancel

$$\frac{\partial r}{\partial t} + u \cdot \nabla_x r = 0, \tag{B3}$$

$$2r\sigma \frac{\partial u}{\partial t} + \nabla_x r + 2r\sigma u \nabla_x u + r\sigma \nabla_x(|x|^2) = 0, \tag{B4}$$

$$-r\frac{\partial \sigma}{\partial t} - ru \cdot \nabla_x \sigma + 2r\sigma \nabla_x u(h,h) = 0, \tag{B5}$$

$$r\nabla_x \sigma = 0, \tag{B6}$$

for any $h \in S^2$. Assume first that $r > 0$ almost everywhere. Then Equation (B6) gives

$$\sigma(t,x) = \sigma_0(t),$$

and Equation (B5) is now

$$-r\frac{\partial \sigma}{\partial t} + 2r\sigma \nabla_x u(h,h) = 0.$$

According to L. Desvillettes (see [De]), $(\nabla \otimes u - \frac{\sigma_0'(t)}{2\sigma_0(t)} Id)$ is an antisymmetric bilinear form: there exist two functions $t \mapsto \omega(t)$ and $t \mapsto u_0(t)$ such that

$$u(t,x) = \frac{\sigma_0'(t)}{2\sigma_0(t)} x + \omega(t) \wedge x + u_0(t).$$

According to Equation (B4),

$$\frac{\partial u}{\partial t} + \frac{\sigma_0'(t)}{\sigma_0(t)} \omega \wedge x = -\frac{1}{2\sigma_0} \nabla_x(\log r) + \frac{1}{2}\nabla_x(|\omega \wedge x|^2) - \frac{1}{2}\left((\frac{\sigma_0'(t)}{2\sigma_0(t)})^2 + 1 \right) \nabla_x(|x|^2) - \left((\frac{\sigma_0'(t)}{2\sigma_0(t)}) u_0 + \omega \wedge u_0 \right)$$

is a gradient. This is possible only if

$$\frac{\partial \omega}{\partial t} = -\frac{\sigma_0'(t)}{\sigma_0(t)} \omega(t),$$

$$\omega(t) = \frac{\omega_0}{\sigma_0(t)}$$

for some fixed $\omega_0 \in I\!R^3$.

With the notation

$$M = \int \int_{I\!R^3 \times I\!R^3} g(t,x,v)\, dxdv \quad \forall\, t \in I\!R,$$

and using again Equation (B4), $r(t,x)$ takes the form

$$r(t,x) = \frac{M}{(2\pi)^{3/2}} \cdot \sqrt{a(t)} \Big(a(t) + |\omega_0|^2 b(t) \Big) \cdot e^{-\left(\frac{a(t)}{2} |x - x_0(t)|^2 + \frac{b(t)}{2} |\omega_0 \wedge (x - x_0(t))|^2 \right)}$$

provided

$$a(t) + |\omega_0|^2 b(t) > 0 \quad \forall\, t \in I\!R$$

and if $a(t)$, $b(t)$ and $x_0(t)$ satisfy the following equations which are obtained by identifying to zero the coefficients of the polynomial in $(x - x_0(t))$ in Equation (B4):

$$a(t) = \sigma_0''(t) - \frac{(\sigma_0'(t))^2}{2\sigma_0(t)} + 2\sigma_0(t),$$

$$b(t) = -\frac{2}{\sigma_0(t)},$$

$$\frac{1}{2\sigma_0(t)}\left(\sigma_0''(t) - \frac{(\sigma_0'(t))^2}{2\sigma_0(t)} + 2\sigma_0(t)\right)x_0(t) + \frac{1}{(\sigma_0(t))^2}\omega_0\wedge(\omega_0\wedge u_0(t)) + \left(u_0'(t) + \frac{\sigma_0'(t)}{2\sigma_0(t)}u_0(t) + \frac{1}{\sigma_0(t)}\omega_0\wedge x_0\right) = 0. \qquad (B7)$$

The solution of Equation (B7) is given by:

$$\begin{aligned} x_0(t) = &-\left(\frac{\sigma_0''(t)}{2\sigma_0(t)} - \frac{(\sigma_0'(t))^2}{4(\sigma_0(t))^2} + 1 - \frac{|\omega_0|^2}{(\sigma_0(t))^2}\right)^{-1} \\ &\cdot\left\{\frac{1}{|\omega_0|^2}\omega_0\wedge\left(\omega_0\wedge\left(u_0'(t) + \frac{\sigma_0'(t)}{2\sigma_0(t)}u_0(t)\right) - \frac{1}{\sigma_0(t)}\omega_0\wedge u_0(t)\right\} \\ &-\left(\frac{\sigma_0''(t)}{2\sigma_0(t)} - \frac{(\sigma_0'(t))^2}{4(\sigma_0(t))^2} + 1\right)^{-1}\cdot\left\{\left(\left(u_0'(t) + \frac{\sigma_0'(t)}{2\sigma_0(t)}u_0(t)\right)\cdot\frac{\omega_0}{|\omega_0|}\right)\frac{\omega_0}{|\omega_0|}\right\} \end{aligned} \qquad (B8)$$

The problem is now reduced to solve Equation (B3):

$$\frac{\partial(\log r(t,x))}{\partial t} + u(t,x)\cdot\nabla_x(\log r(t,x)) = 0,$$

which gives the explicit expressions for $\sigma_0(t)$ and $u_0(t)$ by identifying the coefficients of the polynomial in $(x - x_0(t))$ to 0:

$$\begin{aligned} &\frac{a'(t)}{2a(t)} + \frac{a'(t) + |\omega_0|^2 b'(t)}{a(t) + |\omega_0|^2 b(t)} - \frac{a'(t)}{2}|x - x_0(t)|^2 - \frac{b'(t)}{2}|\omega_0\wedge(x - x_0(t))|^2 \\ &\quad + a(t)(x - x_0(t))\cdot x_0'(t) + b(t)(\omega_0\wedge(x - x_0(t)))\cdot(\omega_0\wedge x_0'(t)) \\ &\quad - \left(\frac{\sigma_0'(t)}{2\sigma_0(t)}x + \frac{1}{\sigma_0(t)}\omega_0\wedge x + u_0(t)\right)\cdot\left(a(t)(x - x_0(t)) - b(t)\omega_0\wedge\left(\omega_0\wedge(x - x_0(t))\right)\right). \end{aligned}$$

The second order terms in $(x - x_0(t))$ provide the identity:

$$a(t) = \frac{a_0}{\sigma_0(t)}$$

for some constant $a_0 \in]0, +\infty[$ and the condition on ω_0 is now

$$|\omega_0|^2 < \frac{a_0}{2}.$$

The first order coefficients in $(x - x_0(t))$ give

$$\begin{aligned} &a_0\left(x_0'(t) - \left(\frac{\sigma_0'(t)}{\sigma_0(t)}x_0(t) + \frac{1}{\sigma_0(t)}\omega_0\wedge x_0(t) + u_0(t)\right)\right) \\ &\quad + 2\omega_0\wedge\left[\omega_0\wedge\left(x_0'(t) - \left(\frac{\sigma_0'(t)}{\sigma_0(t)}x_0(t) + \frac{1}{\sigma_0(t)}\omega_0\wedge x_0(t) + u_0(t)\right)\right)\right] = 0, \end{aligned}$$

which is equivalent to

$$x_0'(t) = \frac{\sigma_0'(t)}{\sigma_0(t)}x_0(t) + \frac{1}{\sigma_0(t)}\omega_0\wedge x_0(t) + u_0(t). \qquad (B9)$$

The zeroth order coefficient corresponds to

$$\frac{\sigma_0'(t)}{\sigma_0(t)} = 0 \ .$$

σ_0 is therefore a constant as well as $a = 2\sigma_0$ and $b = -\frac{2}{\sigma_0}$, and the condition on ω_0 is now

$$|\omega_0| < \sigma_0 \ .$$

According to Equation (B8),

$$x_0(t) = \frac{\sigma_0^2}{\sigma_0^2 - |\omega_0|^2} \left[\frac{1}{|\omega_0|^2} \omega_0 \wedge \big(\omega_0 \wedge u_0'(t)\big) - \frac{1}{\sigma_0} \omega_0 \wedge u_0(t) \right] - \frac{u_0'(t) \cdot \omega_0}{|\omega_0|^2} \omega_0 \ .$$

Plugging this expression for $x_0(t)$ into Equation (B9), we get for $u(t)$ the system

$$\begin{cases} (u_0'' + u_0) \cdot \omega_0 = 0 \\ [(u_0'' + \dfrac{|\omega_0|^2}{\sigma_0^2} u_0) \wedge \omega_0 = 0 \end{cases}$$

which achieves the proof of Proposition 6 in the case $r > 0$ a.e.

Note that the same proof holds on any connected component of

$$\{(t, x) \in I\!R^+ \times I\!R^3 : r(t, x) > 0\},$$

so that r cannot vanish. □

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