FREE ENERGY AND SOLUTIONS OF THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM : EXTERNAL POTENTIAL AND CONFINEMENT (LARGE TIME BEHAVIOR AND STEADY STATES)

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INTRODUCTION

Consider the Vlasov-Fokker-Planck equation

$$\partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = \partial_v \cdot (vf + \theta \partial_v f) \tag{VFP}$$

where the distribution function f is a nonnegative function of $(t,x,v) \in I\!R^+ \times I\!R^N \times I\!R^N$ and where the field $E(t,x)$ is given by the Poisson equation

$$\mathrm{div}_x E(t,x) = \rho(t,x) = \int_{I\!R^N} f(t,x,v)\, dv - n(x) \ . \tag{P}$$

n is here a given nonnegative function. The Vlasov-Poisson-Fokker-Planck system is nonlinear since $E(t,x)$ depends on f through equation (P). In the following, we shall assume that f belongs to $L^1(I\!R^N \times I\!R^N)$ and define the mass by

$$M = \int\int_{I\!R^N \times I\!R^N} f(t,x,v)\, dxdv$$

(it does not depend on t). Another estimate plays a crucial role : if we define the free energy by

$$F[f(t)] = \int\int_{I\!R^N \times I\!R^N} f(t,x,v)\left(\frac{|v|^2}{2} + \frac{1}{2}U(t,x) + U_0(x) + \theta \ln f(t,x,v)\right) dxdv \ ,$$

(see below the definition of U and U_0: $E(t,x) = -\nabla_x[U(t,x) + U_0(x)]$), then

$$\frac{d}{dt}F[f(t)] = -\int\int_{I\!R^N \times I\!R^N} |v\sqrt{f} + 2\theta\partial_v\sqrt{f}|^2\, dxdv \ .$$

Assume that there exists a function U_0 such that

$$\Delta U_0 = n(x) \ ,$$

and that E derives from a potential $V(t,x)$ such that

$$E(t,x) = -\nabla_x V(t,x) \ .$$

If $U = V - U_0$, then the Vlasov-Poisson-Fokker-Planck system is equivalent to

$$\begin{cases} \partial_t f + v \cdot \partial_x f - \nabla_x(U + U_0) \cdot \partial_v f = \partial_v \cdot (vf + \partial_v f) \\ -\Delta U(t,x) = \displaystyle\int_{I\!R^N} f(t,x,v)\, dv \end{cases} \tag{$VPFP$}$$

The goal of this paper is to understand the role of the external potential U_0. It is based on convexity properties. Assume that the free energy functional and the configurational free energy functional are respectively defined by

$$
\begin{aligned}
F : L^1_+(I\!R^N \times I\!R^N) &\longrightarrow I\!R \\
f \mapsto F[f] &= \int\int_{I\!R^N\times I\!R^N} f(x,v)\Big(\frac{|v|^2}{2} + \frac{1}{2}U(x) + U_0(x) + \theta \ln f(x,v)\Big)\, dxdv\ ,\\
G : L^1_+(I\!R^N) &\longrightarrow I\!R \\
\rho \mapsto G[\rho] &= \int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2}U(x) + U_0(x)\big)\, dx\ ,
\end{aligned}
$$

(with U given by the Poisson equation)

The main results can be summarized as follows.

Theorem : *Assume that U_0 belongs to $L^1_{\rm loc}(I\!R^N)$ (with $N \geq 3$) and is bounded from below in a neighborhood of $|x| = +\infty$:*

$$\exists R > 0 \quad \text{such that} \quad U_0^- \in L^\infty(B(R)^c)\ .$$

The following properties are equivalent :

(i) If $U_0 \in Lip(I\!R^N)$, for any solution $f \in C^0(I\!R^+, L^1(I\!R^N \times I\!R^N))$ of the Vlasov-Poisson-Fokker-Planck system such that $F[f(t=0)] < +\infty$ and $(t,x) \mapsto \nabla U(t,x)$ belongs to $L^\infty_{\rm loc}(I\!R^+; L^\infty(I\!R^N))$, $\Big(f(t,.,.)\Big)_{t>0}$ is tight in $L^1(I\!R^N \times I\!R^N)$.

(ii) If $U_0 \in Lip(I\!R^N)$, for any solution $f \in C^0(I\!R^+, L^1(I\!R^N \times I\!R^N))$ of the Vlasov-Poisson-Fokker-Planck system such that $F[f(t=0)] < +\infty$ and $(t,x) \mapsto \nabla U(t,x)$ belongs to $L^\infty_{\rm loc}(I\!R^+; L^\infty(I\!R^N))$, $\Big(f(t,.,.)\Big)_{t>0}$ converges in $L^1(I\!R^N \times I\!R^N)$ as $t \to +\infty$ to a stationary solution.

(iii)

$$I(M) = \inf\{F[f]\ :\ f \geq 0,\ f \in L^1(I\!R^N \times I\!R^N),\ \int\int_{I\!R^N\times I\!R^N} f(x,v)\, dxdv = M\} > -\infty$$

(iv)

$$J(M) = \inf\{G[\rho]\ :\ \rho \geq 0,\ \rho \in L^1(I\!R^N),\ \int_{I\!R^N} \rho(x)\, dx = M\} > -\infty$$

(v) There exists a solution $(f,U) \in L^1(I\!R^N \times I\!R^N) \times L^{\frac{N}{N-1},\infty}(I\!R^N))$ of the stationary (i.e. a solution which does not depend on t) Vlasov-Poisson-Fokker-Planck system.

(vi) $e^{-\frac{U_0}{\theta}}$ belongs to $L^1(I\!R^N)$.

Assertions (i)-(vi) define equivalent notions of confinement. (i) says that that no mass can run away at infinity when one considers the long time behavior. If $\Big(f(t,.,.)\Big)_{t>0}$ is tight, it converges as $t \to +\infty$ to the unique distribution function corresponding to the unique minimizer of G. The key estimate is the free energy $F[f(t)]$, provided it is bounded from below. $G[\rho]$ is — up to a constant — the free energy of a Maxwellian function (which is always below the free energy of any distribution function having the same spatial density ρ). If G is bounded from below, there exists a unique solution of the stationary Vlasov-Poisson-Fokker-Planck system, which is a Maxwellian function, *i.e.* of the form

$$\rho(x) \cdot \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2})} \; .$$

This solution corresponds to

$$\rho(x) = M \cdot \frac{e^{-\frac{U+U_0}{\theta}}}{\int_{I\!R^N} e^{-\frac{U(x)+U_0(x)}{\theta}} \, dx}$$

because of the Vlasov-Fokker-Planck equation, and is uniquely determined by the Poisson-Boltzmann-Emden equation

$$-\Delta U = M \cdot \frac{e^{-\frac{U+U_0}{\theta}}}{\int_{I\!R^N} e^{-\frac{U(x)+U_0(x)}{\theta}} \, dx} \; .$$

The stationary solution corresponds to the unique minimizer of G (U is defined only up to a constant).

$\rho_0 = e^{-\frac{U_0}{\theta}}$ is the asymptotic stationary density corresponding to the limiting spatial density as $\theta \to +\infty$ (or $M \to 0$ as shown by the change of variables $MV(x) = \frac{U(x)}{\theta}$) since

$$\rho(x) = M \cdot \frac{\rho_0 e^{-\frac{U}{\theta}}}{\int_{I\!R^N} \rho_0(x) e^{-\frac{U(x)}{\theta}} \, dx}$$

(see Part I, Section 4 for more details). If $U_0(x) \sim \ln |x|$ (which is the critical growth) as $|x| \to +\infty$, then there exists a critical temperature $\theta_c = \frac{1}{N}$ such that $e^{-\frac{U_0}{\theta}}$ belongs to $L^1(I\!R^N)$ if and only if $\theta < \theta_c$.

The conditions of the theorem are optimal in the sense that if U_0 is not confining, then any solution of the evolution problem is vanishing (in the case where U_0 is bounded from below — if U_0 is not bounded from below, other phenomena may occur). If U_0 is not confining, the stationary problem has no solution.

As a corollary, we may also notice that a solution of the evolution problem is stationary if and only if it is a critical point of the free energy.

For the solution of the evolution problem, the assumption $U_0 \in Lip(I\!R^N)$ is needed for the coherence of the framework (it could be removed in the assertions that do not invoke the evolution problem). The property that G or F are bounded from below is sufficient to prove the weak L^1-convergence : no mass may run away (for the evolution problem, or for a minimizing sequence in the stationary case), but also concentration of mass is impossible (see Part I, Remark 1.3).

The assumption that U_0 is bounded from below in a neighbourhood of $|x| = +\infty$ is used only to prove that the condition $e^{-\frac{U_0}{\theta}} \in L^1(I\!R^N)$ is necessary (when at least $e^{-\frac{U_0}{\theta}} \in L^1_{\text{loc}}(I\!R^N)$). Assumptions (i)-(v) hold without it if $e^{-\frac{U_0}{\theta}}$ belongs to $L^1(I\!R^N)$.

The fact that the distribution functions corresponding to steady states are Maxwellian functions has been established first in [Dr1,2] provided $\liminf_{|x|\to+\infty} \frac{U_0(x)}{|x|} > 0$. Such a property has been extended in [BD] to the case $\liminf_{|x|\to+\infty} \frac{U_0(x)}{\ln|x|} > N\theta$, where it has been proved that this condition is also sufficient to pass to the limit in the evolution problem (assertion (i) of the Theorem). The main ingredient was the fact that the free energy is bounded from below because of the estimate given in Proposition 1.4, Part I (this estimate — due to Carleman — has been used by R.J. DiPerna & P.-L. Lions to get a bound for the entropy for various kinetic equations). For the study of the stationary Maxwellian solutions of the Vlasov-Poisson system, which are the steady states of the Vlasov-Poisson-Fokker-Planck system, one may refer to [GL], [Do1,2]: the condition (vi) of the above theorem is sufficient to prove the existence of a solution (and a uniqueness result). We show that this condition is necessary and sufficient to pass to the limit in the evolution problem and to prove that the steady states are in fact Maxwellian stationary distribution functions. The main ingredient here is the use of an improved Jensen inequality, which replaces the usual estimate for the free energy (or for the entropy).

This paper also contains generalizations of a recent paper by R. Glassey, J. Schaeffer & Y. Zheng ([GSZ]) for the steady states of the Vlasov-Poisson-Fokker-Planck system (nonexistence of solutions for L^1 underlying background densities, existence for asymptotically constant underlying background densities). Direct proof for these two cases are given.

Indications on the physical derivation of the model can be found in [GSZ], and in [BD] when the potential U_0 defined above is a "confining potential", *i.e.* increasing rapidly enough at infinity. Such a model has to be considered when there are two species of particles with opposite sign charges, and when one species (which form the "underlying background density") is already thermalized (see [Bo1] in the collisionless case: the Vlasov-Poisson system). The time-dependant Vlasov-Poisson-Fokker-Planck system describes the evolution of the distribution function of the heaviest ones when they are subject to Brownian random forces (it is an idealized model of the effects of the collisions with the underlying background density) and to a viscous friction force.

For existence results, one has to refer to [Bo2] and [BD] for the Cauchy problem, and to

[Dr1,2], [GL] and [Do1,2] in the stationary case. This paper only deals with the questions of the asymptotic behavior for large time, the factorization result for the steady states (*i.e.* the fact that the steady states are Maxwellian functions, which means that they simultaneoulsy satisfy the stationary Vlasov-Poisson system and belong to the kernel of the linear Fokker-Planck operator) and the role of the free energy.

The paper is divided as follows.

The first part is devoted to the study of the free energy and of its minimum, using bounds obtained by the Jensen inequality and an improved version of it. We prove that the condition that $e^{-\frac{U_0}{\theta}}$ belongs to $L^1(I\!R^N)$ is optimal. Some comments on the behavior of the minimum when the mass or the temperature vary are also given (section 4).

These results are applied in Part II to the evolution problem for the Vlasov-Poisson-Fokker-Planck system: we prove the convergence to the unique stationary solution if $e^{-\frac{U_0}{\theta}} \in L^1(I\!R^N)$, which extends the result given in [BD], and the vanishing of the solution in the other cases (provided U_0 is at least bounded from below in a neighbourhood of $|x| = +\infty$).

Part III is devoted to the steady states. It is proved that these states are Maxwellian functions, an extension of Dressler's results. Direct proofs for generalizations of the cases studied in [GSZ] are also given.

How to derive the free energy estimates for the solutions of the Vlasov-Poisson-Fokker-Planck system has been rejected at the end of the paper.

Notations. The Marcinkiewicz spaces $L^{p,\infty}(I\!R^N)$ and the $L^p_{\mathrm{unif}}(I\!R^N)$ spaces are respectively defined by

$$L^{p,\infty}(I\!R^N) = \{f \in L^1_{\mathrm{loc}}(I\!R^N) \mid \sup_{\lambda>0} \lambda.\mathrm{meas}\{x \in I\!R^N \mid |f(x)| > \lambda\}^{1/p} < \infty\} \ ,$$

$$L^p_{\mathrm{unif}}(I\!R^N) = \{f \in L^1_{\mathrm{loc}}(I\!R^N) \mid \sup_{x\in I\!R^N} \int_{B(x,1)} |f(x)|^p \, dx < \infty\} \ .$$

When the asymptotic boundary conditions for the potential U are not specified, we shall assume that

$$U \longrightarrow 0 \quad \text{in } L^{\frac{N}{N-2},\infty}\big(B^c(I\!R^N)\big) \quad \text{as } R \to +\infty \ .$$

PART I. THE FREE ENERGY

1. Jensen's inequality and related topics

If we apply the Jensen inequality to the convex function $t \mapsto t \ln t$, we obtain

$$\left(\int_\Omega e^{-h(y)/\theta}\, dy\right)^{-1} \cdot \left[\theta \int_\Omega g(y) \ln g(y)\, dy + \int_\Omega g(y)\, h(y)\, dy\right]$$
$$= \theta \int_\Omega \left[g(y) e^{h(y)/\theta}\left[\ln\left[g(y) e^{h(y)/\theta}\right] d\mu(y)\right.\right.$$
$$\geq \theta t \ln t \Big|_{t=\int_\Omega g(y) e^{h(y)/\theta}\, d\mu(y) = \frac{\int_\Omega g(y)\, dy}{\int_\Omega e^{-h(y)/\theta}\, dy}}$$

with $d\mu(y) = \frac{e^{-h(y)/\theta}\, dy}{\int_\Omega e^{-h(y)/\theta}\, dy}$, for any measurable subset Ω in $I\!R^m$. This proves that

$$\theta \int_\Omega g(y) \ln g(y)\, dy + \int_\Omega g(y)\, h(y)\, dy \geq \theta \int_\Omega g(y)\, dy \cdot \ln\left(\frac{\int_\Omega g(y)\, dy}{\int_\Omega e^{-h(y)/\theta}\, dy}\right). \tag{1.1}$$

A Taylor developpement at order two of $t \mapsto t \ln t$ gives Csiszar-Kullback (see [C], [K]) type inequalities. For example, one may state the

Lemma 1.1 : *Assume that Ω is a measurable subset of $I\!R^m$ and that $g \in L^1(\Omega;\ dy)$ is a nonnegative function such that $g(\ln g)^+$ also belongs to $L^1(\Omega;\ dy)$. If $h \in L^1(\Omega;\ g(y)dy)$ is such that*

$$e^{-h/\theta} \quad \text{belongs to } L^1(\Omega;\ dy)$$

for some $\theta > 0$, then

$$g \ln g \quad \text{belongs to } L^1(\Omega;\ dy)$$

and

$$H[g] - H[m_g] \geq \frac{\theta}{2} \int_\Omega \left(\sqrt{g(y)} - \sqrt{m_g(y)}\right)^2 dy\ , \tag{1.2}$$

where

$$H[g] = \theta \int_\Omega g(y) \ln g(y)\, dy + \int_\Omega g(y)\, h(y)\, dy$$

and

$$m_g(y) = \frac{\int_\Omega g(y)\, dy}{\int_\Omega e^{-\frac{h(y)}{\theta}}\, dy} \cdot e^{-\frac{h(y)}{\theta}}\ .$$

Proof of Lemma 1.1 : Consider a Taylor developpement at order two of $\psi(t) = t \ln t$:

$$\psi(t_2) - \psi(t_1) = \psi'(t_1)(t_2 - t_1) + \frac{1}{2}\psi''(t)(t_2 - t_1)^2 \quad \text{for some } t \in]t_1, t_2[\ ,$$

$$\psi'(t_1) = 1 + \ln t_1 \quad \text{and} \quad \psi''(t) = \frac{1}{t} \; .$$

$$\begin{aligned}\int_\Omega g(y)\ln g(y)\,dy - &\int_\Omega m_g(y)\ln m_g(y)\,dy \\ &= \int_\Omega \Big(g(y) - m_g(y)\Big)\Big(1 + \ln\big(m_g(y)\big)\Big)\,dy \\ &\quad + \frac{1}{2}\int_\Omega \frac{\Big(g(y) - m_g(y)\Big)^2}{\tau(y)g(y) + (1-\tau(y))m_g(y)}\,dy\end{aligned}$$

for some function $x \mapsto \tau(x)$ with values between 0 and 1 :

$$H[g] - H[m_g] = \frac{1}{2}\int_\Omega \frac{\Big(g(y) - m_g(y)\Big)^2}{\tau(y)g(y) + (1-\tau(y))m_g(y)}\,dy \; .$$

Consider the linear function $\tau \mapsto j(\tau) = (g-m)^2 - \big(\tau g + (1-\tau)m\big)\big(\sqrt{g} - \sqrt{m}\big)^2$ (g and m are here two positive real numbers) :

$$j(\tau) \geq \min(j(0), j(1)) \quad \text{and} \quad j(1) = m^2 - 3mg + 2m^{1/2}g^{3/2} \; .$$

Consider now $t \mapsto k(t) = m^2 - 3mt + 2m^{1/2}t^{3/2}$:

$$k'(t) = 3m^{1/2}(t^{1/2} - m^{1/2}) \; ,$$

$$k''(t) = \frac{3}{2}\frac{m^{1/2}}{t^{1/2}} \; .$$

For any $t \in I\!R$, $k(t) \geq 0$ and $k(t) = 0$ if and only if $t = m$, which proves that

$$j(1) \geq 0 \; .$$

Exchanging g and m, we also get $j(0) \geq 0$:

$$\frac{(g-m)^2}{\tau g + (1-\tau)m} \geq \big(\sqrt{g} - \sqrt{m}\big)^2$$

which ends the proof. □

As a straightforward consequence of (1.2), we can state the

Corollary 1.2 : *Under the same assumptions as in Lemma 1.1,*

$$K(M) = \inf\{H[g] \;:\; g \geq 0,\; g \in L^1(\Omega),\; \int_\Omega g(y)\,dy = M\}$$

is bounded from below for any $M > 0$ and

$$K(M) = H\Big[\overline{g} = M \cdot \frac{e^{-h/\theta}}{\int_\Omega e^{-h(y)/\theta}\,dy}\Big] = \theta M \ln\Big(\frac{M}{\int_\Omega e^{-h(y)/\theta}\,dy}\Big) \; .$$

Moreover, $\overline{g}$ is the unique minimzer of K(M).

Remark 1.3 : Since $t \mapsto t \ln t$ is strictly convex, H is a strictly convex functional (which proves that the minimum is unique). It is interesting in view of the application to a free energy with a self-consistent potential energy to give a proof of the existence of the minimum using a minimization method. Assume here that $\Omega = I\!\!R^m$.

Consider a sequence $(g_n)_{n\in I\!\!N}$ such that, for any $n \in I\!\!N$,

$$g_n \geq 0,\ g_n \in L^1(I\!\!R^m),\ \int_{I\!\!R^m} g_n(y)\, dy = M\ ,$$

and such that

$$\lim_{n\to+\infty} H(g_n) = K(M)\ .$$

Because of (1.1), the sequence $(g_n)_{n\in I\!\!N}$ does not concentrate. Let us prove it by contradiction. Assume that

$$\exists\ \epsilon > 0 \quad \exists\ (x_n)_{n\in I\!\!N} \in (I\!\!R^m)^{I\!\!N} \quad \text{with} \quad \lim_{n\to+\infty} x_n = x_\infty \in I\!\!R^m\ ,$$

$$\forall\ R > 0 \quad \lim_{n\to+\infty} \int_{|x_n-x|<R} g_n(x)\, dx > \epsilon\ .$$

After the extraction of a subsequence, we may assume that there exists a sequence $(R_n)_{n\in I\!\!N}$ such that

$$\lim_{n\to+\infty} R_n = 0 \quad \text{and} \quad \int_{|x_n-x|<R_n} g_n(x)\, dx = \epsilon \quad \forall\ n \in I\!\!N\ .$$

Applying (1.1) independently to both integrals

$$H[g_n] = \int_{|x_n-x|\geq R_n} g_n(x)\Big(\theta\ln\big(g_n(x)\big) + h(x)\Big)\, dx + \int_{|x_n-x|<R_n} g_n(x)\Big(\theta\ln\big(g_n(x)\big) + h(x)\Big)\, dx\ ,$$

we get

$$H[g_n] \geq \theta(M-\epsilon)\ln\left(\frac{M-\epsilon}{\int_{|x_n-x|\geq R_n} e^{-h/\theta}\, dx}\right) + \theta\epsilon\ln\left(\frac{\epsilon}{\int_{|x_n-x|<R_n} e^{-h/\theta}\, dx}\right)\ .$$

Since $e^{-h/\theta} \in L^1(I\!\!R^m)$,

$$\lim_{n\to+\infty} \int_{|x_n-x|\geq R_n} e^{-h/\theta}\, dx = ||e^{-h/\theta}||_{L^1(I\!\!R^m)} \quad \text{and} \quad \lim_{n\to+\infty} \int_{|x_n-x|>R_n} e^{-h/\theta}\, dx = 0\ ,$$

$$\lim_{n\to+\infty} H[g_n] = +\infty\ ,$$

which provides a contradiction with the assumption that $(g_n)_{n\in I\!\!N}$ is a minimizing sequence.

Also because of (1.1), $(g_n)_{n\in I\!\!N}$ is tight. If this was not the case, up to the extraction of a subsequence, we would have :

$$\exists \epsilon > 0,\ \forall\ R_0 > 0,\ \exists R > R_0 \quad \text{such that} \lim_{n\to+\infty} \int_{B(R)^c} g_n(y)\, dy > \epsilon$$

Using (1.1),

$$\int_{B(R)^c} e^{-h(y)/\theta}\, dy \geq \epsilon \cdot e^{-\frac{K(M)}{\theta\epsilon}} \ ,$$

which is obviously in contradiction with

$$\lim_{R_0 \to +\infty} \int_{B(R_0)^c} e^{-h(y)/\theta}\, dy = 0 \ .$$

Dunford-Pettis criterion applies: $(g_n)_{n\in I\!N}$ is weakly compact in $L^1(I\!R^m)$. Up to the extraction of a subsequence, there exists a function $g \in L^1(I\!R^m)$ such that $(g_n)_{n\in I\!N}$ weakly converges in $L^1(I\!R^m)$ to g and

$$\int_{I\!R^m} g(y)\, dy = M \ .$$

H is convex:

$$H[g] \leq \lim_{n\to+\infty} H[g_n] = K(M) \ ,$$

which proves that g is a minimizer for $K(M)$.

When more is known on the integrability properties of h, the estimate on $H[g]$ may be improved. We present here an extension of an idea introduced by Carleman and used by R.J. DiPerna and P-L. Lions in their papers [DPL1] on the Boltzmann equation and on the Vlasov-Poisson system [DPL2] to get an estimate of the entropy, and give a more detailed version of this result.

This result will not be used in the rest of the paper. It is given here only to complete the picture of the estimates for the free energy. Throughout this section, we will use the following notations: $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$ (f^+ and f^- are therefore always nonnegative.)

Proposition 1.4 : *Let us consider two functions g and h such that g is nonnegative, g, $g(\ln g)^+$, $(h^+ + 1)e^{-h^+}$ and $g.h$ belong to $L^1(I\!R^m)$ for some $m > 1$. Then $g \ln g$ belongs to $L^1(I\!R^m)$ and*

$$\begin{aligned}\int_{I\!R^m} g(y)\ln g(y)\, dy \geq \int_{\substack{h>0\\ 0\leq g<e^{-h}}} \big(g(y) - e^{-h(y)}\big)\, dy - \int_{\substack{h>0\\ 0\leq g<e^{-h}}} h(y)e^{-h(y)}\, dy \\ - \int_{\substack{h<0\\ 1<g<e^{-h}}} h(y)e^{-h(y)}\, dy - \int_{I\!R^m} g(y)h(y)\, dy \ .\end{aligned} \tag{1.3}$$

Proof of proposition 1.4 : The proof of (1.3) is given by a simple computation based on the decomposition of $\int g(y)\ln g(y)\, dy$ into three parts :

$$\int_{0\leq g\leq 1} g(y)\ln g(y)\, dy = \int_{0\leq g<e^{-h^+}} g(y)\ln g(y)\, dy + \int_{e^{-h^+}\leq g\leq 1} g(y)\ln g(y)\, dy + \int_{g>1} g(y)\ln g(y)\, dy \ .$$

1) $t \mapsto |t \ln t| + t$ is increasing on $[0, 1/e]$ and $\frac{g}{e} < \frac{e^{-h^+}}{e} \leq \frac{1}{e}$ on $\{0 \leq g < e^{-h^+}\}$:

$$\frac{1}{e}\int_{0\leq g<e^{-h^+}} \Big(|g(y)\ln g(y)| + g(y)\Big)\, dy = \int_{0\leq g<e^{-h^+}} |\frac{g(y)}{e}\ln\big(\frac{g(y)}{e}\big)|\, dy$$
$$\leq \int_{0\leq g<e^{-h^+}} \big(h^+(y)+1\big)e^{-(h^+(y)+1)}\, dy\ .$$

2) $t \mapsto |\ln t|$ is decreasing on $]0, 1]$:

$$\int_{e^{-h^+}\leq g\leq 1} g(y)\cdot|\ln g(y)|\, dy \leq \int_{e^{-h^+}\leq g\leq 1} g(y)\cdot h^+(y)\, dy\ .$$

Combining 1) and 2), we get

$$0 \geq \int_{0\leq g\leq 1} g(y)\ln g(y)\, dy$$
$$\geq \int_{0\leq g<e^{-h^+}} \big(g(y) - e^{-h(y)}\big)\, dy - \int_{0\leq g<e^{-h^+}} h^+(y)e^{-h^+(y)}\, dy - \int_{e^{-h^+}\leq g\leq 1} g(y)h(y)\, dy\ ,$$

which proves that $g(\ln g)^-$ belongs to $L^1(I\!\!R^m)$.

3) To prove Proposition 1.4, it is enough to notice that

$$\int_{g>1} g(y)\ln g(y)\, dy$$
$$\geq \int_{g\geq e^{h^-}>1} g(y)\ln g(y)\, dy$$
$$\geq \int_{g\geq e^{h^-}>1} g(y)h^-(y)\, dy$$
$$= -\int_{g\geq e^{h^-}>1} g(y)h(y)\, dy\ ,$$

which gives

$$\int_{I\!\!R^N} g(y)\ln g(y)\, dy \geq \int_{\substack{h>0\\ 0\leq g<e^{-h}}} g(y)\, dy - \int_{h>0} \big(h(y)+1\big)e^{-h(y)}\, dy - \int_{g\notin]1,e^{h^-}[} g(y)h(y)\, dy\ ,$$

and (1.3) easily follows because of the identity :

$$\int_{g\in]1,e^{h^-}[} g(y)h(y)\, dy \ = \int_{\substack{h<0\\ 1<g<e^{-h}}} g(y)h(y)\, dy \ \geq \int_{\substack{h<0\\ 1<g<e^{-h}}} h(y)e^{-h(y)}\, dy\ .$$

$\square$

Remark 1.5 : If h is nonnegative, identity (1.3) is clearly optimal (take $g = e^{-h}$).

2. Applications to the free energy

2.1. The linear case

The free energy functional of a nonnegative distribution function $f \in L^1(I\!R^N \times I\!R^N; dxdv)$ is defined by

$$F[f] = \int\int_{I\!R^N \times I\!R^N} f(x,v)(\frac{|v|^2}{2} + U_0(x) + \theta \ln f(x,v))\, dxdv$$

for a temperature $\theta > 0$, when there is no self-consistent potential energy. We assume first that the potential is a fixed (external) given potential, which corresponds to a linear Vlasov-Fokker-Planck equation (without self-consistent potential). We apply Lemma 1.1 and Corollary 1.2 with $m = 2N$, $y = (x,v)$, $F[f] = H[g]$,

$$g(y) = f(x,v) \quad \text{and} \quad h(x,v) = \frac{|v|^2}{2} + U_0(x)\ .$$

Defining m^M as

$$m^M(x,v) = \frac{M}{\int_{I\!R^N} e^{-U_0(x)/\theta}\, dx} \cdot \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \cdot e^{-U_0/\theta} \quad \text{with } M = \int\int_{I\!R^N \times I\!R^N} f(x,v)\, dxdv\ ,$$

we get the

Corollary 2.1 : *Assume that $f \in L^1(I\!R^N \times I\!R^N)$, $f \geq 0$, and $U_0 \in L^\infty_{\text{loc}}(I\!R^N)$ are such that*

$$F[f] = \int\int_{I\!R^N \times I\!R^N} f(x,v)(\frac{|v|^2}{2} + U_0(x) + \theta \ln^+ f(x,v))\, dxdv < +\infty\ ,$$

$$e^{-\frac{1}{\theta}U_0} \in L^1(I\!R^N)\ .$$

Then with the above notations,

$$F[f] - F[m^M] \geq \frac{\theta}{2} \int_{I\!R^N \times I\!R^N} \left(\sqrt{f(x,v)} - \sqrt{m^M(x,v)}\right)^2 dxdv\ ,$$

and m^M is the unique minimizer of

$$I(M) = \inf\{F[f]\ :\ f \geq 0,\ f \in L^1(I\!R^N \times I\!R^N),\ \int\int_{I\!R^N \times I\!R^N} f(x,v)\, dxdv = M\}\ ,$$

$$I(M) = \theta M \ln\left[\frac{M}{(2\pi\theta)^{N/2} \cdot \int_{I\!R^N} e^{-U_0(x)/\theta}\, dx}\right]\ .$$

It is interesting to see how the Jensen inequality applies (which proves that $F[f] - F[m^M] \geq 0$).

Consider

$$m^M(x,v) = \rho(x)\, \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \quad \text{with } \rho(x) = \int_{I\!R^N} f(x,v)\, dv\ ,$$

and apply Jensen's inequality to

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f(x,v)}{m^M(x,v)} \ln\left(\frac{f(x,v)}{m^M(x,v)}\right) d\mu(x,v)$$

($t\mapsto t\ln t$ is a convex function) where

$$d\mu(x,v)=\frac{1}{M}m^M(x,v)\;dxdv \quad \text{and} \quad M=\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x,v)\;dxdv=\int_{\mathbb{R}^N}\rho(x)\;dx\;.$$

$$\begin{aligned}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} &\left(\frac{f(x,v)}{m^M(x,v)}\right) \ln\left(\frac{f(x,v)}{m^M(x,v)}\right) d\mu(x,v)\\ &\geq t\ln t\Big|_{t=\int\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f(x,v)}{m^M(x,v)}\;d\mu(x,v)=\frac{1}{M}\int\int_{\mathbb{R}^N\times\mathbb{R}^N}f(x,v)\;dxdv=1}=0\;,\end{aligned}$$

provides the identity

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x,v)\ln\Big(f(x,v)\Big)\;dxdv\geq\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x,v)\ln\Big(m^M(x,v)\Big)\;dxdv\;.$$

Since

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x,v)U_0(x)\;dxdv=\int_{\mathbb{R}^N}\rho(x)U_0(x)\;dx \quad \text{and} \quad \int\int_{\mathbb{R}^N\times\mathbb{R}^N} m^M(x,v)\frac{|v|^2}{2}\;dxdv=\frac{1}{2}NM\theta\;,$$

we get the identity

$$F[f]\geq F[m^M]\;.$$

The inequality is in fact strict except if $f=m^M$ almost everywhere and we have (using Lemma 1.1)

$$F[f]-F[m^M]\geq\frac{\theta}{2}\int\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\sqrt{f(x,v)}-\sqrt{m^M(x,v)}\right)^2\;dxdv\;.$$

2.2. The self-consistent case

We assume now that the potential is given by a fixed external potential U_0 and a self-consistent nonnegative one due to the Poisson equation

$$-\Delta U=\rho(x)=\int_{\mathbb{R}^N} f(x,v)\;dv\;.$$

The free energy in this case is

$$F[f]=\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(x,v)\Big(\frac{|v|^2}{2}+\frac{1}{2}U(x)+U_0(x)+\theta\ln f(x,v)\Big)\;dxdv\;.$$

We first compare the free energy with the "configurational" free energy and then minimize it.

Applying Lemma 1.1 to

$$h(v)=\theta\ln\rho+\frac{|v|^2}{2}\;,$$

(take $y = v,\ m = N,\ \Omega = I\!R^N$) we get

$$F[f] \geq F[m_f] = G[\rho] - \frac{1}{2} N M \theta \ln(2\pi\theta) \quad \text{where} \quad G[\rho] = \int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2} U(x) + U_0(x)\big)\, dx\ , \quad (2.1)$$

and the equality occurs if and only if

$$f(x,v) = m_f(x,v) = \rho(x)\ \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \qquad (x,v) \in I\!R^N \times I\!R^N \quad \text{a.e.}\ ,$$

with $\rho(x) = \int_{I\!R^N} f(x,v)\, dv$.

Proposition 2.2 : *Assume that $e^{-U_0/\theta}$ belongs to $L^1(I\!R^N)$ with $N \geq 3$.*
(i) The infimum of the free energy

$$I(M) = \inf\{F[f]\ :\ f \geq 0,\ f \in L^1(I\!R^N \times I\!R^N),\ \int\!\!\int_{I\!R^N \times I\!R^N} f(x,v)\, dxdv = M\}$$

and the infimum of the "configurational" free energy

$$J(M) = \inf\{G[\rho]\ :\ \rho \geq 0,\ \rho \in L^1(I\!R^N),\ \int_{I\!R^N} \rho(x)\, dx = M\}\ ,$$

where

$$G[\rho] = \int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2} U(x) + U_0(x)\big)\, dx\ , \quad U \geq 0\ ,$$

are bounded from below (for any $M > 0$) and satisfy

$$I(M) = J(M) - \frac{1}{2} N M \theta \ln(2\pi\theta)\ , \quad J(M) > \theta M \ln\!\left(\frac{M}{\int_{I\!R^N} e^{-U_0(x)/\theta}\, dx}\right)\ . \qquad (2.2)$$

(ii) The infima $I(M)$ and $J(M)$ are realized respectively by

$$\overline{f}(x,v) = \overline{\rho}(x) \cdot \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \quad \text{and} \quad \overline{\rho}(x) = M \cdot \frac{e^{-\left(U(x)+U_0(x)\right)/\theta}}{\int_{I\!R^N} e^{-\left(U(x)+U_0(x)\right)/\theta}\, dx}$$

where U is the unique solution in $L^{\frac{N}{N-1},\infty}(I\!R^N)$ of the Poisson-Boltzmann-Emden equation

$$U = \frac{C_N}{|x|^{N-2}} * M \cdot \frac{e^{-\left(U(x)+U_0(x)\right)/\theta}}{\int_{I\!R^N} e^{-\left(U(x)+U_0(x)\right)/\theta}\, dx}$$

($\frac{C_N}{|x|^{N-2}}$ is the Green function of $-\Delta$ in $I\!R^N$). The minimizing functions exists and are unique, and if U is the solution of the Poisson-Boltzmann-Emden equation, then

$$J(M) = -\theta M \ln\!\left(\frac{1}{M} \int_{I\!R^N} e^{-\frac{U(x)+U_0(x)}{\theta}}\, dx\right) - \frac{M}{2} \frac{\int_{I\!R^N} U(x) \cdot e^{-\frac{U(x)+U_0(x)}{\theta}}\, dx}{\int_{I\!R^N} e^{-\frac{U(x)+U_0(x)}{\theta}}\, dx}\ .$$

Proof of Proposition 2.2 : (i) is a consequence of (1.1) with $m = 2N$, $y = (x, v)$,

$$g(y) = f(x, v) \quad \text{and} \quad h(x, v) = \frac{|v|^2}{2} - \theta \ln \rho(x) \ ,$$

$$\rho(x) = \int_{I\!R^N} f(x, v) \ dv \ .$$

According to Lemma 1.1,

$$F[f] - F[m_f] \geq \frac{\theta}{2} \int \int_{I\!R^N \times I\!R^N} \left(\sqrt{f(x, v)} - \sqrt{m_f(x, v)} \right)^2 \ dxdv \ ,$$

with $m_f(x, v) = \rho(x) \ \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}}$ and

$$F[m_f] = G[\rho] - \frac{1}{2} NM\theta \ln(2\pi\theta)$$

gives the relation between $I(M)$ and $J(M)$. To prove that $J(M)$ is bounded from below, it is enough to notice that (if $\rho \not\equiv 0$)

$$G[\rho] > \int_{I\!R^N} \rho(x) \big(\theta \ln \rho(x) + U_0(x) \big) \ dx = H[\rho] \ ,$$

since $U > 0$ and to apply Corollary 1.2 to H with $h = U_0$.

$$G[\rho] > H\left[M \cdot \frac{e^{-U_0(x)/\theta}}{\int_{I\!R^N} e^{-U_0(x)/\theta} \ dx} \right] = \theta M \ln\left(\frac{M}{\int_{I\!R^N} e^{-U_0(x)/\theta} \ dx} \right) \ .$$

(ii) cannot be proved directly by the same method as for Corollary 2.1, since the minimizer still depends on ρ through U because of the Poisson equation. We use a relaxed energy method : consider

$$\overline{G}[\rho] = G[\frac{\rho + \overline{\rho}}{2}]$$

and minimize it over the set of the $L^1(I\!R^N)$ nonnegative functions such that $\int_{I\!R^N} \rho(x) \ dx = M$ for some fixed constant $M > 0$. $\overline{\rho}$ is the unique solution of

$$\overline{\rho} = M \cdot \frac{e^{-\left(\overline{U}(x) + U_0(x)\right)/\theta}}{\int_{I\!R^N} e^{-\left(\overline{U}(x) + U_0(x)\right)/\theta} \ dx} \ , \quad \overline{U} = \frac{C_N}{|x|^{N-2}} * \overline{\rho}$$

(see [Do1,2]). $\overline{G}$ is a C^1 convex functional, since

$$\int_{I\!R^N} \rho(x) U(x) \ dx = C_N \cdot \int \int_{I\!R^N \times I\!R^N} \frac{\rho(x)\rho(y)}{|x - y|^{N-2}} \ dxdy$$

(the existence of a minimizer holds for the reason given in Remark 1.3). $\overline{G}$ is convex : there exists a unique minimizer ρ which satisfies the associated Euler equation :

$$\theta \ln\left(\frac{\rho + \overline{\rho}}{2} \right) + V + U_0 = \lambda - \theta \ ,$$

where λ is the Lagrange multiplier corresponding to the constraint $\int_{I\!R^N} \rho(x)\, dx = M$ and V is defined by

$$V = \frac{C_N}{|x|^{N-2}} * \left(\frac{\rho+\overline{\rho}}{2}\right) .$$

Solving the Euler equation, we obtain

$$\frac{\rho+\overline{\rho}}{2} = e^{\lambda-\theta} \cdot e^{-\frac{V+U_0}{\theta}} ,$$

and λ is fixed by the constraint $\int_{I\!R^N} \rho(x)\, dx = M$:

$$\rho = \overline{\rho} .$$

On one hand, since $\overline{G}$ is convex,

$$\overline{G}[\rho] = G[\frac{\rho+\overline{\rho}}{2}] \leq \frac{1}{2}(G[\rho] + G[\overline{\rho}]) ,$$

and on the other hand

$$G[\overline{\rho}] = G[\frac{\overline{\rho}+\overline{\rho}}{2}] = \overline{G}[\overline{\rho}] = \inf_{\substack{\rho\geq 0 \\ ||\rho||_{L^1(I\!R^N)}=M}} \overline{G}[\rho]$$

because $\overline{\rho}$ realizes the minimum of $\overline{G}$:

$$G[\overline{\rho}] \leq \inf_{\substack{\rho\geq 0 \\ ||\rho||_{L^1(I\!R^N)}=M}} \frac{1}{2}(G[\rho] + G[\overline{\rho}]) = \frac{1}{2}G[\overline{\rho}] + \frac{1}{2} \inf_{\substack{\rho\geq 0 \\ ||\rho||_{L^1(I\!R^N)}=M}} G[\rho] ,$$

$$G[\overline{\rho}] = \inf_{\substack{\rho\geq 0 \\ ||\rho||_{L^1(I\!R^N)}=M}} G[\rho] .$$

$\square$

Remark 2.3 :
(i) The solution of

$$-\Delta U = M \cdot \frac{e^{-(U+U_0)/\theta}}{\int_{I\!R^N} e^{-(U+U_0)/\theta} dx}$$

is defined up to an additive constant. If we consider the one satisfying

$$U = \frac{C_N}{|x|^{N-2}} * M \cdot \frac{e^{-\left(U(x)+U_0(x)\right)/\theta}}{\int_{I\!R^N} e^{-\left(U(x)+U_0(x)\right)/\theta}\, dx} ,$$

it is nonnegative and unique (see [Do1]) as soon as

$$e^{-U_0/\theta} \in L^1(I\!R^N) .$$

Moreover (we assume $N \geq 3$), U belongs to $L^{\frac{N}{N-2},\infty}(I\!R^N)$ and ∇U belongs to $L^{\frac{N}{N-1},\infty}(I\!R^N)$.

(ii) It has also been proved in [Do1] that ∇U belongs to $L^2(I\!\!R^N)$ if

$$e^{-U_0/\theta} \in L^{\frac{2N}{N+2}}(I\!\!R^N) ,$$

which allows to perform an integration by parts of

$$\frac{1}{2}\int_{I\!\!R^N} \rho(x)U(x)\;dx = \frac{1}{2}\int_{I\!\!R^N} |\nabla U(x)|^2\;dx$$

using the Poisson equation. In this case,

$$\begin{aligned} G\Big[\overline{\rho} = M\cdot \frac{e^{-\big(U(x)+U_0(x)\big)/\theta}}{\int_{I\!\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\;dx}\Big] - M\theta \ln M \\ = -\frac{1}{2}\int_{I\!\!R^N} |\nabla U(x)|^2\;dx - M\theta \ln\Big[\int_{I\!\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\;dx\Big] = -\mathcal{J}[U] , \end{aligned}$$

where $U \mapsto \mathcal{J}[U]$ is a convex functional whose minimum is precisely the solution of the Poisson-Boltzmann-Emden equation. Moreover, (2.2) proves that

$$\min\{\mathcal{J}[U] \;:\; \nabla U \in L^2(I\!\!R^N)\} \leq \theta M \ln\Big(\int_{I\!\!R^N} e^{-U_0(x)/\theta}\;dx\Big) .$$

(iii) if U belongs to $L^\infty(I\!\!R^N)$, then

$$||U||_{L^\infty(I\!\!R^N)} \geq \theta \ln\Big(\frac{\int_{I\!\!R^N} e^{-U_0(x)/\theta}\;dx}{\int_{I\!\!R^N} e^{-(U(x)+U_0(x))/\theta}\;dx}\Big) .$$

This can be shown with the Jensen inequality :

$$\int_{I\!\!R^N} e^{-U(x)/\theta}\;d\mu(x) \geq e^{-\int_{I\!\!R^N} \frac{U(x)}{\theta}\;d\mu(x)} \geq e^{-\frac{1}{\theta}||U||_{L^\infty(I\!\!R^N)}}$$

with

$$d\mu(x) = \frac{e^{-U_0(x)/\theta}}{\int_{I\!\!R^N} e^{-U_0(x)/\theta}\;dx}\;dx .$$

(iv) A direct minimization of G provides an existence result (and also a uniqueness result since G is strictly convex) of a solution of the Poisson-Boltzmann-Emden equation as soon as one can prove that the minimum is a strictly nonnegative function almost everywhere, so that one can write the Euler equation corresponding to the critical point. A relaxation method can be used to overcome this difficulty.
(v) The difference between F and G can be estimated by the Gross' logarithmic Sobolev inequality (see [T1,2]): for any nonnegative function f in $L^1(I\!\!R^N \times I\!\!R^N)$ such that $\rho = \int_{I\!\!R^N} f(.,v)\;dv$ and $M = ||f||_{L^1(I\!\!R^N)}$,

$$F[f] - \Big(G[\rho] + \frac{1}{2}NM\theta \ln(2\pi\theta)\Big) \leq \frac{1}{2}\int\int_{I\!\!R^N \times I\!\!R^N} |v\sqrt{f} + 2\theta\nabla_v(\sqrt{f})|^2\;dxdv .$$

However, up to our knowledge, there is no estimate of $\frac{d}{dt}G[\rho(t)]$ for a solution $f(t,.,.)$ of the Vlasov-Poisson-Foker-Planck system, and it seems therefore much more difficult to give a rate of convergence to the equilibrium for the nonlinear case than for the linear homogeneous Fokker-Planck equation.

3. The L^1-condition is necessary and sufficient for the free energy to be bounded from below

It has been proved above that if $e^{-U_0/\theta} \in L^1(I\!R^N)$, the free energy is bounded from below. Before proving that this condition is necesssary too, let us state a result in the linear case:

Lemma 3.1 : *Assume that $e^{-h/\theta} \in L^1_{\rm loc}(I\!R^N)$ does not belong to $L^1(I\!R^N)$. Then*

$$K(M) = \inf\{H[g] \ : \ g \geq 0, g \in L^1(I\!R^m;\ dy),\ g(\ln^+ g + |h|) \in L^1(I\!R^m;\ dy),\ \int_{I\!R^m} g(y)\, dy = M\}$$
$$= -\infty \quad \text{for any } M > 0 \ ,$$

where

$$H[g] = \theta \int_{I\!R^m} g(y) \ln g(y)\, dy + \int_{I\!R^m} g(y)\, h(y)\, dy \ .$$

Proof of Lemma 3.1 : Since $e^{-h/\theta}$ does not belong to $L^1(I\!R^N)$, there exists some $\epsilon > 0$, an increasing sequence $(R_n)_{n\in I\!N}$ and a sequence of measurable sets $(\Omega_n)_{n\in I\!N}$ such that

$$\Omega_n \subset B(R_{n+1})\backslash B(R_n) \quad \text{and} \quad \int_{\Omega_n} e^{-h(y)/\theta}\, dy > \epsilon \ .$$

Consider now

$$g_n = M \frac{e^{-h/\theta} \sum_{l=1}^n \chi\Omega_l}{\int_{\cup_{l=1}^n \Omega_l} e^{-h(y)/\theta}\, dy} \ .$$

Then

$$H[g_n] = -\theta M \ln\Big(\int_{\cup_{l=1}^n \Omega_l} e^{-h(y)/\theta}\, dy\Big) \leq -\theta M \ln(n\epsilon) \to -\infty \quad \text{as } n \to +\infty \ .$$

□

The nonlinear case is more difficult since the self-consistent term may counterbalance the estimate one gets in the linear case. For this reason, one has to impose a further condition on the behaviour of U_0 at infinity.

Proposition 3.2 : *Assume that $e^{-U_0/\theta} \in L^1_{\rm loc}(I\!R^N)$ (with $N \geq 3$) does not belong to $L^1(I\!R^N)$ and that $U_0(x)$ is bounded from below for $|x|$ large enough :*

$$\exists R > 0\ , \quad \exists K \in I\!R \quad \text{such that} \quad U_0(x) > K \quad x \in B(R)^c \text{ a.e. .}$$

Then $I(M) = J(M) = -\infty$ for any $M > 0$, *where*

$$I(M) = \inf\{F[f] \;:\; f \geq 0,\; f \in L^1(I\!R^N \times I\!R^N),\; \int\int_{I\!R^{2N}} f(x,v)\; dxdv = M\}\;,$$

$$J(M) = \inf\{G[\rho] \;:\; \rho \geq 0,\; \rho \in L^1(I\!R^N),\; \int_{I\!R^N} \rho(x)\; dx = M\}\;,$$

with

$$F[f] = \int\int_{I\!R^{2N}} f(x,v)(\frac{|v|^2}{2} + \frac{1}{2}U(x) + U_0(x) + \theta \ln f(x,v))\; dxdv\;,$$

$$G[\rho] = \int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2}U(x) + U_0(x)\big)\; dx\;.$$

Proof of Proposition 3.2 : Because of (2.1),

$$I(M) = J(M) - \frac{1}{2}NM\theta \ln(2\pi\theta)\;,$$

it is enough to prove that J is not bounded from below.

First case: U_0 is bounded from below. The idea of the proof is the same as in Lemma 3.1. The self-consistent potential energy is bounded by the Hardy-Littlewood-Sobolev inequalities:

$$0 \leq \int_{I\!R^N} \rho(x)U(x)\; dx = \int_{I\!R^N} |\nabla U(x)|^2\; dx \leq C \cdot ||\rho||^2_{L^{\frac{2N}{N+2}}}(I\!R^N)$$

for some constant $C > 0$. The proof is obtained by taking $\rho_n = g_n$ as in the proof of Lemma 3.1:

$$||\rho_n||^2_{L^{\frac{2N}{N+2}}}(I\!R^N) \leq \frac{M^2}{(n\epsilon)^{\frac{N-2}{N}}} e^{\frac{N-2}{N\theta}||U_O^-||_{L^\infty(I\!R^N)}} \to 0 \quad \text{as} \quad n \to +\infty\;.$$

Second case: U_0 is not bounded from below: there exists a constant $k > 0$ such that $U_0^k = \max(U_0, -k)$ satisfies $\rho_0^k = e^{-U_0^k/\theta} \not\in L^1(I\!R^N)$. If this was not the case, then

$$\frac{\rho_0^k}{\int_{I\!R^N} \rho_0^k(x)\; dx}\;,$$

would concentrate, a contradiction with the assumption that $e^{-U_0/\theta} \in L^1_{\text{loc}}(I\!R^N)$.

Applying the results of the first case to

$$G^k[\rho] = \int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2}U(x) + U_0^k(x)\big)\; dx\;,$$

we get the result since

$$G[\rho] \leq G^k[\rho]\;.$$

□

4. Thermodynamics

This section is devoted to the study of the variation of the infimum of the free energy with respect to the mass and the temperature.

4.1 Dependence of the minimum of the free energy in the mass

Proposition 4.1 : *$I(M)=\inf\{F[f]\ :\ f\geq 0,\ f\in L^1(I\!R^N\times I\!R^N),\ \int\int_{I\!R^N\times I\!R^N} f(x,v)\ dxdv=M\}$ (resp. $J(M)=\inf\{G[\rho]\ :\ \rho\geq 0,\ \rho\in L^1(I\!R^N),\ \int_{I\!R^N}\rho(x)\ dx=M\}$) is an increasing function of the mass if and only if*

$$\int_{I\!R^N} e^{-\frac{U+U_0}{\theta}}\ dx\leq\frac{M\ e}{(2\pi\theta)^{N/2}}$$

(resp. $\int_{I\!R^N} e^{-\frac{U+U_0}{\theta}}\ dx\leq M\ e$), where U is the unique solution of the Poisson-Boltzmann-Emden equation

$$U=\frac{C_N}{|x|^{N-2}}*M\cdot\frac{e^{-\big(U(x)+U_0(x)\big)/\theta}}{\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\ dx}\ .$$

A sufficient condition for $I(M)$ (resp. $J(M)$) to be increasing is

$$\int_{I\!R^N} e^{-\frac{U_0}{\theta}}\ dx\leq\frac{M\ e}{(2\pi\theta)^{N/2}}$$

(resp. $\int_{I\!R^N} e^{-\frac{U_0}{\theta}}\ dx\leq M\ e$).

Proof of Proposition 4.1 : It is based on a simple homogeneity argument. For a fixed ρ, we may consider

$$M(\lambda)=\int_{I\!R^N}\rho^\lambda\ dx\quad\text{and}\quad g(\lambda)=G[\rho^\lambda]\quad\text{with}\quad\rho^\lambda(x)=\lambda\rho(x)\ .$$

A straightforward computation gives

$$\frac{d}{d\lambda}M(\lambda)_{|\lambda=1}>0\quad\text{and}\quad\frac{d}{d\lambda}g(\lambda)_{|\lambda=1}=\theta M+g(1)+\frac{C_N}{2}\int\int_{I\!R^N\times I\!R^N}\frac{\rho(x)\rho(y)}{|x-y|^{N-2}}\ dxdy$$
$$=\theta M\Big(1-\ln\big(\frac{1}{M}\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\ dx\big)\Big)\geq 0$$

if and only if

$$\ln\big(\frac{1}{M}\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\ dx\big)\leq 1\ ,$$

which proves the first assertion of Proposition 4.1. The solution of the Poisson-Boltzmann-Emden equation

$$U=\frac{C_N}{|x|^{N-2}}*M\cdot\frac{e^{-\big(U(x)+U_0(x)\big)/\theta}}{\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\ dx}$$

is nonnegative and

$$\int_{I\!R^N} e^{-\frac{U+U_0}{\theta}}\;dx \le \int_{I\!R^N} e^{-\frac{U_0}{\theta}}\;dx\;,$$

which gives the sufficient condition. □

4.2 Dependence of the minimum of the free energy in the temperature

The dependence of the free energy in the temperature is more complicated than the dependence in the mass. Let us first consider the limit case: $\theta \to +\infty$.

Proposition 4.2 : *Assume that $U_0 \in C^1(\Omega)$ and that $U_0 \equiv +\infty$ in Ω^c for some bounded domain Ω in $I\!R^N$ ($N \ge 3$). Then the solution U of the Poisson-Boltzmann-Emden equation is such that when $\theta \to +\infty$*

$$U \to U^\infty = \frac{C_N}{|x|^{N-2}} * M \cdot \frac{\rho^\infty}{\int_{I\!R^N} \rho^\infty(x)\;dx} \quad \text{in } L^{\frac{N}{N-1},\infty}(I\!R^N)$$

$$\rho^\theta = M \cdot \frac{e^{-\big(U(x)+U_0(x)\big)/\theta}}{\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\;dx} \to M \cdot \rho^\infty \quad \text{in } L^1(I\!R^N)$$

where $\rho^\infty = \chi_\Omega$ is the characteristic function of Ω:

$$\rho^\infty \equiv 1 \quad \text{in } \Omega \quad \text{and} \quad \rho^\infty \equiv 0 \quad \text{in } \Omega^c\;.$$

The proof is easily obtained by applying Lebesgue's theorem of dominated convergence to ρ^θ. A special case is the following: $\rho_0 = \rho^\theta = \chi_\Omega$ does not depend on θ. Taking then $V = \frac{U}{\theta}$, we may immediately deduce from Proposition 4.1 the

Corollary 4.3 : *Assume that ρ_0 is the characteristic function of a bounded domain Ω in $I\!R^N$ with $N \ge 3$. With the same notations as above*

$$\inf\{F[f] : f \ge 0,\; f \in L^1(I\!R^N \times I\!R^N),\; \int\int_{I\!R^N \times I\!R^N} f(x,v)\;dxdv = M\}$$

is a decreasing function of the temperature if $|\Omega| \le \frac{M\,e}{\theta\,(2\pi)^{N/2}}$.

The proof immediately follows from the fact that V is a solution of the Poisson-Boltzmann-Emden equation

$$V = \frac{C_N}{|x|^{N-2}} * \frac{M}{\theta} \cdot \frac{\rho_0 e^{-V}}{\int_{I\!R^N} \rho_0(x) e^{-V(x)}\;dx}$$

(with mass $\frac{M}{\theta}$ and temperature 1).

Remarks :

(i) It is more difficult to give a general result (when $\rho_0 \not\equiv \chi_\Omega$) for the dependence of the free energy in the temperature than in the mass. For instance, applying Proposition 4.2,

$$J(M) \sim M\theta \ln\left(\int_{I\!R^N} \rho_0(x)\, dx\right) - \frac{C_N}{2} \int\int_{I\!R^N \times I\!R^N} \frac{\rho_0(x)\rho_0(y)}{|x-y|^{N-2}}\, dxdy$$

as $\theta \to +\infty$: the dependence of $J(M)$ in θ clearly depends on the sign of $\ln\left(\int_{I\!R^N} \rho_0(x)\, dx\right)$.

(ii) If U_0 is bounded from below, the fact that $\rho_0 = e^{-\frac{U_0}{\theta}}$ belongs to $L^1(I\!R^N)$ is completely determined by the asymptotic behavior of $U_0(x)$ as $|x| \to +\infty$. If $U_0(x) = o(\ln|x|)$, ρ_0 does not belong to $L^1(I\!R^N)$. If $\ln|x| = o(U_0(x))$, ρ_0 belongs to $L^1(I\!R^N)$. If $U_0(x) \sim \ln|x|$, there exists a critical temperature $\theta_c = \frac{1}{N} > 0$ such that ρ_0 belongs to $L^1(I\!R^N)$ if $\theta < \theta_c$ and ρ_0 does not belong to $L^1(I\!R^N)$ if $\theta > \theta_c$.

PART II. THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM : THE TIME DEPENDENT PROBLEM

1. Large time behavior in a confining potential

For existence results for the Vlasov-Poisson-Fokker-Planck system (when there is no confining potential), we refer to the papers by F. Bouchut (see [Bo2]) and the references therein. We present here a small extension to [BD], in which we improve the condition on the external potential and give the optimal condition.

Definition : *A solution of the Vlasov-Poisson-Fokker-Planck system is a nonnegative function*

$$f \in C([0,+\infty[; L^1(I\!R^N \times I\!R^N))$$

which is a solution of

$$\begin{cases} \partial_t f + v \cdot \partial_x f - \nabla_x (U + U_0) \cdot \partial_v f = \partial_v \cdot (vf + \theta \partial_v f) \\ -\Delta U = \displaystyle\int_{I\!R^N} f(t,x,v)\, dv \end{cases}$$

such that

$$\int\int_{I\!R^N \times I\!R^N} (|v|^2 + \frac{U(t,x)}{2} + U_0(x) + \theta \ln^+ f(t,x,v)) f(t,x,v)\, dxdv_{|t=0} < +\infty$$

and such that $(t,x) \mapsto \nabla U(t,x)$ belongs to $L^\infty_{\rm loc}(I\!R^+; L^\infty(I\!R^N))$.

If $(t,x) \mapsto \nabla_x U(t,x)$ does not belong to $L^\infty_{\rm loc}(I\!R^+; L^\infty(I\!R^N))$, we have to use the notion of renormalized solutions (see [BD] and also Part III, section 1 in the stationary case).

Proposition 1.1 : *Assume that $U_0 \in Lip(I\!R^N)$, $e^{-U_0/\theta} \in L^1(I\!R^N)$, $N \geq 3$. Given any solution of the Vlasov-Poisson-Fokker-Planck system , then $f(t,.,.) \to m_f$ in $L^1(I\!R^N \times I\!R^N)$ as $t \to +\infty$, where*

$$m_f(x,v) = M \cdot \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \cdot \frac{e^{-(\overline{U}+U_0)/\theta}}{\int_{I\!R^N} e^{-(\overline{U}+U_0)/\theta} dx} \quad \text{with } M = \int\int_{I\!R^N \times I\!R^N} f(x,v)\, dxdv\ ,$$

and where $\overline{U}$ is the unique (up to an additive constant) solution in $L^{\frac{N}{N-2},\infty}(I\!R^N)$ of

$$-\Delta \overline{U} = M \cdot \frac{e^{-(\overline{U}+U_0)/\theta}}{\int_{I\!R^N} e^{-(\overline{U}+U_0)/\theta} dx}\ .$$

Moreover, for any $T > 0$,

$$\lim_{t \to +\infty} \int_t^{t+T} \left(\int\int_{I\!R^N \times I\!R^N} |v|^2 \cdot |f(s,x,v) - m_f(x,v)|\, dxdv \right) ds = 0\ ,$$

$$\lim_{t \to +\infty} \int_t^{t+T} \left(\int\int_{I\!R^N \times I\!R^N} |\nabla_v \sqrt{f(s,x,v)} - \nabla_v \sqrt{m_f(x,v)}|^2\, dxdv \right) ds = 0\ .$$

Proof of Proposition 1.1 : The proof is exactly the same as in [BD] except that the condition

$$\lim_{|x|+\infty} \frac{U_0(x)}{|x|} > N\theta$$

has been replaced by the optimal condition

$$e^{-U_0/\theta} \in L^1(I\!R^N) \ .$$

This last condition is sufficient to prove that :

(i) the free energy functional

$$F[f(t)] = \int\int_{I\!R^N \times I\!R^N} f(t,x,v)(\frac{|v|^2}{2} + \frac{1}{2}U(t,x) + U_0(x) + \theta \ln f(t,x,v))\ dxdv$$

is bounded from below (see Part I, section 2.2),

(ii) that $(f(t))_{t>0}$ is tight (the proof is based on the same idea as Remark 1.3, Part I).

The rest of the proof (passage to the limit, compactness results, use of renormalized solutions, convergence) holds in the same way as in [BD].

The assumption that $U_0 \in Lip(I\!R^N)$ which is needed only for the coherence of the framework, together with the condition $e^{-U_0/\theta} \in L^1(I\!R^N)$ imply that U_0 is bounded from below.

The optimality of the condition $e^{-U_0/\theta} \in L^1(I\!R^N)$ is proved below. □

2. Vanishing when there is an external but non confining potential

Theorem 2.1 : *Assume that $U_0 \in Lip_{\mathrm{loc}}(I\!R^N)$ is bounded from below in a neighborhood of $|x| = +\infty$ and such that there exists a solution f in $C(I\!R^+; L^\infty \cap L^1(I\!R^N \times I\!R^N))$ of the Vlasov-Poisson-Fokker-Planck system. Assume also that there exists some $\varepsilon > 0$ such that $f \log f \in C([0,\varepsilon[; L^1(I\!R^N \times I\!R^N))$. If $e^{-U_0/\theta} \notin L^1(I\!R^N)$, then there exists $\tau \in [\varepsilon, +\infty[$ such that $f \log f$ belongs to $C([0,\tau[; L^1(I\!R^N \times I\!R^N))$ and*

$$f(t,.,.) \to 0 \quad \text{weakly in } L^1(I\!R^N) \quad \text{as } t \to \tau,\ t < \tau \ .$$

Proof : Assume that $(f(t,.,.))_{t>0}$ is not vanishing. We first have to prove the existence of $\tau \in [\varepsilon, +\infty]$ such that

$$\limsup_{\substack{t\to\tau \\ t<\tau}} S[f(t)] = -\infty \ ,$$

$$\text{with } S[f(t)] = \int\int_{I\!R^N \times I\!R^N} f(t,x,v) \log f(t,x,v)\ dxdv \ .$$

If this is not the case,

$$\limsup_{t\to+\infty} F[f(t)] > -\infty \ ,$$

since the free energy $t \mapsto F[f(t)]$ is decreasing with respect to t as long as it is well defined (see Appendix A) and since U_0 is bounded from below. $(f(t+.,.,.))_{t>0}$ weakly converges in $L^1([0,T]\times \mathbb{R}^N_{\text{loc}} \times \mathbb{R}^N; dtdxdv)$ for some $T>0$ – up to the extraction of a subsequence – to some limiting function $\overline{f}$ (with $m = ||\overline{f}(t,.,.)||_{L^1(\mathbb{R}^N\times\mathbb{R}^N;\,dxdv)} > 0$) and $\overline{f}$ is the unique Maxwellian function of mass m which is solution of the Poisson-Boltzmann equation (with mass m). But this is impossible (see section III, Theorem 3.1) and proves therefore that either

$$\limsup_{t\to+\infty} S[f(t)] = -\infty$$

(and that τ exists) or that $(f(t,.,.))_{t>0}$ is vanishing.

We consider now the case $\limsup_{\substack{t\to\tau\\ t<\tau}} S[f(t)] = -\infty$, which is possible only if

$$\limsup_{\substack{t\to\tau\\ t<\tau}} F[f(t)] = -\infty$$

because U_0 is bounded from below in a neighbourhood of $|x| = +\infty$. Define indeed

$$\overline{f}(t,x,v) = \epsilon^{-N}(t)\cdot f(t,\frac{x}{\epsilon(t)},v)$$

where $t \mapsto \epsilon(t)$ is given by the condition that $t \mapsto S[\overline{f}(t)]$ is constant: assume that

$$S[f_0] = S[\overline{f}(t)] = S[f(t)] - N\log\epsilon(t)||f(t)||_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}\ ,$$

$$\epsilon(t) = \exp\left(\frac{S[f(t)]-S[f_0]}{N||f(t)||_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}}\right)\ ,\qquad \limsup_{\substack{t\to\tau\\ t<\tau}} \epsilon(t) = 0\ .$$

$(\overline{f}(t,.,.))_{t\in]0,\tau[}$ is weakly compact in $L^1(\mathbb{R}^N_{\text{loc}}\times\mathbb{R}^N)$, which proves the vanishing result. □

Remarks :

(i) According to [Bo2], if $U_0 \equiv Const$, there exists a strong solution f which satisfies the conditions of Theorem 2.1 if f_0 is regular enough: assume for example that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} f_0(x,v)\Big(\log^+(f_0(x,v)) + |x|^2 + |v|^2\Big)\,dxdv + \infty\ .$$

A formal computation (multiply the Vlasov-Fokker-Planck equation respectively by $|x|^2$ and $|v|^2$ and integrate by parts) gives

$$\frac{d}{dt}\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)|v|^2\,dxdv = -2\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)|v|^2\,dxdv + N\theta\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)\,dxdv \le N\theta\int_{\mathbb{R}^N\times\mathbb{R}^N} f_0(x,v)\,dxdv$$

and

$$\frac{d}{dt}\int_{I\!R^N\times I\!R^N} f(t,x,v)|x|^2\,dxdv = 2\int_{I\!R^N\times I\!R^N} f(t,x,v)\;(x\cdot v)\;dxdv$$
$$\leq \left(\int_{I\!R^N\times I\!R^N} f(t,x,v)|x|^2\,dxdv\cdot\int_{I\!R^N\times I\!R^N} f(t,x,v)|v|^2\,dxdv\right)^{1/2}$$

which clearly proves that $\int_{I\!R^N\times I\!R^N} f(t,x,v)\big(|x|^2+|v|^2\big)\,dxdv$ remains finite for any $t>0$.

$$\int_{I\!R^N\times I\!R^N} f(t,x,v)\log^-\big(f(t,x,v)\big)\,dxdv$$

is therefore finite by Lemma 1.1 or Proposition 1.4 of Part I: $\tau=+\infty$ (see below for more details).

(ii) The method still applies when U_0 is not bounded from below but such that

$$\limsup_{\substack{t\to\tau\\ t<\tau}}\int\int_{I\!R^N\times I\!R^N} f(t,x,v)\log f(t,x,v)\,dxdv = -\infty$$

for some $\tau\in]0,+\infty]$.

(iii) As long as $\big((f\log f)(t,.,.)\big)_t$ (respectively $(f(t,.,.))_t$) is uniformly bounded in $L^1(I\!R^N\times I\!R^N)$ (respectively in $L^\infty(I\!R^N\times I\!R^N)$), no concentration may occur and $(\overline{f}(t,.,.))_t$ is weakly compact in $L^1_{\rm loc}(I\!R^N\times I\!R^N)$, but the limit we obtain after extraction of a subsequence may have a non finite free energy. More in general the case when $e^{-U_0/\theta}$ does not belong to $L^1(I\!R^N)$ and U_0 is not bounded from below is not easy to handle without imposing technical conditions since there is no natural notion of solution (a global in time existence result is not clear without further conditions).

(iv) If $\lim_{\substack{t\to\tau\\ t<\tau}} F[f(t)]=-\infty$, the rate of convergence is given by $\epsilon(t)$: consider (like in the proof of Theorem 2.1 $\overline{f}$ such that

$$\overline{f}(t,x,v)=\epsilon^{-N}(t)\cdot f(t,\frac{x}{\epsilon(t)},v)\;.$$

For any ball B,

$$||f(t)||_{L^1(B\times I\!R^N)}=||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}$$

and because of Jensen's inequality

$$\Big(||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}\Big)\ln\left(\frac{||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}}{(\epsilon(t))^N}\right)\leq\int\int_{\epsilon(t)B\times I\!R^N}\overline{f}(t,x,v)\;\ln\overline{f}(t,x,v)\,dxdv$$
$$+\ln(|B|)\cdot\int\int_{\epsilon(t)B\times I\!R^N}\overline{f}(t,x,v)\,dxdv\;.$$

Since

$$||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}+\left|\int\int_{\epsilon(t)B\times I\!R^N}\overline{f}(t,x,v)\;\ln\overline{f}(t,x,v)\,dxdv\right|\to 0\quad\text{as }t\to\tau,\;t<\tau\;,$$

$$\left(||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}\right)\ln\left(\frac{||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)}}{(\epsilon(t))^N}\right)\leq 1$$

for t in a neighborhood of τ^-, large enough.

Consider $h(x) = x\ln(\frac{x}{\eta}) - 1$ for some $\eta > 0$ small enough. $h(0) = -1$, h is convex and $\lim_{x\to+\infty} h(x) = +\infty$: $h(x) = 0$ has a unique solution $x = x(\eta) \leq \frac{1+e^{-1}}{|\ln(\eta)|}$ since $h\left(\frac{1+e^{-1}}{|\ln(\eta)|}\right) > 0$.

$$||f(t)||_{L^1(B\times I\!R^N)} = ||\overline{f}(t)||_{L^1(\epsilon(t)B\times I\!R^N)} \leq \frac{1+e^{-1}}{N|\ln(\epsilon(t))|} \quad \text{as } t\to\tau\ , \quad t<\tau\ .$$

PART III. THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM : STEADY STATES

A stationary solution of the Vlasov-Poisson-Fokker-Planck system, *i.e.* a solution of

$$\begin{cases} v\cdot\partial_x f - \nabla_x(U+U_0)\cdot\partial_v f = \partial_v\cdot(vf+\theta\partial_v f) \\ -\Delta U = \int_{I\!R^N} f(x,v)\,dv \end{cases}$$

in the renormalized sense has to be of the form

$$f(x,v) = M\cdot\frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}}\cdot\frac{e^{-(U+U_0)/\theta}}{\int_{I\!R^N} e^{-(U+U_0)/\theta}dx}\quad\text{with } M=\int\int_{I\!R^N\times I\!R^N} f(x,v)\,dxdv$$

provided $e^{-U_0/\theta}\in L^1(I\!R^N)$ (see [Dr1,2] for comments on such a factorization result). The Vlasov-Fokker-Planck equation is then satisfied if $\nabla_x\rho = -\frac{1}{\theta}(U+U_0)$ and the problem is reduced to solve the Poisson-Boltzmann-Emden equation

$$-\Delta U = M\cdot\frac{e^{-(U+U_0)}}{\int_{I\!R^N} e^{-(U+U_0)}\,dx}\,. \tag{PBE}$$

This has been proved by K. Dresler when $\liminf_{|x|\to+\infty}\frac{U_0(x)}{|x|}>0$ in [Dr1,2] and extended in [BD] to the case $\liminf_{|x|\to+\infty}\frac{U_0(x)}{\ln|x|}>N\theta$. Actually the result holds as well under the condition $e^{-U_0/\theta}\in L^1(I\!R^N)$, $U_0^-\in L^\infty$ in a neighbourhood of $|x|=+\infty$. This result is stated in Proposition 1.1 provided $\nabla U\in L^\infty(I\!R^N)$ and is generalized without this assumption using the notion of renormalized solutions (see remark 4 — a definition of renormalized solutions — and Appendix A, Proposition A.1).

1. Some remarks

A first method to handle the stationary solutions is to consider solutions of the evolution problem which do not depend on t (see 1). Since the free energy is a convex functional, such solutions are also characterized as critical points of appropriate functionals (see 2 and 3). A direct approach of the stationary solutions of the Vlasov-Poisson-Fokker-Planck system (without assumption on ∇U) can also be done with renormalized solutions (see remark 4 and Appendix A).

1) Proposition 1.1 : *Let $N \geq 3$. Assume that*

$$e^{-\frac{U_0}{\theta}} \in L^1(I\!\!R^N) \tag{1.1}$$

and that $f \in C^0(I\!\!R^+, L^1(I\!\!R^N \times I\!\!R^N))$ is such that

$$\int\int_{I\!\!R^N \times I\!\!R^N} \big(\frac{|v|^2}{2} + U_0(x) + \ln^+ f(t,x,v)\big) f(t,x,v)\, dxdv\,_{|t=0} + \frac{1}{2}\int_{I\!\!R^N} |\nabla U|^2(t,x)\, dx\,_{|t=0} < +\infty \tag{1.2}$$

and

$$\nabla_x U(t,x) \in L^\infty(I\!\!R^+_{\text{loc}} \times I\!\!R^N)\ , \tag{1.3}$$

f is a stationary solution if and only if f realizes the minimum of the functional

$$f \mapsto F[f] = \int\int_{I\!\!R^N \times I\!\!R^N} \big(\frac{|v|^2}{2} + \frac{U}{2} + U_0 + \theta \ln f(x,v)\big) f(x,v)\, dxdv\ .$$

under the constraint

$$M = \int\int_{I\!\!R^N \times I\!\!R^N} f(t,x,v)\, dxdv\ . \tag{1.4}$$

The proof of Proposition 1.1 relies as for Theorem 3.1 on Proposition 2.2 (Part I, Section 2.2).

2) If f is a stationary solution of the Vlasov-Poisson-Fokker-Planck system, then

$$-F[f] = \frac{1}{2}\int_{I\!\!R^N} |\nabla U|^2\, dx + \int_{I\!\!R^N} \rho_0 U\, dx + \theta M \ln\Big(\frac{1}{M} \cdot \int_{I\!\!R^N} e^{-\frac{U+U_0}{\theta}}\, dx\Big) + \frac{1}{2} N M \theta \ln(2\pi\theta)$$

and the functional $U \mapsto \frac{1}{2}\int_{I\!\!R^N} |\nabla U|^2\, dx + \int_{I\!\!R^N} \rho_0 U\, dx + \theta M \ln\Big(\frac{1}{M} \cdot \int_{I\!\!R^N} e^{-\frac{U+U_0}{\theta}}\, dx\Big)$ is strictly convex. The Euler-Lagrange equation associated to the unique critical point of this functional is precisely the Poisson-Boltzmann-Emden equation, provided $\rho_0 = e^{-\frac{U_0}{\theta}}$ is regular enough, $\rho_0 \in L^{\frac{2N}{N+2}}(I\!\!R^N)$ with $N \geq 3$ for instance (this approach has been used in [Do1]).

3) Instead of F, let us consider the functional

$$G[f,U] = \int\int_{I\!\!R^N \times I\!\!R^N} \big(\frac{|v|^2}{2} + U_0(x) + U(t,x) + \theta \ln f(t,x,v)\big) f(t,x,v)\, dxdv - \frac{1}{2}\int_{I\!\!R^N} |\nabla U(t,x)|^2\, dx\ .$$

Of course, $F[f] = G[f,U]$ provided U is a solution of the Poisson equation. But one can also notice that this functional is strictly convex in f, strictly concave in U, and that (f,U) *is formally a critical point of* G *under constraint* (1.4) *if and only if*

$$\frac{|v|^2}{2} + U_0 + U + \theta \ln f = Constant, \quad -\Delta U = \int_{I\!\!R^N} f(t,x,v)\, dv\ .$$

The Poisson-Boltzmann-Emden equation is then automatically satisfied, because of equation (1.4). *The unique steady state of the Vlasov-Poisson-Fokker-Planck system is therefore formally characterized as*

$$\sup_{\nabla U\in L^\infty(\mathbb{R}^+_{\text{loc}}\times\mathbb{R}^N)}\left(\inf_{\substack{f\in L^1(\mathbb{R}^N\times\mathbb{R}^N),\ f\geq 0,\\ f\ln f\in L^1(\mathbb{R}^N\times\mathbb{R}^N)\\ \iint_{\mathbb{R}^N\times\mathbb{R}^N} f\,dxdv=M}} G[f,U]\right)\ .$$

4) Renormalized stationary solutions such that

$$\sqrt{f}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\quad\text{and}\quad \partial_v\sqrt{f}\in L^2(\mathbb{R}^N\times\mathbb{R}^N)\ ,$$

are solutions of the stationary Vlasov-Poisson-Fokker-Planck system in the sense that the following equations hold true in the sense of the distributions for any $\varepsilon>0$

$$\frac{v\cdot\partial_x f}{\sqrt{f+\varepsilon}}+\frac{E(x)\cdot\partial_v f}{\sqrt{f+\varepsilon}}-\partial_v\cdot(v\sqrt{f+\varepsilon}+\theta\partial_v\sqrt{f+\varepsilon})=N\left(\frac{\varepsilon}{\sqrt{f+\varepsilon}}-\sqrt{\varepsilon}\right)+\frac{\theta}{2}\frac{|\partial_v f|^2}{(f+\varepsilon)^{3/2}}$$

(See [BD] for more details on renormalized solutions for the Vlasov-Poisson-Fokker-Planck system).

Such a notion of solution is useful to handle the general case (without *a priori* assumption on ∇U). A generalization of Proposition 1.1 is given in Appendix A, Proposition A.1 in that case. The proof relies on standard regularization techniques and a compactness result which is the analogue for stationary solutions of the results stated in [BD].

5) The question of the existence of solutions without imposing a normalization of mass has been addressed in [GSZ]. This can be viewed as a special case of the present framework (if a solution f exists, choose $M=||f||_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}$). The asymptotic boundary conditions for U is taken to be

$$U\longrightarrow 0\quad\text{in } L^{\frac{N}{N-2},\infty}\big(B^c(\mathbb{R}^N)\big)\quad\text{as } R\to+\infty\ .$$

A general framework would be that U behaves like a te sum of an harmonic function and a function satisfying the above condition (see [GSZ] for a discuss of this point).

2. Two examples

We first give two examples with direct proofs. These examples extend the results given in [GSZ].

There are no stationary solutions of the Vlasov-Poisson-Fokker-Planck system

$$\begin{cases} \partial_t f + v \cdot \partial_x f + E(t,x) \cdot \partial_v f = \partial_v \cdot (vf + \theta \partial_v f) \\ \mathrm{div}_x E(t,x) = \rho(t,x) = \displaystyle\int_{I\!R^N} f(t,x,v)\, dv - n(x) \end{cases}$$

with finite total mass $M = \int\int_{I\!R^3 \times I\!R^3} f(t,x,v)\ dxdv$ given by the Poisson-Boltmann-Emden equation if n belongs to $L^1(I\!R^3)$ (and has a compact support) whatever the value of $\int_{I\!R^3} n(x)\,dx$ is (this result is easily deduced from the general nonexistence result: $e^{-U_0/\theta} = e^{-\frac{1}{4\pi\theta|x|} * n(x)}$ does not belong to $L^1(I\!R^N)$).

Proposition 2.1 : *Assume that n is a nonnegative $L^1(I\!R^3)$ function and has a compact support. If*

$$U, U_0 \to Constant \quad \text{as} \quad |x| \to +\infty \quad \text{in } L^{3,\infty}(I\!R^3),$$

then the Poisson-Boltzmann-Emden equation

$$-\Delta U = M \cdot \frac{e^{-(U+U_0)/\theta}}{\int_{I\!R^N} e^{-(U+U_0)/\theta}\, dx} \tag{PBE}$$

has no solution such that

$$\int_{I\!R^3} f(t,x,v)\, dv = e^{-(U+U_0)/\theta} \in L^1(I\!R^3)\ .$$

Proof : First of all, there is no restriction to assume

$$\lim_{|x|\to+\infty} U(x) = \lim_{|x|\to+\infty} U_0(x) = 0 \quad \text{in } L^{3,\infty}(I\!R^3)\ ,$$

since adding a constant to U or U_0 does not change the (PBE) equation. Let

$$C = \frac{M}{4\pi \int_{I\!R^N} e^{-(U+U_0)/\theta}\, dx}\ .$$

We will assume that (PBE) has a solution and find a contradiction.

If $e^{-(U+U_0)/\theta}$ belongs to $L^1(I\!R^3)$, then

$$U(x) = \frac{C}{|x|} * e^{-(U+U_0)/\theta} \geq 0\ ,$$

U belongs to $L^{3,\infty}(I\!R^3)$ and

$$1 - \frac{U}{\theta} + \frac{U^2}{2\theta^2} \leq e^{-U/\theta} \leq 1 - \frac{U}{\theta}\ .$$

Since $L^1_{\text{unif}}(I\!R^3) \cap L^2_{\text{unif}}(I\!R^3) \subset L^{3,\infty}(I\!R^3)$,

$$\lim_{|x|\to+\infty} e^{-U/\theta} = 1 \quad \text{in } L^1_{\text{unif}}(I\!R^3)\ .$$

On the other side

$$0 \le U_0 = -\frac{1}{4\pi|x|} * n(x) \le -\frac{1}{4\pi(|x|-R)} \int_{I\!R^3} n(x)\, dx \quad \forall\, x \in B^c(0,R)\ ,$$

if n is supported in $B(0,R)$. $U_0(x)$ goes to 0 uniformly as $|x| \to +\infty$, which proves that

$$\lim_{|x|\to+\infty} e^{-(U+U_0)/\theta} = 1 \quad \text{in } L^1_{\text{unif}}(I\!R^3)\ .$$

This is obviously in contradiction with the assumption that

$$e^{-(U+U_0)/\theta} \in L^1(I\!R^3)\ . \qquad \square$$

Usually n is a background density of particles with charges of sign opposite to the sign of the charges of the particles that are described by f, but there is no difficulty to extend this proof to the case where n may be not everywhere positive: the negative part of n has to be taken into account with U (replace $U(x)$ by $(\frac{1}{4\pi|x|} * n^-(x)) + U(x)$, and the positive part of n can be treated in the same way as before. The nonexistence result still holds under the more general condition $n^+ \in L^1(I\!R^3)$. The condition that U has a compact support can also be replaced by the weaker assumption that U has enough moments in $|x|$.

2.2. Existence results for asymptotically constant or decaying underlying background densities

We assume here that the underlying background densities are asymptotically constant

$$n(x) = 1 + \eta(x) \quad \text{with } \eta(x) \to 0 \quad \text{as } |x| \to +\infty\ . \tag{2.1}$$

If we define U_0 as

$$U_0(x) = V_0(x) + \frac{|x|^2}{2N} \quad \forall\, x \in I\!R^N\ ,$$

then

$$\Delta V_0 = \eta(x)\ ,$$

and U is solution of the Poisson-Boltzmann-Emden equation

$$-\Delta U = M \cdot \frac{\rho_0(x) e^{-U/\theta}}{\int_{I\!R^N} \rho_0(x) e^{-U/\theta}\, dx} \tag{PBE}$$

with

$$\rho_0(x) = e^{-(\frac{|x|^2}{2N} + V_0(x))/\theta}\ .$$

The following result is an immediate consequence of the results contained in [Do1] (again the general theory applies and a necessary and sufficient condition for a solution to exist is: $e^{-U_0/\theta} \in L^1(I\!R^N)$).

Proposition 2.2 : *Assume that condition* (2.1) *is satisfied and* $N \geq 3$. *Then* (PBE) *has a unique solution in*

$$D^{1,2}(I\!R^N) = \{ v \in L^{\frac{2N}{N-2}}(I\!R^N) \mid \nabla v \in L^2(I\!R^N) \}$$

provided one of the two following conditions is satisfied:

(i) General case: assume that there exists a constant $\kappa \in [0,1[$ *such that* $(\kappa - \eta)^+$ *(i.e. its positive part) belongs to* $L^1 \cap L^P(I\!R^N)$ *for some* $p > N$.

(ii) Radial case ($N \geq 3$)*: assume that there exists a function* $\tilde{\eta} \in L^1_{\text{loc}}(I\!R^+)$ *such that* $\eta(x) = \tilde{\eta}(|x|)$*, such that*

$$V_0(r) = \int_0^r ds \int_0^s t^2 \tilde{\eta}(t)\, dt$$

is defined for all $r > 0$*, and such that for some* $\epsilon > 0$

$$\liminf_{r \to +\infty} \frac{V_0(r)}{r^2 - (N+\epsilon)\theta \ln(r)} \geq 0 \,.$$

Proof : According to [Do1], one has to check that ρ_0 belongs to $L^1(I\!R^N)$. In the general case (i), $(V_0 - \kappa \frac{|x|^2}{2N})^-$ is bounded in $L^\infty(I\!R^N)$. In the radial case (ii), the problem is solved by studying the asymptotic behaviour of $\frac{r^2}{2N} + V_0(r)$: $\rho_0 \in L^1(I\!R^N)$, and according to [Do1], the (PBE) has then a unique solution in $L^{\frac{N}{N-2},\infty}(I\!R^N)$ such that $\lim_{|x| \to +\infty} U(x) = 0$. It is not difficult to see that this solution is nonnegative (using for example the maximum principle). $U \mapsto Ue^{-U}$ is bounded for $U > 0$, which implies that U belongs to $D^{1,2}(I\!R^N)$. □

Proposition 2.1 applies for instance to any radial perturbation η such that there exists $\kappa \in [0,1[$ satisfying $\eta(x) - \kappa \leq \frac{1}{(|x|+1)^\alpha}$ as $|x| \to +\infty$ (for some $\alpha > 0$). Condition (ii) can also be refined: the optimal condition is in fact (see Theorem 3.1 below)

$$e^{-r^2 \cdot V_0(r)} \in L^1([0, +\infty[\,;\, r^2\, dr) \,.$$

3. A necessary and sufficient condition

Theorem 3.1 : *Let* $\theta > 0$, $M > 0$, $U_0 \in L^\infty_{\text{loc}}(I\!R^N)$, $N \geq 3$. *A necessary and sufficient condition for the existence of a solution* $U \in L^{\frac{N}{N-2},\infty}$ *of the Poisson-Boltzmann-Emden equation*

$$-\Delta U = \rho, \quad \rho = M \cdot \frac{e^{-(U+U_0)/\theta}}{\int_{I\!R^N} e^{-(U+U_0)/\theta} dx} \qquad (PBE)$$

such that ρ belongs to $L^1(I\!R^N)$ is

$$e^{-U_0/\theta} \in L^1(I\!R^N)\ .$$

The solution if it exists is unique.

Proof : Existence and uniqueness when $e^{-U_0/\theta} \in L^1(I\!R^N)$ have been proved in [Do1] (see also [Ba], [GL]).

The existence of a solution of the Poisson-Boltzmann-Emden equation means the existence of a critical point $\overline{\rho} \in L^1(I\!R^N)$ of the functional $\overline{G}$ defined in the proof of Proposition 2.2 in Part I, Section 2.2. Since $\overline{G}$ is convex, $\overline{G}$ reaches therefore its minimum (and is therefore bounded from below). The functional G is also convex :

$$2\overline{G}[\overline{\rho}] - G[\overline{\rho}] \leq G[\rho] \quad \forall\, \rho \in L^1(I\!R^N),\ \rho \geq 0 \quad \text{ such that } ||\rho||_{L^1(I\!R^N)} = M\ .$$

But $G[\overline{\rho}] = \overline{G}[\overline{\rho}]$ provides

$$G[\overline{\rho}] \leq G[\rho] \quad \forall\, \rho \in L^1(I\!R^N),\ \rho \geq 0 \quad \text{ such that } ||\rho||_{L^1(I\!R^N)} = M\ ,$$

which also proves that G is bounded from below.

According to Corollary 3.2 (Part I, Section 3), this is possible only if $e^{-U_0/\theta}$ belongs to $L^1(I\!R^N)$. □

Remark 3.2 : One may also prove the result directly by using the Jensen inequality when ρU belongs to $L^1(I\!R^N)$: as in Part I, Section I,

$$\begin{aligned} G[\rho] = &\int_{I\!R^N} \rho(x)\big(\theta \ln \rho(x) + \frac{1}{2}U(x) + U_0(x)\big)\ dx \\ &\qquad\qquad \geq \theta \int_{I\!R^N} \rho(x)\ dx \cdot \ln\left(\frac{\int_{I\!R^N} \rho(x)\ dx}{\int_{I\!R^N} e^{-\big(U(x)+U_0(x)\big)/\theta}\ dx}\right) - \frac{1}{2}\int_{I\!R^N} \rho(x)U(x)\ dx \end{aligned}$$

(apply Equation (1.1) of Part I with $y = x$, $g(y) = \rho(x)$ and $h(y) = U(x) + U_0(x)\big)$.

APPENDIX A. THE FREE ENERGY OF A SOLUTION OF THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM

In order to make this paper as self-consistent as possible, we present here some computations for the free energy of a solution of the Vlasov-Poisson-Fokker-Planck system. We first of all state the main estimate for the stationary case. Indications on the time-dependant case are given after.

A.1 The stationary case

Proposition A.1 : *Assume that $f \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$ is a nonnegative function such that*

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} (|v|^2 + \frac{U(x)}{2} + U_0(x) + \theta \ln^+ f(x,v)) f(x,v)\, dxdv < +\infty \ ,$$

where $U_0 \in L^\infty_{\rm loc}(\mathbb{R}^N)$ is such that $e^{-U_0/\theta} \in L^1(\mathbb{R}^N)$. If f is a solution of the stationary Vlasov-Poisson-Fokker-Planck system

$$\begin{cases} v \cdot \partial_x f - \nabla_x (U + U_0) \cdot \partial_v f = \partial_v \cdot (vf + \partial_v f) \\ -\Delta U = \displaystyle\int_{\mathbb{R}^N} f(x,v)\, dv \end{cases}$$

in the renormalized sense, then

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |v\sqrt{f} + 2\theta \partial_v \sqrt{f}|^2\, dxdv = 0 \ ,$$

$$f(x,v) = M \cdot \frac{e^{-\frac{|v|^2}{2\theta}}}{(2\pi\theta)^{N/2}} \cdot \frac{e^{-(U+U_0)/\theta}}{\int_{\mathbb{R}^N} e^{-(U+U_0)/\theta} dx} \quad \text{with } M = \int\int_{\mathbb{R}^N \times \mathbb{R}^N} f(x,v)\, dxdv \ ,$$

where U is a solution of the Poisson-Boltzmann-Emden equation

$$-\Delta U = M \cdot \frac{e^{-(U+U_0)}}{\int_{\mathbb{R}^N} e^{-(U+U_0)}\, dx} \ . \qquad (PBE)$$

Proof : Assume first that f belongs to $\mathcal{S}(\mathbb{R}^N \times \mathbb{R}^N)$, that $\rho = \int_{\mathbb{R}^N} f\, dv$ belongs to $\mathcal{S}(\mathbb{R}^N)$ and that

$$\liminf_{|x| \to +\infty} \frac{U_0(x)}{|x|^2} > 0 \ .$$

These assumptions on the regularity of f and ρ are consistent with the assumption on the asymptotic behaviour of U_0, and allow integration by parts.

Let us multiply the (VFP) equation by $(|v|^2/2+U+U_0+\ln f)$ and integrate by parts. Each term of the left side of the equation which does not cancel can be put in a divergence form

$$\int\int_{I\!R^N\times I\!R^N}(\frac{1}{2}|v|^2+U+U_0)\Big(v\cdot\partial_x f-\nabla_x(U+U_0)\cdot\partial_v f\Big)\,dxdv$$
$$=\int\int_{I\!R^N\times I\!R^N}\partial_x\cdot\Big(\frac{1}{2}v|v|^2f(x,v)\Big)\,dxdv-\frac{1}{2}\int\int_{I\!R^N\times I\!R^N}\partial_v\cdot\Big(\partial_x\cdot(U+U_0)\;f(x,v)\Big)\,dxdv$$
$$=0\;,$$

$$\int\int_{I\!R^N\times I\!R^N}\Big(1+\ln f\Big)\Big(v\cdot\partial_x f-\nabla_x(U+U_0)\cdot\partial_v f\Big)\,dxdv$$
$$=\int\int_{I\!R^N\times I\!R^N}\Big(\partial_x\cdot\big(v(f\ln f-f)\big)-\partial_v\big(\nabla_x(U+U_0)\cdot(f\ln f-f)\big)\Big)\,dxdv$$
$$=0\;,$$

and the right hand side gives

$$-\int\int_{I\!R^N\times I\!R^N}(vf+\theta\partial_v f)\;\partial_v\cdot(\theta\ln f+\frac{|v|^2}{2})\;dxdv$$
$$=-\int\int_{I\!R^N\times I\!R^N}(vf+\theta\partial_v f)\frac{1}{f}(vf+\theta\partial_v f)\;dxdv$$
$$=-\int\int_{I\!R^N\times I\!R^N}|v\sqrt{f}+2\theta\partial_v\sqrt{f}|^2\;dxdv\;.$$

A standard regularization method allows to extend the results to distribution functions satisfying the regularity assumptions of Proposition A.1 (the passage to the limit is possible because of the compactness properties of the Vlasov-Fokker-Planck operator). Actually, the result also follows from the time dependant case. □

Remark : In the stationary case, there is an other method ([P]) to prove that the steady states are Maxwellian functions. Formally multiply left and right hand sides of the Vlasov equation by

$$f\;e^{(\frac{|v|^2}{2}+U(x)+U_0(x))/\theta}$$

and integrate by parts :

$$0=-\int\int_{I\!R^N\times I\!R^N}\frac{f^2}{2}\Big[\Big(v\cdot\partial_x-\nabla_x(U+U_0)\cdot\partial_v\Big)\cdot e^{(\frac{|v|^2}{2}+U(x)+U_0(x))/\theta}\Big]\;dxdv$$
$$=-\frac{1}{\theta}\int\int_{I\!R^N\times I\!R^N}|vf+\theta\partial_v f|^2\cdot e^{(\frac{|v|^2}{2}+U(x)+U_0(x))/\theta}\;dxdv\;.$$

The estimates for the time-dependant case are formally obtained in the same way. Let us assume for more simplifications that in the Vlasov-Fokker-Planck equation, the field $E(t,x)$ is coupled with f through a smooth kernel $K(x) \in \mathcal{D}(R^N)$:

$$U(t,x) = K(x) * \int_{\mathbb{R}^N} f(t,x,v)\, dv\ ,$$

instead of the kernel $-\frac{1}{4\pi}\frac{1}{|x|}$ (in dimension $N=3$; see [BD]). We just present here a formal computation.

First of all, a direct integration w.r.t. x and v gives

$$\frac{d}{dt}\int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(t,x,v)\, dxdv = 0\ . \qquad (Mass\ Conservation)$$

We can also multiply the (VFP) equation by $|v|^2$ and integrate again w.r.t. x and v. Integrations by parts provide for the left hand side

$$\frac{d}{dt}\int\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|v|^2}{2}f(t,x,v)dxdv - \int_{\mathbb{R}^N} dx\ (U+U_0)\ \partial_x\cdot\int_{\mathbb{R}^N} vf(t,x,v)\, dv\ .$$

Using again the (VFP) equation (multiplied by v and integrated w.r.t. v), we get

$$\partial_x\cdot\int_{\mathbb{R}^N} vf(t,x,v)\, dv = -\frac{\partial}{\partial t}\int_{\mathbb{R}^N} f(t,x,v)\, dv$$

and can evaluate the last term :

$$\begin{aligned}&\int_{\mathbb{R}^N} dx\ (U+U_0)\ \partial_x\int_{\mathbb{R}^N} vf(t,x,v)\, dv\\ &\qquad = -\int_{\mathbb{R}^N} dx\ (U+U_0)\ \frac{\partial}{\partial t}\int_{\mathbb{R}^N} f(t,x,v)\, dv\\ &\qquad = -\frac{\partial}{\partial t}\int_{\mathbb{R}^N} dx\ (\frac{U}{2}+U_0)\int_{\mathbb{R}^N} f(t,x,v)\, dv\ ,\end{aligned}$$

which provides the

$$(Free\ Energy\ Estimate)$$

$$\frac{d}{dt}\int\int_{\mathbb{R}^N\times\mathbb{R}^N}(\frac{|v|^2}{2}+\frac{U}{2}+U_0+\theta\ln f)f(t,x,v)\, dxdv = -\int\int_{\mathbb{R}^N\times\mathbb{R}^N}|v\sqrt{f}+2\theta\partial_v\sqrt{f}|^2\, dxdv\ .$$

The free energy is finite and bounded from below as soon as

$$\int\int_{\mathbb{R}^N\times\mathbb{R}^N}(|v|^2+\frac{U}{2}+U_0+\ln^+ f)f(t,x,v)\, dxdv_{|t=0} < +\infty$$

(see Part I) if $e^{-U_0/\theta}\in L^1(\mathbb{R}^N)$. As in the stationary case, this result can be completely justified in a general setting (see [BD] for a detailed proof) :

Lemma A.2 : *Assume that* $f \in C(][0,+\infty[; L^1(\mathbb{R}^N \times \mathbb{R}^N))$ *is a nonnegative function such that*

$$\int\int_{\mathbb{R}^N \times \mathbb{R}^N} (|v|^2 + \frac{U(t,x)}{2} + U_0(x) + \theta \ln^+ f(t,x,v)) f(t,x,v)\, dxdv_{|t=0} < +\infty \,,$$

where $U_0 \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *is such that* $e^{-U_0/\theta} \in L^1(\mathbb{R}^N)$*. If* f *is a solution of the Vlasov-Poisson-Fokker-Planck system*

$$\begin{cases} \partial_t f + v \cdot \partial_x f - \nabla_x (U + U_0) \cdot \partial_v f = \partial_v \cdot (vf + \theta \partial_v f) \\ -\Delta U = \int_{\mathbb{R}^N} f(t,x,v)\, dv \end{cases}$$

in the renormalized sense, then

$$t \mapsto F[f(t)] = \int\int_{\mathbb{R}^N \times \mathbb{R}^N} (|v|^2 + \frac{U(t,x)}{2} + U_0(x) + \theta \ln^+ f(t,x,v)) f(t,x,v)\, dxdv$$

is decreasing and for any $T > 0$*,*

$$\lim_{t \to +\infty} \int_t^{t+T} \left(\int\int_{\mathbb{R}^N \times \mathbb{R}^N} |v \sqrt{f(s,x,v)} + 2\theta \partial_v \sqrt{f(s,x,v)}|^2\, dxdv \right) ds = 0 \,.$$

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