# An introduction to kinetic equations: the Vlasov-Poisson system and the Boltzmann equation

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The purpose of kinetic equations is the description of dilute particle gases at an intermediate scale between the microscopic scale and the hydrodynamical scale. By dilute gases, one has to understand a system with a large number of particles, for which a description of the position and of the velocity of each particle is irrelevant, but for which the decription cannot be reduced to the computation of an average velocity at any time $t \in \mathbb{R}$ and any position $x \in \mathbb{R}^d$: one wants to take into account more than one possible velocity at each point, and the description has therefore to be done at the level of the phase space – at a statistical level – by a distribution function $f(t,x,v)$.

This course is intended to make an introductory review of the literature on kinetic equations. Only the most important ideas of the proofs will be given. The two main examples we shall use are the Vlasov-Poisson system and the Boltzmann equation in the whole space.

## 1 Introduction

### 1.1 The distribution function

The main object of kinetic theory is the *distribution function* $f(t,x,v)$ which is a nonnegative function depending on the time: $t \in \mathbb{R}$, the position: $x \in \mathbb{R}^d$, the velocity: $v\in\mathbb{R}^d$ or the impulsion $\xi$). A basic requirement is that $f(t,.,.)$ belongs to $L^1_{\rm loc}$ $(\mathbb{R}^d\times\mathbb{R}^d)$ and from a physical point of view $f(t,x,v)\,dxdv$ represents "the probability of finding particles in an element of volume $dxdv$, at time $t$, at the point $(x,v)$ in *the (one-particle) phase space*".

$f$ describes the statistical evolution of the system of particles: $f$ has to be constant along the characteristics $(X(t),V(t))$ in the phase space given by Newton's law:

$$\dot X=\frac{dX}{dt}=V \quad , \quad \dot V=\frac{dV}{dt}=F(t,X(t))=-\partial_x U(t,X)$$

if $F$ derives from a potntial $U$.

$$0=\frac{d}{dt}f\Big(t,X(t),V(t)\Big)=\partial_t f+V(t)\cdot\partial_x f+F\Big(t,X(t)\Big)\cdot\partial_v f$$

and satisfies therefore the *transport equation*:

$$\partial_t f+v.\partial_x f+F(t,x)\cdot\partial_v f=0 \tag{1.1}$$

with the notations: $\partial_t f=\frac{\partial f}{\partial \epsilon}$, $\partial_x f=\nabla_x f=\Big(\frac{\partial f}{\partial x_1}\ ,\ \frac{\partial f}{\partial x_1},\dots\frac{\partial f}{\partial x_d}\Big)$, $\partial_v f=\nabla_v f=\Big(\frac{\partial f}{\partial v_1},\frac{\partial f}{\partial v_2},\dots\frac{\partial f}{\partial v_d}\Big)$.

## 1.2 Mean field approximation and collisions

A mean field approximation corresponds to the case where the force itself depends on some average of the distribution function, for instance

$$F(t,x)=(\partial_x V_0 *_x \rho)(t,x)\,, \quad \rho(t,x)=\int_{\mathbb{R}^d} f(t,x,v)\,dv\,.$$

The *Vlasov-Poisson system* is given by $V_0(z)=\frac{1}{4\pi|z|}$ (in dimension $d=3$), or ${\rm div}_x F=\rho$ in general.

Another limit corresponds to short range two-body potentials, for which the effects of the interaction can be considered as a collision: it occurs at a fixed time $t$ for a given position $x$ and acts only on the velocities (in the thermodynamical limit). For dilute gases,no more than two particles are involved in a collision. The fundamental example is the *Boltzmann equation*:

$$\partial_t f+v.\partial_x f \ = \ Q(f,f) \quad (t,x,v)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d \tag{$BE$}$$

where the collision kernel takes the form

$$Q(f,f) \ = \ \int\!\!\int_{\mathbb{R}^d\times S^{d-1}} B(v-v_*,\omega)(f'f'_*-ff_*)\,dv_*d\omega\,, \tag{1.2}$$

$$f=f(t,x,v)\ ,\ f_*=f(t,x,v_*)\ ,\ f'=f(t,x,v'_*)\ ,\ f'_*=f(t,x,v'_*)\,,$$

$v$ and $v_*$ are the velocities of the incoming particles (before collision), $v'$ and $v'_*$ are the velocities of the outgoing particles (after collision) and are given in terms of $v$ and $v_*$ by

$$v' = v - \Big((v - v_*).\omega\Big)\omega \ ,$$
$$v'_* = v_* + \Big((v - v_*).\omega\Big)\omega \ ,$$

for some $\omega \in S^{d-1}$ which parametrizes the set of admissible outgoing velocities under the constraints $v + v_* = v' + v'_*$ and $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$ and $B(v - v_*, \omega)$ is the differential cross-section, which measures the probability of the collision process $(v, v_*) \mapsto (v', v'_*) = T_\omega(v, v_*)$. Note that the collision operator is local in $(t, x)$ and has two parts:

- "the incoming part": $Q_-(f,f) = \int\int B(v - v_*, \omega) f f_* dv_* d\omega$,
- "the outgoing part": $Q_+(f,f) = \int\int B(v - v_*, \omega) f' f'_* \, dv_* d\omega$,

and we may write: $Q(f,f) = Q_+(f,f) - Q_-(f,f)$.

## 1.3 Conservation of mass

Consider a solution $f(t,x,v)$ of the linear transport equation (1.1) or of the Boltzmann equation (BE) and formally perform an integration w.r.t. $v$: if the mass flux is defined by

$$j(t,x) \ = \ \int_{\mathbb{R}^d} f(t,x,v) v \, dv \ ,$$

then one obtains:

$$\partial_t \rho(t,x) + \mathrm{div}_x(j(t,x)) \ = \ 0 \tag{1.3}$$

since the force term is in divergence in $v$ form or since $\int Q(f,f)\, dv = 0$ (of course, one has to assume a sufficient decay of $f$ to justify this computation). This expresses the *local conservation of mass* (or of the number of the particles).

If the problem is stated in the whole space ($x \in \Omega = \mathbb{R}^d$), performing one more integration w.r.t. $x$ and provided $f$ has a sufficient decay in $x$ too, then:

$$\frac{d}{dt} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} f(t,x,v)\, dx dv = \frac{d}{dt} \int_{\mathbb{R}^d} \rho(t,x)\, dx \ = \ 0$$

This relation is the *global conservation of the mass*.

## 1.4 A priori energy estimates

Consider a solution of

$$\partial_t f \ + \ v.\partial_x f \ - \ \partial_x U.\partial_v f \ = \ 0 \ . \tag{1.4}$$

Multiplying equation (1.4) by $\frac{|v|^2}{2}$ and integrating w.r.t. $x$ and $v$, we get

$$\frac{d}{dt}\int\int \frac{|v|^2}{2} f(t,x,v)\,dxdv - \int\int \partial_x U.\partial_v f\,\frac{|v|^2}{2}\,dxdv$$

because $v.\partial_x f\frac{|v|^2}{2} = \mathrm{div}_x\Big(v\frac{|v|^2}{2}f\Big)$ is in divergence form in $x$. Performing one integration by parts w.r.t. $v$ and an other w.r.t. $v$, we get

$$\frac{d}{dt}\int\int \frac{|v|^2}{2} f(t,x,v)\,dxdv - \int U(t,x)\mathrm{div}_x\Big[\int f(t,x,v)v dv\Big] = 0 \tag{1.5}$$

which combined with (1.3) gives the conservation of the energy

$$\frac{d}{dt}\int\int f(t,x,v)\Big(\frac{|v|^2}{2} + U(t,x)\Big)\,dxdv = 0\,. \tag{1.6}$$

Let us consider now the following simple nonlinear Vlasov equation, where the potential $U$ (which may depend on $t$) is given in the mean field approach by the convolution of $\rho$ with some smooth compactly supported kernel $K(x)$:

$$U(t,x) \;=\; K *_x \rho(t,x) = \int K(x-y)\rho(t,y)\,dy\,.$$

The Vlasov equation is now nonlinear (quadratic) and nonlocal:

$$\partial_t f + v.\partial_x f - \partial_x(K*\rho).\partial_v f \;=\; 0 \tag{1.7}$$

and this can also be seen at the level of the energy: exactly as before, combining (1.3) and (1.5), we obtain

$$\begin{aligned}\frac{d}{\partial_t}\int\int \frac{|v|^2}{2} f(t,x,v)\,dxdv &= -\int U(t,x)\partial_t\rho(t,x)\,dx\\ &= -\int\int dxdy K(x-y)\rho(t,y)\partial_t\rho(t,x)\\ &= -\frac{1}{2}\frac{d}{dt}\int\int dxdy K(x-y)\rho(t,y)\rho(t,x)\end{aligned}$$

which provides the conservation of the energy:

$$\frac{d}{dt}\Big[\int\int f(t,x,v)\frac{|v|^2}{2}\,dxdv + \frac{1}{2}\int \rho(t,x)U(t,x)\Big] \;=\; 0\,.$$

Note here the factor $\frac{1}{2}$ in front of the potential energy term.

## 1.5 Velocity averaging lemmas

Velocity averaging lemmas are a basic tool to obtain some compactness in the framework of kinetic equations with distribution functions in $C([0,T],L^1(\mathbb{R}^d\times\mathbb{R}^d))$. We follow the presentation given in [21] and [7], but the basic reference is [23] and also more recent papers by Lions and al. These results together with the notion of renormalization are two crucial steps in the construction of the renormalized solutions for the Boltzmann equation by DiPerna and Lions.

**Lemma 1.1** *Let $f\in L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$ with $f$ having – uniformly in $(t,x)$ – a compact support in $v$ and assume that*

$$Tf \;=\; \partial_t f + v.\partial_x f$$

*belongs to $L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)$. Then $\rho=\int f(.,.,v)\,dv$ is bounded in $H^{1/2}(\mathbb{R}^d\times\mathbb{R}^d)$.*

**Proof:** Consider $\hat f(\tau,z,v)$ the Fourier transform of $f$ w.r.t. $t$ and $x$ and consider $A=\sqrt{\tau^2+z^2}$, $(\tau_0,z_0)=\frac{1}{A}(\tau,z)\in S^d$. $\int \hat f(\tau,z,v)\,dv=I_1+I_2$ where $I_1=\int_{|\tau_0+vz_0|<\frac{1}{A}}\hat f(\tau,z,v)\,dv$ and $I_2=\int_{|\tau_0+vz_0|\geq\frac{1}{A}}\hat f(\tau,z,v)\,dv$. Because of the assumption on the support of $f$, there exists $C>0$ such that

$$I_1^2\leq\int|\hat f|^2(\tau,z,v)\,dv.\mathrm{meas}\Big\{v:|z_0+z_0v|<\frac{1}{A}\Big\}\leq\frac{C}{A}\,,$$

$$I_2^2\leq\frac{1}{A^2}\int|\tau_0+z_0v|^{-2}\,dv\cdot\int|\tau+zv|^2\,|\hat f|^2(\tau,z,v)\,dv\,,$$

where the integral $\frac{1}{A^2}\int|\tau_0+z_0v|^{-2}\,dv$ has to be taken over the set

$$\Big\{u\in\mathrm{supp}(f):|\tau_0+z_0.v|\geq\frac{1}{A}\Big\}$$

and is of order $A$. Putting $I_1$ and $I_2$ together, we get

$$\int_{\mathbb{R}\times\mathbb{R}^d}\sqrt{1+\tau^2+|z|^2}\cdot|\int_{\mathbb{R}^d}\hat f(\tau,z,v)dv|^2\;dzd\tau<+\infty.$$

□

We can also state the result in the form which is appropriate for solutions to kinetic equations in $L^1$ (see [21] or [7] for a proof).

**Corollary 1.2** *Assume that $(g_n)_{n\in\mathbb{N}}$ converges weakly in $L^1([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ and that $(Tg_n)_{n\in\mathbb{N}}$ is weakly relatively compact in $L^1([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$. Then if $\psi_n$ is a bounded sequence in $L^\infty([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ that converges a.e. to some function $\psi$,*

$$\lim_{n\to+\infty}\Big\|\int_{\mathbb{R}^d}(g_n\psi_n)(t,x,v)\,dv-\int_{\mathbb{R}^d}(g\psi)(t,x,v)\,dv\Big\|_{L^1([0,T]\times\mathbb{R}^d)}=0\,. \qquad (1.8)$$

## 1.6 Interpolation lemmas

In the two following lemmas, the relations between the norms and the exponents are easily recovered using scalings in $x$ and $v$. The first lemma can be found for instance in [31, 32]. The second one is a generalization of the first lemma to higher moments. These lemmas are related to the estimates used by B. Perthame [34] or R. Illner & G. Rein [27] for the study of the dispersion of the Vlasov-Poisson system in dimension three. See [15] for a complete proof.

**Lemma 1.3** *Let $f$ be a nonnegative function belonging to $L^p(\mathbb{R}^d\times\mathbb{R}^d)$ for some $p\in]1,+\infty]$ such that $x\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\,|v|^k\,dv$ belongs to $L^r(\mathbb{R}^d)$ for some $(r,k)\in[1,+\infty[\times]0,+\infty[$. Then the function $x\mapsto\rho(x)=\int_{\mathbb{R}^d} f(x,v)\,dv$ belongs to $L^q(\mathbb{R}^d)$ with $q=r\cdot\frac{d(p-1)+kp}{d(p-1)+kr}$ and satisfies*

$$\|\rho\|_{L^q(\mathbb{R}^d)}\leq C(d,p,k)\cdot\|f\|^{\alpha}_{L^p(\mathbb{R}^d\times\mathbb{R}^d)}\cdot\|\int_{\mathbb{R}^d} f(x,v)\,|v|^k\,dv\|^{1-\alpha}_{L^r(\mathbb{R}^d)}\,,$$

*with $\alpha=\frac{kp}{d(p-1)+kp}$, $r\in(1,+\infty)$, $q\in(1+\frac{k(p-1)}{d(p-1)+k},p+\frac{d(p-1)}{k})$ and*

$$C(n,p,k)=\Big(|S^{d-1}|\Big)^{\frac{k(p-1)}{d(p-1)+kp}}\cdot\left(\frac{\left(\frac{kp}{p-1}\right)^{\frac{d(p-1)}{d(p-1)+kp}}}{d^{\frac{(p-1)(d(p-2)+kp)}{p(d(p-1)+kp)}}}+\frac{d^{\frac{k}{d(p-1)+kp}}}{\left(\frac{kp}{p-1}\right)^{\frac{kp}{d(p-1)+kp}}}\right).$$

**Proof:** Assume to simplify that $p<+\infty$ and consider the integral defining $\rho$:

$$\rho(x)=\int_{|v|<R} f(x,v)\,dv+\int_{|v|\geq R} f(x,v)\,dv\,,$$

$$\int_{|v|<R} f(x,v)\,dv\leq\left(\frac{1}{d}|S^{d-1}|R^d\right)^{1-1/p}\cdot\left(\int_{\mathbb{R}^d}|f(x,v)|^p\,dv\right)^{1/p},$$

$$\int_{|v|\geq R} f(x,v)\,dv\leq\frac{1}{R^k}\int_{\mathbb{R}^d} f(x,v)\,|v|^k\,dv\,.$$

If we optimize on $R$, then we get

$$\rho(x)\leq C(d,p,k)\cdot\left(\int_{\mathbb{R}^d}|f(x,v)|^p\,dv\right)^{\frac{k}{d(p-1)+kp}}\cdot\left(\int_{\mathbb{R}^d} f(x,v)\,|v|^k\,dv\right)^{\frac{d(p-1)}{d(p-1)+kp}}.$$

The $L^q$-norm of $\rho$ is now bounded and using Hölder's inequality, we obtain the result for a convenient choice of the exponents. □

**Lemma 1.4** *Let $f$ be a nonnegative function belonging to $L^p(\mathbb{R}^d\times\mathbb{R}^d)$ for some $p\in]1,+\infty]$ such that $x\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x,v)\,|v|^k\,dv$ belongs to $L^r(\mathbb{R}^d)$ for some $(r,k)\in[1,+\infty[\times]0,+\infty[$. Let $m\in[0,k]$ and assume that $m<\frac{p-1}{p-r}\cdot(d(r-1)+kr)$*

*if* $r<p$. *Then the function* $x \mapsto \int_{\mathbb{R}^d} f(x,v)|v|^m\,dv$ *belongs to* $L^u(\mathbb{R}^d)$ *with* $u = r \cdot \frac{d(p-1)+kp}{d(p-1)+m(p-r)+kr}$ *and satisfies for* $\beta = \frac{(k-m)p}{d(p-1)+kp}$

$$\| \int_{\mathbb{R}^d} f(x,v)\,|v|^m\,dv\|_{L^u(\mathbb{R}^d)} \le K \cdot \|f\|^{\beta}_{L^p(\mathbb{R}^d\times\mathbb{R}^d)} \cdot \| \int_{\mathbb{R}^d} f(x,v)\,|v|^k\,dv\|^{1-\beta}_{L^r(\mathbb{R}^d)}\,,$$

$$K = \left(|S^{d-1}|\right)^{\frac{(k-m)(p-1)}{d(p-1)+kp}} \cdot \left( \frac{\left(\frac{kp}{p-1}\right)^{\frac{d(p-1)}{d(p-1)+kp}}}{d^{\frac{(p-1)(d(p-2)+kp)}{p(d(p-1)+kp)}}} + \frac{d^{\frac{k}{d(p-1)+kp}}}{\left(\frac{kp}{p-1}\right)^{\frac{kp}{d(p-1)+kp}}} \right)^{(k-m)/k}.$$

# 2 The Vlasov-Poisson system

In this section we consider the Vlasov-Poisson system

$$\partial_t f + v.\partial_x f - \partial_x U.\partial_v f \;=\; 0 \quad t>0, \quad x,v \in \mathbb{R}^d \tag{2.1}$$

$$\Delta U \;=\; \gamma\rho \;=\; \gamma \int_{\mathbb{R}^d} f(t,x,v)\,dv \tag{2.2}$$

in dimension $d$ (with $d=3$ unless it is specified; for $d=2$, see [15]) and with $\gamma=-1$ (plasma physics or eletrostatic case) or $\gamma=+1$ (gravitational case). The global existence of weak solutions goes back to Arsen'ev [3] and is now known under weak assumptions like:

$$f \in L^1 \cap L^\infty(\mathbb{R}^6), \; \int\!\!\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|v|^2\,dxd\eta < +\infty$$

Here we will rather focuse on strong solutions – solutions for which the characteristics are defined in a classical sense – or even classical $C^1$ solutions for which each of the terms makes sense as a continuous function (and $\partial_x U$ as a Lispchitz function). For stationary solutions, see [4], [15], [16], [19].

## 2.1 Classical solutions and characteristics

We present here in dimension $d=3$ a result which has been established first by K. Pfaffelmoser [35] and then improved by several authors, in the version given by R. Glassey in [22] (initially given by Schaeffer in [37]). The main ingredient of this approach is to start with a solution which is initially compactly supported and to control the growth of the size of the support. Let

$$Q(t) = 1 + \sup\Big\{ |v| : \exists (t,x) \in (0,t)\times\mathbb{R}^3 s.t.\; f(t,x,v) \neq 0 \Big\}$$

**Theorem 2.1** *Let $f_0$ be a non negative $C^1$ compactly supported function. Then the Cauchy problem for (2.1) has a unique $C^1$ solution and*

$$Q(t) \;\leq\; C_p(1+t)^p \quad \text{with} \quad p > \frac{33}{17}\,.$$

Note that the rate of growth has been improved but its optimal value is still unknown.

**Proof:** The proof relies on the iteration scheme

$$\begin{cases} \partial_x f_{n+1} + v.\partial_x f_{n+1} - \partial_x U_n.\partial_v f_{n+1} = 0 \\ +\Delta U_n = \gamma \int_{\mathbb{R}^3} f_n \, dv \\ f_{n+1}(t=0,.,.) = f_0 \end{cases}$$

which is solved at each step by the characteristics method. Passing to the limit is easy after proving the right uniform bounds (energy estimates, bounds on the field and its derivatives, bounds on the derivatives of $f$) which are easily obtained as soon as one has a uniform estimate of the size of the support of $f$ (whatever it is).

To simplify the notation, we shall forget the index $n$ and work directly with a solution. The main step to estimate $Q(t)$ is then to compute for any $t>0$, $\Delta \in ]0,t[$ the quantity:

$$\begin{aligned} \int_{t-\Delta}^{t} \left| E(s,\bar{X}(s)) \right| ds &= c \int_{t-\Delta}^{t} ds \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(s,y,w)}{|\bar{X}(s)-y|^2} \, dydw \\ &= c \int_{t-\Delta}^{t} ds \int\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(s,y,v)}{|\bar{X}(s)-X(s,t,y,v)|^2} \, dydv \end{aligned} \tag{2.3}$$

using the fact that the map $(x,v) \mapsto (X(s,t,x,v),\ V(s,t,x,v))$ given by

$$\frac{dX}{ds} = V\,, \quad \frac{dV}{ds} = -\partial_x U(s,X)\,, \quad (X,V)(t,t,x,v) = (x,v)$$

is measure preversing (here $\bar{X}(s)$) denotes any fixed given characteristics): it is indeed deriving from the flow of an hamiltonian system.

The nexlastt step is to split the integral in (2.3) into the integral over three sets (usually called the "good", the "bad" and the "ugly") and to optimize on the parameters defining these sets, thus obtaining

$$\frac{1}{\Delta} \int_{t-\Delta}^{t} |E(s,\bar{X}(s))| \, ds \;\leq\; CQ(t)^{16/33} |\log Q(t)|^{1/2}\,.$$

$\square$

**Remark 2.2** *The proof is valid in the gravitational case as well as in the plasma physics case since both parts of the energy (kinetic and self consistent potential parts) are uniformly bounded (for some fixed term interval $[0,T]$) even in the gravitational case, where they enter in the energy with opposite signs. The reason is the following.*

*According to Hardy-Littlewood-Sobolev inequalities*

$$\Big\| |x|^{-\lambda} * \phi \Big\|_{L^p(\mathbb{R}^N)} \leq C \, \Big\| \phi \Big\|_{L^q(\mathbb{R}^N)} ,$$

*with* $0 < \frac{1}{p} = \frac{1}{q} + \frac{\lambda}{N} - 1$, *we can control* $||\partial_x U||_{L^2}$ *by*

$$\Big\| \partial_x U \Big\|_{L^2(\mathbb{R}^3)} \leq C \Big\| \frac{1}{|x|^2} * \rho \Big\|_{L^2(\mathbb{R}^3)} \leq C \Big\| \rho \Big\|_{L^{6/5}(\mathbb{R}^3)} .$$

*Then, using Hölder's inequality, we get*

$$\|\rho\|_{L^{6/5}(\mathbb{R}^3)} \leq \|\rho\|^{7/12}_{L^1(\mathbb{R}^3)} \cdot \|\rho\|^{5/12}_{L^{5/3}(\mathbb{R}^3)}$$

*and the* $L^{5/3}$*-norm is controlled by the following interpolation identity (which is a limit case of Lemma 1.3):*

$$\Big\| \rho \Big\|_{L^{5/3}(\mathbb{R}^3)} \leq C . \Big\| f \Big\|^{2/5}_{L^\infty([0,T]\times\mathbb{R}^3\times\mathbb{R}^3)} \Big( \int\int f(t,x,v)|v|^2 \, dxdv \Big)^{3/5} .$$

*If* $K(t) = \int\int f(t,x,v)\frac{|v|^2}{2}\, dxdv$ *and* $P(t) = \frac{1}{2}\int |\partial_x U(t,x)|^2 \, dx$ *are the kinetic energy and the potential energy respectively, then the total energy is*

$$\text{Const} = K(t) - \gamma P(t) \geq K - CK^{10/12}$$

*proving therefore that* $K$ *and also* $P$ *are uniformly bounded (in* $t$*) in terms of* $f_0$.

## 2.2 The Lions and Perthame approach for strong solutions

An alternating approach to find strong solutions in dimension $d = 3$ when the initial data is not compactly supported has been developpped by Lions and Perthame in [32]. It is mainly based on *a priori* estimates for the field $\partial_x U$ and for moments of order $m > 3$.

**Theorem 2.3** *Let* $f_0 \geq 0$ *be a function in* $L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|v|^{m_0} \, dxdv < +\infty$$

*for some* $m_0>3$. *Then there exists a solution of (2.1)-(2.2) in* $C(\mathbb{R}^+, L^p(\mathbb{R}^3\times\mathbb{R}^3))\cap L^\infty(\mathbb{R}^+\times\mathbb{R}^3\times\mathbb{R}^3)$ *for any* $p\in[1,+\infty[$ *satisfying*

$$\sup_{t\in(0,T)}\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)|v|^{m_0}\,dxdv \ \le\ C(T) \quad \text{for any} \quad T>0\,,$$

$$\rho(t,x)=\int_{\mathbb{R}^3} f(t,x,v)\,dv\in C(\mathbb{R}^+,L^q(\mathbb{R}^3))\,,\quad 1\le q<\frac{3+m_0}{3}\,,$$

$$\partial_x U(t,x)\in C(\mathbb{R}^+,L^q(\mathbb{R}^3))\,,\quad \frac{3}{2}\ <\ q\ <\ 3\frac{3+m_0}{6-m_0}\,.$$

**Proof:** The main estimate is the propagation of moments.

$$\begin{aligned} f(t,x,v) &= \int_0^t \mathrm{div}_v(E)(t-s,x-vs)f(t-s,x-vs,v)ds+f_0(x-vt,v)\\ &= \int_0^t \mathrm{div}_v\Big[Ef(t-s,x-vs,v)\Big]\,ds\\ &\qquad +\int_0^t s\,\mathrm{div}_x\Big[Ef(t-s,x-vs,v)\Big]\,ds+f_0(x-vt,v)\,. \end{aligned}$$

If $\rho_0(t,x)=\int f_0(x-vt,v)\,dv$, then

$$\rho(t,x)=\rho_0(t,x)+\int_0^t s\,\mathrm{div}_x\Big[Ef(t-s,x-vs,v)\Big]\,ds$$

and according to Hardy-Littlewood-Sobolev inequalities with $\frac{1}{p}=\frac{1}{r}-\frac{1}{3}$, $\frac{3}{2}<p<+\infty$,

$$\Big\|E(t,.)\Big\|_{L^p(dx)}\le\Big\|\rho_0(t,.)\Big\|_{L^r}+C\Big\|\int_0^t s\int_{\mathbb{R}^3} Ef(t-s,x-v_s,v)\,dvdx\Big\|_{L^p}\,.$$

For $p=m+3$, $r=\frac{3(m+3)}{m+6}$, $m\ge3$,

$$\Big\|\rho_0(t,.)\Big\|_{L^r}\le C\Big(\int\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)|v|^m\,dxdv\Big)^{\frac{3}{m+3}}\ =\ \text{const.}$$

and $\Big\|E(t)\Big\|_{L^{m+3}(\mathbb{R}^3)}\ \le\ C\Big(1+\Big\|\int_0^t s\int_{\mathbb{R}^3} Ef(t-s,x-vs,v)\,dvds\Big\|_{L^{m+3}(\mathbb{R}^3)}\Big)$. But

$$\frac{d}{dt}\Big(\int\int|v|^k f(t,x,v)\,dxdv\Big)\le C\Big\|E(t)\Big\|_{L^{k+3}}\Big(\int\int|v|^k f(t,x,v)\,dxdv\Big)^{\frac{k+2}{k+3}}$$

which (roughly spoken) closes the system of Gronwall estimates. $\square$

## 2.3 Time-dependent rescalings and dispersion

In this section, we introduce as in [20] the time-dependent rescalings for kinetic equations on the example of the Vlasov-Poisson system (2.1)-(2.2) (see also [17], [18]) and show in dimension $d=3$ how this provides the dispersion estimates found independently by Perthame and Illner & Rein (see [34] and [27]).

Consider the Vlasov-Poisson system and compute the transformation of variables given by $A(t)$, $R(t)$, $G(t)$ as follow:

$$dt = A^2(t)d\tau,\ x = R(t)\xi\,.$$

Assuming that $t\mapsto x(t)$ and $\tau\mapsto\xi(\tau)$ respectively satisfy $\frac{dx}{dt}=v$ and $\frac{d\xi}{d\tau}=\eta$, the new velocity variable $\eta$ has to satisfy

$$v=\frac{dx}{dt}=\dot R(t)\xi+R(t)\frac{d\xi}{d\tau}\frac{d\tau}{dt}=\dot R(t)\xi+\frac{R(t)}{A^2(t)}\eta\,.$$

Here $\dot{}$ always denotes derivative with respect to $t$. Let $F$ be the rescaled distribution function: $f(t,x,v)=G(t)F(\tau,\xi,\eta)$. The aim is to choose this transformation in such a way that the rescaled Vlasov equation is still a transport equation on the phase space and contains a given, external force and a friction term. If the rescaled potential is given by

$$-\Delta W=\int F\,d\eta\,,$$

the Vlasov equation transforms into

$$\partial_\tau F+\eta\cdot\partial_\xi F+2A^2(\frac{\dot A}{A}-\frac{\dot R}{R})\eta\cdot\partial_\eta F-\frac{\ddot R A^4}{R}\xi\cdot\partial_\eta F-\frac{R^dG}{A^{2d-4}}\partial_\xi W\cdot\partial_\eta F+A^2\frac{\dot G}{G}F=0.$$

We want $F$ to be a conservation law on $(\xi,\eta)$-space (preservation of the $L^1$-norm), so we require $\frac{\dot A}{A}-\frac{\dot R}{R}=\frac{1}{2d}\frac{\dot G}{G}$ which holds if and only if $G=\left(\frac{A}{R}\right)^{2d}$ (up to a multiplicative constant) and the Vlasov equation becomes

$$\partial_\tau F+\eta\cdot\partial_\xi F+\mathrm{div}_\eta\left[\left(\frac{1}{d}A^2\frac{\dot G}{G}\eta-\ddot R\frac{A^4}{R}\xi-R^dGA^{4-2d}\partial_\xi W\right)F\right]=0.$$

Next we require that the external force in the above Vlasov equation becomes time independent and that there is no time-dependent factor in front of the nonlinear term. We therefore require

$$\ddot R\frac{A^4}{R}=-\gamma c_0\,,\quad R^dGA^{4-2d}=1,$$

where $c_0>0$ is an arbitrary constant. Thus we get $A=R^{d/4}$, $G=R^{\frac{d-4}{2}d}$ and $R$ has to solve

$$\ddot R=-\gamma c_0R^{1-d}\,.$$

Without any restriction, we may asume that $c_0 = 1$, $R(0) = 1$ and $\dot{R}(0) = 0$:

$$F(\tau = 0, \xi, \eta) = f(t = 0, \xi, \eta) = f_0(\xi, \eta)\,.$$

By considering for $F$ the derivative of the energy

$$E(\tau) = \frac{1}{2} \int\!\!\int \Big( |\eta|^2 + W(\tau, \xi) - \gamma |\xi|^2 \Big) F(\tau, \xi, \eta)\, d\eta\, d\xi\,.$$

with respect to $\tau$:

$$\frac{dE}{d\tau} = (d-4)\, R^{\frac{d}{2}-1} \dot{R} \cdot \frac{1}{2} \int\!\!\int |\eta|^2 F(\tau, \xi, \eta)\, d\eta\, d\xi, \,,$$

and writing $L(t) = E(\tau(t))$ in terms of the original variables, we obtain the

**Proposition 2.4** *The function $t \mapsto L(t)$ given by*

$$L(t) = R^{d-2}(t) \int\!\!\int_{\mathbb{R}^d \times \mathbb{R}^d} \left| v - \frac{\dot{R}}{R} x \right|^2 f\, dv\, dx + R^{d-2}(t) \int_{\mathbb{R}^d} \left( U - \gamma \frac{|x|^2}{R^d(t)} \right) \rho\, dx$$

*is decreasing for $d = 2$, 3, 4.*

In dimension $d = 3$, if $\gamma = -1$, $R(t)$ behaves as $t \to \infty$ as $t$, which essentially proves that $\int\!\int f(t, x, v) |x - vt|^2\, dx dv = O(t)$. By an interpolation between this moment and the $L^\infty$-norm of $f$, Perthame and Illner & Rein (see [34] and [27]) proved the following decay estimate.

**Corollary 2.5** *Consider a solution of the Vlasov-Poisson system in the electrostatic case ($\gamma = -1$) corresponding to a nonnegative initial data $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $\int\!\int f_0(x, v) [|x|^2 + |x|^2]\, dx dv$ is bounded. Then*

$$\|\rho(t, .)\|_{L^{5/3}(\mathbb{R}^3)} = O(t^{-3/5})\,.$$

Further estimates on $\partial_x U$ for instance can also be obtained. The method introduced in [20] provides refined estimates and explain how to obtain Lyapunov functionals using time-dependent rescalings in various related systems of fluid dynamics or quantum mechanics, and what is the relation with the pseudo-conformal law.

# 3 Introduction to the Boltzmann equation

For Sections 3.1 and 3.3, we essentially follow the presentation of B. Perhame in [9]. For a detailed study of the hard spheres case we shall refer to [7] and for a more classical theory of perturbations, to [22]. The results on the homogeneous case (and the limit of grazing collisions) are directly collected from the original papers. The dispersion results for renormalized solutions are new results. For the moment, there is no book covering all the mathematical aspects of the Boltzmann equation, the most complete at this time beeing probably the book by Cercignani, Illner and Pulvirenti [7] (hard sphere case only).

## 3.1 The Boltzmann equation

The *non homogeneous Boltzmann equation* (BE) in $\mathbb{R}^d$ describes a cloud of particles expanding in the vacuum. It is an integro-differential equation where the integral part is the *Boltzmann collision operator* is given by (1.2). We are assuming that the particles have the same mass and are affected only by (binary) elastic collisions, so that the conservation of the impulsion and of the kinetic energy respectively give

$$v' + v'_* = v + v_* , \tag{3.1}$$

$$|v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 , \tag{3.2}$$

where $v$ and $v_*$ are the incoming velocities, $v'$ and $v'_*$ the outgoing velocities. These relations can be solved into

$$\begin{aligned} v' &= v - \Big[(v - v_*)\cdot\omega\Big]\omega , \\ v'_* &= v - * + \Big[(v - v_*)\cdot\omega\Big]\omega . \end{aligned}$$

for any $\omega \in S^{d-1}$. We denote by $T_\omega$ the operator acting on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$(v', v'_*) = T_\omega(v, v_*)$$

The *differential cross-section* $B$ is a measurement of the probability of a collision corresponding to a given $\omega$. Physical considerations (microreversibility, galilean invariance) allow to consider $B$ depending only on $|v - v_*|$ and $(v - v_*)\cdot\omega$, and further formal considerations show that for power-like two body potentials

$$u_0(r) \;=\; k\, r^{1-s} \;,$$

$B$ takes the form

$$B(z,\omega) = |z|.\beta(\cos\theta)$$

with $\cos\theta = \frac{z}{|z|}\cdot\omega$ and $\gamma = \frac{s-5}{s-1}$. $\beta$ has a singularity for $\theta = \frac{\pi}{2}$:

$$\beta(\cos\theta) \;\sim\; \left(\frac{\pi}{2} - \theta\right)^{-\frac{s+1}{s-1}}$$

(see [5], [25], [38]). The limit case $s = +\infty$ corresponds to the *hard-spheres model* and it is customary to speak of *hard potentials* for $s > 5$ ($\gamma > 0$) and *soft* *potentials* for $2 \le s \le 5$ ($-3 \le \gamma \le 0$), the limits $s=2$ and $s=5$ corresponding to the Coulomb potential and to the "*Maxwellian molecules*" (no dependence in $|z|$) respectively.

The operator $T_\omega$ has the following properties, which are usually referred as "detailed balance":

i) $T_\omega \circ T_\omega = \mathrm{Id}$: Microreversibility of the collisions,

ii) det $(T_\omega)=1$: $dv' dv'_* = dv dv_*$,

iii) $T_\omega(v_*, v) = (v'_*, v')$,

iv) The collision invariants, *i.e.* the functions $\varphi$ such that

$$\varphi + \varphi_* = \varphi' + \varphi'_*$$

where $\varphi_*$, $\varphi'$ and $\varphi'_*$ respectively stand for $\varphi(v_*), \varphi(v')$ and $\varphi(v'_*)$, are given by:

$$\varphi(v) = a + b.v + c|v|^2 \quad \text{for some constants} \quad (a,b,c) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\ .$$

provided $\varphi$ is continuous at one point: see [22] for instance for a detailed proof (we shall see another proof in Section 3.3).

Unless it is explicitely specified, we shall assume that $d=3$. The classical assumptions on the collision kernel are:

- " *weak angular cut-off*" (Grad): $B \in L^1_{\mathrm{loc}}(\mathbb{R}^3 \times S^2)$
- " *mild growth condition*"

$$\lim_{|z| \to +\infty} \frac{1}{1+|z|^2} \int_{|z-v|<R} \Big( \int_{S^d} B(v,\omega)\, d\omega \Big) dv = 0 \quad \forall R > 0$$

- positivity of $B$ almost everywhere in $(z,\omega) \in \mathbb{R}^d \times S^{d-1}$.

In the case of a power-law interaction, the first two assumptions respectively mean:

$$\Big( \gamma > -3 \text{ (or } s > 2) \text{ and } \ b \in L^1(S^2) \Big) \quad \text{and} \quad \Big( \gamma < 2 \text{ or } \ s > 1 \Big)$$

## 3.2 Conservation laws and $H$-theorem

For any functions $f$, $\varphi$ such that all the involved quantities are well defined,

$$\int_{\mathbb{R}^d} Q(f,f) \varphi(v)\, dv = -\frac{1}{4} \int\!\!\int\!\!\int B(f' f'_* - f f_*)(\varphi' + \varphi'_* - \varphi - \varphi_*)\, dv dv_* d\omega \quad (3.3)$$

As a consequence, we have the

**Lemma 3.1** *(i) Conservation of mass:*

$$\int_{\mathbb{R}^d} Q(f,f)\, dv \;=\; 0$$

*(ii) Conservation of impulsion:*

$$\int Q(f,f) v\, dv \;=\; 0$$

*(iii) Conservation of kinetic energy:*

$$\int Q(f,f)|v|^2\, dv \;=\; 0$$

*(iv) Production of entropy:*

$$\int Q(f,f)\log f\, dv = -\frac{1}{4}\iiint B(v-v_*,\omega)(f'f'_*-ff_*)\log\Big(\frac{f'f'_*}{ff_*}\Big) dv dv_* d\omega \leq 0$$

These identities are easily proved by applying (3.3) with $\varphi=1$, $v$, $|v|^2$, $\log f$, and using identities (3.1) and (3.2) for (ii) and (iii). The last estimate proves the decay of the entropy, since $(x-y)\log(\frac{x}{y})\leq 0\ \forall (x,y)\in ]0,+\infty]^2$. It is known as Boltzmann's H-theorem: for a solution $f$ of (BE),

$$\frac{d}{dt}\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\log f(t,x,v) dx dy \;\leq\; 0$$

Consider now a solution $f(t,x,v)$ of (BE) and the following "macroscopic" quantities, which describe the system at the fluid mechanics level:

- *spatial density*: $\rho(t,x)=\int_{\mathbb{R}^d} f(t,x,v)\, dv$,
- *momentum density*: $u(t,x)=\frac{1}{\rho(t,x)}\int_{\mathbb{R}^d} f(t,x,v) v\, dv$,
- *stress tensor*: $p_{ik}(t,x)=\int_{\mathbb{R}^d}\widetilde{u}_i(t,x,v)\widetilde{u_k}(t,x,v) f(t,x,v)\, dv$ where $\widetilde{u}(t,x,v)=u(t,x)-v$,
- *energy density*: $\frac{1}{2}\int_{\mathbb{R}^d} f(t,x,v)|v|^2\, dv$,
- *internal energy*: $e(t,x)=\frac{1}{2\rho(t,x)}\displaystyle\sum_{i=1}^{d} p_{ii}(t,x)$,
- *heat flux tensor*: $q=-\frac{1}{2}\int_{\mathbb{R}^d}\widetilde{u_k}|\widetilde{u}|^2 f(t,x,v) dv$,

*The pressure*: $p(t,x)=\frac{1}{3}\sum_{i=1}^{d}p_{ii}(t,x)$ obeys to the *equation of state*:

$$p \;=\; \frac{2}{3}\rho e\,. \tag{3.4}$$

The multiplication of (BE) by $1,v$ and $|v|^2$ and an integration w.r.t. $v$ gives at least formally

$$\begin{cases} \partial_t\rho+\partial_x(\rho u) \;=\; 0 \\ \partial_t(\rho u)+\partial_x(p+\rho u\otimes u) \;=\; 0 \\ \partial_t\Big[p(\rho+\frac{1}{2}|u|^2)+\partial_x.\Big[\rho u(e+\frac{1}{2}|u|^2)+up+q\Big] \;=\; 0 \end{cases}$$

These equations and the equation of state (3.4) provide 6 scalar equations for 13 unknowns and we need to impose "*constitutive equations*" to relate those quantities and to close the system. We may for instance consider the following cases:

- *Euler equations for ideal fluids*: $p_{ij}(t,x)=p(t,x)\delta_{ij}\,,\; q_i=0$
- *Navier Stokes equations for viscous fluids* :

$$\begin{aligned} &p_{ij}(t,x)=p(t,x)\delta_{ij}-\mu\Big(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\Big)-\lambda(\partial_x.u)\delta_{ij}\,,\\ &q_i=-k\frac{\partial T}{\partial x_i}\,. \end{aligned}$$

- *Grad hierarchy*: $f(t,x,v)=M_{\rho,u,T}P(v)$ where $M_{\rho,u,T}$ is a local Maxwellian having the same moments in $1,v,|v|^2$ as $f$, and $P$ is a well choosen polynomial. However, this system is not hyperbolic (see [8]).
- *Levermore hierarchy*: $f(t,x,v)=e^{p_{t,x}(v)}$. The closure of this hierarchy is not explicit, but the first nontrivial system (with 17 moments) is hyperbolic (see [28]).

Note that we may derive a macroscopic entropy inequality

$$\begin{aligned} &\frac{\partial}{\partial_t}(S(t,x))+\mathrm{div}(\eta(t,x)) \;\le\; 0\\ &S(t,x) \;=\; \int f(t,x,v)\log\, f(t,x,v)\,dv\,,\\ &\eta(t,x) \;=\; \int f(t,x,v)\log\, f(t,x,v)v\,dv\,, \end{aligned}$$

which is fundamental to describe the shocks in the fluid limit.

## 3.3 Equilibriums are Maxwellian

**Proposition 3.2** *Let $f(v)$ satisfy $\int_{\mathbb{R}^d}(1+|v|^2)f(v)\,dv<+\infty$ and assume that:*

$$f'f'_* = f f_* \quad \forall \omega \in S^{d-1}\,.$$

*Then $f$ is a Maxwellian: $\exists(\rho,T,u)\in\mathbb{R}^+\times]0,+\infty[\times\mathbb{R}^3$*

$$f(v) \;=\; \frac{\rho}{(2\pi T)^{d/2}}e^{-\frac{|v-u|^2}{2T}}\,.$$

**Proof:** Assume that $\int_{\mathbb{R}^d} f(v)\,dv=1$ and $\int_{\mathbb{R}^d} f(v)v\,dv=0$. Consider $g(k)=\int_{\mathbb{R}^d} f(v)e^{iv\cdot k}\,dv$:

$$0=g(k)g(k_*)-\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(v)f(v_*)e^{i(kv+k_*.v_*)}\;e^{i[(k-k_*).\omega][(v-v_*).\omega]}\,dv_*dv$$

If $\omega=\frac{\omega_0+\epsilon\eta}{\sqrt{1+\epsilon^2}}$ for some $(\eta,\omega_0)\in(S^{d-1})^2$ such that $\omega_0.(k-k_*)=0$, $\omega_0.\eta=0$, then a development at the first order gives: $\omega_0.\Big(\nabla_k(gg_*)-\nabla_{k_*}(gg_*)\Big)=0$ and $\omega_0.\nabla_k g=0$. Assume that $k_*=0$: $g$ is radially symmetric and $\nabla_k\log g-\nabla_{k_*}\log g_*$ is proportional to $k-k_*: g(k)=e^{-\beta|k|^2}$. $\square$

## 3.4 Stability, existence of renormalized solutions to the Boltzmann equation

In this section, we consider the Boltzmann equation under the weak (Grad) angular cut-off, the mild growth condition and the positivity (and symmetry) assumptions of Section 3.1. The global existence of solution to the Cauchy problem for arbitrarily large initial data has been proved by DiPerna and Lions in [13] and [14] (see also [21] and [29]) in the framework of the *renormalized solutions*. We assume that the initial data $f_0$ is a nonnegative $L^1(\mathbb{R}^d\times\mathbb{R}^d)$ function such that

$$(x,v)\mapsto f_0(x,v)(|x|^2+\log|v|^2+|\log f_0|)\,dxdv \text{ belongs to } L^1(\mathbb{R}^d\times\mathbb{R}^d). \quad (3.5)$$

As a consequence of the *a priori* estimate

$$\int\int f(t,x,v)|x-tv|^2\,dxdv \;=\; \int\int f_0(x,v)|v|^2\,dxdv$$

which holds because the Boltzmann collision kernel is local in $(t,x)$, and because of the H-theorem:

$$S(t)=\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\log\Big(f(t,x,v)\Big)dxdv \text{ is decreasing },$$

$S(t)$ is bounded from below. To prove it, we may use Jensen's inequality

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\Big[\log f(t,x,v)+|v|^2+|x-tv|^2\Big]\,dxdv$$
$$\geq \ \Big\|f\Big\|_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}\cdot\log\left(\frac{||f||_{L^1(\mathbb{R}^d\times\mathbb{R}^d)}}{\int\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-(|v|^2+|x-tv|^2)}\,dxdv}\right)>-\infty\quad \forall t>0\,.$$

The main difficulty of the Boltzmann equation is to give a sense to the products $f(t,x,v)f(t,x,v_*)$ and to $f(t,x,v')f(t,x,v'_*)$ when $f$ is only a $L^1$ function. Even for a bounded collision kernel $B$, if we write the simplest possible estimate:

$$\int_{\mathbb{R}^d} Q_+(f,f)\,dv = \int_{\mathbb{R}^d} Q_-(f,f)\,dv$$
$$= \Big\|\int^{S^{d-1}} B(z,\omega)\,d\omega\Big\|_{L^\infty(\mathbb{R}^d,dz)}\cdot\Big(\int_{\mathbb{R}^d} f(t,x,v)\,dv\Big)^2\ ,$$

we can see that $(t,x)\mapsto\Big(\int_{\mathbb{R}^d} f(t,x,v)\,dv\Big)^2$ still does not make much sense.

The main idea of renormalized solutions is to replace the equation by a renormalized equation and write that

$$\frac{Q^+(f,f)}{1+f}\quad\text{belongs to}\quad L^\infty(R^+,L^1(\mathbb{R}^d\times K))$$

for any compact set $K$ in $\mathbb{R}^d_v$. A nonnegative distribution function $f$ is said to be a *renormalized solution of the Boltzmann equation* if $f\in C^0(\mathbb{R}^+,L^1(\mathbb{R}^d\times\mathbb{R}^d))$ is such that

$$\int\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\Big(1+|v|^2+|x-tv|^2+|\log\Big(f(t,x,v)\Big)|\Big)dxdv<+\infty$$

for any $t>0$, and if for any $\beta\in C^1(\mathbb{R}^+,\mathbb{R}^+)$ such that $\beta'(t)(1+t)$ is bounded in $\mathbb{R}^+$,

$$\Big(\frac{\partial}{\partial t}+v.\partial_x\Big)\beta(f)=\beta'(f)Q(f,f)\ \text{in}\ \ \mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d)\qquad (RBE)$$

**Theorem 3.3** *(DiPerna & Lions) Under the above assumptions, there exists a global in time renormalized solution to the Boltzmann equation.*

This result is obtained through compactness arguments and appropriate regularization, so that an almost equivalent result is the following stability result.

**Theorem 3.4** *(DiPerna & Lions) Consider a sequence of initial data $f_0^n$ converging in $L^1(\mathbb{R}^d\times\mathbb{R}^d)$ to some $f_0$ such that (3.5) is uniformly satisfied. Then the corresponding renormalized solutions $f^n$ converge up to the extraction of a subsequence to a renormalized solution to the Cauchy problem associated to $f_0$.*

These results are uncomplete from several points of view: the conservation of the energy is not established, the $H$-theorem holds as an inequality and the question of the uniqueness is open.

The boundary problem has been studied by Hamdache [26], Arkeryd and Cercignani [1]. One should also mention the study of the large time asymptotics by Desvillettes [11] and Cercignani [6] (in a bounded domain). For the theory of small classical solutions or perturbations of a stationary solution, we refer to [9] and [22]. An overview of the results in the homogeneous case will be given the next Section. Let us finally mention the existence results recently given by Arkeryd and Nouri in [2] for stationary solutions in a bounded domain.

## 3.5 The homogeneous Boltzmann equation

In the case where the distribution function does not depend on $x$, the situation is much simpler and better results have been proved for already a long time. The general framework is given by $L^1$-spaces with weights: consider $L^1_s$ and $L\log L$ such that

$$f \in L^1_s(\mathbb{R}^d) \Longleftrightarrow f \in L^1(\mathbb{R}^d) \text{ and } \|f\|_{L^1_s} = \int_{\mathbb{R}^d} |f(v)| \left(1+|v|^2\right)^{s/2} dv < +\infty,$$

$$f \in L\log L(\mathbb{R}^d) \Longleftrightarrow f \in L^1(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d} |f(v)\log(f(v))|\, dv < +\infty\,.$$

Consider now the Cauchy problem for the homogeneous Boltzmann equation

$$\begin{cases} \partial_t f = Q(f,f) \\ f(t=0,.,.) = f_0 \end{cases} \qquad (t,v) \in \mathbb{R}\times\mathbb{R}^d \qquad (HBE)$$

### 3.5.1 $L^1$ theory for the hard spheres case ($d=3$)

The hard spheres collision kernel is

$$Q(f,f) = \int\int_{\mathbb{R}^d \times S^{d-1}} \Big|(v-v_*)\cdot\omega\Big| (f'f'_* - ff_*)\, dv_* d\omega$$

We follow here the presentation of [7].

**Theorem 3.5** *Let $f_0 \geq 0$ be an initial data such that $f_0 \in L^1_4 \cap L\log L$. Then there exists a unique solution $f$ in $C^0(\mathbb{R}^+, L^1(\mathbb{R}^3))$. Moreover, $f \in L^1_4(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} f(t,v)\log f(t,v)\, dv \leq \int_{\mathbb{R}^3} f_0(v)\log f_0(v)\, dv\,.$$

**Proof:** First one considers the collision kernel truncated for large velocities:

$$Q_M(f,g)=\frac{1}{2}\int\int \chi_M(v-v_*)|(v-v_*)\cdot\omega|(f'g'_*+g'f'_*-fg_*-f_*g)\,dv_*\,d\omega$$

where $\chi_M$ is the characteristic function of $[0,M]$.

$$\begin{aligned}\Big\|Q^M(f,f)\Big\|_{L^1(\mathbb{R}^3)} &\leq CM\Big\|f\Big\|^2_{L^1(\mathbb{R}^3)},\\ \Big\|Q^M(f,f)-Q^M(g,g)\Big\|_{L^1(\mathbb{R}^3)} &\leq C\Big\|f+g\Big\|_{L^1(\mathbb{R}^3)}\cdot\Big\|f-g\Big\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

With these inequalities, the iteration scheme given by

$$f^M_{n+1}(t,v)=f_0(v)+\int_0^t Q\Big(f^M_n,f^M_n\Big)(S,v)\,ds$$

converges to a function $f^M$ in $C^1\big([0,T],L^1(\mathbb{R}^3)\big)$ provided $T$ is small enough. Next, one has to prove that $f^M$ is nonnegative, which is obtained by proving that $f^M$ solves

$$\begin{cases}\partial_t g+\mu g &= \Gamma^M g\\ g(t=0,.) &= f_0\end{cases}$$

where $\Gamma^M(g)=Q^M(g,g)+\mu g\int g(v)\,dv$ is a positive monotone operator for $\mu$ large.

**Lemma 3.6** *(Povzner [36]) Suppose that* $s\geq 2$, $f,g\in L^1_s$, $f\geq 0$, $g\geq 0$. *Then* $\int_{\mathbb{R}^3}(1+|v|^2)^{s/2}Q(f,g)\,dv\leq c(s)\Big(\|f\|_{1,s}\|g\|_{1,2}+\|f\|_{1,2}\|g\|_{1,s}\Big)$.

By a Gronwall inequality and the conservation of the energy,

$$\Big\|f^M(t)\Big\|_{1,s}\text{ is bounded for any } t\in[0,T]$$

and an elementary computation shows that:

$$\|Q^M_+(f,f)\|_{1,2}\leq C\Big(\|f\|_{1,4}\cdot\|f\|_{L^1}+\|f\|^2_{1,2}\Big).$$

By Dunford-Pettis' criterion, $f^M(t)$ converges up to the extraction of a subsequence to $f(t)$ for all $t\in[0,T]$ and a direct computation shows the convergence of the collision term. Uniqueness follows from a Gronwall argument and the $H$-theorem is given by the convexity of the entropy. □
Note that the assumption $f_0\in L^1_2$ is sufficient for the conservation of the energy [33].

### 3.5.2 Soft potentials

To illustrate the case opposite to the hard spheres case, we consider the case of the soft potentials in $\mathbb{R}^3$ $(2<s\leq 5 \Longleftrightarrow -3<\gamma<0)$:

$$\begin{aligned} B(z,\omega) &= |s|^{\gamma(s)}\cdot\zeta(\theta) \\ \zeta(\theta) &= \beta(\cos\theta) \text{ with the notations of Section 3.1} \end{aligned}$$

This case corresponds to potentials like $r^{1-s}$. We shall consider

- weak solutions for $0\geq\gamma\geq -2$ $(1\geq s\geq \frac{7}{3})$: see the independent papers by Goudon and Villani [24] and [39].
- $H$-solutions (introduced by Villani) for $-3\leq\gamma<2$.

**Theorem 3.7** *Let $f_0$ be a nonnegative function such that*

$$v\mapsto f_0[v(1+|v|^2+\log f_0(v)] \text{ belongs to } L^1(\mathbb{R}^3)\,.$$

*Then there exists a weak solution of (HBE) in the sense that $f$ is nonnegative, belongs to $L^\infty(\mathbb{R}^+(L^1_2\cap L\ \log L)\cap C^0(\mathbb{R}^+,L^1(\mathbb{R}^3)\cap L^1([0,T],L^1_2,\gamma(\mathbb{R}^3))$ and*

$$\int_{\mathbb{R}^3} f(t,v)\log f(t,v)\,dv \;\leq\; \int_{\mathbb{R}^3} f_0(v)\log f_0(v)\,dv \quad \forall t>0$$

*and for any $\varphi\in\mathcal{D}(\mathbb{R}^3)$, for any $s,t\geq 0$*

$$\int f(t,v)\varphi(v)\,dv-\int f(s,v)\varphi(v)\,dv \;=\; \int_s^t d\tau\int Q(f,f)(\tau,v)\varphi(v)\,dv\,. \tag{3.6}$$

The main point is to notice that in the weak formulation, one may write

$$\int Q(f,f)\varphi\,dv=-\frac{1}{4}\int\int\int B(v-v_*,\omega)(f'f'_*-ff_*)(\varphi'+\varphi'_*-\varphi-\varphi_*)\,dvdv_*d\omega. \tag{3.7}$$

and that for $z=v-v_*$ close to 0, the term $(\varphi'+\varphi'_*-\varphi-\varphi_*)$ provides a term of order $|z|^2$ after averaging on $\phi\in S^1$ (if we write $d\omega=\sin\theta\,d\theta d\phi$).

N.B. the right hand side in (3.6) has to be understood in the sense (3.7).

To go further, *i.e.* to soft potentials corresponding to $-2\leq s\leq\frac{7}{3}$ Villani introduced in [39] the notion of $H$-solutions which is based on the following remark. Denoting by $F$ and $F'$ the tensor function $ff_*$ and $f'f'_*$, we may write:

$$\int Q(f,f)\,dv=\frac{1}{4}\int\sqrt{B}(\sqrt{F'}-\sqrt{F}).\sqrt{B}(\sqrt{F'}+\sqrt{F})(\varphi'+\varphi'_*-\varphi-\varphi_*)\,dvdv_*d\omega \tag{3.8}$$

Using the inequality $(x-y)\log(\frac{x}{y})\geq 4(\sqrt{x}-\sqrt{y})$, the entropy dissipation term indeed controls $\sqrt{B}(\sqrt{F'}-\sqrt{F})$ in $L^2(\mathbb{R}^3)$, and the cancellations in $\varphi'+\varphi'_*-$

$\varphi - \varphi_*$ allow to give a sense to the weak formulation whenever the right hand side in (3.6) is replaced by expression (3.8).

A similar study can be done for the Landau equation and this framework is very convenient to consider the limit of grazing collisions which corrresponds to concentration of $\zeta(\theta)$ at $\theta = 0$. The Landau equation appears then as a Taylor development at order 2 of the Boltzmann equation (see [10], [24], [39]).

### 3.5.3 Gain of moments and regularizing effects for collision kernel without cut-offs

Consider the case of hard potentials and assume that the initial data is bounded in $L^1_{2+\delta}$ with $\delta > 0$. According to Povzner inequality

$$\int_0^T d\tau \int f(\tau,v)|v|^{2+\gamma+\delta}\,dv \le C_1 \int f_0(v)\,|v|^{2+\delta}\,dv + C_2 T\Big(\int f_0(v)\,|v|^2\,dv\Big)^2$$

(see [12], [33]). In other words, $f \in L^1([0,T],\ L^1_{2+\delta+\gamma})$ for any $\delta > 0$. Thus by iteration any moment becomes finite for any positive time.

More interesting probably are the regularization properties of the Boltzmann collision kernel. For forces with an infinite range, and especially for inverse power laws, the weak angular cut-off assumption is not satisfied: if $B(z,\omega) = |z|^\gamma\,\zeta(\theta)$, then $\zeta$ has a singularity of order $\frac{s+1}{s-1} = 1+\nu$. P.-L. Lions proved in [30] that

$$\sqrt{f}(t) \in H^r_{\mathrm{loc}}(\mathbb{R}^d) \quad \forall r < \frac{\nu}{2}\Big(\frac{1}{1+\frac{\nu}{d-1}}\Big)$$

using the smoothing properties of $Q_+$. Recently, further results have been given by Villani, and Desvillettes and Wennberg.

## 3.6 Dispersion for the renormalized solutions

We conclude this introduction to the Boltzmann equation by giving a dispersion result for the renormalized solutions. A preliminary result has been obtained by B. Perthame in [34], but we follow here the approach of [17] based on Jensen's inequality.

**Theorem 3.8** *Under the same assumptions as in Section 3.4, consider a renormalized solution $f \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ corresponding to an initial datum $f_0 \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $f_0(|x|^2 + |v|^2 + |\log f_0|)$ is bounded in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Then*

$$\begin{aligned}(1+t^2)\ \int\!\!\int f\Big|v - \frac{t}{1+t^2}x\Big|^2\,dxdv + \frac{1}{1+t^2}\int\!\!\int f|x|^2\,dxdv + \int\!\!\int f\log f\,dxdv \\ \le \int\!\!\int f_0(|x|^2 + |v|^2)\,dxdv + \int\!\!\int f_0 \log f_0\,dxdv\,, \qquad (3.9)\end{aligned}$$

*and there exists a positive constant* $C = C(f_0)$ *which depends only on* $f_0$ *such that for any* $r > 0$,

$$m(r,t) := \int_{|x|<r} [\int_{\mathbb{R}^3} f(t,x,v)\, dv]\, dx \leq \frac{C(f_0)}{\log\left(\frac{\sqrt{1+t^2}}{r}\right)} . \tag{3.10}$$

**Proof:** We may first notice that

$$(1+t^2)\int\int f|v - \frac{t}{1+t^2}x|^2\, dxdv + \frac{1}{1+t^2}\int\int f|x|^2\, dxdv$$
$$= \int\int f|v|^2\, dxdv + \int\int f|x - vt|^2\, dxdv$$

We now use (3.10) to obtain a dispersion relation via an interpolation which is in a sense the limit case (see Section 1.6) as $p \to 1$ of an interpolation between moments and an $L^p$-norm. The result is obtained using several times Jensen's inequality: *if* $f$ *and* $g$ *are two nonnegative* $L^1(\Omega)$ *solutions such that* $f(|\log f| + |\log g|)$ *belongs to* $L^1(\Omega)$, *Jensen's inequality applied to* $t \mapsto t \log t = s(t)$ *with the measure* $d\mu(y) = \frac{g(y)dy}{\int_\Omega g(y)\, dy}$ *gives*

$$\frac{\int_\Omega f \log(\frac{f}{g})\, dy}{\int_\Omega g(y)\, dy} = \int_\Omega s(\frac{f}{g})\, d\mu(y) \geq s\left(\int_\Omega \frac{f}{g}\, d\mu(y)\right) = \frac{\int_\Omega f(y)\, dy}{\int_\Omega g(y)\, dy} \log\left(\frac{\int_\Omega f(y)\, dy}{\int_\Omega g(y)\, dy}\right) . \tag{3.11}$$

Applying first this inequality to $g = e^{-(1+t^2)|v - \frac{t}{1+t^2}x|^2}$ with $y = v$, $\Omega = \mathbb{R}^3$, and then integrating w.r.t. $x$, we get

$$\int \rho \log \rho\, dx \tag{3.12}$$
$$\leq \int\int f \log f\, dxdv + (1+t^2)\int\int f|v - \frac{t}{1+t^2}x|^2\, dxdv - \frac{3M}{2}\log\left(\frac{1+t^2}{\pi}\right)$$

where $M = m(\infty, t) = \int\int_{\mathbb{R}^3 \times \mathbb{R}^3} f(t,x,v)\, dxdv = \int_{\mathbb{R}^3} \rho(t,x)\, dx$. Applying now (3.11) to $\Omega = B(0,R)$, $g \equiv 1$, $y = x$, we find

$$m(r,t) \log m(r,t) \leq m(r,t) \log(\frac{4\pi}{3} r^3) + \int_{|x|<r} \rho(t,x) \log \rho(t,x)\, dx . \tag{3.13}$$

But

$$\int_{|x|<r} \rho(t,x) \log \rho(t,x)\, dx = \int_{\mathbb{R}^3} \rho(t,x) \log \rho(t,x)\, dx - \int_{|x|>r} \rho(t,x) \log \rho(t,x)\, dx . \tag{3.14}$$

Applying again (3.11) to $\Omega = \mathbb{R}^3 \setminus B(0,r)$, $y=x$, we find for $m=m(r,t)$

$$-\int_{|x|>r} \rho \log \rho \, dx \ \leq \int_{|x|>r} \rho \frac{|x|^2}{1+t^2} \, dx$$
$$-(M-m)\log(M-m) + \frac{3}{2}(M-m)\log[\pi(1+t^2)] \, .$$

Combining (3.12), (3.13), (3.14) and (3.15), we obtain (3.10). □

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