Monokinetic charged particle beams : Qualitative behavior of the solutions of the Cauchy problem and $2d$ time-periodic solutions of the Vlasov-Poisson system

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Let (f, U) be a solution of the $2d$ Vlasov-Poisson system for charged particles :

$$\left\{\begin{array}{ll} \partial_t f + v \cdot \partial_x f - (\partial_x U(t,x) + \partial_x U_0(x)) \cdot \partial_v f = 0 & (V) \\ -\Delta U = \rho(t,x) = \int_{\mathbb{R}^2} f(t,x,v)\, dv & (P) \end{array}\right.$$

$f = f(t,x,v)$ is the distribution function, defined on the phase space $\mathbb{R}^2_x \times \mathbb{R}^2_v$ (x represents the position and v the velocity of the particles) and $U = U(t,x)$ is the self-consistent potential given by the Poisson equation. U_0 is an external potential which depends only on $|x|$.

Such a simplified problem appears when one considers the free propagation of a particle beam in $\mathbb{R}^3$ and when the particles are monokinetic in the $z-$direction of the axis of the beam and when one considers a distribution function averaged other the velocities in the $z-$direction. The beam is assumed to be infinite in that direction, with a spatial density not depending on z (see Appendix A for a formal derivation of the model). This model may be used in high energy plasma physics (see [Do5]) to describe 3-dimensional electron beams with high density. It takes the dispersion of the velocities in the orthogonal to the z-direction plane as well as the self-consistent nonlinear interaction into account, which seems to be crucial for the understanding of the experiments and of the numerical simulations.

The two main motivations are indeed the following: first to understand how the support of a beam grows when it is initially compactly supported and more generally, what is its behaviour, how the tail (which corresponds to large velocities and is not easily computed by particle simulation methods) of the distribution function evolves, what kind of dispersion is obtained... and second, to give a systematic method for the study of the time-periodic solutions that clearly appear for large times in numerical simulations.

A natural asymptotic boundary condition for such a model would be to consider only a logarithmic growth of the potential $U(t,x)$ as $|x| \to +\infty$, but since we are interested in the local behavior of such a solution, we will relax this assumption and consider more general cases in part II. In part I, we will assume that

$$U(t,x) = -\frac{1}{2\pi} \ln|x| * \rho(t,x)$$

and study the problem in a framework for which this formula makes sense.

For the same reason, we will also frequently assume throughout the paper (but in most cases it is not essential) that the external potential – when there is one – is harmonic : there exists some $\rho_0 \in]0, +\infty[$ such that

$$U_0(x) = \frac{\rho_0}{2} \cdot |x|^2 \quad \forall\, x \in \mathbb{R}^2\ . \qquad (H)$$

The paper is divided into two parts. The first one is concerned with the evolution problem (existence, regularity or uniqueness as well as more qualitative aspects, like dispersion results, growth of the support for compactly supported distribution functions or equipartition of the energy). The main estimate is a Lyapunov functional which is used to control the energy (and provides an existence result for non compactly supported distribution functions) and to give an estimate for the dispersion ($N = 2$ appears to be the critical dimension and we have to use logarithmic estimates). In the second part, we focus on time-periodic (and stationary) solutions and present results in two directions: first, we give an (uncomplete) classification of the solutions that are time-periodic (but radially symmetric if they are averaged over one time period); then we study a special class of solutions which is the counterpart of the class of solutions of the Vlasov-Poisson system in dimension three that satisfy the so-called Ehlers & Rienstra ansatz.

Three sections, which are of general interest but rather technical, are rejected at the end of the paper: Appendix A deals with a formal derivation of the two dimensional model, which main interest is to show that the model is local, with two consequences: it is not restrictive to take the confining potential harmonic, and there is no *a priori* natural boundary condition for U. In Appendix B are stated two interpolation lemmas, with an explicit computation of the constants. Appendix C provides explicit and detailed statements for Jeans' theorem.

Following Perthame's definitions (see [Pe2]), a *weak solution* is a solution in the sense of the distributions such that the energy is bounded but not necessarily constant. A *strong solution* (in dimension $N = 2$) is a solution that has moments of order $2+\epsilon$ in v and x (at least when the external potential is harmonic) and such that the energy does not depend on t. In this paper, we will make use of an intermediate notion of solutions corresponding to the case $f(U_0 + |v|^2)$ in L^1 with U_0 growing at least logarithmically at infinity (confinement case) or such that f has a moment in x of order $m \in [1, 2]$. For such solutions, the self-consistent potential energy is continuous w.r.t. the time (in dimension $N = 2$), but the kinetic energy and the external potential energy are only bounded w.r.t. the time.

Since the number of references in this work is quite huge and concerns very different subjects, the references will be mentioned throughout the paper.

For general results on the evolution problem, one has to mention at least the papers by S. Ukai & T. Okabe [UO] and S. Wollman [W1,2] in dimension two, and the papers by C. Bardos & P. Degond [BaD], P-L. Lions & B. Perthame [LP2] and K. Pfaffelmoser [Pf] in dimension three. The dispersion relations are strongly related to the work of R. Illner & G. Rein [IR], P-L. Lions & B. Perthame [LP3] and B. Perthame [Pe2].

What concerns the time-periodic solutions and the Jeans' theorem is directly inspired respectively by J. Batt, H. Berestycki, P. Degond & B. Perthame [BBDP] and by J. Batt, W. Faltenbacher, & E. Horst [BFH].

Part I :
Qualitative behavior of the solutions of the Cauchy problem

Introduction

The first part of this paper is devoted to various results on the initial value problem for the two dimensional Vlasov-Poisson system in the presence (or not) of a confining potential.

In section 1, we introduce two notions of Lyapunov functionals. In section 2 (see the complete list of references therein), an existence result is given, in a more general framework than what had been established by S. Ukai & T. Okabe in [UO] and S. Wollman in [W1,2] . This result essentially benefits of recent papers on the (more difficult) theory for the three dimensional problem. The questions of the regularity and of the uniqueness are treated with the approach developped by P.-L. Lions and B. Perthame in [LP2].

A dispersion result is given in section 3: if there is no confining potential, the solution is vanishing for large time. This result is obtained using the same methods as R. Illner & G. Rein in [IR] or B. Perthame in [Pe2] for the case of the dimension three. Dimension two corresponds to a limit case for this method (use of logarithmic estimates).

The question of the growth of the support of an initially compactly supported distribution function is studied in section 4. One has to mention that the method, which is strongly dependant of the potential when it is applied to the computation of the size of the support in the phase space, gives the growth in the velocity space even if there is no confining potential, as in the paper [R] by G. Rein, but with a different method.

An equipartition of the energy result is given in section 5. The estimate obtained there is in fact the keypoint of Part I since it allows the computations on the Lyapunov functional. Section 5 also contains the moment estimates that are needed to compute the Lyapunov functional as well as to define the notion of solutions.

1. A Lyapunov functional and *a priori* estimates

Following the same idea as in B. Perthame [Pe2] and R. Illner and G. Rein in [IR], we first derive a Lyapunov functional for the Vlasov-Poisson system in dimension 2 with an external potential. In the following, we shall assume that f and U are smooth, and that f is compactly supported, in order to perform any integration by part that is needed. We shall see later how to handle the non smooth case.

Multiplying the Vlasov equation by $|v|^2$, $(x \cdot v)$ and $|x|$, we can obtain respectively

$$\frac{d}{dt}\left[\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot |x|^2\;dxdv\right] = 2\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot (x\cdot v)\;dxdv\;, \tag{1.1}$$

$$\frac{d}{dt}\left[\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(x\cdot v)\;dxdv\right] = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot|v|^2\;dxdv - \int_{I\!R^2}(x\cdot\nabla U_0)\rho(t,x)\;dx + \frac{M^2}{4\pi}\;, \tag{1.2}$$

$$\begin{aligned}\int\int_{I\!R^2\times I\!R^2} & f(t,x,v)\cdot(|v|^2 + U(t,x) + 2U_0(x))\;dxdv \\ &= \int\int_{I\!R^2\times I\!R^2} f(t=0,x,v)\cdot(|v|^2 + U(t=0,x) + 2U_0(x))\;dxdv = E_0\;,\end{aligned} \tag{1.3}$$

using the fact that

$$\begin{aligned}\int\int_{I\!R^2\times I\!R^2} & |v|^2\cdot\nabla U(t,x)\cdot\partial_v f(t,x,v)\;dxdv \\ &= 2\int_{I\!R^2} dx\;U(t,x)\;\mathrm{div}\int_{I\!R^2} dv\;vf(t,x,v) \\ &= -2\int_{I\!R^2} dx\;U(t,x)\frac{\partial\rho}{\partial t}(t,x) \\ &= -\int_{I\!R^2}\rho(t,x)U(t,x)\;dx \\ &= -\int\int_{I\!R^2\times I\!R^2} f(t,x,v)U(t,x)\;dxdv\;,\end{aligned}$$

and that

$$\begin{aligned}\int\int_{I\!R^2\times I\!R^2} & (x\cdot v)\cdot\nabla U(t,x)\cdot\partial_v f(t,x,v)\;dxdv \\ &= -\int_{I\!R^2} dx\;x\cdot\nabla U(t,x)\rho(t,x) \\ &= \frac{M^2}{4\pi}\;,\end{aligned}$$

because of Poisson's equation (see section 5 for the proof of this identity).

In dimension $N = 3$ (see [Pe2] and [IR]), it is enough to compute (for some $\alpha > 0$)

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{t+\alpha}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot|x - v(t+\alpha)|^2\;dxdv + (t+\alpha)\int_{I\!R^2}|\nabla U(t,x)|^2\;dx\right) \\ &= -\frac{1}{(t+\alpha)^2}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x - v(t+\alpha)|^2\;dxdv\end{aligned}$$

to get a decay estimate since the Lyapunov functional is a positive quantity. In dimension $N = 2$ and in the presence of an external potential U_0, the analogous computation would give

$$\begin{aligned}\frac{d}{dt}&\left(\frac{1}{(t+\alpha)^2}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\big(|x - v(t+\alpha)|^2 + U(t,x) + 2U_0(x)\big)\;dxdv + \frac{M^2}{2\pi}\ln(t+\alpha)\right) \\ &= -\frac{2}{(t+\alpha)^3}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x - v(t+\alpha)|^2\;dxdv + \frac{2}{t+\alpha}\int_{I\!R^2}(x\cdot\nabla U_0(x))\;\rho(t,x)\;dx\;,\end{aligned} \tag{1.4}$$

which is not enough to conclude (even if $x \cdot \nabla U_0(x) \le 0$) since $\int\int f(t,x,v)U(t,x)\;dxdv$ is not necessarily a positive quantity. We will distinguish two cases:

- $U_0(x)$ is growing when $|x| \to +\infty$ like at least $\ln|x|$ (confinement case)
- $x \mapsto \frac{x\cdot\nabla U_0}{1+|x|^2}$ is bounded (and, for instance, $x \cdot \nabla U_0 \le 0$ – dispersive case. In this case, L_α is decreasing)

Case 1: (confinement case)

Proposition 1.1 : *Assume that f_0 is a nonnegative function in $L^1 \cap L^\infty(I\!R^2 \times I\!R^2)$ such that*

$$E_0 = \int\int_{I\!R^2\times I\!R^2} f_0(x,v)\cdot(|v|^2 + U(t=0,x) + U_0(x))\;dxdv < +\infty$$

with $U_0 \ge 0$, $\nabla U_0 \in W^{1,\infty}_{\rm loc}(I\!R^2)$ and $U(t=0,x)$ given by the Poisson equation. Assume also that

$$L = \liminf_{|x|\to+\infty} \frac{U_0(x)}{\ln|x|} > \frac{M}{2\pi}$$

with $M = ||f_0||_{L^1(I\!R^2\times I\!R^2)}$. If f is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data f_0, then :

(i) There exists a constant $C \in]0, 2-\frac{M}{\pi L}[$ such that

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(U(t,x)+2U_0(x))\;dxdv \ge C\cdot\int\int_{I\!R^2\times I\!R^2} f(t,x,v)U_0(x)\;dxdv \ge 0\ .$$

(ii) If

$$\exists m\in[1,2] \quad \forall\, t>0 \qquad \int\int_{I\!R^2\times I\!R^2} f(t,x,v)(|v|^2+|x|^m)\;dxdv < +\infty\ ,$$

then

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(U(t,x)+(2-C)U_0(x))\;dxdv \le E_0\ ,$$

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(|v|^2+CU_0(x))\;dxdv \le E_0\ ,$$

and

$$L_\alpha(t) = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\big(\frac{|x-v(t+\alpha)|^2}{(t+\alpha)^2} + U(t,x) + 2U_0(x)\big)\;dxdv + \frac{M^2}{2\pi}\ln(t+\alpha)$$

is bounded from below for any $\alpha>0$, $t>0$. Moreover, if f is a strong solution, i.e. a solution such that

$$\exists \varepsilon>0 \quad \forall\, t>0 \qquad \int\int_{I\!R^2\times I\!R^2} f(t,x,v)(|v|^{2+\varepsilon}+|x|^{2+\varepsilon})\;dxdv < +\infty\ ,$$

then

$$\frac{dL_\alpha}{dt}(t) = -\frac{2}{(t+\alpha)^3}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x-v(t+\alpha)|^2\;dxdv + \frac{2}{t+\alpha}\int_{I\!R^2}(x\cdot\nabla U_0(x))\,\rho(t,x)\;dx\ . \quad (1.4)$$

Proof of Proposition 1.1 : We have to prove that the self-consistent potential energy is from below

$$-\int_{\mathbb{R}^2} \rho(t,x)U(t,x)\;dx \le \frac{M^2}{\pi}\ln 2 + \frac{M}{\pi}\int_{\mathbb{R}^2} \rho(t,x)\ln|x|\;dx$$

since

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} \rho(t,x)\rho(t,y)\;\ln|x-y|\;dxdy \le 2\int\int_{|y|\le|x|} \rho(t,x)\rho(t,y)\;\ln(2|x|)\;dxdy\;.$$

Then

$$\frac{M}{\pi}\int_{\mathbb{R}^2} \rho(t,x)\ln|x|\;dx \le \frac{M}{\pi}\int_{|x|<k} \rho(t,x)\ln|x|\;dx + \frac{M}{\pi}\int_{|x|\ge k} \rho(t,x)\ln|x|\;dx$$
$$\le \frac{M^2}{\pi}\ln k + \frac{M}{\pi}\int_{\mathbb{R}^2} \rho(t,x)U_0(x)\;dx\cdot\sup_{|x|\ge k}\frac{\ln|x|}{U_0(x)}$$

for k large enough.

$$\int_{\mathbb{R}^2} \rho(t,x)U(t,x)\;dx \ge -\frac{M^2}{\pi}(\ln 2+\ln k) - \frac{M}{\pi\inf_{|x|\ge k}\frac{U_0(x)}{\ln|x|}}\int_{\mathbb{R}^2} \rho(t,x)U_0(x)\;dx\;.$$

The proof of (ii) is contained in Remark 1.2 below (see also Lemma 5.4 in Section 5). □

Remark 1.2 :

(i) We will prove in section 5 that the condition on f holds as soon as the initial data has moments of order m and 2 respectively in x and v. f is a strong solution if the initial data has moments of order $2+\epsilon$.

(ii) For a solution $f\in L^1\cap L^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ with moments of order 2 in v and such that
- either (case a)

$$\lim_{|x|\to+\infty}\frac{U_0(x)}{\ln|x|} = +\infty\;,$$

- or (case b) f has a moment of order m in x,

no concentration or vanishing of the self-consistent potential energy may occur: if we split

$$\int_{\mathbb{R}^2} \rho(t,x)U(t,x)\;dx = -\frac{1}{2\pi}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} \rho(t,x)\rho(t,y)\;\ln|x-y|\;dxdy$$

into three parts corresponding to $|x-y|<\epsilon$, $\epsilon<|x-y|<\epsilon^{-1}$ and $|x-y|<\epsilon^{-1}$, we just have to control the first one and the third one:

$$\int\int_{|x-y|<\epsilon} \rho(t,x)\rho(t,y)\;\ln|x-y|\;dxdy \le ||\rho(t,.)||_{L^1(\mathbb{R}^2)}\cdot||\rho(t,.)||_{L^2(\mathbb{R}^2)}\cdot\left(\int_0^\epsilon 2\pi(\ln r)^2 r\;dr\right)^{1/2}\;,$$

and (case a), using the symmetry $(x,y)\mapsto(y,x)$,

$$\begin{aligned}
&\int\int_{|x-y|>\epsilon^{-1}} \rho(t,x)\rho(t,y)\ \ln|x-y|\ dxdy \\
&= 2\int\int_{\substack{|x-y|>\epsilon^{-1}\\ |y|<\theta|x|}} \rho(t,x)\rho(t,y)\ \ln|x-y|\ dxdy + 2\int\int_{\substack{|x-y|>\epsilon^{-1}\\ \theta|x|<|y|<|x|}} \rho(t,x)\rho(t,y)\ \ln|x-y|\ dxdy \\
&\le 2M\int_{|x|>\frac{1}{\epsilon(1-\theta)}} \rho(t,x)\ln(|x|(1+\theta))\,dx + 2\int_{|x|>\frac{1}{2\epsilon}}\rho(t,x)\ln(2|x|)\,dx\int_{|y|>\frac{\theta}{2\epsilon}}\rho(t,y)\,dy \\
&\qquad \to 0 \quad \text{as } \epsilon\to 0_+
\end{aligned}$$

for any $\theta\in]0,1[$.

In (case b), if $\rho(t,.)$ has a moment of order $m>0$, using again the symmetry $(x,y)\mapsto(y,x)$,

$$\int\int_{|x-y|>\epsilon^{-1}} \rho(t,x)\rho(t,y)\ \ln|x-y|\ dxdy \le \epsilon^m\ln(\frac{1}{\epsilon})\int\int_{|x-y|>\epsilon^{-1}}\rho(t,x)\rho(t,y)\ |x-y|^m\ dxdy$$

for $\epsilon>0$ small enough, and the conclusion holds using the identity $|x-y|^m\le 2^m(|x|^m+|y|^m)$. (Case a) as well as (Case b) correspond to cases where the self-consistent potential energy is continuous w.r.t. the time.

(iii) If $U_0(x)=K\ln(1+|x|)$ for some $K<\frac{M}{2\pi}$, for any $\alpha>0$, we can get the following dispersion-type estimate:

$$\liminf_{t\to+\infty}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\frac{|x-(t+\alpha)v|^2}{(t+\alpha)^2}\ dxdv \le 2M(K-\frac{M^2}{4\pi})\ .$$

Let us prove it: according to Proposition 1.2 (i), with $L=K$, $C=2-\frac{M}{\pi K}$, as $t\to+\infty$,

$$\begin{aligned}
&\frac{M^2}{2\pi}\ln(t+\alpha)\ (1+o(1)) \\
&\quad\le L_\alpha(t)\le L_\alpha(0)-\int_0^t\frac{2}{s+\alpha}\Big(\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(s,x,v)\frac{|x-(s+\alpha)v|^2}{(s+\alpha)^2}\ dxdv+2KM\Big)\ ds\ ,
\end{aligned}$$

$$\frac{M^2}{2\pi}\ln(t+\alpha)\le\Big(2KM-\liminf_{t\to+\infty}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\frac{|x-(t+\alpha)v|^2}{(t+\alpha)^2}\ dxdv\Big)\ln(t+\alpha)+o(\ln(t+\alpha))\ .$$

Case 2: Assume that $x\mapsto\frac{x\cdot\nabla U_0}{1+|x|^2}$ belongs to $L^\infty(\mathbb{R}^2)$. Let us compute (with $\alpha>1$ and $t>0$)

$$\begin{aligned}
H_\alpha(t) = \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\cdot\Big(&\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2+\frac{1}{(t+\alpha)^2}|x-v(t+\alpha)|^2 \\
&+U(t,x)+U_0(x)+\frac{M}{2\pi}\ln(t+\alpha)\Big)\ dxdv\ ,
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
\frac{d}{dt}H_\alpha(t) = &-\frac{2}{(t+\alpha)^3}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\cdot|(1+\frac{1}{2\ln(t+\alpha)})x-v(t+\alpha)|^2\ dxdv \\
&-\frac{1}{(t+\alpha)^3\big(\ln(t+\alpha)\big)^2}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|x|^2+\frac{2}{t+\alpha}\int_{\mathbb{R}^2}(x\cdot\nabla U_0)\rho(t,x)\ dx\ .
\end{aligned} \tag{1.6}$$

The reason why such a quantity is decreasing if $x \cdot \nabla U_0(x) \leq 0$ for almost all $x \in I\!R^2$ and why one has to introduce a term $\int\int f(t,x,v)|x|^2\, dxdv$ is related to the notion of asymptotic dispersion profile. This is the subject of a paper in preparation with G. Rein [DR].

The Lyapunov functional is finite for any $t > 0$: we have indeed

$$\int_{I\!R^2} \rho(t,x)U(t,x)\, dx = -\frac{1}{2\pi}\int\int_{I\!R^2\times I\!R^2} \rho(t,x)\rho(t,y)\ln|x-y|\, dxdy$$

and

$$\int\int_{|x-y|<k} \rho(t,x)\rho(t,y)\ln|x-y|\, dxdy \leq M^2 \ln k\ ,$$

$$\int\int_{|x-y|\geq k} \rho(t,x)\rho(t,y)\ln|x-y|\, dxdy \leq \int\int_{|x-y|\geq k} \rho(t,x)\rho(t,y)|x-y|^2\, dxdy \cdot \frac{\ln k}{k^2}$$

provided $k \geq \sqrt{e}$, since $k \mapsto \frac{\ln k}{k^2}$ is decreasing on $[\sqrt{e}, +\infty]$. Then

$$\int\int_{|x-y|\geq k} \rho(t,x)\rho(t,y)|x-y|^2\, dxdy \leq 4M\int \rho(t,x)|x|^2\, dx$$

because $|x-y|^2 \leq 2(|x|^2+|y|^2)$. Thus

$$\int_{I\!R^2} \rho(t,x)U(t,x)\, dx \geq -\frac{M^2}{2\pi}\ln k - \frac{2M}{\pi}\int_{I\!R^2} \rho(t,x)|x|^2\, dx \cdot \frac{\ln k}{k^2}\ ,$$

and

$$\begin{aligned}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\left(\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2 + U(t,x) + \frac{M}{2\pi}\ln(t+\alpha)\right) dxdv \\ \geq \int_{I\!R^2} \rho(t,x)|x|^2\, dx \cdot \left(\frac{1}{(t+\alpha)^2\ln(t+\alpha)} - \frac{2M}{\pi}\cdot\frac{\ln k}{k^2}\right) \\ + \frac{M^2}{2\pi}\cdot\Big(\ln(t+\alpha) - \ln k\Big)\ .\end{aligned}$$

Let $k = k(t)$ be such that

$$\frac{1}{(t+\alpha)^2\ln(t+\alpha)} = \frac{2M}{\pi}\cdot\frac{\ln k}{k^2}\ .$$

As $t \to +\infty$,

$$k(t) = \sqrt{\frac{M}{2\pi}}(t+\alpha)\ln(t+\alpha)\cdot(1+o(1))$$

and then

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\left(\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2 + U(t,x) + \frac{M}{2\pi}\ln(t+\alpha)\right) dxdv \geq -C\Big(1+\ln\big(\ln(t+\alpha)\big)\Big) \quad (1.7)$$

for some constant $C > 0$. We can summarize these properties in the

Proposition 1.3 : *Assume that f_0 is a nonnegative function in $L^1 \cap L^\infty(I\!R^2\times I\!R^2)$ such that*

$$\int\int_{I\!R^2\times I\!R^2} f_0(x,v)\cdot(|x|^2+|v|^2)\, dxdv < +\infty$$

with $U_0 \geq 0$, $\nabla U_0 \in W^{1,\infty}_{\rm loc}(I\!R^2)$. *Assume also that* $x \mapsto \frac{x\cdot\nabla U_0}{1+|x|^2}$ *is bounded. Then, if* f *is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data* f_0,

(i) there exists some $\alpha_0 > 0$ *and a constant* $C > 0$ *such that for any* $\alpha > \alpha_0$,

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\Big(\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2 + U(t,x) + \frac{M}{2\pi}\ln(t+\alpha)\Big)\,dxdv \geq -C\Big(1+\ln\big(\ln(t+\alpha)\big)\Big) \tag{1.9}$$

(ii) For any $\alpha > 1$, *if* $x\cdot\nabla U_0(x) \leq 0$ *for almost all* $x \in I\!R^2$, *then*

$$\begin{aligned} H_\alpha(t) = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\Big(&\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2 + \frac{1}{(t+\alpha)^2}|x-v(t+\alpha)|^2 \\ &+ U(t,x) + U_0(x) + \frac{M}{2\pi}\ln(t+\alpha)\Big)\,dxdv \;\leq H_\alpha(0) \quad \forall t>0\,, \end{aligned} \tag{1.8}$$

and, if f *is a strong solution (i.e. has a moment of order* $2+\epsilon$ *in* x *and* v*), then*

$$\begin{aligned} \frac{d}{dt}H_\alpha(t) = &-\frac{2}{(t+\alpha)^3}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot|(1+\frac{1}{2\ln(t+\alpha)})x - v(t+\alpha)|^2\,dxdv \\ &- \frac{1}{(t+\alpha)^3\big(\ln(t+\alpha)\big)^2}\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2 + \frac{2}{t+\alpha}\int_{I\!R^2}(x\cdot\nabla U_0)\rho(t,x)\,dx\,. \end{aligned} \tag{1.6}$$

Proof of Proposition 1.3 : To prove (1.8), one may use a regularized problem (*i.e.* a smooth approximation of f_0 and a regularized kernel instead of Poisson's kernel) and pass to the limit:

$$\begin{aligned} H_\alpha(t) \leq H_\alpha(0) - \Big\{\int_0^t &\frac{2}{(s+\alpha)^3}\int\int_{I\!R^2\times I\!R^2} f(s,x,v)\cdot|(1+\frac{1}{2\ln(s+\alpha)})x - v(s+\alpha)|^2\,dxdv \\ &+ \frac{1}{(s+\alpha)^3\big(\ln(s+\alpha)\big)^2}\int\int_{I\!R^2\times I\!R^2} f(s,x,v)|x|^2 \\ &+ \frac{2}{s+\alpha}\int_{I\!R^2}(x\cdot\nabla U_0)\rho(s,x)\,dx\Big\}\,ds\,. \quad \forall\; t>0 \end{aligned} \tag{1.10}$$

(1.9) then holds without any further computations. We refer to section 5 for the justifications of (1.6) and (1.8) depending on the moments one knows to exist. □

Remark 1.4 : For a strong solution corresponding to an external potential U_0 such that $x \mapsto \frac{x\cdot\nabla U_0}{1+|x|^2}$ is bounded but for which $x\cdot\nabla U_0$ changes its sign, Equation (1.6) still holds.

2. Existence, regularity and uniqueness results

Existence results of the Cauchy problem for the Vlasov-Poisson system are now well known (see [Pf], [R], [S], [Pe2], [H], [HH]) in dimension three without external potential.

The situation in dimension two is easier than in dimension three (see [A1], [UO], [W1,2], [BaD]). However, since we consider here the case with an external potential and since we assume weaker assumptions on f_0 than in the previous papers, let us give an existence result.

Theorem 2.1 : *Assume that U_0 is nonnegative and that f_0 is a nonnegative function in $L^1 \cap L^\infty(I\!R^2 \times I\!R^2)$ such that $(x,v) \mapsto f_0(x,v)\cdot(|v|^2+U_0(x)+U(t=0,x))$ belongs to $L^1(I\!R^2 \times I\!R^2)$. If ∇U_0 belongs to $W^{1,\infty}_{\rm loc}(I\!R^2)$, then there exists a nonnegative solution $f \in C^0(I\!R^+; L^1(I\!R^2 \times I\!R^2))$ in the sense of the distributions of the Vlasov-Poisson system with initial data f_0 such that*

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(|v|^2+|U(t,x)|+U_0(x)+U(t=0,x))\,dxdv < +\infty \tag{2.1}$$

provided

- either (confinement)

$$\lim_{|x|+\infty} \frac{U_0(x)}{\ln|x|} = +\infty$$

- or

$$x \mapsto \frac{\nabla U_0(x)}{1+|x|} \in L^\infty(I\!R^2)\ ,$$

and

$$\exists m \in [1,2] \quad \int\int_{I\!R^2\times I\!R^2} f_0(x,v)\,(|x|^m+|v|^2)\,dxdv < +\infty\ ,$$

- or

$$x \mapsto \frac{x\cdot\nabla U_0(x)}{1+|x|^2} \in L^\infty(I\!R^2)\ ,$$

and

$$\int\int_{I\!R^2\times I\!R^2} f_0(x,v)\,(|x|^2+|v|^2)\,dxdv < +\infty\ .$$

Note that the energy estimate holds as an inequality

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot(|v|^2+U(t,x)+U_0(x))\,dxdv$$
$$\leq \int\int_{I\!R^2\times I\!R^2} f_0(x,v)\cdot(|v|^2+U(t=0,x)+U_0(x))\,dxdv < +\infty\ ,\quad \forall\, t>0 \tag{2.2}$$
$$\left|\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\,U(t,x)\,dxdv\right| < +\infty$$

in both cases.

These solutions are weak solutions but it is easy to prove that they are in fact strong as soon as f_0 satisfies the condition

$$\exists \epsilon > 0 \quad \int\int_{I\!R^2\times I\!R^2} f_0(x,v)\,(|x|^{2+\epsilon}+|v|^{2+\epsilon})\,dxdv < +\infty$$

since the momenta of order $2+\epsilon$ are finite for any $t>0$ (see lemma 5.2). In that case (2.2) becomes an equality, while estimates like bounds on the energy are enough for weak solutions.

The conditions on f could be weakned using the notion of renormalized solutions (see [DPL1-2]) while the notion of solutions in the sense of the characteristics still holds (in the renormalized sense) as soon as $\nabla U + \nabla U_0$ at least belongs to $W^{1,1}_{\rm loc}(I\!R^+ \times I\!R^2)$ (see Remark 2.2 below). Since the proof of such results rely on now classical methods, let us only mention the main ingredients.

Sketch of the proof :

1) a priori estimates : Assume first that f is a classical solution, smooth enough to justify the integrations by parts.

a) for any C^2 convex funtion s defined on $]0,+\infty[$

$$\frac{d}{dt}\int\int_{I\!R^2\times I\!R^2} s(f(t,x,v))\;dxdv = 0$$

since

$$\begin{aligned}
0 &= \int\int_{I\!R^2\times I\!R^2} \frac{\partial f}{\partial t}(t,x,v)\cdot s'(f(t,x,v))\;dxdv + \int\int_{I\!R^2\times I\!R^2} v\cdot\partial_x f(t,x,v)\cdot s'(f(t,x,v))\;dxdv \\
&\quad + \int\int_{I\!R^2\times I\!R^2} \nabla(U(t,x)+U_0(x))\cdot\partial_v f(t,x,v)\cdot s'(f(t,x,v))\;dxdv \\
&= \int\int_{I\!R^2\times I\!R^2} \frac{\partial s(f)}{\partial t}(t,x,v)\;dxdv + \int\int_{I\!R^2\times I\!R^2} \nabla_x\cdot\Big(v\;s(f)(t,x,v)\Big)\;dxdv \\
&\quad + \int\int_{I\!R^2\times I\!R^2} \nabla_v\cdot\Big(\nabla(U(t,x)+U_0(x))\;s(f)(t,x,v)\Big)\;dxdv\;.
\end{aligned}$$

(Of course the result also holds for a non convex function s, but the result does not pass to the limit: see below).

This proves for example that

$$\frac{d}{dt}||f(t,.,.)||_{L^p(I\!R^2\times I\!R^2)} = 0$$

for any $p\in[1,+\infty[$ and

$$||f(t,.,.)||_{L^\infty(I\!R^2\times I\!R^2)} \le ||f_0||_{L^\infty(I\!R^2\times I\!R^2)} \quad \forall\, t\in[0,+\infty[\;. \tag{2.3}$$

For $p=1$, this proves the conservation of the mass:

$$||\rho(t,.)||_{L^1(I\!R^2)} = ||f(t,.,.)||_{L^1(I\!R^2\times I\!R^2)} = ||f_0||_{L^1(I\!R^2\times I\!R^2)} \quad \forall\, t\in[0,+\infty[\;. \tag{2.4}$$

b) conservation of the energy:

$$\frac{d}{dt}\int\int_{I\!R^2\times I\!R^2} (|v|^2+U(t,x)+2U_0(x))\;f(t,x,v)\;dxdv = 0\;. \tag{2.5}$$

(multiply the Vlasov equation by $|v|^2$, integrate by parts and use the Poisson equation.)

c) Lyapunov's functional: equations (1.3) and (1.5) hold as in section 1 (see also sections 3 and 5). They are needed only in case 2.
d) an interpolation lemma (see Appendix B for more details): with the notation $\rho(t,x) = \int_{I\!R^2} f(t,x,v)\,dv$,

$$||\rho(t,.)||_{L^2(I\!R^2)} \leq 2\sqrt{\pi}\cdot ||f(t,.,.)||^{1/2}_{L^\infty(I\!R^2\times I\!R^2)}\cdot |||v|^2\,f(t,.,.)||^{1/2}_{L^1(I\!R^2\times I\!R^2)}\;.$$

We can indeed split the integral defining ρ into two integrals and evaluate these integrals in different ways

$$\rho(t,x) = \int_{|v|<R} f(x,v)\,dv + \int_{|v|\geq R} f(x,v)\,dv\;,$$

$$\int_{|v|<R} f(x,v)\,dv \leq \pi R^2\cdot ||f(t,.,.)||_{L^\infty(I\!R^2\times I\!R^2)}\;,$$

$$\int_{|v|\geq R} f(x,v)\,dv \leq \frac{1}{R^2}\int_{I\!R^2} f(x,v)\,|v|^2\,dv\;.$$

If we optimize on $R = R(t,x)$, then we get

$$\rho(t,x) \leq 2\sqrt{\pi}\cdot ||f(t,.,.)||^{1/2}_{L^\infty(I\!R^2\times I\!R^2)}\cdot\left(\int_{R^N} f(x,v)\,|v|^2\,dv\right)^{1/2}\;,$$

which easily proves the estimate. Using now (2.3) and (2.5), we get

$$||\rho(t,.)||_{L^2(I\!R^2)} \leq 2\sqrt{\pi}\cdot ||f_0||^{1/2}_{L^\infty(I\!R^2\times I\!R^2)}\cdot ||f(t,x,v)|v|^2||^{1/2}_{L^1(I\!R^2_x\times I\!R^2_v)}\quad \forall\, t\in[0,+\infty[\;. \tag{2.6}$$

e) Hardy-Littlewood-Sobolev inequality : there exist a constant $C>0$ depending only on $p\in]1,2[$ such that

$$||\nabla U(t,.)||_{L^q(I\!R^2;\,dx)} \leq C\cdot ||\rho(t,.)||_{L^p(I\!R^2;\,dx)}\;, \tag{2.7}$$

if $\frac{1}{p}-\frac{1}{q}=\frac{1}{2}$.

For any sequence of smooth solutions f^n of the Vlasov equation corresponding to the approximating Poisson problem

$$\nabla U^n = (\frac{1}{2\pi|x|}*\phi^n)*\int_{I\!R^2} f^n\,dv\;,$$

with $\phi^n(x) = n^2\phi(nx)$ a regularizing function and to an initial data f_0^n which is also a regularization of f_0, (1.3), (1.5) and (2.3-2.7) hold. Passing to the limit, (2.5) has to be replaced by (2.2) while (2.3), (2.4), (2.6) and (2.7) still hold. (1.3), (1.5), (2.5) of course hold for strong solutions.

2) compactness: A simple method to pass to the limit in the equation is to notice that as soon as $\int f^n(t,.,v)\psi(v)\,dv$ strongly converges in L^2 for any L^∞ with compact support function

ψ, then the limit f is a solution of the Vlasov-Poisson system in the distribution sense. This is easily obtained using the averaging lemmas (see [GLPS], [G], [GG], [DPLM]) since

$$\frac{\partial f}{\partial t} + v \cdot \partial_x f = \nabla_v \cdot \left[\nabla (U + U_0) \cdot f \right] ,$$

and since $\nabla(U+U_0) \cdot \int f(t,.,v)\psi(v)\, dv$ belongs to L^1 for any L^∞ function ψ (take for example $p = \frac{6}{5}$ and $q = 3$ in (2.7)).

3) the existence result for the regularized problem is easily obtained using the characteristics and a fixed point method. □

Remark 2.2 : The assumption $f_0 \in L^\infty(I\!R^2 \times I\!R^2)$ can also be removed and replaced by the condition $f_0 \ln f_0 \in L^1(I\!R^2 \times I\!R^2)$ giving an existence result for renormalized solutions as in [DPL2].

Regularity and uniqueness results are obtained in a classical way (see [A2], [UO] and [W1,2] or [BaD] and [LP2] for results respectively in dimension two and three). The results we present here follow the strategy of proof used by P.-L. Lions & B. Perthame [LP2], except that in dimension $N = 2$, there is no need of high order moments. These results extend the ones obtained by Ukai & Okabe [UO] and S. Wollman [W1,2] (without external potential) to a more general class of solutions (with eventually an external potential).

Proposition 2.3 : *Let f_0 and U_0 satisfy the same assumptions as in Theorem 2.1 and assume that $t \mapsto (X_0(s,t,x,v), V_0(s,t,x,v))$ are the characteristics in the external potential U_0 defined by*

$$\frac{d}{ds} X_0(s,t,x,v) = V_0(s,t,x,v) \ , \qquad \frac{d}{ds} V_0(s,t,x,v) = -\nabla_x U_0(X_0(s,t,x,v)) \ ,$$
$$X_0(t,t,x,v) = x, \qquad V(t,t,x,v) = v \ .$$

(i) Regularity :

If for any $T > 0$, there exists an $R_0 > 0$ such that

$$\begin{aligned} &(t,x,v) \mapsto \text{supess}\{f_0(y,w) \ : \ |y - X_0(0,t,x,v)| < \frac{Rt^2}{2}, \ |w - V_0(0,t,x,v)| < Rt\} \\ &\qquad \text{belongs to } L^\infty((0,T) \times I\!R^2_x; L^1(I\!R^2_v)) \quad \forall\, R > R_0 \\ &\lim_{R \to +\infty} (\frac{\epsilon}{1+\epsilon} R)^{-\frac{1+\epsilon}{\epsilon}} ||\text{supess}\{f_0(y,w) \ : \ |y - X_0(0,t,x,v)| < \frac{Rt^2}{2}, \\ &\qquad |w - V_0(0,t,x,v)| < Rt\}||_{L^\infty((0,T)\times I\!R^2_x; L^1(I\!R^2_v))} = 0 \end{aligned} \tag{2.8}$$

for some $\epsilon > 0$ which may depend on R (and T), then for any $T > 0$, there exists a solution f satisfying the same properties as in Theorem 2.1 and such that $(t,x) \mapsto \rho(t,x) = \int_{I\!R^2} f(t,x,v)\, dv$ belongs to $L^\infty((0,T) \times I\!R^2_x)$. If moreover ∇U_0 belongs to C^2 and

$$\begin{aligned} &\forall\, R,T > 0 \ , \quad \sup\{|\nabla_{x,v} f_0(y + vt, w)| \ : \ |y - X_0(0,t,x,v)| \le R, \ |w - V_0(0,t,x,v)| \le R\} \\ &\qquad \text{belongs to} \quad L^\infty((0,T) \times I\!R^2_x; L^1 \cap L^2(I\!R^2_v)) \ , \end{aligned} \tag{2.9}$$

then for any $T > 0$, ρ belongs to $L^\infty((0,T); C^{0,1}(\mathbb{R}^2_x))$ and $(t,x) \mapsto \nabla U(t,x) = -\frac{x}{2\pi|x|^2} *_x \rho(t,.)$ belongs to

$L^\infty((0,T); C^{1,\beta}(\mathbb{R}^2_x))$ for any $\beta \in]0,1[$.

(ii) Uniqueness :

If f_0 satisfies conditions (2.8) and (2.9), then the solution of the Vlasov-Poisson system such that ρ belongs to $L^\infty((0,T) \times \mathbb{R}^2_x)$ is unique(*).

Examples : If $f_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ has a compact support, then (2.8) is automatically satisfied. If it is a Lipschitz function, then (2.9) also holds true. Conditions (2.8) and (2.9) are related to the asymptotic behavior of f_0 as $|(x,v)| \to +\infty$. Assume for example that f_0 is dominated by a gaussian in x and v:

$$f_0(x,v) \le C \cdot e^{-\frac{1}{2\sigma}(|v|^2+|x|^2)} \quad \forall\, (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2 .$$

1) Assume first that $U_0 \equiv 0$. A straightforward computation gives

$$\begin{aligned}\int_{\mathbb{R}^2} dv \left(\text{supess}\{f_0(y+vt,w) \ : |y-x| < \frac{1}{2}Rt^2,\ |v-w| < Rt\} \right) \\ \le C \cdot e^{\frac{t^2(t^2+4)}{8\sigma}R^2} \cdot \int_{\mathbb{R}^2} dv\, e^{-\frac{1}{2\sigma}(|x+vt|^2+|v|^2)} \\ \le \frac{8\pi\sigma C}{t^2+4} \cdot e^{\frac{t^2(t^2+4)}{8\sigma}R^2} ,\end{aligned}$$

which proves (2.8) by taking for instance

$$\epsilon = \frac{8\sigma \ln R}{R^2 T^2 (T^2+4)} .$$

2) If U_0 is an harmonic potential (assume $U_0(x) = |x|^2$ to avoid technicalities), then

$$|V_0(0,t,x,v)|^2 + |X_0(0,t,x,v)|^2 = |v|^2 + |x|^2 .$$

$$\begin{aligned}\int_{\mathbb{R}^2} dv \left(\text{supess}\{f_0(y,w) \ : |y - X_0(0,t,x,v)| < \frac{1}{2}Rt^2,\ |w - V_0(0,t,x,v)| < Rt\} \right) \\ \le C \cdot e^{\frac{t^2(t^2+4)}{4\sigma}R^2} \cdot \int_{\mathbb{R}^2} dv\, e^{-\frac{1}{2\sigma}(|x|^2+|x|^2)} \\ \le 2\pi\sigma C \cdot e^{\frac{t^2(t^2+4)}{8\sigma}R^2} ,\end{aligned}$$

and the result holds again with

$$\epsilon = \frac{8\sigma \ln R}{R^2 \max(T,1)^2 (\max(T,1)^2+4)} .$$

As in [LP2], we could relax the assumption (2.8) and replace it by the condition

$$\forall\, T > 0\,, \quad \forall\, R > 0\,, \quad (t,x,v) \mapsto \text{supess}\{f_0(y,w) \ : \ |y - X_0(0,t,x,v)| < \frac{1}{2}Rt^2,\ |w - V_0(0,t,x,v)| < Rt\}$$
$$\text{belongs to} \quad L^\infty((0,T) \times \mathbb{R}^2_x; L^1(\mathbb{R}^2_v)) ,$$

(*) Of course, the potential U is always defined up to an additive constant.

but in this case, one would have to prove first an estimate on a momentum of order higher than 2. This can be avoided in dimension 2 (see part 1 of the proof below) if (2.8) is satisfied.

Proof of Proposition 2.3 : For the details of the proof, one may refer to [LP2] and check that the proof can be adapted to the dimension $N=2$ in the presence of a external potential. We give here a sketch of that proof and refer to [LP2] for more the details.

1) Since ρ belongs to $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2_x))$, $(t,x)\mapsto E(t,x)+\nabla_x U_0(x) = -\nabla_x U(t,x)$ belongs to $L^\infty((0,T); W^{1,2}_{\rm loc}(\mathbb{R}^2_x))$ by Agmon, Douglis & Nirenberg's theorem. Using characteristics (for the following truncated transport problem)

$$\begin{array}{ll}\dot X(s) = V(s)\ , & \dot V(s) = E^R(s,X(s))\ ,\\ X(t)=x\ , & V(t)=v\ ,\end{array}$$

in the sense of DiPerna and Lions [DPL1] with $E^R(s,x) = -\nabla U_0(x) - \min(R, |\nabla U(s,x)|)\cdot \frac{\nabla U(s,x)}{|\nabla U(s,x)|}$, one proves that

$$|V(0)-V_0(0,t,x,v)| \le Rt\ ,$$

$$|X(0)-X_0(0,t,x,v)+tV_0(0,t,x,v)| \le \frac{Rt^2}{2}\ ,$$

where $t\mapsto (X_0(s,t,x,v), V_0(s,t,x,v))$ are the characteristics in the external potential U_0 defined by

$$\begin{array}{ll}\frac{d}{ds}X_0(s,t,x,v) = V_0(s,t,x,v)\ , & \frac{d}{ds}V_0(s,t,x,v) = -\nabla_x U_0(X_0(s,t,x,v))\ ,\\ X_0(t,t,x,v)=x\ , & V(t,t,x,v)=v\ .\end{array}$$

The solution f^R of

$$\partial_t f^R + v\cdot\partial_x f^R + E^R(t,x)\cdot\partial_v f^R = 0$$

satisfies

$$f^R(t,x,v) = f_0(X(0),V(0)) \le \text{supess}\{f_0(y,w)\ :\ |y-X_0(0,t,x,v)| < \frac{Rt^2}{2},\ |w-V_0(0,t,x,v)|<Rt\}\ .$$

As in the proof of Theorem 2.1, a direct computation shows that, for any $\epsilon>0$,

$$2\pi||E^R(t,.)+\nabla U_0||_{L^\infty(\mathbb{R}^2;\,dx)} \le ||\rho(t,.)||_{L^1(\mathbb{R}^2;\,dx)} + \frac{1}{2\pi}(2\pi\frac{1+\epsilon}{\epsilon})^{\frac{1+\epsilon}{2+\epsilon}}||\rho(t,.)||^{\frac{2}{2+\epsilon}}_{L^2(\mathbb{R}^2;\,dx)}||\rho(t,.)||^{\frac{\epsilon}{2+\epsilon}}_{L^\infty(\mathbb{R}^2;\,dx)}$$

(split the integral $\int\frac{1}{|x-y|}\rho(t,y)\ dy$ into two parts corresponding respectively to $|x-y|>1$ and $|x-y|\le 1$ and use Hölder's inequality with $p=\frac{2+\epsilon}{1+\epsilon}$ and $p'=2+\epsilon$ for the second one). Assumption (2.8) just asserts that $||E^R(t,.)+\nabla U_0||_{L^\infty(\mathbb{R}^2_x)}$ does not depend on R for R large enough. Taking then $f=f^R$ proves that ρ belongs to $L^\infty((0,T)\times\mathbb{R}^2_x)$.

2) There exist an $\alpha>0$ ($\alpha=e^{-CT}$ for some $C>0$) such that the characteristic curves

$$\begin{array}{lll}\dot X(s,t,x_i,v_i) = V(s,t,x_i,v_i)\ , & \dot V_i(s) = E(s,X_i(s))\ , & \\ X(t,t,x_i,v_i)=x_i\ , & V(t,t,x_i,v_i)=v_i\ , & i=1,2\ ,\end{array}$$

are uniquely defined and satisfy

$$|X(s,t,x_1,v_1)-X(s,t,x_2,v_2)|+|V(s,t,x_1,v_1)-V(s,t,x_2,v_2)| \le C\,\left(|x_1-x_2|^\alpha+|v_1-v_2|^\alpha\right) \quad \forall\, t,s\in[0,T]\ ,$$

whenever $E+\nabla_x U_0 = -\nabla U$ is given by ρ through the Poisson equation.

The proof relies on the estimate

$$|(E(t,x)+\nabla_x U_0(x))-(E(t,y)+\nabla_x U_0(y))| \le C|x-y|\ln\frac{1}{|x-y|}$$

for any $x,y\in I\!R^2$ such that $|x-y|<1/2$, which is proved exactly in the same way as in [LP2]: since

$$\left|\frac{x-z}{|x-z|^2}-\frac{y-z}{|y-z|^2}\right| = |x-y|\cdot\frac{1}{|x|\cdot|y|}\ ,$$

$$\begin{aligned}
2\pi\left|\nabla U(t,x)-\nabla U(t,y)\right| =& 2||\rho(t)||_{L^\infty(I\!R^2)}\int_{|x-z|<\epsilon}\frac{dz}{|x-z|}+\int_{\substack{\epsilon<|x-z|<1\\ \epsilon<|y-z|<1}}\left|\frac{x-z}{|x-z|^2}-\frac{y-z}{|y-z|^2}\right|\rho(t,z)\,dz\\
&+\int_{\substack{|x-z|>1\\ |y-z|>1}}\rho(t,z)\,dz\\
&\le 4\pi\epsilon||\rho(t)||_{L^\infty(I\!R^2)}+|x-y|\cdot\left(\int_{\epsilon<|z|<1}\frac{dz}{|z|^2}\cdot||\rho(t)||_{L^\infty(I\!R^2)}+||\rho(t)||_{L^1(I\!R^2)}\right.\\
&= 4\pi\epsilon||\rho(t)||_{L^\infty(I\!R^2)}+|x-y|\cdot\left(2\pi|\ln\epsilon|||\rho(t)||_{L^\infty(I\!R^2)}+||\rho(t)||_{L^1(I\!R^2)}\right.\ ,
\end{aligned}$$

and the result holds with $\epsilon=|x-y|\cdot|\ln(|x-y|)|$. Then

$$\ln\left(|X_1(s)-X_2(s)|^2+|V_1(s)-V_2(s)|^2\right)\le\left(|X_1(s)-X_2(s)|^2+|V_1(s)-V_2(s)|^2\right)e^{-CT}\ ,$$

which gives the result with $\alpha=e^{-CT}$.

3) Since

$$\begin{aligned}
|\rho(t,x_1)-\rho(t,x_2)| \le& \int_{I\!R^2}|f_0(X(0,t,x_1,v_1),V(0,t,x_1,v_1))-f_0(X(0,t,x_2,v_2),V(0,t,x_2,v_2))|\,dv\\
\le& \int_{I\!R^2}\left(\sup\{|\nabla_{x,v}f_0(y,w))|\ :\ |y-X_0(0)|\le R,\ |w-V_0(0)|\le R\}\right)dv\\
&\cdot\sup_{v\in I\!R^2}\{|X(0,t,x_1,v_1)-X(0,t,x_2,v_2)|+|V(0,t,x_1,v_1)-V(0,t,x_2,v_2)|\}\ ,
\end{aligned}\tag{2.10}$$

(for some R eventually depending on t) again like in [LP2], this proves that ρ belongs to $L^\infty((0,T);C^{0,\alpha}(I\!R^2_x))$, which ensures that E belongs to $L^\infty((0,T);C^{1,\alpha}(I\!R^2_x))$ using Schauder's theorem (see [ADN], [GT]): the characteristic curves are Lipschitz continuous and (2.10) written with $\alpha=1$ proves that ρ belongs to $L^\infty((0,T);C^{0,1}(I\!R^2_x))$ and that E belongs to $L^\infty((0,T);C^{1,\beta}(I\!R^2_x))$ for any $\beta\in]0,1[$.

4) the uniqueness result can be shown in the same way as in [LP2] (with the simplification that we dont have to care about the field term coming from the initial data since we

only have to consider the difference between two solutions - the argument is not used for getting estimates on "higher moments"). Assume that there exist two solutions f_1 and f_2 corresponding to the same initial data f_0 and define

$$D(t) = \sup_{0\leq s\leq t} ||f_1(s,.,.) - f_2(s,.,.)||_{L^2(I\!R^2\times I\!R^2;\, dx\, dv)} \ .$$

Then

$$\frac{d}{dt}D^2(t) \leq 2C(T)\cdot \sup_{0\leq s\leq t} ||E_1(s,.) - E_2(s,.)||_{L^2(I\!R^2;\, dx)} \cdot D(t)$$

since condition (2.9) provides

$$\nabla_{x,v} f \in L^\infty((0,T)\times I\!R^2_x; L^2(I\!R^2_v)) \ .$$

Then

$$\frac{d}{dt}D(t) \leq C(T)\cdot \Delta(t) \tag{2.11}$$

where

$$\Delta(t) = \sup_{0\leq s\leq t} ||E_1(s,.) - E_2(s,.)||_{L^2(I\!R^2;\, dx)} \ .$$

f_i $(i=1,2)$ can be represented by

$$f_i(t,x,v) = f_0(x-vt,v) + \int_0^t E_i(t-s,x-vs)\cdot \partial_v f_i(t-s,x-vs,v)\ ds \ .$$

Performing an integration by parts, this gives the following expressions for ρ_i and $E_1 - E_2$

$$\rho_i(t,x) = \int_{I\!R^2} f_0(x-vt,v)\ dv + \mathrm{div}_x \int_0^t [\int_{I\!R^2} \big(E_i(t-s,x-vs)\cdot f_i(t-s,x-vs,v)\big)\ dv]\ s\ ds \ ,$$

$$E_1(t,x) - E_2(t,x) = \frac{x}{2\pi|x|^2} *_x \mathrm{div}_x \int_0^t [\int_{I\!R^2} \big((E_1-E_2)f_1 + E_2(f_1-f_2)\big)(t-s,x-vs,v)\ dv]\ s\ ds \ .$$

Using the fact that $\mathrm{div}_x(\frac{x}{2\pi|x|^2})$ is the Dirac distribution,

$$||E_1(t,.) - E_2(t,.)||_{L^2(I\!R^2;\, dx)} \leq || \int_0^t [\int_{I\!R^2} \big((E_1-E_2)f_1 + E_2(f_1-f_2)\big)(t-s,x-vs,v)\ dv]\ s\ ds||_{L^2(I\!R^2;\, dx)} \ .$$

Applying the Cauchy-Schwarz inequality, we get

$$\int_0^t \int_{I\!R^2} \big(|E|f\big)(t-s,x-vs,v)\ dv\ s\ ds$$
$$\int_0^t s\ ds \Big(\int_{I\!R^2} |E|^2(t-s,x-vs)\ dv\Big)^{1/2} \cdot \Big(\int_{I\!R^2} f^2(t-s,x-vs,v)\ dv\Big)^{1/2}$$
$$\int_0^t ds||E(t-s,.)||_{L^2(I\!R^2;\, dx)} \cdot \Big(\int_{I\!R^2} f^2(t-s,x-vs,v)\ dv\Big)^{1/2} \ .$$

Applying successively this computation to $\big(E = E_1 - E_2,\ f = f_1\big)$ and $\big(E = E_2,\ f = f_1 - f_2\big)$, we get

$$||E_1(t,.) - E_2(t,.)||_{L^2(I\!R^2;\, dx)} \leq t \sup_{0\leq s\leq t} ||(E_1-E_2)(t-s,.)||_{L^2(I\!R^2;\, dx)} ||f_0||_{L^2(I\!R^2\times I\!R^2;\, dx\, dv)} + C\cdot D(t) \ ,$$

where C is a constant which only depends on the energy : this computation is summarized in the identity

$$\Delta(t) \leq t \cdot \Delta(t) \cdot ||f_0||_{L^2(I\!R^2 \times I\!R^2)} + C \cdot D(t) ,$$

$$\Delta(t) \leq \frac{C}{1 - t \cdot ||f_0||_{L^2(I\!R^2 \times I\!R^2)}} \cdot D(t) .$$

Combining this with (2.11), we get

$$\frac{d}{dt} D(t) \leq t \cdot \frac{C}{1 - t \cdot ||f_0||_{L^2(I\!R^2 \times I\!R^2)}} \cdot D(t)$$

which proves that $D(t) \equiv 0$ for any $t \in]0, t_0[$ with

$$t_0 = \Big(||f_0||_{L^2(I\!R^2 \times I\!R^2)} \Big)^{-1} . \qquad \square$$

Remark 2.4 : As in [LP2], condition (2.9) is satisfied as soon as for example the L^1 bound is satisfied and f_0 is Lipschitz continuous in x and v.

3. A dispersion result

When there is no confining potential, the solution of the Vlasov-Poisson system is vanishing for large time. We present here a dispersion result which is the analogous in dimension 2 of the result obtained by R. Illner & G. Rein and B. Perthame in dimension 3 (see [IR], [Pe2]). It is essentially based on the computation of the Lyapunov functional of section 1.

Proposition 3.1 : *Assume that f is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to the initial data f_0, where f_0 is a nonnegative function in $L^1 \cap L^\infty(I\!R^2 \times I\!R^2)$ such that*

$$\int\int_{I\!R^2 \times I\!R^2} f_0(x,v) \cdot (|x|^2 + |v|^2 + U_0(x)) \, dxdv < +\infty$$

with $U_0 \geq 0$, $\nabla U_0 \in W^{1,\infty}_{\rm loc}(I\!R^2)$, $x \cdot \nabla U_0 \leq 0$ a.e. Then there exists some $\alpha_0 > 0$ and a constant $C_\alpha > 0$ (for any $\alpha > \alpha_0$) such that

(i) for any $t > 0$,

$$-C_\alpha \Big(1 + \ln\big(\ln(t+\alpha)\big) \Big) \leq H_\alpha(t) = \int\int_{I\!R^2 \times I\!R^2} f(t,x,v) \cdot \Big(\frac{1}{(t+\alpha)^2 \ln(t+\alpha)} |x|^2 + \frac{1}{(t+\alpha)^2} |x - v(t+\alpha)|^2 + U(t,x) + U_0(x) + \frac{M}{2\pi} \ln(t+\alpha) \Big) \, dxdv \ \leq H_\alpha(0) ,$$

(ii) for any $T > 0$, for a strong solution,

$$\int_0^T \frac{1}{(t+\alpha)^3 \big(\ln(t+\alpha)\big)^2} \left(\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2 \, dxdv \right) dt \leq C_\alpha \left(1 + \ln\big(\ln(T+\alpha)\big) \right) , \tag{3.2}$$

$$\int_0^T \frac{1}{t+\alpha} \left(\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v - \frac{x}{t+\alpha}|^2 \, dxdv \right) dt \leq C_\alpha \left(1 + \ln\big(\ln(T+\alpha)\big) \right) . \tag{3.3}$$

As a straightforward consequence, we have the

Corollary 3.2 : *With the same assumptions as in Proposition 3.1, there exists some $\alpha_0 > 0$ and a constant $C_\alpha > 0$ (for any $\alpha > \alpha_0$) such that :*

(i) For any $t > 0$,

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^2 \, dxdv \leq C_\alpha \left(1 + \ln(t+\alpha) \ln\big(\ln(t+\alpha)\big) \right)$$

and

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2 \, dxdv \leq C_\alpha \left(1 + (t+\alpha)^2 \ln(t+\alpha) \ln\big(\ln(t+\alpha)\big) \right) , \qquad (3.4) ,$$

$$\frac{1}{(t+\alpha)^2} \int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x - (t+\alpha)v|^2 \, dxdv \leq C_\alpha \cdot \left(1 + \ln\big(\ln(t+\alpha)\big) \right) .$$

If f is a strong solution, then

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v) U(t,x) \, dxdv \sim -\frac{M^2}{2\pi} \ln(t+\alpha) .$$

If moreover $U_0 \equiv 0$, then

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^2 \, dxdv \sim \ln(t+\alpha) ,$$

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2 \, dxdv \sim (t+\alpha)^2 \ln(t+\alpha) .$$

(ii) for any $T > 1$,

$$\int_0^t \frac{1}{t+\alpha} \int\int_{I\!R^2\times I\!R^2} f(t,x,v) \cdot |(1 + \frac{1}{2\ln(t+\alpha)}) \frac{x}{t+\alpha} - v|^2 \, dxdv \leq C_\alpha \left(1 + \ln\big(\ln(T+\alpha)\big) \right) , \tag{3.5}$$

$$\frac{1}{\ln\big(\ln(T+\alpha)\big)} \int_0^T \frac{1}{t+\alpha} ||\rho(t,.)||^2_{L^2(I\!R^2)} \, dt < C_\alpha \tag{3.6}$$

and

$$\frac{1}{\ln\big(\ln(T+\alpha)\big)} \int_0^T \frac{1}{t+\alpha} ||\nabla U(t,.)||^{\frac{2q}{q-2}}_{L^q(I\!R^2)} \, dt < C_\alpha \tag{3.7}$$

for any $q \in]2, +\infty[$.

(iii) There exists a sequence $(t_n)_{n\in I\!N}$ with $\lim_{n\to+\infty} t_n = +\infty$ such that, for any $\epsilon > 0$,

$$\lim_{n\to+\infty} \int\int_{I\!R^2\times I\!R^2} f(t_n,x,v)|v - \frac{x}{t_n+\alpha}|^2 \, dxdv \cdot \big(\ln(t_n)\big)^{1-\epsilon} = 0 , \tag{3.8}$$

$$\lim_{n\to+\infty} ||\rho(t_n,.)||^2_{L^2(I\!R^2)} \cdot \big(\ln(t_n)\big)^{\frac{1-\epsilon}{2}} = 0 \ , \tag{3.9}$$

$$\lim_{n\to+\infty} ||\nabla U(t_n,.)||_{L^q(I\!R^2)} \cdot \big(\ln(t_n)\big)^{\frac{q-2}{2q}\cdot(1-\epsilon)} = 0 \ , \tag{3.10}$$

for any $q \in]2,+\infty[$.

(iv) if f_0 *is compactly supported, then for any* $t>0$ $f(t,.,.)$ *is compactly supported too. If* $R(t)$ *is the minimal radius of the balls centered at the origin and containing the support of* $f(t,.,.)$*, then*

$$R(t) \geq C_\alpha (t+\alpha). \tag{3.11}$$

Proof of Corollary 3.2 : In this proof we will denote all the positive constants by the same symbol C.

(i) is obtained through a refinement of the proof of (1.3) in section 1: let $k=k(t)$ be such that

$$\frac{1}{2}\cdot\frac{1}{(t+\alpha)^2\ln(t+\alpha)} = \frac{2M}{\pi}\cdot\frac{\ln k}{k^2} \ .$$

As $t\to+\infty$,

$$k(t) = \sqrt{\frac{M}{4\pi}}(t+\alpha)\ln(t+\alpha)\cdot(1+o(1))$$

and then

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\Big(\frac{1}{(t+\alpha)^2\ln(t+\alpha)}|x|^2 + U(t,x) + \frac{M}{2\pi}\ln(t+\alpha)\Big)\,dxdv$$
$$\geq \frac{1}{2}\cdot\frac{1}{(t+\alpha)^2\ln(t+\alpha)}\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2\,dxdv - C\Big(1+\ln\big(\ln(t+\alpha)\big)\Big) \ ,$$

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^2\,dxdv \leq C_\alpha\Big(1+(t+\alpha)^2\ln(t+\alpha)\ln\big(\ln(t+\alpha)\big)\Big) \ .$$

On the other side, because of (i) in Proposition 3.1,

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot\frac{1}{(t+\alpha)^2}|x-v(t+\alpha)|^2\,dxdv \leq C_\alpha\Big(1+\ln\big(\ln(t+\alpha)\big)\Big) \ .$$

Thus

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^2\,dxdv \leq C_\alpha\Big(2+\ln(t+\alpha)\ln\big(\ln(t+\alpha)\big)\Big) \ ,$$

which proves (3.4).

Proof of (ii): because of (1.5), for any $T>1$,

$$\int_0^t \frac{1}{t+\alpha}\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)\cdot|(1+\frac{1}{2\ln(t+\alpha)})\frac{x}{t+\alpha} - v|^2\,dxdv \leq C\Big(1+\ln\big(\ln(T+\alpha)\big)\Big) \ . \tag{3.5}$$

Since

$$\rho(t,x) = \int_{|v-(1+\frac{1}{2\ln(t+\alpha)})\frac{x}{t+\alpha}|<R} f(t,x,v)\, dv + \int_{|v-(1+\frac{1}{2\ln(t+\alpha)})\frac{x}{t+\alpha}|>R} f(t,x,v)\, dv\ ,$$

the classical method of interpolation provides

$$\rho(t,x) \le \pi R^2 ||f(t,.,.)||_{L^\infty(I\!R^2\times I\!R^2)} + \frac{1}{R^2}\int_{R^2} f(t,x,v)\cdot |v-(1+\frac{1}{2\ln(t+\alpha)})\frac{x}{t+\alpha}|^2\, dv\ .$$

Optimizing on R,

$$||\rho(t,.)||^2_{L^2(I\!R^2)} \le 2\sqrt{\pi}||f(t,.,.)||_{L^\infty(I\!R^2\times I\!R^2)}\cdot \int_{R^2\times I\!R^2} f(t,x,v)\cdot |v-(1+\frac{1}{2\ln(t+\alpha)})\frac{x}{t+\alpha}|^2\, dxdv\ ,$$

we get easily estimate (3.6).

(3.7) is then deduced from the Hardy-Littlewood-Sobolev inequality :

$$||\nabla U(t,.)||_{L^q(I\!R^2;\,dx)} \le c(q)||\rho(t,.)||_{L^p(I\!R^2;\,dx)} \quad \text{with} \quad \frac{1}{p}-\frac{1}{q}=\frac{1}{2} \quad \text{for any} \quad q\in]2,+\infty[\ ,$$

and the Hölder inequality for ρ :

$$||\rho(t,.)||_{L^p(I\!R^2;\,dx)} \le ||\rho(t,.)||^{2/p-1}_{L^1(I\!R^2;\,dx)}\cdot ||\rho(t,.)||^{2(1-1/p)}_{L^2(I\!R^2;\,dx)} = ||\rho(t,.)||^{2/q}_{L^1(I\!R^2;\,dx)}\cdot ||\rho(t,.)||^{1-2/q}_{L^2(I\!R^2;\,dx)}\ ,$$

$$||\nabla U(t,.)||^{\frac{2q}{q-2}}_{L^q(I\!R^2;\,dx)} \le (c(q))^{\frac{2q}{q-2}} M^{\frac{q}{q-2}} ||\rho(t,.)||_{L^2(I\!R^2;\,dx)}\ ,$$

which proves (3.7).

Proof of (iii): the method is the same for (3.8), (3.9) and (3.10). Let us prove for example (3.9). Assume that

$$\liminf_{t\to+\infty} ||\rho(t,.)||^2_{L^2(I\!R^2)}\cdot \big(\ln(t)\big)^{\frac{1-\epsilon}{2}} = k > 0\ .$$

For $T\to+\infty$,

$$\frac{1}{\ln\big(\ln(T+\alpha)\big)}\int_0^T \frac{1}{t+\alpha}||\rho(t,.)||^2_{L^2(I\!R^2)}\, dt \ge \frac{k}{\big(\ln(t_n)\big)^{\frac{1-\epsilon}{2}}\cdot \ln\big(\ln(T+\alpha)\big)}\int_0^T \frac{1}{t+\alpha}||dt = \frac{k}{\epsilon}\left[\big(\ln(T+\alpha)\big)^\epsilon\right]_0^T$$

giving a contradiction with (3.6).

Proof of (iv): the fact that $f(t,.,.)$ is compactly supported if f_0 is compactly supported will be proved in section 4. f is then a strong solution:

$$-\frac{M^2}{2\pi}\ln\big(2R(t)\big)\big) \le \int\int_{I\!R^2\times I\!R^2} f(t,x,v)U(t,x)\, dxdv = -\frac{M^2}{2\pi}\ln\big(t+\alpha\big)\,(1+o(t))\ .$$

$\square$

4. Growth of the support

This part is devoted to the special case corresponding to an initial data with compact support. The following theorem gives an upper bound for the growth of the support of the distribution function with respect to the time. Since it remains of finite size for any positive time, the moments in v are finite (which also means that the distribution function is a strong solution: see [Pe2]). We assume here that U_0 is harmonic (see Appendix A for the justifications of the model):

$$U_0(x) = \frac{\rho_0}{2}|x|^2 \quad \forall\, x \in \mathbb{R}^2 \ .$$

Theorem 4.1 : *Assume that f is a solution in the sense of the distributions of the Vlasov-Poisson system corresponding to an initial data f_0 in $L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ with compact support. Then f is a strong solution (in Perthame's sense) and for any $t > 0$, $f(t,.,.)$ has a compact support, and for any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$R(t) = \mathrm{diam}(\mathrm{supp}\big(f(t,.,.)\big)) \le \mathrm{diam}(\mathrm{supp}\big(f_0\big)) + C(\varepsilon)\big(1+t\big)^{1+\varepsilon} \ . \tag{4.1}$$

Proof : As before the proof is established in the context of classical smooth solutions. The result is then obtained by passing to the limit for a well choosen approximating sequence. It is not very difficult to check that the estimates on the support may be evaluated uniformly (for a well choosen approximating sequence).

Consider $E(t,x,v) = \frac{|v|^2}{2} + U(t,x) + U_0(x)$:

$$\frac{d}{dt}E(t,x(t),v(t)) = \frac{\partial U}{\partial t}(t,x(t)) \tag{4.2}$$

for any characterictics $t \mapsto (x(t),v(t))$. The main idea is to evaluate the L^∞-norm of $\frac{\partial U}{\partial t}$ using the Poisson equation (derived with respect to the time):

$$-\Delta(\frac{\partial U}{\partial t}) = \frac{\partial \rho}{\partial t} \ . \tag{4.3}$$

An integration of the Vlasov equation with respect to v shows that (local conservation of the mass)

$$\int_{\mathbb{R}^2} dv\ \big[\partial_t f + v\cdot\partial_x f - (\partial_x U(t,x) + \partial_x U_0(x))\cdot\partial_v f\big] = 0 \ ,$$

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot \int_{\mathbb{R}^2} f(t,x,v)\ v\ dv = 0 \ . \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$-\Delta(\frac{\partial U}{\partial t}) = -\nabla_x \cdot \int_{\mathbb{R}^2} f(t,x,v)\ v\ dv \ ,$$

$$|\frac{\partial U}{\partial t}| \le \frac{dy}{2\pi} \int_{I\!R^2} \frac{1}{|x-y|} \int_{I\!R^2} f(t,y,v)\ v\ dv\ ,$$

$$||\frac{\partial U}{\partial t}(t,.)||_{L^\infty(I\!R^2)} \le \frac{1}{2\pi}\left(2\pi \int_0^{R(t)} \frac{r\ dr}{r^{2-\epsilon}}\right)^{\frac{1}{2-\epsilon}} \cdot \left(\int_{I\!R^2} dx\ |\int_{I\!R^2} f(t,x,v)\ v\ dv|^{\frac{2-\epsilon}{1-\epsilon}}\right)^{\frac{1-\epsilon}{2-\epsilon}}$$
$$= \frac{1}{2\pi}\left(\frac{2\pi}{\epsilon}(R(t))^\epsilon\right)^{\frac{1}{2-\epsilon}} \cdot ||\int_{I\!R^2} f(t,.,v)\ v\ dv||_{L^{\frac{2-\epsilon}{1-\epsilon}}(I\!R^2;\ dx)}\ .$$

Using now the interpolation lemma given in Appendix B (Lemma B.2), we get

$$||\int_{I\!R^2} f(t,.,v)\ v\ dv||_{L^{u=\frac{2-\epsilon}{1-\epsilon}}(I\!R^2;\ dx)}$$
$$\le K(2,\infty,\frac{4-\epsilon}{1-\epsilon},1) \cdot ||f(t,.,.)||^\beta_{L^\infty(I\!R^2\times I\!R^2)} \cdot |\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\ |v|^{\frac{4-\epsilon}{1-\epsilon}}\ dxdv|^{1-\beta}\ .$$

with

$$\beta = \frac{1}{2-\epsilon} \quad \text{and} \quad 1-\beta = \frac{1-\epsilon}{2-\epsilon}\ .$$

This finally proves that

$$||\frac{\partial U}{\partial t}(t,.)||_{L^\infty(I\!R^2)} \le \frac{K(2,\infty,\frac{4-\epsilon}{1-\epsilon},1)}{2\pi}\left(\frac{2\pi}{\epsilon}(R(t))^\epsilon\right)^{\frac{1}{2-\epsilon}}$$
$$\cdot ||f_0||^{\frac{1}{2-\epsilon}}_{L^\infty(I\!R^2\times I\!R^2)} \cdot \left(\int\int_{I\!R^2\times I\!R^2} f(t,x,v)\ |v|^{\frac{4-\epsilon}{1-\epsilon}}\ dxdv\right)^{\frac{1-\epsilon}{2-\epsilon}}\ , \tag{4.5}$$

$$||\frac{\partial U}{\partial t}(t,.)||_{L^\infty(I\!R^2)} \le C(\epsilon)\cdot \left(R(t)\right)^{\frac{\epsilon}{2-\epsilon}+(\frac{4-\epsilon}{1-\epsilon}-2)\cdot\frac{1-\epsilon}{2-\epsilon}=\frac{2(1+\epsilon)}{2-\epsilon}}\ ,$$

for some constant

$$C(\epsilon) = \epsilon^{\frac{\epsilon}{2-\epsilon}} \cdot ||f_0||^{\frac{1}{2-\epsilon}}_{L^\infty(I\!R^2\times I\!R^2)} \cdot \left(\sup_{t>0}\int\int_{I\!R^2\times I\!R^2} f_0(x,v)\ (|v|^2 + U(t=0,x) + 2U_0(x))\ dxdv\right)^{\frac{1-\epsilon}{2-\epsilon}}$$

which only depends on the initial data.

Using the energy estimate,

$$E(t,x(t),v(t)) \le E(t,x(t),v(t))_{|t=0} + C(\epsilon)\cdot\left(R(t)\right)^{\frac{2(1+\epsilon)}{2-\epsilon}} \cdot t\ ,$$

and the fact that U is bounded from below on $\text{supp}(\rho)$ by $-\frac{M}{2\pi}\ln(2R(t))$ and that U_0 is harmonic,

$$\min(\rho_0,1)\cdot\left(|x(t)|^2+|v(t)|^2\right) - \frac{M}{2\pi}\ln(2R(t)) \le E(t,x(t),v(t)) \le E(0,x(0),v(0)) + C(\epsilon)\cdot\left(R(t)\right)^{\frac{2(1+\epsilon)}{2-\epsilon}}\cdot t\ ,$$
$$\min(\rho_0,1)\cdot\left(R(t)\right)^2 - \frac{M}{2\pi}\ln(2R(t)) \le E(t,x(t),v(t)) \le \sup_{x_0,v_0)\in\text{supp}(f_0)} E(0,x_0,v_0) + C(\epsilon)\cdot\left(R(t)\right)^{\frac{2(1+\epsilon)}{2-\epsilon}}\cdot t\ , \tag{4.6}$$

which essentially proves the result for any $\varepsilon > \frac{3\epsilon}{2(1-2\epsilon)}$. □

Remark 4.2 :

1) The precise form of the confining potential U_0 is used in the proof of Theorem 4.1 only to prove (4.6). The proof is easily extended to the case when U_0 satifies an other explicit behavior like $U_0(x) \sim |x|^\alpha$ as $|x| \to +\infty$ with $\alpha > 0$, $\alpha \neq 2$. Whatever U_0 behaves like, the estimate holds for the velocity:

$$\tilde{R}(t) = \sup\{|v| \ : \ \exists x \in I\!R^2 \quad \text{s.t. } (x,v) \in \text{supp}(f(t,.,.)) \ \}$$

also satisfies

$$\tilde{R}(t) \leq C(\varepsilon) \cdot (1+t)^{1+\varepsilon}$$

for some constant $C(\varepsilon) > 0$ (and for any $\varepsilon > 0$) as soon as U_0 is bounded from below and satisfies the other conditions of Theorem 2.1 (one has to add a term $\ln(\ln(t+\alpha))$ in the estimate of $\int\int_{I\!R^2 \times I\!R^2} f(t,x,v)|v.vert^2 \ dxdv$ which essentially does not change the result if for instance $x \cdot \nabla U \leq 0$ a.e. – dispersive case). Even if U_0 is not bounded from below, the method may still apply but the estimate may be much worse. For example, if there exists some direction $\nu \in S^1$ such that $\nabla U_0(t\nu) \cdot \nu < 0$ for any $t > 0$ large enough, and $\lim_{t \to +\infty} U_0(t\nu) = -\infty$, the growth of the size of the support will surely be given by the linear motion in the potential U_0.

2) In a recent work, G. Rein [R] showed that in three dimensions, the growth (in the velocities) of the support is of order $(1+t)^{2/3}$ when $U_0 \equiv 0$, but the method is rather different. The bound obtained here is probably too large (since it corresponds – roughly spoken – to the growth given by the free motion, *i.e.* what one can get when $U_0 \equiv 0$, for the velocities), but as well as in dimension three, the question of the optimal growth is clearly open. One may conjecture that the optimal growth is at least logarithmic.

In dimension $N = 1$, the method also applies:

$$||\frac{\partial U}{\partial t}||^2_{L^\infty(I\!R)} \leq ||j||^2_{L^1(I\!R)} \leq ||f||_{L^1(I\!R)} \int\int_{I\!R^2 \times I\!R^2} f(t,x,v)|v|^2 \ dxdv \leq C(1+R(t)) \ ,$$

$$(R(t))^2 - R(t) \leq C\big(1 + t\sqrt{1+R(t)}\big)$$

which gives for $R(t)$ an estimate of order $(1+t)^{2/3}$ as $t \to +\infty$.

One can notice that the method we apply here fails in dimension $N = 3$.

3) One could ask if there were other interpolation inequalities that would improve the result. This does not seem very clear. Let us try to optimize the estimate of the L^2-norm of $j(t,x) = \int_{I\!R^2} f(t,x,v) \ v \ dv$. In the following computations, C is a bound for different constants depending only on the initial data.

$$||j(t,.)||_{L^2(I\!R^2; \ dx)} \leq ||j(t,.)||^{1-u/2}_{L^\infty(I\!R^2; \ dx)} \cdot ||j(t,.)||^{u/2}_{L^u(I\!R^2; \ dx)} \ .$$

A direct computation of $||j(t,.)||_{L^\infty(I\!R^2;\,dx)}$ gives

$$||j(t,.)||_{L^\infty(I\!R^2;\,dx)} \le \int_{|v|<R(t)} f(t,x,v)\,|v|\,dv \le C\cdot\Big(R(t)\Big)^3 .$$

Using lemma B.2, we can interpolate $||j(t,.)||_{L^u(I\!R^2;\,dx)}$ between $||f(t,.,.)||_{L^p(I\!R^2\times I\!R^2)}$ and

$||f(t,.,.)|v|^k||_{L^1(I\!R^2\times I\!R^2)}$:

$$||j(t,.)||_{L^u(I\!R^2;\,dx)} \le C||f(t,.,.)|v|^k||_{L^1(I\!R^2\times I\!R^2)}^{\frac{3p-2}{2(p-1)+kp}} \le C\cdot\Big(R(t)\Big)^{\frac{(k-2)(3p-2)}{2(p-1)+kp}} ,$$

with $u=\frac{2(p-1)+kp}{3(p-1)+k}$. Putting these two estimates together, we get

$$||j(t,.)||_{L^2(I\!R^2;\,dx)} \le C\cdot\Big(R(t)\Big)^{1+\frac{k-1}{3(p-1)+k}} .$$

Optimizing with respect to p and k, we again get

$$||j(t,.)||_{L^2(I\!R^2;\,dx)} \le C\cdot R(t) .$$

4) Assume that $U_0\equiv 0$. Like in [Pe1], it is possible to give an higher moment estimate for f (which depends on $\tilde R(t)$ only through a term of order $\frac{3}{2}\eta$ as $\eta\to 0_+$) provided this moment is bounded for the initial data: for any $\eta>0$, there exists a constant $C(\eta)$ such that

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^3\,dxdv \le \int\!\!\int_{I\!R^2\times I\!R^2} f_0(x,v)|v|^3\,dxdv + C(\eta)\cdot R(t)^{\frac{3\eta}{2-\eta}}\cdot t\,\Big(1+\ln(t+\alpha)\ln\big(\ln(t+\alpha)\big)\Big)^{2\frac{1-\eta}{2-\eta}} , \tag{4.7}$$

$$\int\!\!\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^4\,dxdv \le \int\!\!\int_{I\!R^2\times I\!R^2} f_0(x,v)|v|^4\,dxdv + C(\eta)\cdot \tilde R(t)^{\frac{6\eta}{2-\eta}}\cdot t^2\,\Big(1+\ln(t+\alpha)\ln\big(\ln(t+\alpha)\big)\Big)^{\frac{1-\eta}{2-\eta}} , \tag{4.8}$$

but such estimates are not usefull for estimating the support because of their dependance in t: reinjecting it in equation (4.5) does not improve the estimate on $\tilde R(t)$.

Because of its general interest, let us indicate the idea of the proof of (4.7) and (4.8). In the following, $C\eta)$ denotes various constant which may depend on η and on the initial data f_0. We also assume that the solution f is smooth enough to allow the integrations by parts performed below, but the computation can easily be justified in the general setting.

Since

$$|\nabla U(t,x)| \le \frac{1}{2\pi}\int_{I\!R^2}\frac{1}{|x-y|}\int_{I\!R^2} f(t,y,v)\,dv ,$$

like in the proof of Theorem 4.1,

$$\begin{aligned}||\nabla U(t,.)||_{L^\infty(\mathbb{R}^2)} &\le \frac{1}{2\pi}\left(\frac{2\pi}{\eta}(\tilde{R}(t))^\eta\right)^{\frac{1}{2-\eta}} \cdot ||\int_{\mathbb{R}^2} f(t,.,v)\,dv||_{L^{\frac{2-\eta}{1-\eta}}(\mathbb{R}^2;\,dx)} \\ &= C(\eta)\cdot\left(\tilde{R}(t)\right)^{\frac{\eta}{2-\eta}} \cdot ||\rho(t,.)||_{L^{\frac{2-\eta}{1-\eta}}(\mathbb{R}^2;\,dx)}\ .\end{aligned} \tag{4.9}$$

The $L^{\frac{2-\eta}{1-\eta}}$- norm of $\rho(t,.)$ is then evaluated by the interpolation inequality (see Appendix B)

$$||\rho(t,.)||_{L^{\frac{2-\eta}{1-\eta}}(\mathbb{R}^2;\,dx)} \le C(\eta)\cdot||f(t,.,.)||^\alpha_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2;\,dxdv)}\cdot||f(t,.,.)|v|^{k+2}||^{1-\alpha}_{L^1(\mathbb{R}^2\times\mathbb{R}^2;\,dxdv)}\ , \tag{4.10}$$

with $\alpha=\frac{1}{2-\eta}$ and $k=\frac{2\eta}{1-\eta}$. Putting (4.9) and (4.10) together, we get

$$||\nabla U(t,.)||_{L^\infty(\mathbb{R}^2)} \le C(\eta)\cdot||f(t,.,.)||^{\frac{1}{2-\eta}}_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2;\,dxdv)}\cdot||f(t,.,.)|v|^2||^{\frac{1-\eta}{2-\eta}}_{L^1(\mathbb{R}^2\times\mathbb{R}^2;\,dxdv)}\cdot\left(R(t)\right)^{\frac{3\eta}{2-\eta}}\ .$$

Multiplying (formally) the Vlasov equation by $|v|^3$ and integrating by parts, we get

$$\frac{d}{dt}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,|v|^3\,dxdv \le 3||\nabla U(t,.)||_{L^\infty(\mathbb{R}^2)}\cdot\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,|v|^2\,dxdv$$

which proves (4.7). (4.8) is obtained in the same way using an integration w.r.t. t. This method can also easily be generalized and gives for the moment of order n the estimate

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^n\,dxdv \le C_n(\eta)\cdot(1+t)^{n-2+\eta}\ .$$

It is also possible to give for any $\zeta>0$ an estimate of the momentum $\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^{n-\zeta}\,dxdv$ depending only on the initial data and on t^n (no more dependance in $\tilde{R}(t)$ using the Hardy-Littlewood-Sobolev inequality in the limit case $L^2(\mathbb{R}^2)$.

5) The size of the support grows only because of the tail of the distribution (when U_0 is confining or say, at least, growing to $+\infty$ as $|x|\to+\infty$: let us consider here the case when U_0 is harmonic – for more details on confining potentials see [Do6]). For any large $A>0$ we have indeed the following inequalities

$$\int_{|x|>A}\rho(t,x)\,dx = 0$$

if $A>R(t)$ (and $t>0$ is small enough), and for any $A>0$

$$\int_{|x|>A}\rho(t,x)\,dx \le \frac{1}{A^2}\int_{\mathbb{R}^2}|x|^2\rho(t,x)\,dx \le \frac{C}{A^2\min(\rho_0,1)}$$

if $A<R(t)$, for some constant C (the proof that $\int_{\mathbb{R}^2}|x|^2\rho(t,x)\,dx$ is bounded relies on the energy estimate).

5. Equipartition of the energy and moments

5.1 Stationary solutions

Assume first that f is a smooth classical stationary solution of the Vlasov-Poisson system

$$\begin{cases} v\cdot\partial_x f-(\partial_x U(x)+\partial_x U_0(x))\cdot\partial_v f=0 & (V)\\ -\Delta U=\rho(x)=\int_{I\!R^2} f(x,v)\,dv & (P)\end{cases}$$

Multiplying by $(x\cdot v)$, integrating with respect to x and v and performing integrations by parts, we easily get

$$\int\int_{I\!R^2\times I\!R^2} f(x,v)\,|v|^2\,dxdv=\int_{I\!R^2}\Big(x\cdot(\nabla U+\nabla U_0)\Big)\,\rho(x)\,dx\;.$$

Using now the Poisson equation to compute $\int_{I\!R^2}\Big(x\cdot\nabla U(x)\Big)\,\rho(x)\,dx$, we successively get, with the convention of summation over repeated indices,

$$\begin{aligned}\int_{I\!R^2}\Big(x\cdot\nabla U(x)\Big)\,\rho(x)\,dx&=\int_{I\!R^2}\Big(x\cdot\nabla U(x)\Big)\,(-\Delta U(x))\,dx\\ &=-\int_{I\!R^2}x^i\,\frac{\partial U}{\partial x^i}\,\frac{\partial^2 U}{\partial {x^j}^2}\,dx\\ &=\int_{I\!R^2}|\nabla U(x)|^2\,dx+\int_{I\!R^2}x^i\,\frac{\partial^2 U}{\partial x^i\partial x^j}\,\frac{\partial U}{\partial x^j}\,dx\\ &=\int_{I\!R^2}|\nabla U(x)|^2\,dx+\int_{I\!R^2}(x\cdot\nabla)\,(\frac{|\nabla U|^2}{2})\,dx\\ &=0\end{aligned}$$

provided U is smooth enough and such that $|x|\nabla U(x)$ tends to 0 as $|x|\to+\infty$ (this identity is known as Rellich's identity and is frequently used in Pohozaev's method for elliptic problems).

In dimension two, the assumption on the decay of ∇U is not satisfied. One has therefore to take asymptotic boundary terms into account. Assume that $x\mapsto(1+|x|)\rho(x)$ is bounded in $L^1(I\!R^2)$.

$$\begin{aligned}\Big|\nabla U(x)+\frac{x}{2\pi|x|^2}\int_{I\!R^2}\rho(y)\,dy\Big|&=\frac{1}{2\pi}\Big|\int_{I\!R^2}(\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2})\rho(y)\,dy\Big|\\ &\le\frac{1}{2\pi|x|}\int_{I\!R^2}\frac{|y|\rho(y)}{|x-y|}\,dy\end{aligned}$$

since

$$\Big|\frac{x}{|x|^2}-\frac{x-y}{|x-y|^2}\Big|^2=\frac{|y|^2}{|x|^2\cdot|x-y|^2}\;.$$

Then

$$\nabla U(x)=-\frac{x}{2\pi|x|^2}\int\int_{I\!R^2\times I\!R^2} f(y,v)\,dydv\cdot(1+o(1))$$

as $|x|\to+\infty$ if f has a compact support. Integrating over a ball of radius R and taking the large R limit gives

$$\lim_{R\to+\infty}\int_{\partial B(0,R)}\frac{d\sigma(x)}{2\pi|x|}\int_{I\!R^2}\frac{|y|\rho(y)}{|x-y|}\,dy=0\,, \tag{5.1}$$

$$\begin{aligned}\int_{I\!R^2}\Big(x\cdot\nabla U(x)\Big)\,\rho(x)\,dx&=\lim_{R\to+\infty}\Big(-\int_{\partial B(0,R)}\frac{(x\cdot\nabla U(x))^2}{|x|}\,d\sigma(x)+\int_{\partial B(0,R)}|x|\frac{|\nabla U(x)|^2}{2}\,d\sigma(x)\Big)\\&=-\frac{1}{4\pi}\Big(\int\int_{I\!R^2\times I\!R^2}f(x,v)\,dxdv\Big)^2\end{aligned} \tag{5.2}$$

This proves that

$$\int\int_{I\!R^2\times I\!R^2}f(x,v)\,|v|^2\,dxdv=\int_{I\!R^2}\Big(x\cdot\nabla U_0\Big)\,\rho(x)\,dx-\frac{1}{4\pi}\Big(\int\int_{I\!R^2\times I\!R^2}f(x,v)\,dxdv\Big)^2\,.$$

When U_0 is harmonic,

$$\int\int_{I\!R^2\times I\!R^2}f(x,v)\,|v|^2\,dxdv=2\int_{I\!R^2}U_0\,\rho(x)\,dx-\frac{M^2}{4\pi}\,,$$

with $M=\int\int_{I\!R^2\times I\!R^2}f(x,v)\;dxdv$. These results are still true even if ρ has a non compact support.

Proposition 5.1 : *Let $U_0\geq 0$ be such that $x\mapsto\frac{x\cdot\nabla U_0}{1+|x|^2}$ belongs to $L^\infty(I\!R^2)$. Assume that f is a $L^1\cap L^\infty(I\!R^2)$ nonnegative stationary solution of the Vlasov-Poisson system such that*

$$\int\int_{I\!R^2\times I\!R^2}f(x,v)\,(|x|^2+|v|^2)\,dxdv<+\infty$$

and $\rho(x)=\int_{I\!R^2}f(x,v)\,dv$ satisfies: $x\mapsto|x|\cdot\rho(x)\in L^{2+\epsilon}(I\!R^2)$ for some $\epsilon>0$. Then

$$\int\int_{I\!R^2\times I\!R^2}f(x,v)\,|v|^2\,dxdv=\int_{I\!R^2}(x\cdot\nabla U_0)\,\rho(x)\,dx-\frac{M^2}{4\pi}$$

with $\rho(x)=\int_{I\!R^2}f(x,v)\,dv$ and $M=\int_{I\!R^2\times I\!R^2}f(x,v)\,dxdv$.

Proof of Proposition 5.1 : It relies on Lebesgue's theorem of dominated convergence and on Equation (5.1) which is still true even if f has a non compact support: assume that

$$\begin{aligned}&\int_{\partial B(0,R)}d\sigma(x)\cdot\Big|\nabla U(x)+\frac{x}{2\pi|x|^2}\int_{I\!R^2}\rho(y)\,dy\Big|\\&\qquad\leq\int_{\partial B(0,R)}\frac{d\sigma(x)}{2\pi|x|}\int_{I\!R^2}\frac{|y|\rho(y)}{|x-y|}\,dy=\int_{S^1}d\nu\int_{I\!R^2}\frac{|y|\rho(y)}{|R\nu-y|}\,dy\to 0\end{aligned} \tag{5.3}$$

as $R\to+\infty$. On one hand,

$$\begin{aligned}&\Big|\int_{\partial B(0,R)}\Big(\frac{(x\cdot\nabla U)^2}{|x|}-\frac{(x\cdot\frac{M}{2\pi}\frac{x}{|x|^2})^2}{|x|}\Big)\,d\sigma(x)\Big|\\&\qquad\leq\int_{\partial B(0,R)}\Big|\nabla U+\frac{M}{2\pi}\frac{x}{|x|^2}\Big|\,d\sigma(x)\cdot||x\cdot(\nabla U-\frac{M}{2\pi}\frac{x}{|x|^2})||_{L^\infty(\partial B(0,R))}\,.\end{aligned}$$

This can be evaluated by

$$||x \cdot \frac{M}{2\pi} \frac{x}{|x|^2}||_{L^\infty(\partial B(0,R))} = \frac{M}{2\pi}$$

and

$$\begin{aligned} 2\pi ||x \cdot \nabla U||_{L^\infty(\partial B(0,R))} &= \sup_{|x|=R} \left| \int_{I\!R^2} \frac{x \cdot (x-y)}{|x-y|^2} \rho(y)\ dy \right| \\ &= \sup_{|x|=R} \left| M + \int_{I\!R^2} \frac{y \cdot (x-y)}{|x-y|^2} \rho(y)\ dy \right| \\ &\leq M + \sup_{|x|=R} \int_{I\!R^2} \frac{|y|}{|x-y|} \rho(y)\ dy \Big| \end{aligned}$$

using

$$\begin{aligned} \int_{I\!R^2} \frac{|y|}{|x-y|} \rho(y)\ dy &= \int_{|x-y|\leq 1} \frac{|y|}{|x-y|} \rho(y)\ dy + \int_{|x-y|>1} \frac{|y|}{|x-y|} \rho(y)\ dy \\ &\leq || \frac{1}{|x-y|} ||_{L^{\frac{2+\epsilon}{1+\epsilon}}(B(0,1))} |||y|\rho(y)||_{L^{2+\epsilon} B(0,1)} + |||y|\rho(y)||_{L^1(B(0,1))} \ . \end{aligned}$$

On the other hand,

$$\begin{aligned} &\left| \int_{\partial B(0,R)} |x| \left(\frac{1}{2} |\nabla U|^2 - \frac{1}{2} \left| \frac{M}{2\pi} \frac{x}{|x|^2} \right|^2 \right) d\sigma(x) \right| \\ &\leq \frac{1}{2} \int_{\partial B(0,R)} \left| \nabla U + \frac{M}{2\pi} \frac{x}{|x|^2} \right| d\sigma(x) \cdot |||x| \cdot \left(\nabla U - \frac{M}{2\pi} \frac{x}{|x|^2} \right) ||_{L^\infty(\partial B(0,R))} \\ &\leq \frac{M}{\pi} \int_{\partial B(0,R)} \left| \nabla U + \frac{M}{2\pi} \frac{x}{|x|^2} \right| d\sigma(x) \ . \end{aligned}$$

Equation (5.3)has still to be proved. It relies on the

Lemma 5.2 : *Let $g \in L^1(I\!R^2)$ be a nonnegative function such that*

$$\int_{I\!R^2} \frac{g(y)}{|y|}\ dy < +\infty \ .$$

Then

$$\lim_{R\to\infty} \int_{S^1} d\nu \int_{I\!R^2} \frac{g(y)}{|R\nu - y|}\ dy = 0 \ .$$

Proof of Lemma 5.2 : Since

$$\int_{S^1} \frac{d\nu}{2\pi} \int_{I\!R^2} \frac{g(y)}{|R\nu - y|}\ dy = \int_{I\!R^2} \frac{\overline{g}(y)}{|R\nu_0 - y|}\ dy$$

for any $\nu_0 \in S^1$, provided

$$\overline{g}(y) = \int_{S^1} \frac{d\nu}{2\pi}\ g(\nu|y|) \ ,$$

it is enough to prove the lemma when g is radialy symmetric.

$$V(r) = \int_{I\!R^2} \frac{\overline{g}(y)}{2\pi |r\nu_0 - y|}\ dy$$

is a solution of

$$\begin{cases} -\dfrac{1}{r}\dfrac{d}{dr}(r^2\dfrac{dV}{dr}) = \overline{g} \ , \\ \dfrac{dV}{dr}(0) = 0 \ , \\ V(0) = \displaystyle\int_{\mathbb{R}^2} \dfrac{\overline{g}(y)}{2\pi|y|}\ dy \ . \end{cases}$$

Since

$$\frac{dV}{dr}(r) = -\frac{1}{r^2}\int_0^r s\overline{g}(s)\ ds \ ,$$

there exits a limit

$$V(\infty) = \lim_{r\to+\infty} V(r) \ .$$

Thus

$$\begin{aligned} V(r) =& V(0) - \int_0^r \frac{1}{s^2}\int_0^s t^2\overline{g}(t)\ dt\ ds \\ =& V(\infty) + \frac{1}{r}\int_0^r s\overline{g}(s)\ ds + \int_r^{+\infty} \overline{g}(s)\ ds \ , \end{aligned} \tag{5.4}$$

and

$$\left|\frac{1}{r}\int_0^r s\overline{g}(s)\ ds\right| \leq \left|\int_0^r \overline{g}(s)\ ds\right| = \int_{B(0,r)} \frac{g(y)}{2\pi|y|}\ dy \to 0 \quad \text{as } r \to 0_+ \ .$$

Taking $r = 0$ gives therefore $V(\infty) = 0$. □

Remark 5.3 :

(i) Lemma 5.2 is the analogous in dimension $N = 2$ of Newton's identity. Equation (5.4) can also be written as

$$\int_{\mathbb{R}^2} \frac{\overline{g}(y)}{|x-y|}\ dy = \int_{\mathbb{R}^2} \frac{\overline{g}(y)}{\max(|x|,|y|)}\ dy \quad \forall\, x \in \mathbb{R}^2$$

(provided $\overline{g}$ is radially symmetric).

(ii) If the stationary solution is a strong solution that can be approximated by strong (eventualy time dependant) solutions of the Vlasov-Poisson system with compact support, then the result also holds without the assumption that $x \mapsto |x| \cdot \rho(x) \in L^{2+\epsilon}(\mathbb{R}^2)$.

5.2 The evolution problem

This result may be extended to the evolution problem. On one hand, for a smooth and sufficiently decaying at infinity solution,

$$\frac{d}{dt}\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\ (x\cdot v)\ dxdv - \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\ |v|^2\ dxdv + \int_{\mathbb{R}^2}\Big(x\cdot(\nabla U + \nabla U_0)\Big)\ \rho(t,x)\ dx = 0 \ , \tag{5.5}$$

and on the other hand (apply Cauchy-Schwarz inequality to $x\sqrt{f}$ and $x\sqrt{f}$),

$$\left(\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\ (x\cdot v)\ dxdv\right)^2 \leq \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\ |v|^2\ dxdv \cdot \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\ |x|^2\ dxdv \ ,$$

which is bounded since the kinetic energy and the external potential energy are bounded for any $t > 0$ (we assume here that U_0 is harmonic). This can be proved with the same method as in Section 1, Equation (1.3). The energy estimate gives (see Section 3 and Equations (2.3) and (2.5))

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^2\,dxdv + \rho_0\int_{\mathbb{R}^2}\rho(t,x)|x|^2\,dx + \int_{\mathbb{R}^2}\rho(t,x)U(t,x)\,dx$$
$$\leq \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)(|v|^2+\rho_0|x|^2+U(t=0,x))\,dxdv\ .$$

Then, as for Equation (1.3),

$$\int_{\mathbb{R}^2}\rho(t,x)U(t,x)\,dx \geq -\frac{M^2}{2\pi}\ln k - \frac{2M}{\pi}\frac{\ln k}{k^2}\int_{\mathbb{R}^2}\rho(t,x)|x|^2\,dx$$

for some $k \geq \sqrt{e}$. Taking now $k = k(\rho_0) \geq \sqrt{e}$ such that

$$\frac{2M}{\pi}\frac{\ln k}{k^2} < \frac{\rho_0}{2}\ ,$$

we get

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^2\,dxdv + \frac{\rho_0}{2}\int_{\mathbb{R}^2}\rho(t,x)|x|^2\,dx \leq 2E_0 + \frac{M^2}{2\pi}\ln k(\rho_0)$$

which gives a uniform (in t) bound on $\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)(|v|^2+|x|^2)\,dxdv$.

This proves that

$$\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,(x\cdot v)\,dxdv - \int_0^t\left(\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(s,x,v)\,|v|^2\,dxdv + \int_{\mathbb{R}^2}(2U_0-\frac{M}{4\pi})\,\rho(s,x)\,dx\right)ds$$
$$= \int\int_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\,(x\cdot v)\,dxdv \tag{5.6}$$

and then

$$\lim_{t\to+\infty}\frac{1}{t}\int_0^t\left(\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(s,x,v)\,|v|^2\,dxdv - \int_{\mathbb{R}^2} 2U_0(x)\,\rho(s,x)\,dx\right)ds = \frac{-1}{4\pi}\left(\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\,dxdv\right)^2.$$

For a *strong solution, i.e.* a solution such that (see [Pe2])

$$\exists\varepsilon>0\quad\forall\,t>0\qquad\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)(|v|^{2+\varepsilon}+|x|^{2+\varepsilon})\,dxdv < +\infty\ , \tag{5.7}$$

Equations (5.5) and (5.6) are still true. For example, solutions corresponding to initially compactly supported distribution functions are strong solutions (see Section 4). Property (5.7) is in fact true if it holds for the initial data, and it is also possible to prove the following lower order moment estimate (see [C] for a detailed study in dimension 3):

Lemma 5.4 : *Let $U_0 \in W^{2,\infty}_{\rm loc}(\mathbb{R}^2)$, $U_0 \geq 0$. Assume that f is a weak solution of the Vlasov-Poisson system corresponding to a nonnegative initial data $f_0 \in L^1(\mathbb{R}^2\times\mathbb{R}^2)$.*

(i) If U_0 is such that $x \mapsto \frac{\nabla U_0}{1+|x|}$ belongs to $L^\infty(I\!R^2)$ and if for some $\varepsilon > 0$,

$$\int\int_{I\!R^2\times I\!R^2} f_0(x,v)(|v|^{2+\varepsilon} + |x|^{2+\varepsilon})\, dxdv < +\infty\ ,$$

then for any $t > 0$,

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)(|v|^{2+\varepsilon} + |x|^{2+\varepsilon})\, dxdv < +\infty\ . \tag{5.7}$$

(ii) If U_0 is such that $x \mapsto \frac{\nabla U_0}{1+|x|}$ belongs to $L^\infty(I\!R^2)$ and if for some $m \in [1,2]$,

$$\int\int_{I\!R^2\times I\!R^2} f_0(x,v)(|v|^2 + |x|^m)\, dxdv < +\infty\ ,$$

then for any $t > 0$,

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^m\, dxdv < +\infty$$

provided $t \mapsto \int\int f(t,x,v)|v|^2\, dxdv$ is bounded.

(iii) If $x \mapsto \frac{x\cdot\nabla U_0}{1+|x|^2}$ belongs to $L^\infty(I\!R^2)$ and if

$$\int\int_{I\!R^2\times I\!R^2} f_0(x,v)(|v|^2 + |x|^2)\, dxdv < +\infty\ ,$$

then for any $t > 0$,

$$\int\int_{I\!R^2\times I\!R^2} f(t,x,v)(|v|^2 + |x|^2)\, dxdv < +\infty\ .$$

Proof of Lemma 5.4 : Let

$$I(t) = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^{2+\varepsilon}\, dxdv\ ,$$

$$J(t) = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^{2+\varepsilon}\, dxdv\ ,$$

$$K(t) = \int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^2\, dxdv\ ,$$

$$\rho_0 = ||\frac{\nabla U_0}{1+|x|}||_{L^\infty(I\!R^2)}\ .$$

We will prove the estimates only for smooth solutions. Using truncations (like $|x|^{2+\varepsilon}$ for $|x| < R$, $R^{2+\varepsilon}$ for $|x| \geq R$ instead of $|x|^{2+\varepsilon}$), it will then be easy to check that they still hold for weak solutions. As usual, if we multiply the Vlasov equation respectively by $|x|^{2+\varepsilon}$ and $|v|^{2+\varepsilon}$, we get

$$\begin{aligned}\frac{dI}{dt} &= (2+\varepsilon)\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^\varepsilon (x\cdot v)\, dxdv\\ &\leq (2+\varepsilon)\Big(\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|x|^{2+\varepsilon}\, dxdv\Big)^{\frac{1+\varepsilon}{2+\varepsilon}}\Big(\int\int_{I\!R^2\times I\!R^2} f(t,x,v)|v|^{2+\varepsilon}\, dxdv\Big)^{\frac{1}{2+\varepsilon}} \\ &= (2+\varepsilon)\Big(I(t)\Big)^{\frac{1+\varepsilon}{2+\varepsilon}}\Big(J(t)\Big)^{\frac{1}{2+\varepsilon}}\ ,\end{aligned} \tag{5.8}$$

$$
\begin{aligned}
\frac{dJ}{dt} &= -(2+\varepsilon)\Big[\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^\varepsilon((\nabla U_0(x)+\nabla U(t,x))\cdot v)\,dxdv\Big] \\
&\le (2+\varepsilon)\Big[\rho_0\Big(||f||_{L^1(\mathbb{R}^2)}^{\frac{1}{2+\epsilon}} + \big(I(t)\big)^{\frac{1}{2+\varepsilon}}\Big)\Big(J(t)\Big)^{\frac{1+\varepsilon}{2+\varepsilon}} \\
&\qquad + \int |\nabla U(t,x)|\cdot\Big(\int_{\mathbb{R}^2} f(t,x,v)|v|^{1+\varepsilon}\,dv\Big)\,dx\Big]\,.
\end{aligned} \tag{5.9}
$$

<u>*Case (i)*</u>

$$
\begin{aligned}
\frac{dJ}{dt} &= -(2+\varepsilon)\Big[\int\int_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)|v|^\varepsilon((\nabla U_0(x)+\nabla U(t,x))\cdot v)\,dxdv\Big] \\
&\le (2+\varepsilon)\Big[\rho_0\Big(||f||_{L^1(\mathbb{R}^2)}^{\frac{1}{2+\epsilon}} + \big(I(t)\big)^{\frac{1}{2+\varepsilon}}\Big)\Big(J(t)\Big)^{\frac{1+\varepsilon}{2+\varepsilon}} \\
&\qquad + ||f(t,.)||_{L^{\frac{1+\varepsilon}{1-\varepsilon}}(\mathbb{R}^2\times\mathbb{R}^2)}||\nabla U(t,.)||_{L^\infty(\mathbb{R}^2)}\Big(K(t)\Big)^{\frac{1+\varepsilon}{2}}\Big]\,.
\end{aligned}
$$

Then

$$
||\nabla U(t,.)||_{L^\infty(\mathbb{R}^2)} \le \frac{M}{2\pi} + C(\eta)\,||\rho(t,.)||_{L^{2+\frac{\eta}{2}}(\mathbb{R}^2)}\,,
$$

with

$$
C(\eta) = \Big(2\pi\cdot\frac{2+\eta}{\eta}\Big)^{\frac{2+\eta}{4+\eta}}\,,
$$

and (interpolation inequality – see Appendix B)

$$
||\rho(t,.)||_{L^{2+\frac{\eta}{2}}(\mathbb{R}^2)} \le C\cdot||f(t,.,.)||_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)}^{\frac{2}{4+\eta}}\cdot\Big(\int\int_{R^2\times\mathbb{R}^2} f(t,x,v)\cdot|v|^{2+\eta}\,dxdv\Big)^{\frac{2+\eta}{4+\eta}}\,,
$$

for some $C>0$ (not depending on $\eta>0$, small). Hölder's inequality gives

$$
\int\int_{R^2\times\mathbb{R}^2} f(t,x,v)\cdot|v|^{2+\eta}\,dxdv \le \Big(\int\int_{R^2\times\mathbb{R}^2} f(t,x,v)\cdot|v|^2\,dxdv\Big)^\theta\cdot\Big(\int\int_{R^2\times\mathbb{R}^2} f(t,x,v)\cdot|v|^{2+\varepsilon}\,dxdv\Big)^{1-\theta}
$$

with $\theta = 1-\frac{\eta}{\varepsilon}\in]0,\varepsilon[$. Plugging these estimates into (5.8) and (5.9) and using the bounds on $K(t)$ obtained in Section 1, one can prove that $I(t)$ and $J(t)$ have an at most exponential growth in t.

<u>*Case (ii)* :</u> The same computation holds with $m=2+\epsilon\in[1,2]$, except that we have to estimate (5.9) using directly an interpolation inequality:

$$
\begin{aligned}
&\int|\nabla U(t,x)|\cdot\Big(\int_{\mathbb{R}^2}\big(f(t,x,v)\big)^{1/k}\cdot\big(f(t,x,v)\big)^{1-1/k}|v|^{m-1}\,dv\Big)\,dx \\
&\quad\le\int|\nabla U(t,x)|\cdot\Big(\rho(t,x)\Big)^{1/k}\cdot\Big(\int_{\mathbb{R}^2} f(t,x,v)|v|^{(m-1)\frac{k}{k-1}}\,dv\Big)^{1-1/k}\,dx \\
&\quad\le||\nabla U(t,.)||_{L^p(\mathbb{R}^2)}\cdot||\rho(t,.)||_{L^{q/k}(\mathbb{R}^2)}\cdot||f(t,.,.)|v|^{(m-1)\frac{k}{k-1}}||_{L^{r(1-1/k)}(\mathbb{R}^2\times\mathbb{R}^2)}
\end{aligned}
$$

provided $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Thus (see Appendix B),

$$
||\rho(t,.)||_{L^{q/k}(\mathbb{R}^2)} + ||f(t,.,.)|v|^{(m-1)\frac{k}{k-1}}||_{L^{r(1-1/k)}(\mathbb{R}^2\times\mathbb{R}^2)} \le C\cdot||f_0||_{L^\infty(\mathbb{R}^2)}^{1/2}\cdot||f(t,.,.)|v|^2||_{L^1(\mathbb{R}^2\times\mathbb{R}^2)}^{1/2}
$$

for some constant $C > 0$, provided

$$\frac{q}{k} = 2 \quad \text{and} \quad \frac{1}{r} = \frac{m+1}{4} - \frac{1}{2k} \in]0,1[$$

(take for instance $k = \frac{2}{3-m} + 1$). Since $\rho(t,.) \in L^1 \cap L^2(I\!R^2)$, $\nabla U \in L^p(I\!R^2)$ for any $p \in]2,+\infty[$: $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ if and only if $m < 3$ (which is of course true if $m \in [1,2]$). The conclusion then holds as before:

$$\frac{dI}{dt}(t) \le m\, I^{1-1/m}\, J^{1/m} \,,$$

$$\frac{dI}{dt}(t) \le m \Big[\rho_0 \Big(||f_0|_{L^1(I\!R^2 \times I\!R^2)} + I^m \Big) J^{1-1/m} + CK^{\frac{1+m}{4}} \Big] \,.$$

Case (iii) : $m = 2$, $\varepsilon = 0$. Estimate (5.9) is now replaced by the usual energy estimate:

$$J(t) \le \int\!\!\int_{I\!R^2 \times I\!R^2} f_0(x,v)(|v|^2 + U(t=0,x) + 2U_0(x))\, dxdv$$

and like in Section 1 (Remark 1.2, (ii)), we get

$$J(t) \le C(1 + I(t))$$

for some constant $C > 0$ which depends only on f_0. □

As a consequence, we have for strong solutions the following proposition, which generalizes one of the properties obtained by J. Batt in [B].

Proposition 5.5 : *Let $U_0 \in W^{2,\infty}_{\text{loc}}(I\!R^2)$, $U_0 \ge 0$. Assume that f is a nonnegative strong solution of the Vlasov-Poisson system. Then*

$$\frac{d}{dt}\int\!\!\int_{I\!R^2 \times I\!R^2} f(t,x,v)\,(x\cdot v)\, dxdv - \int\!\!\int_{I\!R^2 \times I\!R^2} f(t,x,v)\,|v|^2\, dxdv + \int_{I\!R^2} \big(x \cdot \nabla U_0\big)\, \rho(t,x)\, dx = \frac{M^2}{4\pi}\,, \quad (5.5)$$

$$\int\!\!\int_{I\!R^2 \times I\!R^2} f(t,x,v)\,(x\cdot v)\, dxdv - \int_0^t \Big(\int\!\!\int_{I\!R^2 \times I\!R^2} f(s,x,v)\,|v|^2\, dxdv + \int_{I\!R^2} \big((x\cdot\nabla U_0) - \frac{M}{4\pi}\big)\, \rho(s,x)\, dx \Big)\, ds$$
$$= \int\!\!\int_{I\!R^2 \times I\!R^2} f_0(x,v)\,(x\cdot v)\, dxdv\,. \quad (5.6)$$

Remark 5.6 : Other moment estimates are easily obtained (see also [LP3], [Pe2], [C]): for example, if U_0 is harmonic, then

$$\int_0^{+\infty} dt \int\!\!\int_{I\!R^2 \times I\!R^2} \Big(\frac{|x|^2|v|^2 - (x\cdot v)^2}{|x|^3} - \frac{x}{|x|} \cdot \big(\nabla U(t,x) + \nabla U_0(x)\big) \Big)\, f(t,x,v)\, dxdv < +\infty\,,$$

which is easily proved by multiplying the Vlasov equation by $\frac{(x\cdot v)}{|x|}$ and integrating with respect to t, x and v.

Part II :
$2d$ time-periodic solutions

1. Introduction : some classifications results and a model for time-periodic solutions

Since the problem for stationary solutions is easier than for time-periodic solutions, we present first some classification results for the solutions of the stationary Vlasov-Poisson system. We extend then the ideas developped for these solutions to the time-periodic case. A detailed version of these result is given in Appendix C.

Section 2 of this part is devoted to the study of a subclass of these time-periodic solutions.

1.1. A classification result for stationary solutions

We first explicit the special class of solutions satisfying the weak Ehlers & Rienstra ansatz (see [ER], [BBDP]), and then give a factorization result which proves that these solutions are in fact the generic ones that have a radial spatial density.

It is easy to realize that any distribution function depending on x and v only through the quantities

$$E(x,v) = \frac{1}{2}|v|^2 + U(x) + U_0(x) \quad \text{and} \quad F(x,v) = x \wedge v$$

(we shall say that f satisfies the weak Ehlers & Rienstra ansatz) *i.e.* such that

$$f(x,v) = h(E(x,v), F(x,v)) \qquad (weak\ ER)$$

for some function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, is a solution of the stationary Vlasov equation,

$$v \cdot \partial_x f - (\partial_x U(x) + \partial_x U_0(x)) \cdot \partial_v f = 0 \qquad (sV)$$

provided U is radially symmetric. The problem is then reduced to the Poisson equation

$$-\Delta U = H(U,x)\ , \qquad (P)$$

where $H(U,x) = \int_0^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2\ h(\frac{s_1^2+s_2^2}{2} + U, s_2|x|)$. It is then not difficult to give existence results for such a class of solutions. An interesting point is the fact that these solutions are the most general one can construct (up to some regularity assumptions and a technical non-resonance criterion) that have a radial spatial density.

To be more precise, assume that (f,U) does not depend on t and that U is radially symmetric, *i.e.* that there exists a function $u : \mathbb{R}^+ \to \mathbb{R}$ such that

$$U(x) = u(|x|) \quad \forall\, x \in \mathbb{R}^2\ . \qquad (S1)$$

The factorization result, known as Jeans' theorem (see [BFH] and [Do1-5] for various applications to kinetic equations in plasma physics), gives the form of the generic smooth solution.

The weak form of the result says that the averaged distribution function $\overline{f}$ defined by $\overline{f}(x,v) = \int_{\nu\in S^1} f(|x|\cdot\nu, v)\, d\nu$ for all $(x,v)\in \mathbb{R}^2\times\mathbb{R}^2$ satisfies (locally w.r.t x and v) the weak Ehlers & Rienstra ansatz: for any $(x_0,v_0)\in \mathrm{supp}(f)$, there exist a neighbourhood $\mathcal{V}$ and a function $h:\mathbb{R}^2\to\mathbb{R}^+$ such that

$$\overline{f}(x,v) = h(E(x,v),F(x,v)) \quad \forall\,(x,v)\in\mathcal{V}\,. \qquad (weak\ ER)$$

If moreover a non resonance condition is satisfied (see Appendix C for a precise statement), and provided f is continuous, then the factorization result holds for f:

$$f(x,v) = \overline{f}(x,v) \quad \forall\,(x,v)\in\mathcal{V}\,.$$

A sufficient condition for these two results to be global is $r\mapsto r^3(\frac{du}{dr}+\frac{du_0}{dr})$ is monotone increasing : in this case, one may take $\mathcal{V}=\mathbb{R}^2\times\mathbb{R}^2$.

1.2. Time-periodic solutions

For time-periodic solutions, we may proceed exactly in the same way. Consider now

$$E(t,x,v) = \frac{1}{2}|v|^2 + U(t,x) + U_0(x) \quad \text{and} \quad F(x,v) = x\wedge v\,.$$

The same kind of factorization result as for stationary solutions (under regularity and non resonance assumptions) shows that a time-periodic solution of period T such that the average over one period of the self-consistent potential is radially symmetric satisfies a factorization property, which is global w.r.t. t and local in the phase space $\mathbb{R}^2_x\times\mathbb{R}^2_v$. If it is global, then the factorization result may be written as: there exists a function $g:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^+$ such that for almost all $(t,x,v)\in[0,T]\times\mathrm{supp}(f)$,

$$f(t,x,v) = g(t, E(t,x,v), F(x,v))\,.$$

A detailed version of this result is given in Appendix C. In the following, we will make one more assumption and assume that g does not depend on t. The class of solutions we shall consider is therefore defind by: there exist constant $T>0$ and a function $g:\mathbb{R}^2\to\mathbb{R}^+$ such that for almost all $(t,x,v)\in\mathbb{R}\times\mathbb{R}^2_x\times\mathbb{R}^2_v$,

$$f(t,x,v) = g(E(t,x,v),F(x,v)) \quad \text{and} \quad f(t+T,x,v) = f(t,x,v) \quad . \qquad (weak\ ER)$$

In the next section we will prove that under a technical but generic assumption, the solution satisfies the (strong) Ehlers & Rienstra ansatz

$$f(t,x,v) = g(E(t,x,v) - \omega F(x,v)) \tag{ER}$$

for some function g and for $\omega = \frac{2\pi}{T}$. In that case, exactly as in the paper by J. Batt, H. Berestycki, P. Degond & B. Perthame [BBDP] (devoted to the study of the $3d$-solutions of the gravitational Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz) the Vlasov equation is reduced to

$$\partial_t U(t,x) - \omega(Ax \cdot \nabla_x U(t,x)) = 0 \ ,$$

where the linear operator A is such that $v \cdot Ax = x \wedge v$ for any $(x,v) \in I\!R^2 \times I\!R^2$: $A(x_1,x_2) = -(x_2,x_1)$ in a cartesian system of coordinates.

f and U are in a solid motion of rotation around the z-axis with a constant angular velocity ω and take the following form

$$f(t,x,v) = g\left(\frac{|v-\omega Ax|^2}{2} + U_0(x) + w\left(e^{\omega tA}x\right) - \frac{\omega^2}{2}|x|^2\right) \quad \text{and} \quad U(t,x) + U_0(x) - \frac{\omega^2}{2}|x|^2 = w\left(e^{\omega tA}x\right) \ ,$$

and the problem is reduced to the nonlinear Poisson equation for w

$$-\Delta w + 2(\rho_0 - \omega^2) = G(w)$$

with $G(w) = \pi \int_w^{+\infty} g(s)\, ds$ (we assume here that U_0 is an harmonic potential).

We may first mention that such a formulation provides a very simple way for constructing time-periodic solutions to the Vlasov-Poisson system: since the equation for w does not depend on x, for any solution w, w_τ defined by $x \mapsto w_\tau(x) = w(x+\tau)$ is also a solution for any $\tau \in I\!R^2$, which clearly does not have the same symmetry properties as w (see Remark 3.2). Thus the potential U and the distribution function f are time-periodic solutions which are not radially symmetric and depend therefore explicitely of t. We will also consider the case where the confining potential U_0 is not radially symmetric, and exhibit a branch of solutions that have a logarithmic growth, starting from the solutions that are radially symmetric up to a translation. These solutions are time-periodic (and generically explicitely time-dependent (*i.e.* non stationary) solutions. Adequate conditions on G ensure that they have a finite mass.

But this part of the paper will be mainly devoted to the class of nonisotropic solutions with quadratic growth, *i.e.* the solutions such that

$$w \sim \delta(\theta x_1^2 + (1-\theta)x_2^2) \quad \text{as } |x| \to +\infty$$

for some $\theta \in [0,1]$, (x_1,x_2) beeing a system of cartesian coordinates of $x \in I\!R^2$. In [BBDP], the solutions that were considered were $3d$-solutions of the gravitational Vlasov-Poisson system corresponding in the $2d$-case to solutions such that $\theta = 0$ or 1 ($1d$-solutions), or such that $\theta = \frac{1}{2}$ (radially symmetric solutions). We will adapt their results to the $2d$-electrostatic Vlasov-Poisson system with a confining potential (Section 3, Proposition 3.4: existence of $1d$-solutions, and Proposition 3.5: existence of radially symmetric solutions), but also study the general case: $\theta \in (0,1)$, $\theta \neq \frac{1}{2}$.

The spatial density $(t,x) \mapsto \int_{I\!R^2} f(t,x,v)\,dv$ is, up to a rotation of angle ωt given by

$$G\Big(u(x) + \frac{\delta}{2}|x|^2\Big) = G\Big(v(x) + \delta(\theta x_1^2 + (1-\theta)x_2^2)\Big) \sim G\Big(\delta(\theta x_1^2 + (1-\theta)x_2^2)\Big) \quad \text{as} \quad |x| \to +\infty \, .$$

It belongs to $L^1(I\!R^2_x)$ if $s \mapsto G(s)$ is sufficiently decreasing for $s \to +\infty$. Note that the asymptotic behaviour of u for such solutions is

$$u(x) = \delta(\theta - 1/2)(x_1^2 - x_2^2) + o(|x|^2) \quad \text{as} \quad |x| \to +\infty \, , \tag{1.1}$$

for any θ belonging to $[0,1]$, which corresponds to a non standard asymptotic behaviour (see Theorem 3.3: asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences).

For this study, our main mathematical reference is a paper by J. Batt, H. Berestycki, P. Degond & B. Perthame [BBDP] (three-dimensional Vlasov-Poisson system in the gravitational case). Compared to it the main results of this paper are the following:
- the class of solutions is larger ($\theta \in [0,1]$ instead of $\theta = 0, \frac{1}{2}$ or 1) and the symmetry assumptions are weaker ("weak" Ehlers and Rienstra ansatz instead of "strong" Ehlers and Rienstra ansatz) – roughly spoken, we prove that it corresponds to the class of the solutions that are in a solid motion of rotation with a constant angular velocity and such that the self-consistent potential has an at most quadratic growth,
- since the confinement of the particles is due to an external potential and not to the self-consistent potential (the force between the particles is repulsive), it is possible to perturb it and build a branch of solutions starting from the radially symmetric solutions (up to a translation),
- the asymptotic boundary conditions are systematically explored; choosing a quadratic growth for the self-consistent potential (like in BBDP]) is not absurd (one has to keep in mind that the model corresponds to the study of a beam locally near its axis, as shown in Appendix A).

The study of the stationary solutions has been neglected. Most of the results for time-periodic solutions are easily extended to the stationary case by simply taking $\omega = 0$. One has also to refer to J. Batt, W. Faltenbacher, & E. Horst [BFH] for this point. An attempt of a general classification of the time periodic solutions by the mean of Jeans' theorem is presented in Appendix C. Ideas , which are very popular among astrophysicists (see [Do1-2] for a review) have been introduced from a mathemati cal point of view in [BFH], from which some notations are taken.

2. The Ehler-Rienstra ansatz for time-dependant solutions

In the rest of the paper, we will consider the special class of solutions of the Vlasov-Poisson system that are such that

$$f(t,x,v) = g(E(t,x,v) - \omega F(x,v)) \qquad (ER)$$

(this ansatz will be referred as the Ehlers & Rienstra ansatz) for some $\omega \in I\!R$, and where E and F are defined by

$$E(t,x,v) = \frac{1}{2}|v|^2 + U(t,x) + U_0(x) \ , \qquad (2.1)$$

$$F(x,v) = x \wedge v \ . \qquad (2.2)$$

Such solutions are also called in the physical literature "locally isotropic solutions" (see [ER]). Before studying these solutions in details, we will notice that this class of solutions corresponds to a *a priori* larger class of solutions, the class of solutions satisfying only the weak Ehlers & Rienstra ansatz *i.e.* such that

$$f(t,x,v) = h(E(t,x,v), F(x,v)) \qquad (weak\ ER)$$

provided they are explicitely time-dependant. This result will be important in view of an attempt of classification of all the time-periodic solutions of the Vlasov-Poisson system given in Appendix C.

Theorem 2.1 : *Assume that (f,U) is a solution of the Vlasov-Poisson system satisfying the weak Ehlers & Rienstra ansatz with h a nonnegative function of class C^2 defined on $I\!R^2$ such that $\{(E,F) \in I\!R^2 \ : \ \frac{\partial h}{\partial E}(E,F) = 0\}$ is a finite union of 1-d C^1 manifolds, and such that $x \mapsto \rho(t,x) = \int_{I\!R^2} f(t,x,v)\, dv$ is a nonnegative continuous function with meas$\Big(\{x \in I\!R^2 \ : \ \rho(t,x) = \rho_0\}\Big) = 0$ (U_0 is an harmonic potential such that $U_0(x) = \frac{\rho_0}{2}|x|^2$, where ρ_0 is positive real constant). Assume that $\partial_t U \not\equiv 0$. Then there exists an $\omega \in I\!R$ such that f is time-periodic of period $\frac{2\pi}{\omega}$, f satisfies the Ehlers & Rienstra ansatz, and there exists a C^2 function g such that*

$$h(E,F) = g(E - \omega F) \ .$$

Moreover f and U may be written in the following form

$$f(t,x,v) = g\bigg(\frac{|v - \omega A x|^2}{2} + U_0(x) + w\Big(e^{\omega t A} x\Big) - \frac{\omega^2}{2}|x|^2\bigg) \ . \qquad (2.3)$$

$$U(t,x) = w\Big(e^{\omega t A}x\Big) \ , \tag{2.4}$$

where w is a solution of the nonlinear Poisson equation

$$-\Delta w + 2(\rho_0 - \omega^2) = \rho \ , \tag{2.5}$$

and A is the linear operator such that $v \cdot Ax = x \wedge v$ for any $(x,v) \in I\!R^2 \times I\!R^2$.

Proof :

1) A simple computation shows that

$$\Big(\partial_t + v\cdot\partial_x - (\partial_x U(t,x) + \partial_x U_0(x))\cdot\partial_v\Big)E(t,x,v) = \frac{\partial U}{\partial t}(t,x) \ .$$

2) Let (x_1, x_2) be cartesian coordinates of $x \in I\!R^2$, so that $x \wedge v = x_1 v_2 - x_2 v_1$, and denote by A the linear operator such that Ax is represented by $(-x_2, x_1)$. Let us define F by :

$$F(x,v) = x \wedge v = (v \cdot Ax) \ .$$

Then, with these notations

$$\Big(\partial_t + v\cdot\partial_x - (\partial_x U(t,x) + \partial_x U_0(x))\cdot\partial_v\Big)F(x,v) = -(Ax\cdot\nabla_x U(t,x)) \ .$$

3) If f satisfies the weak Ehlers & Rienstra ansatz, it is therefore a solution of the Vlasov equation if and only if

$$\frac{\partial h}{\partial E}\Big(E(t,x,v),F(x,v)\Big)\cdot\partial_t U(t,x) - \frac{\partial h}{\partial F}\Big(E(t,x,v),F(x,v)\Big)\cdot(Ax\cdot\nabla_x U(t,x)) = 0 \ . \tag{2.6}$$

If $\omega(E,F) = \Big(\frac{\partial h}{\partial F}/\frac{\partial h}{\partial E}\Big)(E,F)$, then

$$\partial_t U(t,x) - \omega(E,F)(Ax\cdot\nabla_x U(t,x)) = 0 \ .$$

Since U is a solution of the Poisson equation and does not depend on v,

$$(x,v) \mapsto \omega(E(t,x,v),F(x,v))$$

is locally a constant on $I\!R^2 \times I\!R^2 \backslash \Sigma$, with $\Sigma = \{(x,v) \ : \ \frac{\partial h}{\partial E}(E(t,x,v),F(x,v)) = 0\}$, and $(I\!R^2 \times I\!R^2)\backslash\Sigma$ is connected. Let us prove it.

4) According to the assumptions on h, one may assume either that there locally exists a C^1 function $E \mapsto F(E)$ such that

$$\frac{\partial h}{\partial E}\Big(E,F(E)\Big) = 0 \ , \qquad (Case\ 1)$$

or that there locally exists a C^1 function $F \mapsto E(F)$ such that

$$\frac{\partial h}{\partial E}\Big(E(F), F\Big) = 0 \,, \qquad (Case\ 2)$$

on $\{(E,F) \in I\!\!R^2 \ : \ \frac{\partial h}{\partial E}(E,F) = 0\}$. Let us look at the first case. According to the expressions of E and F given by equations (2.1) and (2.2), either

$$x \cdot v = 0 \,,$$

or one can (locally) find a function $E \mapsto v(E) = (v_1(E), v_2(E))$ (with values in $I\!\!R^2_v$) such that

$$\begin{cases} v_1 \dfrac{dv_1}{dE} + v_2 \dfrac{dv_2}{dE} = 1 \,, \\ x_2 \dfrac{dv_1}{dE} - x_1 \dfrac{dv_2}{dE} = \dfrac{dF}{dE} \,. \end{cases}$$

and Σ is therefore tangent to

$$\mathrm{Vect}(\frac{dv}{dE}) \times (\nabla U + \nabla U_0)^{\perp} \,,$$

provided $\nabla U + \nabla U_0 \neq 0$, or to

$$\mathrm{Vect}(\frac{dv}{dE}) \times \mathrm{Vect}(\tau)$$

if $\nabla U + \nabla U_0 = 0$. Here τ is the unit tangent vector to the set $\{x \in I\!\!R^2 \ : \ \nabla U(x) + \nabla U_0(x) = 0\}$. It is well defined since it is a unit vector belonging to $\mathrm{Ker}(D^2(U+U_0))$ if $D^2(U+U_0) \neq 0$, or is defined by continuity if $D^2(U+U_0) = 0$. If $D^2(U+U_0) \equiv 0$ on a neighbourhood $\mathcal{V}$, then

$$\frac{\partial^2 U}{\partial x_1^2} + \rho_0 = 0 \quad \text{and} \quad \frac{\partial^2 U}{\partial x_2^2} + \rho_0 = 0 \quad \text{on } \mathcal{V} \,,$$

which would imply that $\rho \equiv \rho_0$ on $\mathcal{V}$, in contradiction with the assumptions on ρ.

In case 2, the proof is exactly the same except that one has to exchange the roles of $E(t,x,v)$ and $F(x,v)$, $E(F)$ and $F(E)$.

Σ is therefore contained in a finite union of manifolds of dimension 2: $(I\!\!R^2 \times I\!\!R^2) \backslash \Sigma$ is connected.

5) For all $t \in I\!\!R$ $(x,v) \mapsto \omega(E(t,x,v), F(x,v))$ is a constant. Obviously, ω does not depend either on t: $\omega = \omega(E,F)$ is therefore almost everywhere w.r.t. (E,F) equal to a constant, which we still denote by ω, when E, F belong to the set

$$\mathcal{X} = \{(E,F) \ : \ E = E(t,x,v) \,, \quad F = F(x,v) \,, \quad (t,x,v) \in I\!\!R \times I\!\!R^2 \times I\!\!R^2\} \,.$$

On $\mathcal{X}$,

$$\frac{\partial h}{\partial F} + \omega \frac{\partial h}{\partial E} = 0 \,,$$

so that there exists a function $g : \mathbb{R} \mapsto \mathbb{R}$ such that

$$h(E,F) = g(E - \omega F) .$$

6) Equation (2.6) now reads

$$\partial_t U - (\omega A x) \cdot \nabla_x U = 0 ,$$

which is solved by

$$U(t,x) = w\Big(e^{\omega t A} x\Big) , \tag{2.4}$$

where w is a solution of the Poisson equation

$$-\Delta w + 2(\rho_0 - \omega^2) = \rho , \tag{2.5}$$

and replacing E and F by their values expressed in terms of ω and w :

$$E(t,x,v) = \frac{|v|^2}{2} + U_0(x) + w\Big(e^{\omega t A} x\Big) .$$

$$F(x,v) = (Ax \cdot v) .$$

gives for f the expression

$$f(t,x,v) = g\bigg(\frac{|v - \omega A x|^2}{2} + U_0(x) + w\Big(e^{\omega t A} x\Big) - \frac{\omega^2}{2}|x|^2\bigg) . \tag{2.3}$$

□

3. The nonlinear Poisson equation and time-periodic solutions

This section contains the main results of the paper. Theorem 3.3 (Asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences) and Theorem 3.7 (Existence of time-periodic anisotropic solutions with finite mass) are completely new results. They present *a priori* considerations on the asymptotic behaviour of the solutions, and existence results for a new class of solutions, which includes the solutions given by J. Batt, H. Berestycki, P. Degond & B. Perthame in [BBDP]. These solutions have the property that they are explicitely time-dependant and may have a finite L^1-norm (mass), which was not the case for the solutions given in [BBDP].

The other results of this section (Proposition 3.1: equivalence with a nonlinear Poisson equation, Proposition 3.4: existence of radially symmetric solutions and Proposition 3.5: existence of $1d$-solutions) are more or less an adaptation of some of the results given in

[BBDP] (for $3d$-solutions of the gravitational Vlasov-Poisson system) to the $2d$-solutions of the Vlasov-Poisson system with a confining potential.

First of all, if f satisfies the Ehlers & Rienstra ansatz, we can give a complete characterization of the time-dependant solutions of the Vlasov-Poisson system in terms of an equivalent nonlinear Poisson equation.

Proposition 3.1 : *(Equivalence with a nonlinear Poisson equation) Assume that g belongs to $L^1 \cap W^{1,\infty}(\mathbb{R})$, that g is nonnegative and not identically equal to 0. (f, U) is a solution in $C^0(\mathbb{R}; L^1(\mathbb{R}^2_{x,\text{loc}} \times \mathbb{R}^2_v)) \times C^1(\mathbb{R}; W^{2,1}_{\text{loc}}(\mathbb{R}^2_x))$ of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (f, U are weak solutions respectively in the sense of the characteristics as defined by R. DiPerna & P.-L. Lions in [DPL1], and in the sense of the distributions) if and only if there exists a solution $w \in W^{2,1}_{\text{loc}}(\mathbb{R}^2)$ of*

$$-\Delta w + 2(\rho_0 - \omega^2) = G(w) , \tag{NLP}$$

with $G(w) = \pi \int_w^{+\infty} g(s)\ ds$. The relation between (f, U) and w is given by Equations (2.3) and (2.4). $x \mapsto \rho(t, x)$ is locally Lipshitz, and U belongs to $C^1(\mathbb{R} \times \mathbb{R}^2)$.

Proof :

1) Assume first that (f, U) is a solution of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz. According to this ansatz

$$g'(E - \omega F) \cdot \Big(\partial_t U - \omega(Ax) \cdot \nabla U\Big) = 0 ,$$

with

$$(E - \omega F)(t, x, v) = \frac{|v - \omega Ax|^2}{2} + U_0(x) + U(t, x) - \frac{\omega^2}{2}|x|^2 .$$

Assume that t and x are fixed, such that x belongs to the support of $\rho(t, .)$, *i.e.* the support of

$$x \mapsto \rho(t, x) = \int_{\mathbb{R}^2} f(t, x, v)\ dv .$$

Because of the assumptions on g,

$$\{v \in \mathbb{R}^2\ :\ g'\big(E - \omega F\big)(t, x, v) \not\equiv 0\}$$

has a strictly positive measure :

$$\frac{d}{dt} U(t, e^{-\omega t A} x) = 0 ,$$

so that $w(x) = U(t, e^{-\omega t A} x) + U_0(x) - \frac{\omega^2}{2}|x|^2$ does not depend on t as long as x belongs to the support of $\rho(t, .)$. Since

$$\rho(t, x) = \int_{\mathbb{R}^2} g\Big((E - \omega F)(t, x, v)\Big)\ dv = \pi \int_0^{+\infty} g\Big(s + U(t, x) + U_0(x) - \frac{|\omega|^2}{2}\Big)\ ds ,$$

we get

$$\rho(t, e^{-\omega tA}x) = \pi \int_0^{+\infty} g\Big(s + w(x)\Big)\, ds = G(w(x)) \ .$$

w is therefore a solution of

$$-\Delta w + 2(\rho_0 - \omega^2) = G(w) \ .$$

Let us define w on $I\!R^2 \backslash \mathrm{supp}(\rho(t,.))$ by

$$-\Delta w + 2(\rho_0 - \omega^2) = -\Delta U_{|(t, e^{-\omega tA}x)} = 0 \ .$$

w does not depend on t for any $x \in I\!R^2$, and the first part of the theorem holds.

The other side of the proof is obvious. □

Remark 3.2 : Since the (NLP) equation does not depend on x, it immediately follows that for any solution w , w_τ defined by

$$w_\tau(x) = w(x + \tau) \quad \forall\, x \in I\!R^2$$

is still a solution for any $\tau \in I\!R^2$. This very simple fact allows us to exhibit explicitely time-dependant priodic solutions as soon as w is not constant, even in the case when w is spherically symmetric, which means that the corresponding solution (f, U) of the Vlasov-Poisson system is rotationally symmetric and stationary (see Proposition 3.4). It is possible to consider such solutions because the domain $I\!R^2$ is of course translation invariant and because the boundary conditions are specified only at infinity. In the following, we will not consider such a cause of time-dependance (except for the study of 1-d solutions, see Proposition 3.5) since it is a consequence of the assumption that the confining potential takes the very special form

$$U_0(x) = \frac{\rho_0}{2} \cdot |x|^2 \quad \forall\, x \in I\!R^2 \ , \tag{H}$$

but we will concentrate our attention on an other cause of time-dependance, which is much more fundamental: the anisotropy of the solutions (see Proposition 3.5 and Theorem 3.7) which is clearly related to the asymptotic boundary conditions (see Theorem 3.3) and to the asymptotic behaviour of U_0.

The asymptotic condition on the behaviour of the density

$$\limsup_{|x| \to +\infty} \rho(t, x) = 0$$

gives a lower bound on

$$\liminf_{|x| \to +\infty} \frac{U(t, x)}{|x|^2} \ .$$

We will see in Proposition 3.5 (Existence of 1-d solutions) that this bound is optimal. Such an asymptotic boundary condition is far from the usual one

$$\lim_{|x|\to+\infty} U(t,x) = Constant \;,$$

but we will see that there are no solutions satisfying such a condition.

Theorem 3.3 : *(Asymptotic behaviour, necessary conditions for the existence of time-periodic solutions and consequences)*

(i) *Under the same assumptions as in Proposition 3.1, there exists a nontrivial (i.e. $f \geq 0$, $f \not\equiv 0$) solution (f, U) of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz such that $\limsup_{|x|\to+\infty} \rho(t,x) = 0$ if one of the two following conditions is satisfied:*

$$\begin{array}{ll} \text{either} & \liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2} \quad \forall\, t \in I\!R \;, \\ \text{or} & \omega^2 \leq \rho_0 \quad \text{and} \quad \liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} = -\frac{\delta}{2} \quad \forall\, t \in I\!R \;, \end{array}$$

where $\delta = \rho_0 - \omega^2$.

(ii) *If $\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2}$, then $\omega^2 \leq \rho_0$, and $(t,x) \mapsto \rho(t,x) = \int_{I\!R^2} f(t,x,v)\, dv$ belongs to $C^0(I\!R, L^1(I\!R^2))$ if and only if $\int_0^{+\infty} G(s)\, ds < +\infty$.*

(iii) *A necessary condition for the existence of a nontrivial solution (f, U) of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz and such that $\limsup_{|x|\to+\infty} \rho(t,x) = 0$ is $\omega^2 \leq \rho_0$.*

(iv) *Assume moreover that for some $\epsilon > 0$, $(t,x) \mapsto \rho(t,x) = (1 + |x|^\epsilon) \int_{I\!R^2} f(t,x,v)\, dv$ belongs to $C^0(I\!R, L^1(I\!R^2))$, i.e. $\int_0^{+\infty} G(s)(1 + s^{\epsilon/2})\, ds < +\infty$. Then there is no solution such that $\lim_{|x|\to+\infty} U(t,x) = U_\infty$ for all $t \in I\!R$, for some constant $U_\infty \in I\!R$.*

(v) *If $\limsup_{|x|\to+\infty} \rho(t,x) = 0$ and $\limsup_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} < +\infty$, or $\limsup_{|x|\to+\infty} \frac{w(x)}{|x|^2} < +\infty$, where w and U are related by $U(t,x) = w\Big(e^{\omega t A} x\Big)$, then there exist $\theta \in [0,1]$ and a system of cartesian coordinates such that*

$$w(x) = U\Big(t, e^{-\omega t A} x\Big) = \delta(\theta x_1^2 + (1-\theta) x_2^2) + o(|x|^2) \quad \text{as} \quad |x| \to +\infty \;. \tag{1.1}$$

Proof :

Since G is positive decreasing and not identically equal to zero, and since

$$\rho(t,x) = G(U(t,x) + \frac{\delta}{2}) \quad \text{with} \quad \delta = \rho_0 - \omega^2 \;,$$

the condition

$$\limsup_{|x|\to+\infty} \rho(t,x) = 0$$

is equivalent to

$$\overline{w} \leq \liminf_{|x|\to+\infty} w(x) = \liminf_{|x|\to+\infty} \left(U(t, e^{-\omega t A} x) + \frac{\delta}{2} |x|^2 \right) ,$$

with

$$\overline{w} = \inf\{w \in I\!R \ : \ G(w) = 0\} \in]-\infty, +\infty] \ ,$$

a condition that is obviously violated if

$$\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} < -\frac{\delta}{2} \quad \forall\, t \in I\!R \ .$$

If $\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} = -\frac{\delta}{2} > 0$, then $w(x) = U(t, e^{-\omega t A}x) + \frac{\delta}{2}|x|^2$ is such that

$$-\Delta(\frac{\delta}{2}|x|^2 - w) = -G(w) \leq 0 \ ,$$

and

$$\limsup_{|x|\to+\infty} \frac{1}{|x|^2}(\frac{\delta}{2}|x|^2 - w) = \frac{\delta}{2} < 0$$

implies that

$$h(R) = \sup_{|x|=R} (\frac{\delta}{2}|x|^2 - w) = -R^2 \inf_{|x|=R} \frac{1}{|x|^2}(\frac{\delta}{2}|x|^2 - w)$$

is such that there exists a sequence $(R_n)_{n\in I\!N}$ satisfying

$$\lim_{n\to+\infty} R_n = +\infty \quad \text{and} \quad h(R_n) = \frac{\delta}{2}R_n^2 \to -\infty \ .$$

Applying the Maximum Principle, we obtain for all $n \in I\!N$,

$$\frac{\delta}{2}|x|^2 - w(x) \leq h(R_n) \quad \forall\, x \in B(0,R) \ ,$$

which is impossible :

$$\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} = -\frac{\delta}{2} \quad \Longrightarrow \quad \omega^2 \leq \rho_0 \ ,$$

which proves (i).

Assume now that $\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2} > 0$.

$$\limsup_{|x|\to+\infty} \frac{1}{|x|^2}(\frac{\delta}{2}|x|^2 - w) = -\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} < 0$$

and

$$-\Delta(\frac{\delta}{2}|x|^2 - w) = -G(w) \leq 0$$

again imply that

$$w(x) \geq \frac{\delta}{2}|x|^2 + R^2 \inf_{|x|=R} \frac{U(t,x)}{|x|^2} \quad \forall\, x \in B(0,R) \ , \quad \forall\, R > 0 \ ,$$

which also gives a contradiction, and proves that

$$\liminf_{|x|\to+\infty} \frac{U(t,x)}{|x|^2} > -\frac{\delta}{2} \quad \Longrightarrow \quad \omega^2 \leq \rho_0 \ ,$$

which proves (ii).

(iii) is an immediate consequence of (i) and (ii).

(iv) is easuly deduced from the asymptotic equivalence

$$U(t,e^{-\omega tA}x)=w(x)-\frac{\delta}{2}|x|^2\sim-\frac{M}{2\pi}\ln|x| \quad \text{with } M=\int_{\mathbb{R}^2}\rho(t,x)\,dx\;\forall\,t\in\mathbb{R}\;.$$

Assume that a system of polar coordinates (r,φ) is given, and define $\tilde{w}$ by

$$\tilde{w}(r,\varphi)=r^2w(r^{-1}\cos\varphi,r^{-1}\sin\varphi)$$

for all $(r,\varphi)\in]0,+\infty[\times[0,2\pi[$. $\tilde{w}$ is solution of

$$4\tilde{w}-3r\frac{\partial\tilde{w}}{\partial r}+r^2\frac{\partial^2\tilde{w}}{\partial r^2}+\frac{\partial^2\tilde{w}}{\partial\varphi^2}=2\delta-G(\frac{\tilde{w}}{r^2})\sim2\delta \quad \text{as } r\to0+\;.$$

Because of the asssumption

$$\limsup_{|x|\to+\infty}\frac{w(x)}{|x|^2}=Const\;,$$

the function $a=\tilde{w}(0,.):[0,2\pi[\to\mathbb{R}$ such that

$$\tilde{w}(r,\varphi)\sim a(\varphi) \quad \text{as } r\to0+$$

satisfies the equation

$$4a+\frac{d^2a}{d\varphi^2}=2\delta\;.$$

Up to a rotation

$$w(r^{-1}\cos\varphi,r^{-1}\sin\varphi)\sim r^{-2}(a_0\cos(2\varphi)+\frac{\delta}{2}) \quad \text{as } r\to0+\;,$$

for some $a_0\in\mathbb{R}$, and because of (i), the condition

$$\liminf_{|x|\to+\infty}\frac{U(t,x)}{|x|^2}\geq-\frac{\delta}{2} \quad \forall\,t\in\mathbb{R}$$

implies that

$$a_0\leq\frac{\delta}{2}\;.$$

Thus, there exists a constant $\theta\in[0,1]$ such that

$$w(x)=\delta(\theta x_1^2+(1-\theta)x_2^2)+o(|x|^2) \quad \text{as} \quad |x|\to+\infty$$

in a well choosen system of cartesian coordinates.

□

We will now study the problem of the existence for different classes of solutions characterized by their symmetries and the corresponding asymptotic behaviour, and give some of the properties of these solutions.

We begin with radially symmetric. These solutions are time-independant (except if they are translated: see Remark 3.2), but may have a finite total mass

$$M = \int\int_{I\!R^2 \times I\!R^2} f(t,x,v)\, dxdv\ .$$

Exactly as in the paper by J. Batt, H. Berestycki, P. Degond & B. Perthame ([BBDP]) we will also study 1-d solutions that may be non stationary time-periodic solutions (even when they are not translated), but are always of infinite mass (Proposition 3.5).

The last result of this section will be devoted to solutions intermediate between the ones of Proposition 3.4 (radially symmetric solutions) and the ones of Proposition 3.5 (1-d solutions), which appear to form (Theorem 3.7) a new class of solutions (as far as the author knows). These solutions have the interesting property that they are time-periodic non stationary solutions, and that they may have a finite total mass. They moreover have a non standard asymptotic behaviour :

$$U(t, e^{-\omega t A}x) \sim \delta(\theta - 1/2)(x_1^2 - x_2^2) + o(|x|^2) \quad \text{as} \quad |x| \to +\infty\ .$$

in a well choosen system of cartesian coordinates.

Proposition 3.4 : *(Existence of radially symmetric solutions)*
Assume that G is a Lipshitz decreasing function, such that

$$\lim_{w \to +\infty} G(w) = 0 \quad \text{and} \quad \lim_{w \to -\infty} G(w) = G^\infty < \infty\ .$$

For any $\delta > 0$,

$$\begin{cases} -\dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{dw}{dr}\right) + 2\delta = G(w) \\ w(0) = w_0 \\ \dfrac{dw}{dr}(0) = 0 \end{cases}$$

has a unique solution $r \mapsto w(r)$ for $r \in]0, +\infty[$. Let us define $\overline{w}$ by

$$\overline{w} = \sup\{w \in I\!R\ :\ G(w) > 2\delta\} \quad \text{if} \quad G^\infty \geq 2\delta\ ,$$

$$\overline{w} = -\infty \quad \text{if} \quad G^\infty < 2\delta\ .$$

Three cases may occur:

(i) $w_0 = \overline{w}$: then $w(r) = w_0$ for all $r > 0$.

(ii) $w_0 > \overline{w}$*: then* w *is strictly increasing on* $]0,+\infty[$ *and*

$$\frac{dw}{dr}(r) \sim \delta r \quad \text{and} \quad w(r) \sim \frac{\delta}{2} r^2 \quad \text{as} \quad r \to +\infty \ .$$

(iii) $w_0 < \overline{w}$*: then* w *is strictly decreasing on* $]0,+\infty[$ *and*

$$\frac{dw}{dr}(r) \sim \frac{1}{2}(2\delta - G^\infty) r \quad \text{and} \quad w(r) \sim \frac{1}{4}(2\delta - G^\infty) r^2 \quad \text{as} \quad r \to +\infty \ .$$

If $g = -\frac{1}{\pi} G'$*, then* f *belongs to* $C^0(\mathbb{R}, L^1(\mathbb{R}^2_{\text{loc}} \times \mathbb{R}^2))$*,* U *belongs to* $C^1(\mathbb{R}, C^2(\mathbb{R}^2))$ *and*

$$f(t,x,v) = g\left(\frac{1}{2}|v - \omega A x|^2 + w(|e^{\omega t A} x|)\right) ,$$

$$U(t,x) = w(|e^{\omega t A} x|) ,$$

with $\omega = \pm\sqrt{\rho_0 - \delta}$*, are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that* f *is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions).* f *and* U *are in fact stationary solutions :*

$$U(t,x) = w(|x|) - U_0(x) + \frac{1}{2}\omega^2 |x|^2 = U(x)$$

and

$$f(t,x,v) = g(\frac{1}{2}|v|^2 + w(|x|) - \omega(v \cdot Ax) + \frac{1}{2}\omega^2|x|^2) = f(x,v)$$

do not depend on t *:* f *(resp.* U*) depend only on* $|x|$*,* $|v|$ *and* $(v \cdot Ax) = x \wedge v$ *(resp.* $|x|$*). They are rotationnally invariant: for any* $\tau \in \mathbb{R}$*,*

$$U(x) = U(e^{\tau A} x) \quad \forall\, x \in \mathbb{R}^2 ,$$

(U *is radially symmetric)*

and

$$f(x,v) = f(e^{\tau A} x, e^{\tau A} v) \quad \forall\, x \in \mathbb{R}^2 .$$

f *belongs to* $L^1(\mathbb{R}^2 \times \mathbb{R}^2)$ *if and only if* G *belongs to* $L^1(\mathbb{R}^+)$ *and* $w_0 > \overline{w}$*.*

Proof :

The existence of a solution for any $r \in [0,+\infty[$ is obvious since G is bounded, and the other properties follow from the formula

$$\frac{dw}{dr} = \frac{1}{r} \int_0^r r\Big(2\delta - G\big(w(s)\big)\Big)\, ds ,$$

which implies that $\frac{dw}{dr}$ has the same sign as $w - w_0$, and that

$$0 \leq (2\delta - G(w_0))\frac{r}{2} \leq \frac{dw}{dr} \leq \delta r \quad \text{if} \quad w_0 \geq \overline{w} ,$$

and

$$0 \leq (2\delta - G^\infty)\frac{r}{2} \leq \frac{dw}{dr} \leq (2\delta - G(w_0))\frac{r}{2} \quad \text{if} \quad w_0 \leq \overline{w} \ .$$

The behaviour when $r \to +\infty$ then follows:

if $w_0 > \overline{w}$, then $\lim_{r\to+\infty} w(r) = +\infty$ and $G(w(r)) - 2\delta \sim -2\delta$.

if $w_0 < \overline{w}$, then $\lim_{r\to+\infty} w(r) = -\infty$ and $G(w(r)) - 2\delta \sim G^\infty - 2\delta$.

if $w_0 = \overline{w}$, then $w(r) \equiv w_0$.

The rest of the proof is a simple computation. □

Remarks :

1) for radially symmetric solutions, U has a logarithmic growth:

$$U\Big(t, e^{-\omega t A}x\Big) = w(|x|) - \frac{\delta}{2}|x|^2 \equiv -\frac{M}{2\pi}l, |x| \quad \text{as } |x| \to +\infty$$

since $-\nabla U\Big(t, e^{-\omega t A}x\Big) = \equiv \frac{1}{|x|^2}\int_0^{|x|} rG(w(r))\ dr$.

This property is characteristic of the radially symmetric solutions among the class of the solutions considered in this paper (see Theorem 3.7 and Remark 3.8).

2) Using the fact that the equation $-\Delta w + 2\delta = G(w)$ is obvoiusly translation invariant, *i.e.* that for any solution w and for any $x_0 \in I\!\!R^2$, $x \mapsto (w(x{+}x_0)$ is also a solution, it is therefore easy to find solutions which are not radially symmetric and therefore explicitely time-dependent. But the translation invariance is of course strongly related to the fact U_0 has been choosen to be an harmonic potential. When this is not the case, see Remark 3.8.

We will now consider 1-d solutions in the following sense: look for a function $s \mapsto w(s)$, $(s \in I\!\!R)$ solution of

$$U(t, e^{-\omega t A}x) + \frac{\delta}{2}(x_1^2 - x_2^2) = w(x_1) \quad \forall\, x = (x_1, x_2) \in I\!\!R^2 \ ,$$

with $\delta = \rho_0 - \omega^2$. U is a solution of the Poisson equation if and only if

$$-\frac{d^2 w}{ds^2} + 2\delta = G(w(s)) \ . \tag{3.1}$$

Note again that this solution is invariant under translations: if w is a solution, then $s \mapsto w(s{+}\tau)$ is also a solution for any fixed $\tau \in I\!\!R$.

Proposition 3.5 : *(Existence of 1-d solutions)*

Assume that G is a Lipshitz decreasing function, such that

$$\lim_{w\to+\infty} G(w) = 0 \quad \text{and} \quad \lim_{w\to-\infty} G(w) = G^\infty < \infty \ .$$

Let us define $\overline{w}$ *by*

$$\overline{w} = \sup\{w \in I\!R \ : \ G(w) > 2\delta\} \quad \text{if} \quad G^\infty \geq 2\delta \ ,$$

$$\overline{w} = -\infty \quad \text{if} \quad G^\infty < 2\delta \ .$$

For any $\delta > 0$*, Equation*

$$-\frac{d^2w}{ds^2} + 2\delta = G(w(s)) \tag{3.1}$$

has a unique solution for $(w(0), \frac{dw}{ds}(0)) \in I\!R^2$ *given. For any* $\delta > 0$*, any solution of Equation (3.1) satisfies one of the two following properties :*

(i) There exists $s_0 \in I\!R$ *such that* $\frac{dw}{ds}(s_0) = 0$*. If* $w_0 = w(s_0)$*, then 3 cases may occur*

$$w_0 = \min_{s \in I\!R} w(s) \quad \text{if} \quad w_0 > \overline{w} \qquad (Case\ 1.a)$$

$$w(s) = w_0 \quad \forall\, s \in I\!R \quad \text{if} \quad w_0 = \overline{w} \qquad (Case\ 1.b)$$

$$w_0 = \max_{s \in I\!R} w(s) \quad \text{if} \quad w_0 < \overline{w} \qquad (Case\ 1.c)$$

w is convex in Case 1.a and concave in case 1.c.

(ii) There exists $s_0 \in I\!R$ *such that* $w(s_0) = \overline{w}$ *and either*

$$\frac{dw}{ds} \equiv 0 \quad \text{and} \quad w(s) \equiv \overline{w} \quad \forall\, s \in I\!R \qquad (Case\ 2.a)$$

or

$$\frac{dw}{ds}(s) \neq 0 \quad \forall\, s \in I\!R \quad \text{and} \quad (\frac{dw}{ds}(s) - \frac{dw}{ds}(s_0)) \cdot (w(s) - \overline{w}) > 0 \quad \forall\, s \neq s_0 \qquad (Case\ 2.b)$$

In Case 2.b, w *is convex on* $\{s \in I\!R \ : \ w(s) > \overline{w}\}$ *and concave on* $\{s \in I\!R \ : \ w(s) < \overline{w}\}$*.*
Case (ii) occurs only if $G^\infty > 2\delta$ *or if*

$$G^\infty = 2\delta \quad \text{and} \quad G(w) = 2\delta \qquad (Case\ 3)$$

for some $w \in I\!R$*. In both cases (and when* $w \not\equiv \overline{w}$ *for case 3),* $\lim_{s\to\pm\infty} w(s) \in \{\pm\infty\}$*, and as* $s \to \pm\infty$*,*

$$\text{if} \quad w \to +\infty \ , \quad \text{then} \quad w(s) \sim \delta s^2 \ ,$$

$$\text{and if} \quad w \to -\infty \ , \quad \text{then} \quad w(s) \sim \frac{1}{2}(2\delta - G^\infty)s^2 \ .$$

In case 3, the same result occurs as $s \to +\infty$*, but as* $s \to -\infty$*,* $s \mapsto \frac{dw}{ds}(s)$ *is constant.*

If $g = -\frac{1}{\pi}G'$*, then* f *and* U *respectively defined by*

$$f(t,x,v) = g\left(\frac{1}{2}|v - \omega Ax|^2 + w(|e^{\omega tA}x|)\right) ,$$

and

$$U(t,x) = w(|e^{\omega tA}x|) \ ,$$

belong to $C^0(I\!R, L^1_{loc}(I\!R^2 \times I\!R^2)$ and $C^1(I\!R, C^2(I\!R^2))$, with $\omega = \pm\sqrt{\rho_0 - \delta}$, are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that f is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions).

This solution is explicitely time-dependant, even if w is a constant function (Cases 1.b / Case 2.a), and time-periodic of period $T = \frac{\pi}{\omega}$ in Case (i), when w is even (i.e. when $\frac{dw}{ds}(0) = 0$) and of period $T = \frac{2\pi}{\omega}$ in the other cases. $f \not\equiv 0$ never belongs to $L^1(I\!R^2 \times I\!R^2)$.

Remark 3.6 : The same results hold in Case (i) even if the condition $G^\infty < +\infty$ is not satisfied. When $G^\infty = +\infty$, it is not very difficult to extend some of the results of Proposition 3.4.

Proof of Proposition 3.5 : The formula

$$\frac{dw}{ds}(s) = \frac{dw}{ds}(s_0) + 2\delta(s - s_0) - \int_0^s G(w(\sigma))\, d\sigma \quad \forall\ (s, s_0) \in I\!R^2$$

and concavity or convexity properties are enough to prove directly the results on w. The rest of the proof is a simple computation again. □

Let us notice that the solutions corresponding to Case (i) of Proposition 3.5 are "radially symmetric" 1-d solutions, *i.e.* even, up to a translation such that $s_0 = 0$. We will now exhibit a third class of solutions, that will be intermediate between 1-d solutions and 2-d radially symmetric solutions, so that they will be time-periodic (with period $\frac{\pi}{\omega}$) and have a finite total mass. Before giving the result, let us introduce some notations (that have already been used in the introduction): assume that (x_1, x_2) are cartesian coordinates of $x \in I\!R^2$. Let us choose $\theta \in [0, 1]$ and consider a solution v of

$$-\Delta v = G\Big(v(x) + Q_\theta\Big) \quad \forall\ x = (x_1, x_2) \in I\!R^2\ , \tag{3.1}$$

with

$$Q_\theta(x) = \delta\big(\theta x_1^2 + (1 - \theta)x_2^2\big)\ .$$

Then u defined by

$$u(x) + \frac{\delta}{2}|x|^2 = u(x) + \frac{\delta}{2}(x_1^2 + x_2^2) = v(x) + Q_\theta(x)$$

is also a solution of

$$-\Delta u = G(u(x) + \frac{\delta}{2}|x|^2)$$

since

$$-\Delta Q_\theta(x) = -\delta(2\theta + 2(1 - \theta)) = -2\delta\ .$$

It immediately follows that

$$u(x) = v(x) + \frac{\delta}{2}(2\theta - 1)(x_1^2 - x_2^2) \quad \forall\ x = (x_1, x_2) \in I\!R^2\ .$$

We can notice that if $\theta = 0$ or $\theta = 1$ these solutions correspond to the $1d$-solutions, and that if $\theta = 1/2 = 1 - \theta$, then they correspond to radially symmetric $2d$-solutions.

Theorem 3.7 : *(Existence of time-periodic anisotropic solutions with finite mass)*
Assume that G is a continuous decreasing function, such that $\lim_{w\to+\infty} G(w) = 0$. Let us define $\overline{w}$ by

$$\overline{w} = \sup\{w \in I\!R \;:\; G(w) > 2\delta\} \quad \text{if} \quad G^\infty \geq 2\delta \quad \text{and} \quad \overline{w} = -\infty \quad \text{if} \quad G^\infty < 2\delta\ ,$$

and assume that G belongs to $L^q(I\!R^+)$ for some $q \in [1, 2[$ and is such that $\int_1^{+\infty} G(s) \ln s\, ds < +\infty$. For any $\delta > 0$, any C^2 solution of

$$-\Delta v = G(v + Q_\theta) \tag{3.1}$$

such that $\inf_{x\in I\!R^2}\Big(v(x) + Q_\theta(x)\Big) > \overline{w}$ and $\liminf_{|x|\to+\infty} \frac{v(x)}{|x|^2} = 0$ is also a solution of

$$T(v + Q_\theta) = v + Q_\theta\ ,$$

where $Q_\theta(x) = Q_\theta((x_1, x_2)) = \delta(\theta x_1^2 + (1-\theta)x_2^2)$ for all $x = (x_1, x_2) \in I\!R^2$, the operator $T : L^\infty_{\rm loc}(I\!R^2) \to L^\infty_{\rm loc}(I\!R^2)$ being defined by

$$Tw(x) = -\frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\, dy + Q_\theta(x) + w_0 - I(w) \quad \forall\, x \in I\!R^2\ ,$$

with $I(w) = \inf_{x\in I\!R^2}\Big(-\frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\, dy + Q_\theta(x) - q_\theta(x)\Big)$, $\min_{x\in I\!R^2}\big(w(x) - q_\theta(x)\big) = w_0$, and

$$q_\theta(x) = \frac{1}{2}(2 - \frac{G(w_0)}{\delta})Q_\theta(x) = \frac{1}{2}\big(2\delta - G(w_0)\big)(\theta x_1^2 + (1-\theta)x_2^2) \quad \forall\, x = (x_1, x_2) \in I\!R^2\ .$$

Then for any $w_0 > \overline{w}$, for any $\theta \in]0, 1[$, there exists a solution v such that

$$\inf_{x\in I\!R^2}\Big(v(x) + Q_\theta(x) - q_\theta(x)\Big) = w_0\ .$$

Let us define g by $g = -\frac{1}{\pi}G'$ and assume that it belongs to $L^1_{\rm loc}(I\!R)$. f defined by

$$f(t, x, v) = g\left(\frac{1}{2}|v - \omega Ax|^2 + w(|e^{\omega tA}x| + U_0(x) - \frac{1}{2}\omega^2|x|^2)\right)\ ,$$

and

$$U(t, x) = w(|e^{\omega tA}x|)\ ,$$

respectively belong to $C^0(I\!R, L^1_{\rm loc}(I\!R^2 \times I\!R^2)$ and $C^1(I\!R, C^2(I\!R^2))$, with $\omega = \pm\sqrt{\rho_0 - \delta}$, and are solutions of the Vlasov-Poisson system satisfying the Ehlers & Rienstra ansatz (in the sense that f is solution in the sense of the characteristics as defined by R.J. DiPerna and P.-L. Lions).

This solution is explicitely time-dependant provided $\theta \neq \frac{1}{2}$ and time-periodic of period $T = \frac{2\pi}{\omega}$ (or $T = \frac{\pi}{\omega}$). Under the above conditions, f belongs to $C^0(I\!R; L^1(I\!R^2 \times I\!R^2))$ if and only if $G_{[w_0,+\infty[}$ belongs to $L^1([w_0, +\infty[)$.

Proof :

The first part of the proposition is obvious. The last part does not present any difficulty: the time-dependance property of f is easily derived from the fact that the asymptotic form of $w = v + Q_\theta$ (as $|x| \to +\infty$) is clearly not invariant under rotations. The solution is by construction $\frac{2\pi}{\omega}$ periodic. In fact the proof below also works on the subset of the functions which are symmetric with respect to the origin, and proves that (f, U) can be choosen $\frac{\pi}{\omega}$ time-periodic. We therefore only have to prove the existence of a solution. We will do it in four steps.

1^{st} *step : more definitions and immediate consequences :*

Let us consider the space

$$X = \{w \in C^0(\mathbb{R}^2;\mathbb{R}) \ : \ \limsup_{|x|\to+\infty} \frac{w(x)}{|x|^2} < +\infty\} ,$$

with the norm

$$||w||_X = ||\frac{w(x)}{1+|x|^2}||_{L^\infty(\mathbb{R}^2)} ,$$

and the subset

$$K_{w_0,C} = \{w \in X \ : \ w(x) \sim Q_\theta(x) \quad \text{as} \quad |x| \to +\infty, \\ \inf_{x\in\mathbb{R}^2}\Big(w(x) - q_\theta(x)\Big) = w_0, \quad w(x) \le C + Q_\theta(x) \quad \forall\, x \in \mathbb{R}^2\} .$$

The operators $T_0, T : X \to X$ defined by

$$(T_0 w)(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x-y|)G(w(y))\, dy + Q_\theta(x) ,$$

and

$$(Tw)(x) = (T_0 w)(x) + w_0 - I(w) \quad \forall\, x \in \mathbb{R}^2 ,$$

with $I(w) = \inf_{x\in\mathbb{R}^2}\Big((T_0 w)(x) - q_\theta(x)\Big)$, are continous on $K_{w_0,C}$, which is a convex closed subset of X.

$(Tw)(x) \sim Q_\theta(x)$ as $|x| \to +\infty$, because $\int_{\mathbb{R}^2} \ln(|x-y|)G(w(y))\, dy = o(|x|^2)$ as $|x| \to +\infty$: $I(w)$ is therefore well defined and

$$\inf_{x\in\mathbb{R}^2}\Big((Tw)(x) - q_\theta(x)\Big) = w_0 .$$

2^{nd} *step : estimates :*

First of all, for any $w \in K_{w_0,C}$, since G is decreasing and $w_0 \le w_0 + q_\theta(y) \le w(y)$,

$$\begin{aligned} k(w_0) = -\frac{1}{2\pi}\int_{|x-y|>1} &\ln(|x-y|)G(w_0 + q_\theta(y))\, dy \\ &\le -\frac{1}{2\pi}\int_{|x-y|>1} \ln(|x-y|)G(w(y))\, dy \\ &\quad \le -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x-y|)G(w(y))\, dy \\ &\qquad \le -\frac{1}{2\pi}\int_{|x-y|<1} \ln(|x-y|)G(w(y))\, dy \le \frac{G(w_0)}{4} , \end{aligned}$$

which gives for $\frac{1}{2\pi}\ln|x| * G(w(x))$ the estimate

$$k(w_0) \leq -\frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\,dy \leq \frac{G(w_0)}{4}\,.$$

Moreover

$$\begin{aligned} I(w) &= \inf_{x\in I\!R^2}\left(-\frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\,dy + Q_\theta(x) - q_\theta(x)\right) \\ &\geq \inf_{x\in I\!R^2}\left(-\frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\,dy\right) \geq k(w_0)\,, \end{aligned}$$

and

$$Tw - Q_\theta \leq w_0 - I(w) - \frac{1}{2\pi}\int_{I\!R^2} \ln(|x-y|)G(w(y))\,dy \leq w_0 - k(w_0) + \frac{G(w_0)}{4} = C(w_0)\,.$$

For any $C \geq C(w_0)$, $K_{w_0,C}$ is non-empty and stable under the action of T. In the following, we shall consider $K = K_{w_0,C(w_0)}$ and get a fixed point result on $T(K) \subset K$ using Schauder's theorem.

3^{rd} *step : a compactness result :*

According to its definition, K is bounded for the norm $||\cdot||_X$. For the moment, let us forget the problem of the non-compactness of $I\!R^2$ and try to apply Ascoli's lemma. We have to prove that $T(K)$ is uniformly equicontinous. As in [Dr1] or in [BoD], it is a consequence of the inequality

$$\begin{aligned} ||\nabla Tw||_{L^\infty(I\!R^2)} &\leq \frac{1}{2\pi}\int_{|x-y|\leq 1} \frac{G(w_0+q_\theta(y))}{|x-y|}\,dy + \frac{1}{2\pi}\int_{|x-y|>1} \frac{G(w_0+q_\theta(y))}{|x-y|}\,dy \\ &\leq G(w_0) + \frac{1}{2\pi}\int_{|x-y|>1} \frac{G(w_0+q_\theta(y))}{|x-y|}\,dy \end{aligned}$$

$$||\nabla Tw||_{L^\infty(I\!R^2)} \leq G(w_0) + c\int_0^{+\infty} G(w_0+s)\,ds$$

$\big($with $c = 2(\delta - G(w_0))\cdot\min(\theta,(1-\theta))\big)$ if $q=1$, and

$$||\nabla Tw||_{L^\infty(I\!R^2)} \leq G(w_0) + \frac{c^{1/q}}{2\pi(p-2)^{1/p}}\left(\int_0^{+\infty} |G(w_0+s)|^q\,ds\right)^{1/q}$$

with $\frac{1}{p}+\frac{1}{q}=1$ if $q\in]1,2[$.

Applying Ascoli's lemma, $T(K)$ would be relatively compact if the functions of K had been defined on a compact set.

4^{th} *step : conclusion :*

Since the set K is a set of functions defined on $I\!R^2$, one has to be careful. Let R be a strictly positive real number.

1) The operator $T^R : C^0(B(0,R)) \to C^0(I\!R^2; I\!R)$ defined by

$$(T^R w)(x) = (T_0^R w)(x) + w_0 - I^R(w) \quad \forall\, x\in B(0,R)\,,$$

$$(T_0^R w)(x) = -\frac{1}{2\pi}\int_{B(0,R)} \ln(|x-y|)G(w(y))\,dy + Q_\theta(x) \quad \forall\, x \in B(0,R)\ ,$$

with $I^R(w) = \inf_{x\in B(0,R)}\Big((T_0 w)(x) - q_\theta(x)\Big)$, is continous on the set K^R defined by

$$K^R = \{w_{|B(0,R)}\ :\ w \in K\}\ ,$$

and applying Ascoli's lemma, one proves that its restriction $\tilde{T}^R$ defined by

$$\tilde{T}^R w = (Tw)_{|B(0,R)}$$

is such that $T^R(K^R)$ is precompact : Schauder's theorem then implies that T^R has a fixed-point on K^R in the following sense: there exists a function w^R of K such that

$$(T^R w^R)(x) = w^R(x) \quad \forall\, x \in B(0,R)\ .$$

2) We just have to compute the difference between w^R and Tw^R:

$$||w^R - Tw^R||_X \le ||w^R - Tw^R||_{L^\infty(B(0,R))} + ||(w^R)_{|B^c(0,R)} - (Tw^R)_{|B^c(0,R)}||_X\ .$$

But on one side

$$||(w^R)_{|B^c(0,R)} - (Tw^R)_{|B^c(0,R)}||_X = O(\frac{\ln R}{1+R^2})$$

uniformly in $w^R \in K^R$ as $R \to +\infty$, because

$$(w^R)_{|B^c(0,R)}(x) - (Tw^R)_{|B^c(0,R)}(x) = O(\frac{\ln|x|}{1+|x|^2})$$

uniformly in $w^R \in K^R$ as $|x| \to +\infty$, and on the other side

$$\begin{aligned} ||w^R - Tw^R||_{L^\infty(B(0,R))} &= ||T^R w^R - Tw^R||_{L^\infty(B(0,R))} \\ &\le \frac{1}{2\pi}\sup_{x\in B(0,R)}\int_{|y|>R} |\ln(|x-y|)|G(w_0 + q_\theta(y))\,dy \\ &\le \frac{1}{2\pi}\int_{|y|>R} |\ln(|y|+R)|G(w_0+q_\theta(y))\,dy \to 0 \end{aligned}$$

as $R \to +\infty$. Because of the uniform estimate on $||\nabla Tw||_{L^\infty(I\!\!R^2)}$ for all $w \in K$, up to the extraction of a subsequence $(R_n)_{n\in I\!\!N}$ such that

$$\lim_{n\to+\infty} R_n = +\infty\ ,$$

there exists a function $w \in K$ such that

$$w^{R_n} \to w \quad \text{in } X \quad \text{as } n \to +\infty\ ,$$

and using Lebesgue's theorem of dominated convergence,

$$Tw^{R_n} \to Tw \quad \text{in } X \quad \text{as } n \to +\infty\ .$$

Passing then to the limit, w is such that

$$||w - Tw||_X = 0 ,$$

which gives the result. □

Remark 3.8 :
1) The solutions of Theorem 3.7 may have a finite total energy

$$\frac{1}{2} \int\int_{I\!R^2 \times I\!R^2} (|v|^2 + U(t,x) + 2U_0(x)) f(t,x,v) \, dxdv$$

provided $s \mapsto g(s)$ is sufficiently decreasing as $s \to +\infty$. 2)
Last part of the proof shows that $T(K)$ is precompact. An alternative method would be to apply directly Schauder's fixed-point theorem to T and K.
3) The question of the uniqueness seems to be more technical than difficult. For example, assuming that g is decreasing and such that

$$\sup_{x \in I\!R^2} \frac{1}{1+|x|^2} \int_{I\!R^2} \big(1+|y|^2\big) \, \big|\big(\ln|x-y|\big)\big| \, g\big(w_0 + q_\theta(y)\big) \, dy < 2\pi ,$$

the operator T is then contracting on the set K. Of course, this only proves the uniqueness of the solution of the fixed point equation since the asymptotic boudary condition is not sufficient to determine the location of the minimum. It is for example clear that the solution of the following fixed point equation

$$T(v + \tilde{Q}_\theta) = v + \tilde{Q}_\theta \ , \quad \limsup_{|x| \to +\infty} \frac{v(x)}{|x|^2} = 0 ,$$

where $\tilde{Q}_\theta(x) = Q_\theta(x + x_0)$ for some $x_0 \in I\!R^2$ provides a solution to

$$-\Delta \tilde{v} = G(\tilde{v} + \delta \frac{\delta}{2}|x|^2)$$

with $\tilde{v}(x) = v(x + x_0)$ and v solution to $-\Delta v = G(v + Q_\theta)$ whivh gives a solution to $-\Delta u = G(u + \frac{\delta}{2}|x|^2)$ satisfying the asymptotic boundary condition provided $u(x) + \frac{\delta}{2}|x|^2 = \tilde{v}(x) + \tilde{Q}_\theta(x)$.
4) Using a developpement of $w(x)$ for $|x| \to +\infty$, and the techniques developped in [GNN] and adapted in [BoD], it is probably not very difficult to prove that up to a translation, the solution w with the anisotropic asymptotic boundary condition is, up to a translation, symmetric with respect to the origin and therefore periodic of period $\frac{\pi}{\omega}$.
5) Instead of $q_\theta(x) = \frac{1}{2}(2\delta - G(w_0))(\theta x_1^2 + (1-\theta)x_2^2)$, we may prove the theorem with

$$q_\theta(x) = q_\theta^\epsilon(x) = \epsilon(\theta x_1^2 + (1-\theta)x_2^2) ,$$

for any $\epsilon \in]0, G(w_0)[$, $w_0 > \overline{w}$. This allows to give an existence result of a solution w such that $W_0 > \inf_{x\in I\!R^2} w(x) > \overline{w}$. There exists indeed a solution w^ϵ for which $w_0 - \epsilon = \inf_{x\in I\!R^2}\Big(w(x) + q^\epsilon_\theta(w)\Big)$, for any $\epsilon > 0$ small enough. But the passage to the limit $\epsilon \to 0+$ is not clear since

$$k^\epsilon(w_0) = -\frac{1}{2\pi}\int_{|x-y|>1} \ln|x-y| G(w_0 + q^\epsilon_\theta(y))\, dy$$

is obviously not bounded from below. For the same reason, we cannot apply directly Lebesgue's theorem of dominated convergence.

Note that the meaning of w_0 in Theorem 3.7 is the same as in Propositions 3.4 and 3.5 if $\epsilon \in [\frac{1}{2}(G(w_0) + 2\delta), G(w_0)[$. The proof is easy. If we denote by w^ϵ

$$w^\epsilon = w - q^\epsilon_\theta\ ,$$

then $-\Delta w^\epsilon = G(w) - 2\delta + 2\epsilon$: $-\Delta w^\epsilon \geq 0$ if $G(w_0) - 2\delta + 2\epsilon \geq 0$ (this implies indeed $G(w(x)) - 2\delta + 2\epsilon \geq 0$ for any $x \in I\!R^2$). Thus, if $\epsilon \in [\delta - \frac{G(w_0)}{2}, \delta[$, then $w_0 = \inf_{x\in I\!R^2} w^\epsilon(x) = \inf_{x\in I\!R^2} w(x)$ for any solution w in the sense of Proposition 3.4 or 3.5.

6) The method also applies when the confining otential is not radially symmetric (but has an asymptotic quadratic growth). For instance, we may consider the case where

$$U_0(x) = \frac{\rho_0}{2}|x|^2 + \lambda v_0(x)$$

where V_0 is a continuous compactly supported radially symmetric perturbation. When the solution (obtained by the fixed pont method) is unique, this allows to prove the existence of a branch of solutions (parametrized by λ - this can be seen as an implicit function therorem) and gives an existence result when U_0 is not an harmonic potential.

It is worth to notice that this provides a branch of solutions starting from the radial solutions (plus a translation) which may have finite mass, a logarithmic growth, and are generically time-periodic non stationary solutions.

5. Conclusion and some open questions

First, let us summarize the results obtained in part II.

The Jeans' Theorem given in Proposition C.2 proves that generically any time-periodic solution such that its average is radially symmetric in fact satisfies the weak Ehlers & Rienstra ansatz, provided it does not explicitely depends on time.

But according to Theorem 2.1, any solution satisfying the weak Ehlers & Rienstra ansatz is time-periodic and obeys to the usual (strong) Ehlers & Rienstra ansatz. The dependance in time is obtained through a rotation with a constant angular velocity. Solving the Vlasov-Poisson system is completely equivalent to solving a nonlinear Poisson equation (Proposition 3.1), whose nonlinearity can be choosen in an arbitrary way.

Whatever the nonlinearity is, if the spatial density belongs to L^1, the angular velocity is bounded by a quantity depending on the confining potential. The asymptotic behaviour of the potential at infinity is such that the level curves are (asymptotically) ellipses, provided the potential is at most quadratic at infinity, which is a reasonable assumption (Theorem 3.3). It is also proved that the potential cannot be constant at infinity.

Special solutions have then been constructed (Proposition 3.4: radially symmetric solutions, and Proposition 3.5: $1d$-solutions) using the ideas of [BBDP]. The most general class of solutions compatible with the natural asymptotic boundary conditions is studied in Theorem 3.7: these solutions are nonisotropic solutions and therefore naturally time-periodic; they may have a finite mass and are explicitely time-dependant, which was not the case for the radially symmetric solutions.

In view of simplifying the computations, the confining potential was supposed to be harmonic. This very special form allows to introduce an other method for giving time-dependant solutions: one can indeed use the invariance by translation of the equation for w (from which the potential is deduced) to get nonsymmetric solutions that will therefore be explicitely time-dependant, but this is a very special property of the harmonic potential. However this allows to build a branch of non trivial solutions by perturbing the harmonic potential. On the other side, the nonisotropic solutions given in Theorem 3.7 are independant of the local form of the confining potential provided its asymptotic behaviour is quadratic.

A complete study of general confining potentials has still to be done. The question of the uniqueness of the solutions (up to a translation, and up to the addition of a constant to the potential) and of their stability would be of the greatest interest. This last question may not be out of reach in view of the characterization that has been given for the time-periodic solutions, but probably implies first the removal of the technical assumptions used in this paper, and a new approach for the time-dependant Vlasov-Poisson system. It is also probably possible to prove symmetry results, as mentioned in Remark 3.8 (this is related to the questions on the uniqueness).

As a concluding remark, let us note that the Ehlers & Rienstra ansatz and the nonlinear Poisson equation have been (from a mathematical point of view) used in many mathematical papers for the study of the $3d$ Vlasov-Poisson gravitational system and that the results given in the paper may easily be adapted to this case.

Appendix A : Formal derivation of a simple $2d$ model for monokinetic charged particle beams

Consider a solution F^λ of the Vlasov-Poisson equation in $\mathbb{R}^3$

$$\partial_t F^\lambda + V\cdot\partial_X F^\lambda - \partial_x U_0^\lambda(x)\cdot\partial_v F^\lambda + \frac{1}{4\pi}\int\int_{\mathbb{R}^3\times\mathbb{R}^3}\frac{X-X'}{|X-X'|^3}\,F^\lambda(t,X',V')\,dX'dV'\cdot\partial_V F^\lambda = 0\ ,$$

with the notations $X=(x,z)\in\mathbb{R}^2\times\mathbb{R}$ and $V=(v,w)\in\mathbb{R}^2\times\mathbb{R}$, and assume that F^λ obeys to the scaling

$$F^\lambda(t,X,V) = \lambda^{\alpha-2}\cdot G^\lambda\Big(t;(\frac{1}{\lambda}x,\lambda z);(\frac{1}{\lambda}v;\lambda^{\alpha-2}(w-w_0))\Big)$$

with $\alpha\in]0,2[$ and $w_0\in\mathbb{R}$. We look for the limit $\lambda\to+\infty$, which corresponds to a beam with a symmetry under translations in the $z-$direction, monokinetic with a velocity w_0 in that direction. The limit, if it exists, has then an infinite mass since $||F^\lambda||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} = \lambda||G^\lambda||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}$. G^λ is solution of

$$\begin{aligned}&\partial_t G^\lambda + v\cdot\partial_x G^\lambda + \lambda^{\alpha-1}w\cdot\partial_z G^\lambda - \partial_x U_0^\lambda(\lambda x)\cdot\partial_v G^\lambda\\ &\quad+\frac{1}{4\pi}\int\int_{(\mathbb{R}^2\times\mathbb{R})\times(\mathbb{R}^2\times\mathbb{R})}\Bigg(\frac{\lambda(x-x')}{\big(\lambda^2|x-x'|^2+\frac{(z-z')^2}{\lambda^2}\big)^{3/2}}\cdot\partial_v G^\lambda\\ &\qquad+\frac{\frac{(z-z')^2}{\lambda^2}}{\big(\lambda^2|x-x'|^2+\frac{(z-z')^2}{\lambda^2}\big)^{3/2}}\cdot\lambda^{2-\alpha}\partial_w G^\lambda\Bigg)\,G^\lambda(t,X',V')\,dx'\frac{dz'}{\lambda}\,dv'dw' = 0\ .\end{aligned}$$

Assume now that each term in this equation remains bounded, that there exists a potential U_0 such that

$$\partial_x U_0(\lambda x) = \lim_{\lambda\to+\infty}\partial_x U_0^\lambda(x)\quad\forall\,x\in\mathbb{R}^2\ ,$$

and consider

$$g(t,x,v,w) = \lim_{\lambda\to+\infty}\frac{1}{2\lambda}\int_{-\lambda}^{+\lambda}G^\lambda(t;(x,z);(v;w))\,dz\ .$$

It is not very difficult to check that – provided it is well defined – g is then solution of

$$\partial_t g + v\cdot\partial_x g + \frac{1}{2\pi}\int\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{x-x'}{|x-x'|^2}\big(\int_{\mathbb{R}} g(t,x',v',w')\,dw'\big)\cdot\partial_v g = 0\ ,$$

since

$$\int_{-\infty}^{+\infty}\frac{\lambda(x-x')}{\Big(\lambda^2|x-x'|^2+\frac{Z}{\lambda^2}\Big)^{3/2}}\,dZ \quad=\quad 2\,\frac{x-x'}{|x-x'|^2}\ .$$

Here w represents the asymptotic dispersion (of order $\frac{1}{\lambda^{2-\alpha}}$) of the velocities around w_0 of the original distribution function, which justifies the word "monokinetic". w is a conserved quantity of the microscopic dynamics and the averaged density function

$$f(t,x,v) = \int_{\mathbb{R}} g(t,x,v,w)\,dw$$

is a solution of the $2d$ Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \partial_x f - (\partial_x U(t,x) + \partial_x U_0(x)) \cdot \partial_v f = 0 \\ -\Delta U = \rho(t,x) = \int_{\mathbb{R}^2} f(t,x,v)\, dv \end{cases}$$

It makes sense to ask that f belongs to $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$, which means that one looks for beams with a finite mass per unit length along the axis, but the behavior in x modelizes the local distribution function and there is therefore no *a priori* natural boundary condition for U. For the same reason, it is also assumed that the exact form of U_0 is not important and that one can take an harmonic potential for modelizing it near its bottom.

Appendix B : two interpolation lemmas

In the two following lemmas, the relations between the norms and the exponents are easily recovered using scalings in x and v. The first lemma can be found for instance in [LP1-2]. The second one is a generalization of the first lemma to moments higher than one. These lemmas are related to the estimates used by Perthame [Pe2] or Illner & Rein [IR] for the question of the dispersion in dimension three (see concluding remark). Detailed proofs are given here, including the explicit form of the constants which appear in the inequalities.

Lemma B.1: *Let f be a nonnegative function belonging to $L^p(\mathbb{R}^N \times \mathbb{R}^N)$ for some $p \in]1,+\infty]$ such that*

$$x \mapsto \int_{\mathbb{R}^N \times \mathbb{R}^N} f(x,v)\, |v|^k\, dv$$

belongs to $L^r(\mathbb{R}^N)$ for some $(r,k) \in [1,+\infty[\times]0,+\infty[$. Then the function

$$x \mapsto \rho(x) = \int_{\mathbb{R}^N} f(x,v)\, dv$$

belongs to $L^q(\mathbb{R}^N)$ with

$$q = r \cdot \frac{N(p-1)+kp}{N(p-1)+kr}$$

and satisfies

$$||\rho||_{L^q(\mathbb{R}^N)} \leq C(N,p,k) \cdot ||f||^{\alpha}_{L^p(\mathbb{R}^N \times \mathbb{R}^N)} \cdot || \int_{\mathbb{R}^N} f(x,v)\, |v|^k\, dv||^{1-\alpha}_{L^r(\mathbb{R}^N)} ,$$

with

$$\alpha = \frac{kp}{N(p-1)+kp} \quad \textit{and} \quad 1-\alpha = \frac{N(p-1)}{N(p-1)+kp} ,$$

and

$$C(n,p,k) = \Big(|S^{N-1}|\Big)^{\frac{k(p-1)}{N(p-1)+kp}} \cdot \left(\frac{\left(\frac{kp}{p-1}\right)^{\frac{N(p-1)}{N(p-1)+kp}}}{N^{\frac{(p-1)(N(p-2)+kp)}{p(N(p-1)+kp)}}} + \frac{N^{\frac{k}{N(p-1)+kp}}}{\left(\frac{kp}{p-1}\right)^{\frac{kp}{N(p-1)+kp}}} \right) .$$

When r varies between 1 and $+\infty$, q varies between $1 + \frac{k(p-1)}{N(p-1)+k}$ and $p + \frac{N(p-1)}{k}$.

Proof : Assume to simplify that $p < +\infty$. Let ρ be defined by

$$\rho(x) = \int_{\mathbb{R}^N} f(x,v)\, dv \quad \forall\, x \in \mathbb{R}^N .$$

We can split the integral defining ρ into two integrals and evaluate these integrals in different ways

$$\rho(x) = \int_{|v|<R} f(x,v)\, dv + \int_{|v|\geq R} f(x,v)\, dv ,$$

$$\int_{|v|<R} f(x,v)\, dv \leq \left(\frac{1}{N}|S^{N-1}|R^N\right)^{1-1/p} \cdot \left(\int_{\mathbb{R}^N} |f(x,v)|^p\, dv\right)^{1/p} ,$$

$$\int_{|v|\geq R} f(x,v)\, dv \leq \frac{1}{R^k} \int_{R^N} f(x,v)\, |v|^k\, dv\ .$$

If we optimize on R, then we get

$$\rho(x) \leq C(N,p,k) \cdot \left(\int_{\mathbb{R}^N} |f(x,v)|^p\, dv \right)^{\frac{k}{N(p-1)+kp}} \cdot \left(\int_{R^N} f(x,v)\, |v|^k\, dv \right)^{\frac{N(p-1)}{N(p-1)+kp}} ,$$

with

$$C(n,p,k) = \left(|S^{N-1}| \right)^{\frac{k(p-1)}{N(p-1)+kp}} \cdot \left(\frac{\left(\frac{kp}{p-1}\right)^{\frac{N(p-1)}{N(p-1)+kp}}}{N^{\frac{(p-1)(N(p-2)+kp)}{p(N(p-1)+kp)}}} + \frac{N^{\frac{k}{N(p-1)+kp}}}{\left(\frac{kp}{p-1}\right)^{\frac{kp}{N(p-1)+kp}}} \right) .$$

The L^q-norm of ρ is now bounded by

$$||\rho||_{L^q(\mathbb{R}^N)} \leq C(N,p,k) \cdot \left[\int_{R^N} \int_{R^N} |f(x,v)|^p\, dv \right)^{\frac{kq}{N(p-1)+kp}} \left(\int_{R^N} f(x,v)\, |v|^k\, dv \right)^{\frac{N(p-1)q}{N(p-1)+kp}} dx \Bigg]^{1/q} ,$$

and using Hölder's inequality, we obtain

$$||\rho||_{L^q(\mathbb{R}^N)} \leq C(N,p,k) \cdot \left[\int_{R^N} \left(\int_{R^N} |f(x,v)|^p\, dv \right)^{\frac{kqs}{N(p-1)+kp}} dx \right]^{1/qs} \cdot$$

$$\left[\left(\int_{R^N} f(x,v)\, |v|^k\, dv \right)^{\frac{N(p-1)qt}{N(p-1)+kp}} dx \right]^{1/qt} ,$$

with

$$\frac{1}{s} + \frac{1}{t} = 1\ .$$

If we assume that

$$\frac{kqs}{N(p-1)+kp} = 1$$

$$\frac{N(p-1)qt}{N(p-1)+kp} = r\ ,$$

then

$$\frac{1}{qs} = \frac{k}{N(p-1)+kp}$$

and

$$\frac{1}{qt} = \frac{N(p-1)}{N(p-1)+kp} \cdot \frac{1}{r}$$

with

$$q = r \cdot \frac{N(p-1)+kp}{N(p-1)+kr}\ ,$$

which proves the lemma:

$$||\rho||_{L^q(\mathbb{R}^N)} \leq C(N,p,k) \left[\int_{R^N} \int_{R^N} |f(x,v)|^p\, dvdx \right]^{\frac{kp}{N(p-1)+kp} \cdot \frac{1}{p}} \cdot \left[\left(\int_{R^N} f(x,v)\, |v|^k\, dv \right)^r dx \right]^{\frac{N(p-1)}{N(p-1)+kp} \cdot \frac{1}{r}}$$

for $p < +\infty$. The same proof holds for $p = +\infty$. □

Lemma B.2: *Let f be a nonnegative function belonging to $L^p(I\!R^N \times I\!R^N)$ for some $p \in]1, +\infty]$ such that*

$$x \mapsto \int_{I\!R^N \times I\!R^N} f(x,v)\ |v|^k\ dv$$

belongs to $L^r(I\!R^N)$ for some $(r,k) \in [1,+\infty[\times]0,+\infty[$. Let $m \in [0,k]$ and assume that

$$m < \frac{p-1}{p-r} \cdot \Big(N(r-1) + kr \Big)$$

if $r < p$. Then the function

$$x \mapsto \int_{I\!R^N} f(x,v)|v|^m\ dv$$

belongs to $L^u(I\!R^N)$ with

$$u = r \cdot \frac{N(p-1)+kp}{N(p-1)+m(p-r)+kr}$$

$$\frac{1}{u} = \frac{k-m}{k} \cdot \frac{1}{r} + \frac{m}{k} \cdot \frac{1}{r} \frac{N(p-1)+kr}{N(p-1)+kp}$$

and satisfies

$$|| \int_{I\!R^N} f(x,v)\ |v|^m\ dv ||_{L^u(I\!R^N)} \leq K(N,p,k,m) \cdot ||f||^{\beta}_{L^p(I\!R^N \times I\!R^N)} \cdot || \int_{I\!R^N} f(x,v)\ |v|^k\ dv ||^{1-\beta}_{L^r(I\!R^N)}\ .$$

with

$$\beta = \frac{(k-m)p}{N(p-1)+kp} \quad \textit{and} \quad 1-\beta = \frac{N(p-1)+mp}{N(p-1)+kp}\ ,$$

and

$$K(n,p,k,m) = \Big(|S^{N-1}| \Big)^{\frac{(k-m)(p-1)}{N(p-1)+kp}} \cdot \left(\frac{\left(\frac{kp}{p-1}\right)^{\frac{N(p-1)}{N(p-1)+kp}}}{N^{\frac{(p-1)(N(p-2)+kp)}{p(N(p-1)+kp)}}} + \frac{N^{\frac{k}{N(p-1)+kp}}}{\left(\frac{kp}{p-1}\right)^{\frac{kp}{N(p-1)+kp}}} \right)^{(k-m)/k}\ .$$

Proof : As in lemma B.1 we assume to simplify that $p < +\infty$.

1^{st} *step :* We prove that

$$|| \int_{I\!R^N} f(x,v)\ |v|^m\ dv ||_{L^u(I\!R^N)} \leq ||\rho||^{(k-m)/k}_{L^q(I\!R^N)} \cdot || \int_{I\!R^N} f(x,v)\ |v|^k\ dv ||^{m/k}_{L^r(I\!R^N)}\ .$$

$$\int_{I\!R^N} f(x,v)\ |v|^m\ dv = \int_{I\!R^N} \big(f(x,v)\big)^{(k-m)/k} \cdot \big(f(x,v)|v|^m\big)^{m/k}\ dv\ .$$

Applying Hölder's inequality, we first get

$$\int_{I\!R^N} f(x,v)\ |v|^m\ dv \leq \left(\int_{I\!R^N} f(x,v)\ dv \right)^{(k-m)/k} \cdot \left(\int_{I\!R^N} f(x,v)\ |v|^k\ dv \right)^{m/k}\ .$$

With the notation $\rho(x) = \int_{I\!R^N} f(x,v)\ dv$,

$$|| \int_{I\!R^N} f(x,v)\ |v|^m\ dv ||^u_{L^u(I\!R^N)} \leq \int_{I\!R^N} \rho(x)^{u\cdot(k-m)/k} \cdot \left(\int_{I\!R^N} f(x,v)\ |v|^k\ dv \right)^{u\cdot m/k}\ dx\ .$$

Applying again Hölder's inequality, we get

$$||\int_{\mathbb{R}^N} f(x,v)\ |v|^m\ dv||^u_{L^u(\mathbb{R}^N)} \leq ||\rho||^{(k-m)/k}_{L^q(\mathbb{R}^N)} \cdot ||\int_{\mathbb{R}^N} f(x,v)\ |v|^k\ dv||^{m/k}_{L^r(\mathbb{R}^N)} ,$$

with

$$q = \frac{(k-m)ur}{kr-um} ,$$

$$\frac{1}{u} = \frac{m}{k} \cdot \frac{1}{r} + \frac{k-m}{k} \cdot \frac{N(p-1)+kp}{N(p-1)+m(p-r)+kr} \cdot \frac{1}{r} .$$

2^{nd} *step :* Applying now Lemma B.1, we get

$$||\rho||^{(k-m)/k}_{L^q(\mathbb{R}^N)} \leq C(N,p,k)^{(k-m)/k} ||f||^{\frac{(k-m)p}{N(p-1)+kp}}_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \cdot ||\int_{\mathbb{R}^N} f(x,v)\ |v|^k\ dv||^{\frac{k-m}{k}\cdot\frac{N(p-1)}{N(p-1)+kp}}_{L^r(\mathbb{R}^N)} ,$$

which gives

$$||\int_{\mathbb{R}^N} f(x,v)\ |v|^m\ dv||^u_{L^u(\mathbb{R}^N)} \leq C(N,p,k)^{(k-m)/k} \cdot ||f||^{\frac{(k-m)p}{N(p-1)+kp}}_{L^p(\mathbb{R}^N\times\mathbb{R}^N)} \cdot ||\int_{\mathbb{R}^N} f(x,v)\ |v|^k\ dv||^{\frac{N(p-1)+mp}{N(p-1)+kp}}_{L^r(\mathbb{R}^N)} .$$

□

Remark : Of course, it is possible to use the method of Lemma B.1 to establish directly the result of Lemma B.2 (but it is more complicated and it does not improve the constants).

The interpolation method can also be used for any moment in $|x-vt|$ (as in [Pe2], [IR] for $|x-vt|^2$). The integration with respect to v of f on $\{|x-vt|<R\}$ instead of the integration of f on $\{|v|<R\}$ provides an explicit dependance in t for the optimal R, which gives a decay for the interpolated quantity. The result is (formally) easily recovered by the change of variables $v \mapsto (x-vt)$).

Appendix C : Jeans' theorem for stationary and time-periodic solutions

Consider first the case of the stationary Vlasov-Poisson system,

$$v \cdot \partial_x f - (\partial_x U(x) + \partial_x U_0(x)) \cdot \partial_v f = 0 \tag{sV}$$

$$-\Delta U = \rho(x) = \int_{I\!R^2} f(x,v)\, dv \ , \tag{P}$$

and assume that (f, U) does not depend on t and that U (and U_0) is radially symmetric, *i.e.* that there exists a function $u : I\!R^+ \to I\!R$ such that

$$U(x) = u(|x|) \quad \forall\, x \in I\!R^2 \ . \tag{S1}$$

Proposition C.1 : Jeans' theorem for stationary solutions

(Weak Formulation) Any $C^1 \cap L^1$ stationary solution (f, U) of (VP) satisfying (S1) is such that $(\overline{f}, U)$ defined by $\overline{f}(x,v) = \int_{\nu \in S^1} f(|x| \cdot \nu, v)\, d\nu$ for all $(x,v) \in I\!R^2 \times I\!R^2$ is also a solution of (VP), $\overline{f}$ satisfies locally the weak Ehlers & Rienstra ansatz: for any $(x_0, v_0) \in I\!R^2 \times I\!R^2$, there exists a neighbourhood $\mathcal{V}$ of (x_0, v_0) and a function $h : I\!R^2 \to I\!R^+$ such that

$$f(x,v) = h(E(x,v), F(x,v)) \quad \forall\, (x,v) \in \mathcal{V} \ , \tag{weak ER}$$

and, if $\mathcal{V} = I\!R^2 \times I\!R^2$, with H defined as in section II.1.1 by $H(U,x) = \int_0^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2\ h(\frac{s_1^2+s_2^2}{2} + U, s_2|x|)$, then

$$-\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) = H(u(r), r) \ .$$

(Strong Formulation) Assume moreover that the following non-resonance condition is satisfied

$$\text{meas}\{(x,v) \in \text{supp}(f) \ : \ \frac{1}{2\pi}\Theta(E(x,v), F(x,v)) \in \mathbb{Q}\} = 0 \tag{NR}$$

with

$$\Theta(E,F) = 2\int_{r_1(E,F)}^{r_2(E,F)} \frac{F}{r^2} \frac{dr}{\sqrt{2E - 2u(r) - 2u_0(r) - \frac{F^2}{r^2}}} \ ,$$

where $[r_1(E,F), r_2(E,F)]$ is the maximal interval such that

$$\forall\, r \in [r_1(E,F), r_2(E,F)], \quad 2E - 2u(r) - 2u_0(r) - \tfrac{F^2}{r^2} \le 0.$$

Then for $(x_0, v_0) \in \text{supp}(f)$ a.e., the support of the trajectory of $(x(t))_{t \in I\!R^+}$ defined by

$$\frac{d^2x}{dt^2} = -\nabla_x (U(x) + U_0(x)) \ , \quad x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = v_0$$

is dense (ergodicity property) in the annulus $C(r_1, r_2)$ with $r_i = r_i(E(x_0,v_0), F(x_0,v_0))$, $i = 1, 2$. As a consequence,

$$f(x,v) = \overline{f}(x,v) \qquad \forall\, (x,v) \in \mathcal{V} \ ,$$

for a neighbourhood $\mathcal{V}$ in $I\!R^2 \times I\!R^2$ of the orbit $t \mapsto ((x(t), \frac{dx}{dt}(t))$.
(Global result) If $r \mapsto r^3(\frac{du}{dr} + \frac{du_0}{dr})$ is monotone increasing, then the above factorization result is global:

$$f(x,v) = h(E(x,v), F(x,v)) \quad \forall\ (x,v) \in \mathcal{V} = I\!R^2 \times I\!R^2\ , \qquad (weak\ ER)$$

and f satisfies the global ($\mathcal{V} = I\!R^2 \times I\!R^2$) weak Ehlers & Rienstra ansatz.

We have to notice that $(\overline{f}, U)$ is a solution of the stationary Vlasov-Poisson system (with the same spatial density) as soon as (f, U) is a solution such that U is radially symmetric.

For the proof of this proposition and for more realistic conditions on U than the (NR) condition, one has to refer to [Do1,2]. The weak formulation of Jeans' theorem was given in [BFH] (for the special class of distribution functions such that $f = \overline{f}$ in the three-dimensional Vlasov-Poisson case – in that case, the result is automatically global). The difficult part of this theory is to check whether there are resonances or not. A classical result (see for example [Arn]) says that if bounded orbits exist and are all closed, then $U + U_0$ is either keplerian or harmonic, which gives an explicit condition for resonances to appear. We can for example state the following result (see [Do1-2] for a proof) :

Proposition : Non Resonance Condition *Assume that U is radially symmetric (i.e. satisfies condition (S1)), that u is at least of class C^4 on $]0, +\infty[$ and that $(E, F) \mapsto \Theta(E, F)$ is analytic. If $U_0(x) = \frac{\rho_0}{2}|x|^2$ and*

$$\text{meas}\{x \in \text{supp}(f)\ :\ \nabla_x(\int_{I\!R^2} f(x,v)\ dv = 0\} = 0\ ,$$

then the non resonance condition (NR) is satisfied.

We now state an equivalent (in the strong formulation) result for time-periodic solutions .

Proposition C.2 : Jeans' theorem for time-periodic solutions
Assume now that (f, U) is a $C^1(I\!R; L^1(I\!R^2 \times I\!R^2)) \times C^1(I\!R \times I\!R^2)$ time-periodic explicitely time-dependant solution of (VP) of period T satisfying the following symmetry assumption

$$\forall\ x \in \bigcup_{t\in[0,T]} \text{supp}(f(t,.,.))\ , \quad x \mapsto \int_0^T U(t,x)\ dt \text{ is radially symmetric.} \qquad (S2)$$

With the same notations as in Proposition C.1, if U satisfies the following ergodicity property,

$$\left(x(t+nT), \text{sgn}(\frac{dx}{dt}(t+nT))\right)_{n\in I\!N} \quad \text{is dense in } C(\tilde{r}_1(t), \tilde{r}_2(t)) \times \{\pm 1\}\ ,$$

with $\tilde{r}_i = r_i(t, E(x_0, v_0), F(x_0, v_0))$, $i = 1, 2$, then for almost all $(t, x, v) \in [0, T] \times \text{supp}(f)$ f satisfies the factorization identity :

$$f(t,x,v) = g(t, E(t,x,v), F(x,v))\ .$$

The factorization identity is not exactly the same as in the weak Ehlers & Rienstra ansatz. In part II, the study has been restricted to the subclass of g which do not depend on t.

Remark : The method applies in the same way when one assumes that

$$U_0(x) \sim \rho_0 \cdot (\theta_0 x_1^2 + (1-\theta_0) x_2^2) \quad \text{as} \quad |x| \to +\infty\ ,$$

for some $\theta_0 \in]0,1[$ (but one has then to make some essentially technical modifications on the assumptions on g: see [BFH], [Do1] for similar problems in dimension 3). It also applies to the more realistic but also more technical case corresponding to the situation when θ_0 depends on t (and is time-periodic of period $T = \frac{2\pi}{\omega}$). The set of time-periodic solutions we have found can be extended to the case when the angular velocity is equal to zero (infinite period) and provides a new class of stationary solutions which seems not to be known up to now. These solutions are such that the asymptotic behaviour (for large $|x|$) of the potential is not isotropic. Among the distribution functions satisfying the Ehlers & Rienstra ansatz, and if we forget about the explicit dependance in t for the factorized distribution function g, the set of stationary solutions (*i.e.* the set of time-dependant solutions of zero angular velocity) is much wider than the set of time-periodic solutions with strictly positive period, even in the limit when ω goes to zero. It is not very difficult indeed to prove the existence of stationary solutions for large classes of h that do not satisfy the (strong) Ehlers & Rienstra ansatz, in the same way as it has been done in [BFH] or [BBDP] (see also references inside).

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