

# Symmetry breaking and weighted Euclidean logarithmic Sobolev inequalities

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**ABSTRACT.** On the Euclidean space, we establish some *Weighted Logarithmic Sobolev* (WLS) inequalities. We characterize a symmetry range in which optimal functions are radially symmetric, and a symmetry breaking range. (WLS) inequalities are a limit case for a family of subcritical *Caffarelli-Kohn-Nirenberg* (CKN) inequalities with similar symmetry properties. A generalized *carré du champ* method applies not only to the optimal solution of the nonlinear elliptic Euler-Lagrange equation and proves a rigidity result as for (CKN) inequalities, but also to entropy type estimates, with the full strength of the *carré du champ* method in a parabolic setting. This is a significant improvement on known results for (CKN). Finally, we briefly sketch some consequences of our results for the weighted diffusion flow.

**KEYWORDS.** Logarithmic Sobolev inequality; Hardy-Sobolev inequality; Caffarelli-Kohn-Nirenberg inequality; symmetry breaking; symmetry; concentration-compactness; optimal functions; optimal constant; carré du champ method

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## 1 Introduction and main results

*Logarithmic Sobolev inequalities* are well known cases of functional inequalities with many applications in various areas of mathematics ranging from information theory to probability theory, functional analysis, differential geometry and mathematical physics. In partial differential equations, these inequalities now appear as fundamental tools for the understanding of rates of convergence, not only for diffusion equations but also, for instance, in kinetic theory. Various settings have been considered depending on the geometry, the presence of a drift or a potential, or the choice of a reference measure. Sharp inequalities, with optimal constants, and equality cases are trickier issues, as the problem is usually difficult to reduce to spectral estimates. Among the few known examples, we can quote the case of the sphere and the characterization [15] by E. Carlen of the set of optimal functions in the Euclidean logarithmic Sobolev inequalities. See [40] for general weights.

In this article, we mainly focus on the case of  $\mathbb{R}^d$  with homogeneous (power-law) weights because of the *symmetry versus symmetry breaking* issue. This is a well-known question for Caffarelli-Kohn-Nirenberg inequalities. Although the weights are invariant under rotations, optimal functions are not necessarily spherically symmetric. V. Felli and M. Schneider gave in [44] a condition for symmetry breaking based on the linear instability of the radial solutions of the Euler-Lagrange equations. Symmetry is a global property. Proving symmetry is therefore a delicate issue and standard methods like moving planes or symmetrization techniques are not sufficient to cover all cases. The problem has recently been fully solved in [35, 38] for some special sub-families of the Caffarelli-Kohn-Nirenberg inequalities using a nonlinear version of the *carré du champ* method introduced by D. Bakry and M. Emery in [3], applied to the Euler-Lagrange equation solved by the optimal functions. The underlying framework is based on entropy methods for nonlinear diffusion equations, but the approach is so far formal by lack of regularity estimates to justify all computations: see [34] for partial results. In the case of (WLS) inequalities, we can use the whole parabolic approach of entropy methods as there is a dense set of (Hermite) polynomials and integrations by parts can be justified. As far as we know, this is the first application of the parabolic *carré du champ* method to the *symmetry versus symmetry breaking* issue for weighted inequalities on  $\mathbb{R}^d$ .

Let  $L^q_\gamma(\mathbb{R}^d)$  with  $d \geq 1$  be the space of all measurable functions  $f$  such that

$$\|f\|_{q,\gamma} := \left( \int_{\mathbb{R}^d} |f|^q |x|^{-\gamma} dx \right)^{1/q}$$

is finite. We also define the space  $H_{\beta,\gamma}^1(\mathbb{R}^d)$  of the functions  $f \in L_\gamma^2(\mathbb{R}^d)$  such that  $\nabla f \in L_\beta^2(\mathbb{R}^d)$  and consider the *weighted logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} \frac{|f|^2}{\|f\|_{2,\gamma}^2} \log \left( \frac{|f|^2}{\|f\|_{2,\gamma}^2} \right) |x|^{-\gamma} dx \leq \mathcal{C}_{\beta,\gamma} + \frac{n}{2} \log \left( \frac{\|\nabla f\|_{2,\beta}^2}{\|f\|_{2,\gamma}^2} \right) \quad \forall f \in H_{\beta,\gamma}^1(\mathbb{R}^d) \quad (\text{WLS})$$

with

$$n := \frac{2(d-\gamma)}{\beta+2-\gamma} \quad (1)$$

and real parameters  $\beta$  and  $\gamma$  satisfying the condition

$$\gamma - 2 < \beta < \frac{d-2}{d} \gamma < d. \quad (2)$$

In (WLS),  $\mathcal{C}_{\beta,\gamma}$  denotes the optimal constant. Let us define the *Felli & Schneider curve*

$$\beta_{\text{FS}}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4(d-1)}, \quad (3)$$

consider the additional parameter

$$\alpha := 1 + \frac{\beta-\gamma}{2}, \quad (4)$$

define the function

$$f_\star(x) := c_{n,d} \sqrt{\alpha} e^{-\frac{1}{4}|x|^{2\alpha}} \quad \text{with} \quad c_{n,d} = \sqrt{\frac{\Gamma(\frac{d}{2})}{2^{\frac{n}{2}} \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})}}$$

such that  $\|f_\star\|_\gamma = 1$  and the constant

$$\mathcal{C}_{\beta,\gamma}^\star := \log \left( \frac{\left(\frac{2}{ne}\right)^{\frac{n}{2}} \Gamma(\frac{d}{2})}{\alpha^{n-1} \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})} \right). \quad (5)$$

Our main result deals with the *symmetry versus symmetry breaking* issue and goes as follows.

**Theorem 1.1 .** *Let  $d \geq 2$ . Assume that  $(\beta, \gamma) \neq (0, 0)$  satisfies (2). Then Inequality (WLS) holds for some constant  $\mathcal{C}_{\beta,\gamma} \leq \mathcal{C}_{\beta,\gamma}^\star$ . Equality in (WLS) is achieved by an optimal function  $f_{\beta,\gamma} \in H_{\beta,\gamma}^{1,1}(\mathbb{R}^d) \setminus \{0\}$  and there are two cases:*

(i) *Symmetry breaking :  $\mathcal{C}_{\beta,\gamma} < \mathcal{C}_{\beta,\gamma}^\star$  and  $f_{\beta,\gamma}$  is not radially symmetric if and only if*

$$\gamma < 0 \quad \text{and} \quad \beta_{\text{FS}}(\gamma) < \beta < \frac{d-2}{d} \gamma. \quad (6)$$

(ii) *Symmetry :  $\mathcal{C}_{\beta,\gamma} = \mathcal{C}_{\beta,\gamma}^\star$  and all optimal functions are given by  $f_\star$  up to a multiplication by an arbitrary real constant and a scaling if and only if*

$$\gamma < d \quad \text{and} \quad \gamma - 2 \leq \beta \leq \beta_{\text{FS}}(\gamma). \quad (7)$$

□

In the symmetry breaking range, the set of optimal functions is generated by  $f_{\beta,\gamma}$  up to rotations, a multiplication by an arbitrary real constant and a scaling. If  $(\beta, \gamma) = (0, 0)$ , optimality is achieved by Gaussian functions corresponding to  $\alpha = 1$  and translations also have to be taken into account according to [15]. If  $d = 1$ , we have the same result as in the symmetry case:  $\mathcal{C}_{\beta,\gamma} = \mathcal{C}_{\beta,\gamma}^\star$  and all optimal functions are given by  $f_\star$  up to a multiplication by an arbitrary real constant and a scaling. At this stage, the driving mechanism responsible for the symmetry breaking phenomenon might still look somewhat mysterious. We will now reformulate Inequality (WLS) into various equivalent forms before coming back to a qualitative explanation of the competition between terms of different nature which explains why symmetry breaking occurs in the range  $\beta > \beta_{\text{FS}}(\gamma)$ .

As in [8], we can reduce (WLS) to the case  $\beta = \gamma$ , at the price of an anisotropy in the gradient term measured by  $\alpha \neq 1$ . Let us consider the *artificial dimension*  $n$  given by (1) and take

$$\nu := d - n.$$

Let us define the operator  $D_\alpha$

$$D_\alpha = \nabla + (\alpha - 1) \frac{x}{|x|^2} (x \cdot \nabla) = \nabla + (\alpha - 1) \omega \partial_r,$$

so that, in spherical coordinates  $(r, \omega) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$ , it writes

$$D_\alpha w = \left( \alpha \partial_r w, \frac{1}{r} \nabla_\omega w \right).$$

By Condition (2), notice that  $n > d$  and  $\nu < 0$  arise from  $\beta < (d-2)\gamma/d$ .

To a function  $f \in H_{\beta, \gamma}^1(\mathbb{R}^d)$ , we associate the function  $g \in H_{\nu, \nu}^1(\mathbb{R}^d)$  such that

$$f(x) = g(|x|^{\alpha-1} x) \quad \forall x \in \mathbb{R}^d. \quad (8)$$

With this change of variables, the function  $f_\star$  is transformed into the *Gaussian* function

$$g_\star(x) = c_{n,d} e^{-\frac{1}{4}|x|^2} \quad \text{with} \quad c_{n,d} = \sqrt{\frac{\Gamma(\frac{d}{2})}{2^{\frac{n}{2}} \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})}}, \quad (9)$$

where the normalization constant  $c_{n,d}$  is such that  $\|g_\star\|_{2,\nu} = 1$ . Let us define

$$\alpha_{\text{FS}} := \sqrt{\frac{d-1}{n-1}} \quad \text{and} \quad \mathcal{K}_{n,\alpha}^\star := \mathcal{C}_{\beta,\gamma}^\star - \log \alpha = \log \left( \frac{(\frac{2}{n e})^{\frac{n}{2}} \Gamma(\frac{d}{2})}{\alpha^n \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})} \right).$$

Using (8), Inequality (WLS) is transformed into the *n-dimensional weighted logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} \frac{|g|^2}{\|g\|_{2,\nu}^2} \log \left( \frac{|g|^2}{\|g\|_{2,\nu}^2} \right) |x|^{-\nu} dx \leq \mathcal{K}_{n,\alpha} + \frac{n}{2} \log \left( \frac{\|D_\alpha g\|_{2,\nu}^2}{\|g\|_{2,\nu}^2} \right) \quad (\text{WLS}_n)$$

where  $n$  plays the role of a dimension at least for scaling properties, even if it is not an integer. Rewritten with the parameters  $\alpha$ ,  $n$  and  $\nu$ , Theorem 1.1 goes as follows.

**Corollary 1.2** . *Let  $n > d \geq 1$ ,  $\nu = d - n$ , and assume that  $\alpha \in (0, 1) \cup (1, +\infty)$ . Then Inequality (WLS<sub>n</sub>) holds for some constant  $\mathcal{K}_{n,\alpha} \leq \mathcal{K}_{n,\alpha}^\star$ . Equality in (WLS<sub>n</sub>) is achieved by an optimal function  $g_{\alpha,n} \in H_{\nu,\nu}^1(\mathbb{R}^d) \setminus \{0\}$  and there are two cases:*

- (i) Symmetry breaking :  $\mathcal{K}_{n,\alpha} < \mathcal{K}_{n,\alpha}^\star$  and  $g_{\alpha,n}$  is not radially symmetric if and only if  $\alpha > \alpha_{\text{FS}}$  and  $d \geq 2$ .
- (ii) Symmetry :  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^\star$  and all optimal functions are given by  $g_\star$  up to a multiplication by an arbitrary real constant and a scaling if and only if either  $d \geq 2$  and  $\alpha \leq \alpha_{\text{FS}}$ , or  $d = 1$ .

□

Since Corollary 1.2 is equivalent to Theorem 1.1 by the change of variables (8), we will use interchangeably the two statements. Any result proved for (WLS) is also true for (WLS<sub>n</sub>) and reciprocally. There are various other equivalent forms of the (WLS) inequalities, exactly as for the standard logarithmic Sobolev inequalities:

▷ The two *non-scale invariant weighted logarithmic Sobolev inequalities*,

$$\|\nabla f\|_{2,\beta}^2 - \sigma \int_{\mathbb{R}^d} |f|^2 \log \left( \frac{|f|^2}{\|f\|_{2,\gamma}^2} \right) |x|^{-\gamma} dx \geq \sigma \left( \frac{n}{2} \log \left( \frac{2e}{n\sigma} \right) - \mathcal{C}_{\beta,\gamma} \right) \|f\|_{2,\gamma}^2, \quad (10a)$$

$$\|D_\alpha g\|_{2,\nu}^2 - \sigma \int_{\mathbb{R}^d} |g|^2 \log \left( \frac{|g|^2}{\|g\|_{2,\nu}^2} \right) |x|^{-\nu} dx \geq \sigma \left( \frac{n}{2} \log \left( \frac{2e}{n\sigma} \right) - \mathcal{K}_{n,\alpha} \right) \|g\|_{2,\nu}^2, \quad (10b)$$

hold for any  $\sigma > 0$  and are equivalent to (WLS) and (WLS<sub>n</sub>). This can be recovered by optimizing the left-hand sides under the scalings  $\lambda \mapsto \lambda^{(d-\gamma)/2} f(\lambda \cdot)$  and  $\lambda \mapsto \lambda^{n/2} g(\lambda \cdot)$ . The equality case in (10b) is achieved by the function  $g_\star^{\alpha,\sigma}(x) := (2\sigma\alpha^{-2})^{n/4} g_\star(\sqrt{2\sigma}x/\alpha)$  if  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^\star$  (symmetry case). Here  $g_\star$  is the Gaussian function given in (9).

- ▷ *Gaussian-like inequalities.* In the case of the standard Sobolev inequality, without weights, the Euclidean form of the inequality is equivalent to the Gaussian form. We have the exact counterpart, which goes as follows. Let us define the probability measure

$$d\nu_\sigma := \nu_\sigma dx \quad \text{with} \quad \nu_\sigma(x) := |x|^{-\nu} (g_\star^{\alpha,\sigma}(x))^2 = c_{n,d}^2 \left(\frac{2\sigma}{\alpha^2}\right)^{\frac{n}{2}} |x|^{-\nu} e^{-\frac{\sigma}{\alpha^2}|x|^2}$$

with  $g_\star^{\alpha,\sigma}$  defined as above. Then (10b) applied to the function  $g = v g_\star^{\alpha,\sigma}$  amounts to

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha v|^2 d\nu_\sigma \geq \sigma \int_{\mathbb{R}^d} |v|^2 \log \left( \frac{|v|^2}{\int_{\mathbb{R}^d} |v|^2 d\nu_\sigma} \right) d\nu_\sigma + \sigma \left( \mathcal{K}_{n,\alpha} - \mathcal{K}_{n,\alpha}^\star \right) \int_{\mathbb{R}^d} |v|^2 d\nu_\sigma \quad (11a)$$

for any  $v \in \mathsf{H}^1(\mathbb{R}^d, d\nu_\sigma)$ , with  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^\star$  in the symmetry range and  $\mathcal{K}_{n,\alpha} - \mathcal{K}_{n,\alpha}^\star < 0$  in the symmetry breaking range. Using the change of variable (8) with  $u(x) = v(|x|^{\alpha-1}x)$  and the probability measure

$$d\mu_\sigma := \mu_\sigma dx \quad \text{with} \quad \mu_\sigma(x) := \alpha \nu_\sigma(|x|^{\alpha-1}x),$$

we also obtain

$$\int_{\mathbb{R}^d} |\nabla u|^2 |x|^{\gamma-\beta} d\mu_\sigma \geq \sigma \int_{\mathbb{R}^d} |u|^2 \log \left( \frac{|u|^2}{\int_{\mathbb{R}^d} |u|^2 d\mu_\sigma} \right) d\mu_\sigma + \sigma \left( \mathcal{C}_{\beta,\gamma} - \mathcal{C}_{\beta,\gamma}^\star \right) \int_{\mathbb{R}^d} |u|^2 d\mu_\sigma. \quad (11b)$$

If  $\mathcal{C}_{\beta,\gamma} = \mathcal{C}_{\beta,\gamma}^\star$  (or equivalently  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^\star$ ), the equality case in (11b) is achieved by the function  $u(x) = 1$  a.e. and in (11a) by  $v(x) = 1$  a.e.

- ▷ *Euclidean logarithmic Sobolev inequalities with Hardy-type correction terms.* We denote by  $L^q(\mathbb{R}^d)$  the standard Lebesgue space with norm  $\|f\|_q := \|f\|_{q,0}$  and consider the function  $h(x) := |x|^{-\nu/2} g(x)$ . An expansion of the square and an integration by parts show that

$$\|\mathsf{D}_\alpha g\|_{2,\nu}^2 = \left\| \mathsf{D}_\alpha h + \frac{1}{2} \alpha \nu \frac{x}{|x|^2} h \right\|_2^2 = \|\mathsf{D}_\alpha h\|_2^2 - \frac{1}{4} \alpha^2 \nu (2(d-2) - \nu) \int_{\mathbb{R}^d} \frac{|h|^2}{|x|^2} dx.$$

For any  $\sigma > 0$ , we can rewrite (10b) in terms of  $h$  as

$$\int_{\mathbb{R}^d} \left( |\mathsf{D}_\alpha h|^2 + V_{\alpha,\nu,\sigma} |h|^2 - \sigma |h|^2 \log \left( \frac{|h|^2}{\|h\|_2^2} \right) \right) dx \geq \sigma \left( \frac{n}{2} \log \left( \frac{2e}{n\sigma} \right) - \mathcal{K}_{n,\alpha} \right) \|h\|_2^2 \quad \forall h \in \mathsf{H}^1(\mathbb{R}^d) \quad (12)$$

where the left-hand side is a Schrödinger energy with an anisotropic kinetic term if  $\alpha \neq 1$ , a logarithmic nonlinearity and a potential

$$V_{\alpha,\nu,\sigma}(x) := -\frac{1}{4} \alpha^2 \nu (2(d-2) - \nu) \frac{1}{|x|^2} - \sigma \nu \log |x| \quad \forall x \in \mathbb{R}^d \setminus \{0\}. \quad (13)$$

We recall that  $\nu$  is a negative parameter: the potential  $V_{\alpha,\nu,\sigma}$  is radially symmetric, with a singularity at  $x = 0$  such that  $\lim_{x \rightarrow 0} V_{\alpha,\nu,\sigma}(x) = +\infty$ , and we also have  $\lim_{|x| \rightarrow +\infty} V_{\alpha,\nu,\sigma}(x) = +\infty$ . An elementary computation shows that  $V_{\alpha,\nu,\sigma}$  achieves its minimum on the centered sphere of radius  $\alpha \sqrt{(2(d-2) - \nu)/(2\sigma)}$ .

Inequality (12) is typical a problem for symmetry breaking. If  $\alpha = 1$  and we omit the potential, a Schwarz symmetrization shows that the minimum of the Schrödinger energy is achieved by a radially symmetric function  $h$ , up to a translation. On the other hand, if we include  $V_{\alpha,\nu,\sigma}$ , in order to minimize the potential energy term, it is favourable to localize as much as possible  $h$  close to a point  $\bar{x}$  in the set of the minima of  $V_{\alpha,\nu,\sigma}$  which, as a curved surface is not compatible with radial symmetry centred at  $\bar{x}$ . A competition between the kinetic and the potential energy terms is taking place, and the arbitrage is gauged by the parameter  $\alpha$ . In the limiting case  $\alpha \rightarrow 0$ , only gradients in the angular variables have to be minimized to make  $\|\mathsf{D}_\alpha h\|_2^2$  small, which favours radially symmetric solutions, but this is not the case for  $\alpha$  large. Going further in this qualitative analysis to decide which one of the two terms wins is difficult. It is the purpose of this paper to give a clear cut answer.

The *carré du champ* method does not only determine the optimal functions in the weighted logarithmic Sobolev inequalities but also characterizes all positive critical points. If  $\sigma = 1$ , let us consider the Euler-Lagrange equations associated with (10a) and (10b), that is,

$$-|x|^\gamma \nabla \cdot (|x|^{-\beta} \nabla f) + f = f \log |f|^2 \quad \text{and} \quad \mathsf{D}_\alpha^\star \mathsf{D}_\alpha g + g = g \log |g|^2 \quad (14)$$

for an appropriate choice of  $\|f\|_{2,\gamma}^2$  and  $\|g\|_{2,\nu}^2$ . We have the following *rigidity* result.

**Corollary 1.3** . *Under the assumptions of Theorem 1.1 or Corollary 1.2, each of the two equations of (14) admits a unique positive solution, given respectively, up to a scaling and a multiplication by a positive constant, by  $f_*$  and  $g_*$ , in the symmetry range. In the symmetry breaking range, each of the two equations admits at least one radially symmetric solution and a continuum of no-radial solutions.*  $\square$

In the limit case  $(\beta, \gamma) = (0, 0)$  corresponding to  $(n, \alpha) = (d, 1)$ , which is not covered in (2), uniqueness is achieved only up to additional translations. We will not give a detailed proof of Corollary 1.3, as it is an elementary consequence of the proof of Corollary 1.2. From the point of view of nonlinear elliptic equations, it amounts to test the equations of (14) by  $-|x|^\gamma \nabla \cdot (|x|^{-\beta} \nabla f)$ . To implement the *carré du champ* method, we use a dynamical version of these test functions given by the weighted heat flows

$$\frac{\partial u}{\partial t} = |x|^\gamma \nabla \cdot (|x|^{-\beta} \nabla u) , \quad (15)$$

Proving (WLS) and (WLS<sub>n</sub>) in the symmetry range is obtained by identifying the optimal decay rate of the *entropy*. The core of the method of D. Bakry and M. Emery is to evolve the entropy by the weighted heat flow: its time-derivative is the *Fisher information*. Reapplying the flow, the key point is to prove the exponential decay of the Fisher information by computing one more  $t$ -derivative.

Let us give a brief review of the literature. For sake of simplicity, results involving powers of  $|\nabla f|^p$  with  $p \neq 2$ , higher order derivatives related for instance to Rellich inequalities, critical weights corresponding to Hardy-type inequalities or results on general manifolds or on Lie groups will not be systematically mentioned, but we will give at least some entry points in the literature. *Logarithmic Sobolev inequalities* have been widely studied, in various settings: see [26, 43, 48, 58, 60] for historical references, [2, 49, 56] for introductory books or lecture notes, and [4, Chapter 5] for a general presentation of CD( $\rho, N$ ) methods applied to functional inequalities. In [5, 45, 46] and [59, Chapter 5], one can find various sufficient conditions for logarithmic Sobolev inequalities to hold. See [5, 53], [59, Chapter 6] and [1] for some results on the interpolation inequalities between Poincaré and logarithmic Sobolev inequalities. Optimal constants and equality cases in logarithmic Sobolev inequalities are tricky issues: beyond observations based on the *carré du champ* in [3], we refer to [15] in the Euclidean and Gaussian cases, to [30] on cylinders (in connection with Caffarelli-Kohn-Nirenberg inequalities), and to [13] for recent considerations on stability in strong norms (see references therein for other stability results measured in, e.g., Wasserstein distance).

In this paper we consider the simple setting of  $\mathbb{R}^d$  with power-law weights, for scaling reasons. Norms other than the standard Euclidean norm could be considered, but the corresponding symmetry results are, to the best of our knowledge, unknown. Our (WLS) inequalities appear as a limit case for a family of subcritical Caffarelli-Kohn-Nirenberg inequalities (CKN), for which *symmetry breaking* is a well known issue that was addressed in various papers: see [8, 9, 17, 23, 25, 35, 38, 39, 44, 50, 57], among others. See Section 3 for some explanations of the mathematical issues. Concerning positive critical points of (CKN), a rigidity result holds as a consequence of a generalized *carré du champ* method applied to the nonlinear elliptic equation solved by the optimal functions. This rigidity result can be rephrased in terms of the properties of branches of solutions of nonlinear elliptic equations depending on a parameter: see [37]. At a formal level, these results for (CKN) can also be interpreted in the framework of entropy methods as strict monotonicity properties deduced from the *carré du champ* method adapted to nonlinear diffusion equations.

The results on sharp functional inequalities in [33, 35] are inspired, on the one hand, by the rigidity results for nonlinear elliptic equations studied in [6, 47], and on the other hand, by entropy and diffusion flows of [3, 29]. The connection is made precise and expanded in [33, 36]: the *carré du champ* method is a central idea for the overall strategy which applies very well to linear diffusion flows with drift potential terms or on compact manifolds. The *carré du champ* method has many aspects, but from the functional inequalities point of view, one can just keep in mind that monotonicity properties through the diffusion flow relate the functional inequality written for an arbitrary initial data to an asymptotic regime, which can be studied using spectral methods. See [10] for an extended presentation applied to a family of Gagliardo-Nirenberg-Sobolev inequalities. Applied to nonlinear flows on the Euclidean space, new difficulties arise as, for instance, integration by parts require precise decay bounds which are not easy to justify. Progress in the absence of singular weights has been achieved in [16, 52]. In presence of weights, the method formally applies but only partial results have been rigorously justified in [11, 34, 42]. For the optimizers of functional inequalities involving singular weights, the difficulty can be bypassed by proving the existence of minimizers and testing directly the solutions of the Euler-Lagrange, which amounts to testing such critical points in the direction corresponding to the flow. This is the simplest interpretation of the method of B. Gidas and J. Spruck in [47]. The issue is then reduced to a rigidity issue for solutions of elliptic equations which, as such, have good regularity and decay properties. So far, all sharp results of symmetry in Caffarelli-Kohn-Nirenberg inequalities have been obtained using such an approach. In the case of logarithmic Sobolev inequalities, we are able to perform the whole parabolic method as there is a dense set

of Hermite functions, in the appropriate version of the inequality, and integrations by parts can be justified. To our knowledge, this is the first result of symmetry *versus* symmetry breaking to be proved with the parabolic version of the *carré du champ* method.

The logarithmic Hardy inequalities studied in [28, 31] correspond to a boundary of the admissible domain of parameters in (WLS). So far, we are not aware of a method that would allow us to deduce results from (WLS) by taking an appropriate limit. For completeness, let us give a few additional reading indications on papers related with ours. Concerning logarithmic Hardy and Sobolev inequalities on Lie groups, we refer to [22] and references therein. See [27] and references therein for logarithmic inequalities involving powers of  $|\nabla f|^p$  with  $p \neq 2$ . We refer to [7, 18, 21] for logarithmic Sobolev inequalities corresponding to non-singular weights of the form  $(1 + |x|^2)^{-\beta/2}$  known as Cauchy measures and their links with the *super Poincaré inequalities*, and to [19, 20, 24] for various related contributions.

This paper is organized as follows. In Section 2, we use the spectral method of V. Felli and M. Schneider to prove the linear instability of the radial optimal functions in the symmetry breaking range of (WLS) and the *carré du champ* method to establish the symmetry in the symmetry range, with self-contained proofs. We use entropy methods in a parabolic setting to prove the symmetry result, which is the first result of this type obtained at non-formal level using a diffusion equation of evolution, in presence of weights. In Section 3, we show how (WLS) can be seen as a limit case for a family of subcritical Caffarelli-Kohn-Nirenberg inequalities. Notice that  $\Gamma$ -convergence methods is expected to provide us with an alternative proof of Corollary 1.2 and Theorem 1.1. Section 4 is devoted to some consequences of our results for the weighted heat flow associated to our weighted logarithmic Sobolev inequalities.

## 2 Optimal functions, symmetry and symmetry breaking in (WLS) inequalities

### 2.1 The weighted logarithmic Sobolev inequality

We start by proving that Inequality (WLS) is well-defined.

**Lemma 2.1 .** *Let  $d \geq 1$ . Assume that  $(\beta, \gamma) \neq (0, 0)$  satisfies (2). Then the inequality*

$$\int_{\mathbb{R}^d} \frac{|f|^2}{\|f\|_{2,\gamma}^2} \log \left( \frac{|f|^2}{\|f\|_{2,\gamma}^2} \right) |x|^{-\gamma} dx \leq \mathcal{C}_{\beta,\gamma}^* + \frac{n}{2} \log \left( \frac{\|\nabla f\|_{2,\beta}^2}{\|f\|_{2,\gamma}^2} \right) \quad \forall f \in \mathbb{H}_{\beta,\gamma}^{1,1}(\mathbb{R}^d)$$

holds with  $\mathcal{C}_{\beta,\gamma}^*$  defined by (5). □

In other words, we prove here that Inequality (WLS) holds for some constant  $\mathcal{C}_{\beta,\gamma} \leq \mathcal{C}_{\beta,\gamma}^*$ . Since (WLS) is subcritical, it is a standard strategy to establish the inequality using an Hölder interpolation and a critical inequality.

**Proof.** Let  $p := 2 \frac{d-\gamma}{d-2-\beta} \in (2, 2^*)$  with  $2^* = +\infty$  if  $d = 1, 2$  and  $2^* = 2d/(d-2)$  if  $d \geq 3$ . Let us consider the critical Caffarelli-Kohn-Nirenberg inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 |x|^{-\beta} dx \geq \mathcal{C}_{\text{CKN}} \left( \int_{\mathbb{R}^d} |u|^p |x|^{-\gamma} dx \right)^{2/p} \quad (16)$$

which has been widely studied, see for instance [14, 17, 35, 51]. Here  $\beta$  and  $\gamma$  satisfy (2) and  $n$  given by (1) is such that

$$\frac{p}{p-2} = \frac{n}{2}.$$

Hölder's inequality

$$\|u\|_{q,\gamma} \leq \|u\|_2^\eta \|u\|_{p,\gamma}^{1-\eta},$$

written with  $\eta = 2 \frac{p-q}{q(p-2)}$  for any  $q \in (2, p)$ , degenerates into an equality as  $q \rightarrow 2_+$ . By differentiating this inequality with respect to  $q$  at  $q = 2$ , we obtain the *logarithmic Hölder inequality*

$$\int_{\mathbb{R}^d} |u|^2 \log \left( \frac{|u|^2}{\|u\|_{2,\gamma}^2} \right) |x|^{-\gamma} dx \leq \frac{p}{p-2} \|u\|_{2,\gamma}^2 \log \left( \frac{\|u\|_{p,\gamma}^2}{\|u\|_{2,\gamma}^2} \right) \quad (17)$$

for any  $p > 2$ . Combined with (16) in the case  $p = 2 \frac{d-\gamma}{d-2-\beta}$ , this establishes the *weighted logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} |u|^2 \log \left( \frac{|u|^2}{\|u\|_{2,\gamma}^2} \right) |x|^{-\gamma} dx \leq \mathcal{A} \|u\|_{2,\gamma}^2 \log \left( \frac{\|\nabla u\|_{2,\beta}^2}{\|u\|_{2,\gamma}^2} \right) + \mathcal{B} \|u\|_{2,\gamma}^2 \quad (18)$$



with  $\mathcal{A} = n/2$ ,  $n$  given by (1) and  $\mathcal{B} = n \log \mathcal{C}_{\text{CKN}}$ . The value of  $\mathcal{A}$  cannot be improved, as shown by the scaling

$$\lambda \mapsto u_\lambda := \lambda^{n/2} u(\lambda).$$

Testing (18) by  $f_*(x) = c_{n,d} \sqrt{\alpha} e^{-\frac{1}{4}|x|^{2\alpha}}$  shows that  $\mathcal{B} \geq \mathcal{C}_{\beta,\gamma}^*$ . The optimal value of  $\mathcal{B}$  is therefore the minimal value for which (18) holds for any  $u \in \mathbf{H}_{\beta,\gamma}^1(\mathbb{R}^d)$ .  $\blacksquare$

## 2.2 Existence of optimal functions

The existence of an optimal function for (WLS) is proved in [13] by concentration-compactness methods when  $n = d$  and  $\alpha = 1$ . A similar proof can be found in [30], which itself relies on an extension of the concentration-compactness method of [54]. The proof in the case  $(n, \alpha) \neq (d, 1)$  can also be done by the same method.

**Proposition 2.2.** *Let  $d \geq 1$  and assume that  $(\beta, \gamma)$  satisfies (2). Equality in (WLS) is achieved by a function  $u \in \mathbf{H}_{\beta,\gamma}^1(\mathbb{R}^d)$  if  $\mathcal{C}_{\beta,\gamma}$  is taken to its optimal value.*  $\square$

**Proof.** We work with the inequality written in the form (12) with  $V_{\alpha,\nu,\sigma}$  defined by (13) and rely on direct variational methods. Since  $V_{\alpha,\nu,\sigma}$  is bounded from below, there is no significant difficulty compared to the proof of the existence of a minimizer for logarithmic Sobolev inequalities without weights or potentials (see for instance [13, 30] for similar results). For completeness, let us give a sketch of a proof.

Using the homogeneity, let us consider a minimizing sequence  $(h_n)_{n \in \mathbb{N}}$  of functions in  $\mathbf{H}^1(\mathbb{R}^d)$  such that  $\|h_n\|_2 = 1$  for any  $n \in \mathbb{N}$  and

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} (|D_\alpha h_n|^2 + V_{\alpha,\nu,\sigma} |h_n|^2 - \sigma |h_n|^2 \log |h_n|^2) dx = \sigma \left( \mathcal{K}_{n,\alpha} + \frac{n}{2} \log \left( \frac{2e}{n\sigma} \right) \right).$$

An optimization under scalings shows that we can choose  $\|D_\alpha h_n\|_2 = \sigma n/2$  with no loss of generality. Using  $\|D_\alpha h_n\|_2 \geq \min\{1, \alpha\} \|\nabla h_n\|_2$  and the standard Euclidean logarithmic Sobolev inequality, we have that  $(|h_n|^2 \log |h_n|^2)_{n \in \mathbb{N}}$  and  $((V_{\alpha,\nu,\sigma})_+^{1/2} h_n)_{n \in \mathbb{N}}$  are bounded in  $L^2(\mathbb{R}^d)$ . For any  $R > 1$  large enough, since

$$\int_{|x|>R} |h_n|^2 dx \leq \frac{1}{\log R} \leq C \int_{\mathbb{R}^d} V_{\alpha,\nu,\sigma} |h_n|^2 dx$$

for some positive constant  $C$  and since concentration is forbidden away from origin by standard Gagliardo-Nirenberg embedding inequalities and (17) while concentration at  $x = 0$  would provide us with an infinite contribution to the potential energy term, the sequence  $(h_n)_{n \in \mathbb{N}}$  is relatively compact in  $L^2(\mathbb{R}^d)$ . Up to the extraction of a subsequence,  $(h_n)_{n \in \mathbb{N}}$  strongly converges in  $L^2(\mathbb{R}^d)$  to some limit  $h$  such that  $\|h\|_2 = 1$ . According to [12, Theorem 2], we have

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} |h_n|^2 \log |h_n|^2 dx = \int_{\mathbb{R}^d} |h|^2 \log |h|^2 dx + \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} |h - h_n|^2 \log |h - h_n|^2 dx.$$

By (12) applied to  $(h - h_n)$  and a convexity argument as in [30],  $\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^d} |h - h_n|^2 \log |h - h_n|^2 dx = 0$  and we conclude that  $h$  realizes the equality case in (12). This completes the proof.  $\blacksquare$

## 2.3 Linear instability and a symmetry breaking range

With  $f(x, y) := x^{1-\frac{2}{n}} e^{\frac{2}{n}\frac{y}{x}}$ , Inequality (WLS<sub>n</sub>) becomes

$$\mathcal{F}[g] := \|D_\alpha g\|_{2,\nu}^2 - e^{-\frac{2}{n}\mathcal{K}_{n,\alpha}} f \left( \int_{\mathbb{R}^d} |g|^2 |x|^{-\nu} dx, \int_{\mathbb{R}^d} |g|^2 \log(|g|^2) |x|^{-\nu} dx \right) \geq 0.$$

We Taylor expand  $\mathcal{F}[g]$  around  $g_*$  by computing  $F[\phi] := \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathcal{F}[g_* + \varepsilon \phi]$  and find that

$$F[\phi] = \|D_\alpha \phi\|_{2,\nu}^2 - e^{-\frac{2}{n}\mathcal{K}_{n,\alpha}} \left( \partial_x f(1, y_*) \|\phi\|_{2,\nu}^2 + \partial_y f(1, y_*) \left( 3 \|\phi\|_{2,\nu}^2 + \int_{\mathbb{R}^d} \log(|g_*|^2) |\phi|^2 |x|^{-\nu} dx \right) \right)$$

for any  $\phi$  such that  $\int_{\mathbb{R}^d} (1, |x|^2) g_* \phi |x|^{-\nu} dx = (0, 0)$ , where

$$\int_{\mathbb{R}^d} |g_*|^2 |x|^{-\nu} dx = 1 \quad \text{and} \quad y_* := \int_{\mathbb{R}^d} |g_*|^2 \log(|g_*|^2) |x|^{-\nu} dx = 2 \log c_{n,d} - \frac{n}{2}.$$

In the symmetry range, we have  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^*$  and

$$e^{-\frac{2}{n} \mathcal{K}_{n,\alpha}} f(1, y_*) = e^{-\frac{2}{n} \mathcal{K}_{n,\alpha}} e^{\frac{2}{n} y_*} = \|D_\alpha g_*\|_{2,\nu}^2 = \frac{n}{4} \alpha^2.$$

Since  $\partial_x f(1, y_*) = (1 - \frac{2}{n} - \frac{2}{n} y_*) f(1, y_*)$  and  $\partial_y f(1, y_*) = \frac{2}{n} f(1, y_*)$ , in that case we obtain

$$F[\phi] = \|D_\alpha \phi\|_{2,\nu}^2 + \frac{1}{4} \alpha^2 \int_{\mathbb{R}^d} |\phi|^2 |x|^{2+n-d} dx - \frac{n}{4} \alpha^2 \left( 1 - \frac{2}{n} - \frac{2}{n} y_* + \frac{6}{n} + \frac{4}{n} \log c_{n,d} \right),$$

so that, under the condition  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^*$ , we have

$$F[\phi] = \|D_\alpha \phi\|_{2,\nu}^2 - \alpha^2 \left( 1 + \frac{n}{2} \right) \|\phi\|_{2,\nu}^2 + \frac{1}{4} \alpha^2 \int_{\mathbb{R}^d} |\phi|^2 |x|^{2+n-d} dx.$$

**Lemma 2.3 .** *Let  $n > 1$  and  $\alpha > 0$  be two real numbers and consider any integer  $d \geq 2$ . If  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^*$ , the lowest nonradial eigenmode associated with the quadratic form  $\phi \mapsto F[\phi]$  is*

$$\lambda_1(\alpha) = \frac{\alpha}{2} \left( \sqrt{4(d-1) + \alpha^2(n-2)^2} - \alpha n \right). \quad (19)$$

□

**Proof.** We use a decomposition into spherical harmonics. Since the lowest eigenvalue of the Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$  is  $(d-1)$ , we have to solve the eigenvalue problem

$$-\alpha^2 \left( \varphi'' + \frac{n-1}{r} \varphi' \right) + \frac{d-1}{r^2} \phi + \frac{\alpha^2}{4} r^2 \varphi = \lambda_1 \varphi$$

for some positive radial function  $r \mapsto \varphi(r)$ ,  $r \in (0, +\infty)$ . Elementary computations show that  $\varphi(r) = r^{1+\delta} e^{-r^2/4}$  solves the equation with  $\delta = \lambda_1(\alpha)/\alpha^2$  and  $\lambda_1(\alpha)$  given by (19). ■

On  $\mathbb{R}^+$ , it is an elementary computation to check that  $\alpha \mapsto \lambda_1(\alpha)$  takes negative values if and only if

$$\alpha > \alpha_{\text{FS}} = \sqrt{\frac{d-1}{n-1}}. \quad (20)$$

Here we find exactly the *Felli & Schneider condition* for symmetry breaking as in [38, 44].

**Proposition 2.4 .** *If (20) holds, then  $\mathcal{K}_{n,\alpha} < \mathcal{K}_{n,\alpha}^*$ .* □

**Proof.** We argue by contradiction. If  $\mathcal{K}_{n,\alpha} = \mathcal{K}_{n,\alpha}^*$ , then  $g_*$  is an optimal function. However, a perturbation of  $g_*$  by an eigenfunction associated with the eigenvalue  $\lambda_1(\alpha) < 0$  given by (19) proves that  $\mathcal{F}$  takes negative values, a contradiction with the definition of  $\mathcal{K}_{n,\alpha}$ . ■

We learn from Propositions 2.2 and 2.4 that Inequality (WLS) admits only non-radial optimal functions if (20) holds. The next step is to prove that all optimal functions for (WLS) are radially symmetric if  $\alpha \leq \alpha_{\text{FS}}$  and given up to a multiplication by a constant and a scaling by  $f_*$ .

## 2.4 A symmetry result by the *carré du champ* method

This section is devoted to the proof of Theorem 1.1, Part (ii), corresponding to the symmetry case, which is the difficult range. The method relies on the *carré du champ* method and it is inspired from [34, Section 3]. However, the presence of the logarithmic nonlinearity imposes various non-trivial changes that are detailed below. Altogether, this is a striking application of the nonlinear *carré du champ* method and we give a complete and self-contained proof. Computations which have already appeared in the context of (CKN) inequalities are clearly indicated. We consider here the *weighted logarithmic Sobolev inequality* written in the form of (10b) with  $\sigma = 1/2$ . If  $D_\alpha^*$  is the adjoint operator of  $D_\alpha$  acting on vector-valued functions  $F$ , with respect to the measure

$$d\mu_n := r^{n-1} dr d\omega,$$

then we have

$$D_\alpha^* F = -|x|^{d-n} \nabla \cdot (|x|^{n-d} F) - (\alpha - 1) r^{1-n} \omega \cdot \partial_r (r^{n-1} F).$$



Moreover we have the useful identity

$$D_\alpha^*(uF) = -D_\alpha u \cdot F + u D_\alpha^* F \quad (21)$$

if  $u$  and  $F$  are respectively scalar- and vector-valued functions. Let us define the operator  $L_\alpha$  by

$$L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 \left( \partial_r^2 + \frac{n-1}{r} \partial_r \right) + \frac{\Delta_\omega}{r^2},$$

where  $\Delta_\omega$  denotes the Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$ , and consider the Fokker-Planck equation

$$\frac{\partial u}{\partial t} = -D_\alpha^*(uF) \quad (22)$$

where the flux  $F$  and the relative pressure variable  $p$  are defined by

$$F(t, x) := D_\alpha \log u + x = D_\alpha \left( \log u + \frac{|x|^2}{2\alpha^2} \right) = D_\alpha p, \quad p := \log u - 2 \log g_\star^{\alpha, \frac{1}{2}}$$

and  $g_\star^{\alpha, 1/2}$  is the normalized optimal function for (10b) with  $\sigma = 1/2$ . Hermite functions are dense and stable under the action of the flow (22) so that we can always work on a finite dimensional space generated by some Hermite functions and argue by density. Since there is no difficulty in integrating by parts, we will do it without further justification. This is the first major difference with [34] where a nonlinear flow is considered and one has to do an approximation procedure on larger and larger balls. To simplify the proof, we divide it in several simple statements. Our main goal is to prove the exponential decay of the *Fisher information*, which goes as follows.

**Lemma 2.5 .** *Assume that  $d \geq 2$ ,  $n > d$  and  $\alpha \in (0, \alpha_{FS}]$ . If  $u$  solves (22) with  $F = D_\alpha p$ , then*

$$\frac{d}{dt} \int_{\mathbb{R}^d} u |F|^2 d\mu_n \leq -2\alpha \int_{\mathbb{R}^d} u |F|^2 d\mu_n.$$

□

It is straightforward to check that

$$\frac{\partial F}{\partial t} + D_\alpha \left( u^{-1} D_\alpha^*(uF) \right) = 0$$

and, as a consequence,

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} u |F|^2 d\mu_n &= \int_{\mathbb{R}^d} \frac{\partial u}{\partial t} |F|^2 d\mu_n + 2 \int_{\mathbb{R}^d} u F \cdot \frac{\partial F}{\partial t} d\mu_n \\ &= - \int_{\mathbb{R}^d} D_\alpha^*(uF) |F|^2 d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha \left( u^{-1} D_\alpha^*(uF) \right) d\mu_n. \end{aligned}$$

A first integration by parts shows that

$$\int_{\mathbb{R}^d} D_\alpha^*(uF) |F|^2 d\mu_n = \int_{\mathbb{R}^d} u F \cdot D_\alpha (|F|^2) d\mu_n.$$

Using (21) and  $D_\alpha \log u = F - x$ , we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} u |F|^2 d\mu_n &= - \int_{\mathbb{R}^d} u F \cdot D_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (D_\alpha^* F - F \cdot D_\alpha \log u) d\mu_n \\ &= - \int_{\mathbb{R}^d} u F \cdot D_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (D_\alpha^* F - |F|^2 + F \cdot x) d\mu_n \\ &= \int_{\mathbb{R}^d} u F \cdot D_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (F \cdot x) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (D_\alpha^* F) d\mu_n. \end{aligned}$$

Using  $uF = D_\alpha u + xu$  and integrating by parts, we obtain

$$\int_{\mathbb{R}^d} u F \cdot D_\alpha (|F|^2) d\mu_n = - \int_{\mathbb{R}^d} u L_\alpha (|F|^2) d\mu_n + \int_{\mathbb{R}^d} u x \cdot D_\alpha (|F|^2) d\mu_n.$$

Hence

$$\frac{d}{dt} \int_{\mathbb{R}^d} u |F|^2 d\mu_n = -2 \int_{\mathbb{R}^d} u K[F] d\mu_n + \int_{\mathbb{R}^d} u x \cdot D_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (F \cdot x) d\mu_n - \frac{2}{n} \int_{\mathbb{R}^d} u (D_\alpha^* F)^2 d\mu_n$$

with

$$K[F] := \frac{1}{2} L_\alpha (|F|^2) + F \cdot D_\alpha (D_\alpha^* F) - \frac{1}{n} (D_\alpha^* F)^2. \quad (23)$$

We recall that  $F = D_\alpha \mathbf{p}$  so that

$$K[D_\alpha \mathbf{p}] = \frac{1}{2} L_\alpha |D_\alpha \mathbf{p}|^2 - D_\alpha \mathbf{p} \cdot D_\alpha L_\alpha \mathbf{p} - \frac{1}{n} (L_\alpha \mathbf{p})^2.$$

Let us state a result inspired by [35, Lemma 5.1], [38, Lemma 4.2 and Lemma 4.3].

**Lemma 2.6 .** *With the above notations, we have the two following estimates:*

(i) *Pointwise estimate:*

$$K[D_\alpha \mathbf{p}] = \alpha^4 \left(1 - \frac{1}{n}\right) \left| \mathbf{p}'' - \frac{\mathbf{p}'}{r} - \frac{\Delta_\omega \mathbf{p}}{\alpha^2 (n-1) r^2} \right|^2 + \frac{2\alpha^2}{r^2} \left| \nabla_\omega \mathbf{p}' - \frac{\nabla_\omega \mathbf{p}}{r} \right|^2 + \frac{k[\mathbf{p}]}{r^4}$$

where

$$k[\mathbf{p}] := \frac{1}{2} \Delta_\omega |\nabla_\omega \mathbf{p}|^2 - \nabla_\omega \mathbf{p} \cdot \nabla_\omega (\Delta_\omega \mathbf{p}) - \frac{1}{n-1} (\Delta_\omega \mathbf{p})^2 - (n-2) \alpha^2 |\nabla_\omega \mathbf{p}|^2.$$

(ii) *Integral estimate on the sphere  $\mathbb{S}^{d-1}$  if  $d \geq 2$ :*

$$\int_{\mathbb{S}^{d-1}} k[\mathbf{p}] u d\omega \geq (n-2) (\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathbf{p}|^2 u d\omega + \delta \int_{\mathbb{S}^{d-1}} |\nabla_\omega \mathbf{p}|^4 d\omega$$

where  $\delta$  is a positive constant depending only on  $n$  and  $d$ .

□

**Proof.** Property (i) can be found in [35, Lemma 5.1] and [38, Lemma 4.2]. See Lemma A.1 in **Appendix A** for a more detailed statement and a proof. The regularity needed in Lemma A.1 is not an issue in our setting, as we consider solutions in spaces generated by a finite number of Hermite polynomials and then argue by density.

Next we focus on the proof of (ii). We go along the lines of the proof of [38, Lemma 4.3], but many details have to be changed to adapt the proof. The results are inspired from [29, 32, 33, 36] and we adopt the presentation of [33]:  $\mathbb{S}^{d-1}$  is considered as a  $(d-1)$ -dimensional compact manifold with metric  $g$  and uniform probability measure  $d\omega$ . We shall indeed assume that it is normalized so that  $\omega(\mathbb{S}^{d-1}) = 1$  to avoid carrying normalization constants. Let us introduce some notation. If  $A_{ij}$  and  $B_{ij}$  are two tensors, then

$$A : B := g^{im} g^{jn} A_{ij} B_{mn} \quad \text{and} \quad \|A\|^2 := A : A.$$

Here  $g^{ij}$  is the inverse of the metric tensor, *i.e.*,  $g^{ij} g_{jk} = \delta_k^i$ . We use the Einstein summation convention and  $\delta_k^i$  denotes the Kronecker symbol. Let us denote the *Hessian* by  $H_\omega \mathbf{p}$  and define the *trace free Hessian* by

$$L_\omega \mathbf{p} := H_\omega \mathbf{p} - \frac{1}{d-1} (\Delta_\omega \mathbf{p}) g.$$

We also consider the following trace free tensor

$$M_\omega \mathbf{p} := \nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p} - \frac{1}{d-1} |\nabla_\omega \mathbf{p}|^2 g,$$

where  $\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p} := (\partial_i \mathbf{p} \partial_j \mathbf{p})_{ij}$  and  $\|\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}\|^2 = |\nabla_\omega \mathbf{p}|^4 = (g^{ij} \partial_i \mathbf{p} \partial_j \mathbf{p})^2$ . Using  $L_\omega : g = M_\omega : g = 0$ , we obtain

$$\|L_\omega \mathbf{p}\|^2 = \|H_\omega \mathbf{p}\|^2 - \frac{1}{d-1} (\Delta_\omega \mathbf{p})^2, \quad (24a)$$

$$\|M_\omega \mathbf{p}\|^2 = \|\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}\|^2 - \frac{1}{d-1} |\nabla_\omega \mathbf{p}|^4 = \frac{d-2}{d-1} |\nabla_\omega \mathbf{p}|^4, \quad (24b)$$

$$L_\omega \mathbf{p} : M_\omega \mathbf{p} = \left( H_\omega \mathbf{p} - \frac{1}{d-1} (\Delta_\omega \mathbf{p}) g \right) : (\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}) = H_\omega \mathbf{p} : (\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}) - \frac{1}{d-1} \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2. \quad (24c)$$

Assume first that  $d \geq 3$ . The Bochner-Lichnerowicz-Weitzenböck formula on  $\mathbb{S}^{d-1}$  takes the simple form

$$\frac{1}{2} \Delta_\omega (|\nabla_\omega \mathbf{p}|^2) = \|\mathbf{H}_\omega \mathbf{p}\|^2 + \nabla_\omega (\Delta_\omega \mathbf{p}) \cdot \nabla_\omega \mathbf{p} + (d-2) |\nabla_\omega \mathbf{p}|^2 \quad (25)$$

where the last term, *i.e.*,  $\text{Ric}(\nabla_\omega \mathbf{p}, \nabla_\omega \mathbf{p}) = (d-2) |\nabla_\omega \mathbf{p}|^2$ , accounts for the Ricci curvature tensor contracted with  $\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}$ . With

$$\mathbf{k}[\mathbf{p}] := \frac{1}{2} \Delta_\omega (|\nabla_\omega \mathbf{p}|^2) - \nabla_\omega \mathbf{p} \cdot \nabla_\omega (\Delta_\omega \mathbf{p}) - \frac{1}{n-1} (\Delta_\omega \mathbf{p})^2 - (n-2) \alpha^2 |\nabla_\omega \mathbf{p}|^2,$$

we compute

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} u \mathbf{k}[\mathbf{p}] d\omega &= \int_{\mathbb{S}^{d-1}} u \left( \|\mathbf{H}_\omega \mathbf{p}\|^2 + (d-2) |\nabla_\omega \mathbf{p}|^2 - \frac{1}{n-1} (\Delta_\omega \mathbf{p})^2 - (n-2) \alpha^2 |\nabla_\omega \mathbf{p}|^2 \right) d\omega \\ &= \int_{\mathbb{S}^{d-1}} u \left( \|\mathbf{L}_\omega \mathbf{p}\|^2 + \left( \frac{1}{d-1} - \frac{1}{n-1} \right) (\Delta_\omega \mathbf{p})^2 + ((d-2) - (n-2) \alpha^2) |\nabla_\omega \mathbf{p}|^2 \right) d\omega \end{aligned}$$

using (25) and (24a).

With  $\mathbf{p} = \log u + \frac{|x|^2}{2\alpha^2}$ , it turns out that

$$\nabla_\omega u = u \nabla_\omega \mathbf{p},$$

whence, applying integrations by parts again and taking into account (24b) and (24c),

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega &= - \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^4 d\omega - 2 \int_{\mathbb{S}^{d-1}} u \mathbf{H}_\omega \mathbf{p} : (\nabla_\omega \mathbf{p} \otimes \nabla_\omega \mathbf{p}) d\omega \\ &= - \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega - 2 \int_{\mathbb{S}^{d-1}} u \mathbf{L}_\omega \mathbf{p} : \mathbf{M}_\omega \mathbf{p} d\omega - \frac{2}{d-1} \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega. \end{aligned}$$

As a consequence, we obtain

$$\int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega = - \frac{d-1}{d+1} \left( \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega + 2 \int_{\mathbb{S}^{d-1}} u \mathbf{L}_\omega \mathbf{p} : \mathbf{M}_\omega \mathbf{p} d\omega \right). \quad (26)$$

On the other hand, integrating (25) on  $\mathbb{S}^{d-1}$  against  $u$  and performing an integration by parts shows that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{S}^{d-1}} u \Delta_\omega (|\nabla_\omega \mathbf{p}|^2) d\omega &+ \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega + \int_{\mathbb{S}^{d-1}} u (\Delta_\omega \mathbf{p})^2 d\omega \\ &= \int_{\mathbb{S}^{d-1}} u \|\mathbf{H}_\omega \mathbf{p}\|^2 d\omega + (d-2) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega \\ &= \int_{\mathbb{S}^{d-1}} u \|\mathbf{L}_\omega \mathbf{p}\|^2 d\omega + \frac{1}{d-1} \int_{\mathbb{S}^{d-1}} u (\Delta_\omega \mathbf{p})^2 d\omega + (d-2) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega \end{aligned}$$

by (24a). Integrations by parts also show that

$$\frac{1}{2} \int_{\mathbb{S}^{d-1}} u \Delta_\omega (|\nabla_\omega \mathbf{p}|^2) d\omega = \frac{1}{2} \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^4 d\omega + \frac{1}{2} \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega,$$

so that, by (24b),

$$\begin{aligned} \frac{1}{2} \frac{(d-1)}{(d-2)} \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega &+ \frac{3}{2} \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega + \frac{d-2}{d-1} \int_{\mathbb{S}^{d-1}} u (\Delta_\omega \mathbf{p})^2 d\omega \\ &= \int_{\mathbb{S}^{d-1}} u \|\mathbf{L}_\omega \mathbf{p}\|^2 d\omega + (d-2) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega. \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} u (\Delta_\omega \mathbf{p})^2 d\omega &+ \frac{1}{2} \frac{(d-1)}{(d-2)} \left( \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega + 3 \int_{\mathbb{S}^{d-1}} u \Delta_\omega \mathbf{p} |\nabla_\omega \mathbf{p}|^2 d\omega \right) \\ &= \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{L}_\omega \mathbf{p}\|^2 d\omega + (d-1) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega. \quad (27) \end{aligned}$$

We can now combine (26) and (27) to get

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} u (\Delta_\omega \mathbf{p})^2 d\omega + \frac{1}{2} \left( \frac{d-1}{d-2} \right)^2 \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega \\ - \frac{3}{2} \frac{(d-1)}{(d-2)} \frac{(d-1)}{(d+1)} \left( \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega + 2 \int_{\mathbb{S}^{d-1}} u \mathbf{L}_\omega \mathbf{p} : \mathbf{M}_\omega \mathbf{p} d\omega \right) \\ = \frac{d-1}{d-2} \int_{\mathbb{S}^{d-1}} u \|\mathbf{L}_\omega \mathbf{p}\|^2 d\omega + (d-1) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega. \end{aligned}$$

This allows us to prove that

$$\int_{\mathbb{S}^{d-1}} u k[\mathbf{p}] d\omega = \int_{\mathbb{S}^{d-1}} u (\mathbf{a} \|\mathbf{L}_\omega \mathbf{p}\|^2 + \mathbf{b} \mathbf{L}_\omega \mathbf{p} : \mathbf{M}_\omega \mathbf{p} + \mathbf{c} \|\mathbf{M}_\omega \mathbf{p}\|^2) d\omega + (n-2) (\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega$$

where we use the fact that

$$(d-2) - (n-2) \alpha^2 + \left( \frac{1}{d-1} - \frac{1}{n-1} \right) (d-1) = (n-2) (\alpha_{\text{FS}}^2 - \alpha^2)$$

with  $\mathbf{a} = \frac{(d-1)(n-2)}{(d-2)(n-1)}$ ,  $\mathbf{b} = \frac{3(d-1)(n-d)}{(n-1)(d+1)(d-2)}$  and  $\mathbf{c} = \frac{(d-1)(n-d)}{(n-1)(d+1)(d-2)}$ . Hence we obtain

$$\int_{\mathbb{S}^{d-1}} u k[\mathbf{p}] d\omega \geq (n-2) (\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{S}^{d-1}} u |\nabla_\omega \mathbf{p}|^2 d\omega + \left( \mathbf{c} - \frac{\mathbf{b}^2}{4\mathbf{a}} \right) \int_{\mathbb{S}^{d-1}} u \|\mathbf{M}_\omega \mathbf{p}\|^2 d\omega$$

because the discriminant  $\mathbf{b}^2 - 4\mathbf{a}\mathbf{c}$  takes negative values. Taking into account (24b), this completes the proof with

$$\delta = \left( \mathbf{c} - \frac{\mathbf{b}^2}{4\mathbf{a}} \right) \frac{d-2}{d-1} = (n-d) \frac{4(d+1)(d-2) + (4d-5)(n-d)}{4(n-1)(n-2)(d+1)^2} > 0.$$

If  $d=2$ , we identify  $\mathbb{S}^1$  with  $[0, 2\pi)$ , denote by  $\theta \in [0, 2\pi)$  the angular variable and by  $u_\theta$  and  $u_{\theta\theta}$  the first and second derivatives of  $u$  with respect to  $\theta$ . As in [35, Lemma 5.3] and [38, Lemma 4.3], we have

$$k[\mathbf{p}] = \frac{n-2}{n-1} |\mathbf{p}_{\theta\theta}|^2 - (n-2) \alpha^2 |\mathbf{p}_\theta|^2 = (n-2) (\alpha_{\text{FS}}^2 |\mathbf{p}_{\theta\theta}|^2 - \alpha^2 |\mathbf{p}_\theta|^2).$$

Let  $w = \sqrt{u}$  and recall that  $u \mathbf{p}_\theta = u_\theta$  so that

$$u |\mathbf{p}_\theta|^2 = 4 |w_\theta|^2 \quad \text{and} \quad u |\mathbf{p}_{\theta\theta}|^2 = 4 \left| w_{\theta\theta} - \frac{|w_\theta|^2}{w} \right|^2.$$

Notice that  $w_{\theta\theta} |w_\theta|^2 = \frac{1}{3} \frac{d}{d\theta} (w_\theta |w_\theta|^2)$ . With one integration by parts we obtain

$$\int_{\mathbb{S}^1} \left| w_{\theta\theta} - \frac{|w_\theta|^2}{w} \right|^2 d\theta = \int_{\mathbb{S}^1} |w_{\theta\theta}|^2 d\theta + \int_{\mathbb{S}^1} \frac{|w_\theta|^4}{w^2} d\theta - 2 \int_{\mathbb{S}^1} w_{\theta\theta} \frac{|w_\theta|^2}{w} d\theta = \int_{\mathbb{S}^1} |w_{\theta\theta}|^2 d\theta + \frac{1}{3} \int_{\mathbb{S}^1} \frac{|w_\theta|^4}{w^2} d\theta.$$

By the Poincaré inequality, we have

$$\int_{\mathbb{S}^1} |w_{\theta\theta}|^2 d\theta \geq \int_{\mathbb{S}^1} |w_\theta|^2 d\theta$$

and conclude that

$$\begin{aligned} \int_{\mathbb{S}^1} u k[\mathbf{p}] d\theta &\geq 4(n-2) (\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{S}^1} |w_\theta|^2 d\theta + \frac{4}{3} \int_{\mathbb{S}^1} \frac{|w_\theta|^4}{w^2} d\theta \\ &= (n-2) (\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{S}^1} u |\mathbf{p}_\theta|^2 d\theta + \frac{1}{12} \int_{\mathbb{S}^1} u |\mathbf{p}_\theta|^4 d\theta. \end{aligned}$$

■

**Lemma 2.7 .** *With the above notations, we have the identity*

$$\int_{\mathbb{R}^d} u x \cdot \mathbf{D}_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot \mathbf{D}_\alpha (F \cdot x) d\mu_n = -2\alpha \int_{\mathbb{R}^d} u |F|^2 d\mu_n.$$

□

Although very elementary, this estimate is fundamental as it establishes the exponential decay of the Fisher information in the symmetry range. Lemma 2.7 is in fact no more than an integration by parts.

**Proof.** Since  $x \cdot D_\alpha = \alpha r \partial_r$ ,  $x \cdot F = \alpha r \partial_r \mathbf{p}$ ,  $x \cdot \nabla_\omega = 0$ , and  $F \cdot \partial_r F = F \cdot \partial_r (D_\alpha \mathbf{p}) = F \cdot D_\alpha \partial_r \mathbf{p} - \frac{1}{r^3} |\nabla_\omega \mathbf{p}|^2$ , we have that

$$\begin{aligned} \int_{\mathbb{R}^d} u x \cdot D_\alpha (|F|^2) d\mu_n - 2 \int_{\mathbb{R}^d} u F \cdot D_\alpha (F \cdot x) d\mu_n \\ = 2\alpha \int_{\mathbb{R}^d} u (r F \cdot \partial_r F - r F \cdot D_\alpha \partial_r \mathbf{p} - \alpha \partial_r \mathbf{p} (\omega \cdot F)) d\mu_n \\ = -2\alpha \int_{\mathbb{R}^d} u \left( \alpha^2 |\partial_r \mathbf{p}|^2 + \frac{|\nabla_\omega \mathbf{p}|^2}{r^2} \right) d\mu_n = -2\alpha \int_{\mathbb{R}^d} u |F|^2 d\mu_n, \end{aligned}$$

which concludes the proof.  $\blacksquare$

**Proof of Lemma 2.5.** Collecting the estimates of Lemmata 2.6 and 2.7 into (23), we have shown that

$$\frac{d}{dt} \int_{\mathbb{R}^d} u |F|^2 d\mu_n + 2\alpha \int_{\mathbb{R}^d} u |F|^2 d\mu_n \leq -2(n-2)(\alpha_{\text{FS}}^2 - \alpha^2) \int_{\mathbb{R}^d} u |\nabla_\omega \mathbf{p}|^2 d\mu_n - 2\delta \int_{\mathbb{R}^d} |\nabla_\omega \mathbf{p}|^4 d\mu_n.$$

Under the assumptions of Lemma 2.5, the right-hand side is nonpositive, which completes the proof.  $\blacksquare$

**Proof of Theorem 1.1.** We learn from Lemma 2.5 that

$$\frac{d}{dt} \left( \int_{\mathbb{R}^d} u |F|^2 d\mu_n - 2\alpha \int_{\mathbb{R}^d} u \mathbf{p} d\mu_n \right) \leq 0.$$

On the other hand,  $t \mapsto \int_{\mathbb{R}^d} u(t, \cdot) |F(t, \cdot)|^2 d\mu_n$  decays exponentially, whence  $\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} u(t, \cdot) |F(t, \cdot)|^2 d\mu_n = 0$ . Any decomposition of  $u$  on a finite dimensional subspace of Hermite functions is exponentially decaying and such that  $\lim_{t \rightarrow +\infty} \int_{\mathbb{R}^d} u(t, \cdot) \mathbf{p}(t, \cdot) d\mu_n = 0$ , thus proving that the inequality  $\int_{\mathbb{R}^d} u |F|^2 d\mu_n \geq 2\alpha \int_{\mathbb{R}^d} u \mathbf{p} d\mu_n$  is always true for any  $t \geq 0$  and, in particular, it holds true for the initial datum, which can be chosen arbitrarily. This amounts to (WLS) in the non scale-invariant form (10b).  $\blacksquare$

### 3 (WLS) inequalities as an endpoint of some Caffarelli-Kohn-Nirenberg inequalities

This section relies on the results of [8, 38] and shows the consistency of our results with the symmetry properties of (CKN) inequalities.

#### 3.1 A brief summary of the symmetry properties of some (CKN) inequalities

On the space  $H_{\beta, \gamma}^{1,p}(\mathbb{R}^d)$  of the functions  $f \in L_{\gamma}^{p+1}(\mathbb{R}^d)$ , such that  $\nabla f \in L_{\beta}^2(\mathbb{R}^d)$ , we consider the special family of *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{2p, \gamma} \leq C_{\beta, \gamma, p} \|\nabla f\|_{2, \beta}^{\vartheta} \|f\|_{p+1, \gamma}^{1-\vartheta} \quad \forall f \in H_{\beta, \gamma}^{1,p}(\mathbb{R}^d) \quad (\text{CKN})$$

with optimal constant  $C_{\beta, \gamma, p}$ , and parameters  $\beta$ ,  $\gamma$  and  $p$  such that

$$\gamma - 2 < \beta < \frac{d-2}{d} \gamma, \quad \gamma \in (-\infty, d), \quad p \in (1, p_*) \quad \text{with} \quad p_* := \frac{d-\gamma}{d-2-\beta}. \quad (28)$$

The exponent

$$\vartheta = \frac{(d-\gamma)(p-1)}{p((d+2+\beta-2\gamma) - p(d-2-\beta))}$$

is determined by the invariance under scalings. The limitation  $p \leq p_*$  in (28) amounts, for a given  $p > 1$  to a restriction to the admissible set of parameters  $(\beta, \gamma)$ , namely

$$\beta \geq d-2 - \frac{d-\gamma}{p}. \quad (29)$$

On the other hand, if  $d \geq 3$ , we notice that the condition  $p_* < d/(d-2)$  is equivalent to

$$\beta < \frac{d-2}{d} \gamma.$$

The range of admissible parameters  $(\beta, \gamma)$  is limited by (28) to a cone in the quadrant  $\beta < d-2$  and  $\gamma < d$  with the additional condition (29). See [38, Fig. 1].

The *symmetry* versus *symmetry breaking* issue is central in Caffarelli-Kohn-Nirenberg inequalities (CKN). Symmetry in (CKN) means that the equality case is achieved by the (generalized) *Aubin-Talenti type functions*

$$g(x) = (1 + |x|^{2+\beta-\gamma})^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d. \quad (30)$$

According to [44], [8, Theorem 2] and in [38, Theorem 1.1] *symmetry breaking* occurs if and only if  $(\beta, \gamma)$  satisfy (6) where  $\gamma \mapsto \beta_{\text{FS}}(\gamma)$  is the *Felli & Schneider* curve defined by (3). In the *symmetry* range determined by (7), the value of  $C_{\beta, \gamma, p}$  is known. According to [8, Appendix A], if (7) holds, we have

$$C_{\beta, \gamma, p} = C_{\beta, \gamma, p}^* \quad (31)$$

where  $\sigma_d := |\mathbb{S}^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$  is the volume of the unit sphere  $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ ,

$$C_{\beta, \gamma, p}^* := \alpha^\zeta K_{\alpha, n, p}^*$$

where  $n$  and  $\alpha$  are given respectively by (1) and (4), and

$$\begin{aligned} \zeta &:= \frac{\vartheta}{2} + \frac{1-\vartheta}{p+1} - \frac{1}{2p} = \frac{(2+\beta-\gamma)(p-1)}{2p(d+2+\beta-2\gamma-p(d-2-\beta))}, \\ \frac{1}{K_{\alpha, n, p}^*} &= \alpha^\vartheta \left( \frac{4n}{p-1} \frac{1}{n+2-p(n-2)} \right)^{\frac{\vartheta}{2}} \left( \frac{2(p+1)}{n+2-p(n-2)} \right)^{\frac{\vartheta}{p+1}} \left( \sigma_d \frac{\Gamma(\frac{n}{2}) \Gamma(\frac{2p}{p-1} - \frac{n}{2})}{2\Gamma(\frac{2p}{p-1})} \right)^\zeta. \end{aligned}$$

### 3.2 (CKN) inequalities, the artificial dimension and the anisotropic gradient

Inequality (CKN) can be recast as an interpolation inequality with the same weight in all integrals which, in terms of scaling properties, amounts to introduce an *artificial dimension*. To a function  $f \in H_{\beta, \gamma}^{1,p}(\mathbb{R}^d)$ , let us associate the function  $F \in H_{\nu, \nu}^{1,p}(\mathbb{R}^d)$  with  $\nu := d-n < 0$  such that  $f(x) = F(|x|^{\alpha-1}x)$  for any  $x \in \mathbb{R}^d$  as in (8). Notice that  $p_* = n/(n-2)$ . With  $\alpha > 0$  and  $p \in (1, p_*)$ , we can rewrite (CKN) as

$$\|F\|_{2p, \nu} \leq K_{\alpha, n, p} \|D_\alpha F\|_{2, \nu}^\vartheta \|F\|_{p+1, \nu}^{1-\vartheta} \quad \forall F \in H_{\nu, \nu}^p(\mathbb{R}^d), \quad (32)$$

for some optimal constant  $K_{\alpha, n, p}$  which is explicitly related to the optimal constant in (CKN): see [8, Proposition 6]. Inequality (32) can be interpreted as a Gagliardo-Nirenberg-Sobolev inequality in the artificial dimension  $n$ . As  $\alpha \neq 1$  unless  $\beta = \gamma$ , notice that symmetry issues in (32) are in no way simpler than in (CKN). A remarkable point is that the Aubin-Talenti type function as defined by (30) is transformed, up to a scaling, into the more standard function

$$x \mapsto \left( 1 + \frac{p-1}{2} |x|^2 \right)^{\frac{1}{1-p}},$$

which converges to the standard gaussian function as  $p \rightarrow 1$ . We refer to [8, Section 2.3] and [35, Section 3.1] for further details. The limit of (32) as  $p \rightarrow 1_+$  is consistent with (WLS<sub>n</sub>). This is what we are going to exploit next.

### 3.3 The limit as $p \rightarrow 1$

Assume that  $\beta > \gamma - 2$  so that (29) is satisfied uniformly in the limit as  $p \rightarrow 1_+$ . Inequality (CKN) can be rewritten in logarithmic form as

$$\log \left( \frac{\|f\|_{2p, \gamma}}{\|f\|_{p+1, \gamma}} \right) \leq \log C_{\beta, \gamma, p} + \vartheta(p) \log \left( \frac{\|\nabla f\|_{2, \beta}}{\|f\|_{p+1, \gamma}} \right). \quad (33)$$



It is clear from (CKN) that  $\lim_{p \rightarrow 1_+} C_{\beta, \gamma, p} = 1$  and both sides in (33) vanish in the limit as  $p \rightarrow 1_+$ , so that the inequality degenerates into an equality. Let us divide both sides of (33) by  $(p-1)$  and consider the limit. Using the identity

$$\frac{d}{dq} \log \|f\|_{q, \gamma} = \frac{1}{q^2} \int_{\mathbb{R}^d} \frac{|f|^q}{\|f\|_{q, \gamma}^q} \log \left( \frac{|f|^q}{\|f\|_{q, \gamma}^q} \right) |x|^{-\gamma} dx,$$

$\lim_{p \rightarrow 1_+} \vartheta(p)/(p-1) = n/4$  where  $n$  is given by (1) and

$$\limsup_{p \rightarrow 1_+} \frac{1}{p-1} \log \left( \frac{\|f\|_{2p, \gamma}}{\|f\|_{p+1, \gamma}} \right) = \frac{1}{4} \int_{\mathbb{R}^d} \frac{|f|^2}{\|f\|_{2, \gamma}^2} \log \left( \frac{|f|^2}{\|f\|_{2, \gamma}^2} \right) |x|^{-\gamma} dx,$$

we can pass to the limit as  $p \rightarrow 1_+$ . The overall picture is consistent with logarithmic Sobolev inequalities (WLS). In the limit as  $p \rightarrow 1_+$ , it is straightforward to see that the conditions that define the symmetry range (7) in (CKN) provide us with the conditions that define the symmetry range in (WLS) as stated in Theorem 1.1. This is also true at the level of the optimal constants in the symmetry range. In fact, these observations provide us with an alternative strategy of proof of Theorem 1.1 based on [38, Theorem 1.1] using  $\Gamma$ -convergence methods in the spirit of [39, 41]. We do not expand on this as we already have a direct proof but for consistency, we state the following result.

**Proposition 3.1 .** *Let  $d \geq 2$  and assume that  $(\beta, \gamma) \neq (0, 0)$  satisfies (2). Then we have*

$$\mathcal{C}_{\beta, \gamma} \leq \mathcal{C}_{\beta, \gamma}^* := 4 \limsup_{p \rightarrow 1_+} \frac{C_{\beta, \gamma, p}^* - 1}{p-1}.$$

□

**Proof.** In view of (31), in the symmetry range for the parameters (7), we can directly differentiate the formula

$$\log C_{\beta, \gamma, p}^* = \zeta(p) \log \alpha + \log K_{\alpha, n, p}^* \quad (34)$$

where  $\vartheta = n \zeta(p)$  and

$$\frac{1}{K_{\alpha, n, p}^*} = \alpha^{n \zeta(p)} \left( \frac{4n}{b(p)} \frac{1}{p-1} \right)^{\frac{n}{2} \zeta(p)} \left( \frac{2(p+1)}{b(p)} \right)^{\frac{n}{p+1} \zeta(p)} \left( \frac{1}{2} \sigma_d \Gamma\left(\frac{n}{2}\right) \frac{\Gamma\left(\frac{2p}{p-1} - \frac{n}{2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)} \right)^{\zeta(p)}$$

and  $b(p) := n + 2 - p(n-2)$ . In particular, note that  $b(1) = 4$  and  $b'(1) = -(n-2)$ . By taking the logarithm, we obtain the identity

$$- \log K_{\alpha, n, p}^* = \zeta(p) \left( n \log \alpha + \frac{n}{p+1} \log \left( \frac{2(p+1)}{b(p)} \right) + \log \left( \frac{1}{2} \sigma_d \Gamma\left(\frac{n}{2}\right) \right) + f(p) \right) \quad (35)$$

where

$$f(p) := \frac{n}{2} \log \left( \frac{4n}{b(p)} \frac{1}{p-1} \right) + \log \left( \frac{\Gamma\left(\frac{2p}{p-1} - \frac{n}{2}\right)}{\Gamma\left(\frac{2p}{p-1}\right)} \right).$$

By using the asymptotic expansion for the *Gamma* function

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(x + \alpha)}{x^\alpha \Gamma(x)} = 1$$

for any  $\alpha \in \mathbb{R}$ , one can compute the limits

$$\lim_{p \rightarrow 1_+} f(p) = -\frac{3}{2} n \log 2.$$

As a result, we can take the derivative with respect to  $p$  and evaluate the limit as  $p \rightarrow 1_+$  in (35) by

$$- \lim_{p \rightarrow 1_+} \frac{\log K_{\alpha, n, p}^*}{p-1} = \frac{1}{4} \log \left( \frac{1}{2} \sigma_d \alpha^n \left(\frac{n\epsilon}{2}\right)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) \right)$$

using the fact that  $\zeta(p) = \alpha \frac{p-1}{pb(p)}$  is such that  $\zeta(1) = \lim_{p \rightarrow 1_+} \zeta(p) = 0$  and  $\zeta'(1) = 1/4$ . With  $n$  and  $\alpha$  given in terms of  $\beta$  and  $\gamma$  respectively by (1) and (4), we deduce from (34) that  $\mathcal{C}_{\beta, \gamma}^* = - \log \left( \frac{1}{2} \sigma_d \alpha^{n-1} \left(\frac{n\epsilon}{2}\right)^{n/2} \Gamma\left(\frac{n}{2}\right) \right)$ . ■

## 4 Some consequences for weighted diffusion flows

### 4.1 Self-similar solutions, intermediate asymptotics and entropy decay rates

Let us consider the self-similar change of variables

$$u(t, x) = R(t)^{\gamma-d} v \left( \log R(t), \frac{x}{R(t)} \right) \quad \text{with} \quad \frac{dR}{dt} = R^{\gamma-\beta-1} = R^{1-2\alpha}$$

which transforms (15) into the *weighted Fokker-Planck equation*

$$\frac{\partial v}{\partial t} = |x|^\gamma \nabla \cdot (|x|^{-\beta} \nabla v + x |x|^{-\gamma} v). \quad (36)$$

A simple stationary solution is given by  $v_\star(x) = c_{n,d} (2/\alpha)^{n/4} \exp(-\frac{1}{2\alpha} |x|^2) = g_\star^{\alpha,\alpha}(x)$  with the notations of Section 1. With  $R(0) = R_0 \geq 0$ , we find that

$$R(t) = (R_0^{2\alpha} + 2\alpha t)^{\frac{1}{2\alpha}} \quad \forall t \geq 0,$$

which shows that  $u_\star(t, x) = R(t)^{\gamma-d} f_\star(\log R(t), x/R(t))$  is simply the Green function associated to (15) if we choose  $R_0 = 0$ . Another interesting choice of  $R_0$  is  $R_0 = 1$  so that the initial datum for (36) is the same as for (15). If  $v$  solves (36), then the function  $w = v/v_\star$  solves the *weighted Ornstein-Uhlenbeck equation*

$$\frac{\partial w}{\partial t} = \frac{|x|^\gamma}{v_\star(x)} \nabla \cdot (|x|^{-\beta} v_\star \nabla w). \quad (37)$$

**Proposition 4.1 .** *In the symmetry range, with  $d\mu_\alpha = |x|^{-\gamma} v_\star(x) dx$  defined as in Section 1, any solution of (37) with nonnegative initial datum  $w_0$  such that  $\int_{\mathbb{R}^d} w_0 d\mu_\alpha = 1$  decays according to*

$$\int_{\mathbb{R}^d} w(t, \cdot) \log w(t, \cdot) d\mu_\alpha \leq \left( \int_{\mathbb{R}^d} w_0 \log w_0 d\mu_\alpha \right) e^{-4\alpha t} \quad \forall t \geq 0.$$

□

**Proof.** We compute  $\frac{d}{dt} \int_{\mathbb{R}^d} w(t, \cdot) \log w(t, \cdot) d\mu_\alpha$  and apply (11b) to  $u = \sqrt{w}$ . ■

By the Csiszár-Kullback-Pinsker inequality

$$\int_{\mathbb{R}^d} |w - 1| d\mu_\alpha \leq \frac{1}{2} \sqrt{\int_{\mathbb{R}^d} w \log w d\mu_\alpha}$$

for any nonnegative function  $w$  such that  $\int_{\mathbb{R}^d} w d\mu_\alpha = 1$ . By undoing the above changes of variables with  $R_0 = 1$ , we can write the following *intermediate asymptotics* result.

**Corollary 4.2 .** *In the symmetry range, any solution of (15) with nonnegative initial datum  $u_0$  such that  $\|u_0\|_{1,\gamma} = 1$  obeys to*

$$\|u(t, \cdot) - u_\star(t, \cdot)\|_{1,\gamma} \leq \frac{1}{2} \sqrt{\int_{\mathbb{R}^d} u_0 \log(u_0/v_\star) |x|^{-\gamma} dx} (1 + 2\alpha t)^{-1} \quad \forall t \geq 0.$$

□

The above results are consistent with the flow

$$\frac{\partial u}{\partial t} = |x|^\gamma \nabla \cdot (|x|^{-\beta} \nabla u^m)$$

with  $m < 1$ , which is adapted to (CKN) inequalities with  $p = 1/(2m - 1)$ . Notice however that parabolic computations as in Section 2.4 are, so far, only formal if  $m < 1$ : see [8, 34, 35, 38] for details.

## 4.2 Hyper-contractivity estimates

Let us measure the gain of regularity by the weighted heat flow (15). the following result generalizes [43, 48, 55].

**Proposition 4.3 .** *Let  $d \geq 1$ ,  $r > q > 1$  and assume that  $\beta$  and  $\gamma$  satisfy (2). If  $u$  is a solution of (15) with initial datum  $u_0 \in L^q_\gamma(\mathbb{R}^d)$ , then*

$$\|u(t, \cdot)\|_{r, \gamma} \leq \mathcal{H}_{\beta, \gamma}^{q, r} \|u_0\|_{q, \gamma} t^{-\frac{n}{2} \frac{r-q}{qr}} \quad \forall t \geq 0 \quad (38)$$

where  $\mathcal{H}_{\beta, \gamma}^{q, r} := t_\star^{\frac{n}{2} \frac{r-q}{qr}}$  and  $t_\star := \frac{n}{8} e^{\frac{2}{n} \mathcal{C}_{\beta, \gamma}} \log\left(\frac{r-1}{q-1}\right)$ .  $\square$

**Proof.** For some exponent  $p$  depending smoothly on  $s$  with  $p'(s) > 0$ , let us consider the function

$$h(s) := \|u(s, \cdot)\|_{p(s), \gamma}.$$

By a standard computation which goes back to [48] we have

$$\frac{h'}{h} = \frac{p'}{p^2} \int_{\mathbb{R}^d} \frac{|u|^p}{h^p} \log\left(\frac{|u|^p}{h^p}\right) |x|^{-\gamma} dx - \frac{1}{h^p} \frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla |u|^{p/2}|^2 |x|^{-\beta} dx \leq \frac{p'}{p^2} \left( \mathcal{C}_{\beta, \gamma} - \frac{n}{2} \log\left(\frac{2e}{n\sigma}\right) \right) \quad (39)$$

where the inequality holds as a consequence of (10a) applied to  $|u|^{p/2}$  with

$$p' = 4\sigma(p-1). \quad (40)$$

With the choice  $\sigma = \sigma_{\beta, \gamma}$  where  $\sigma_{\beta, \gamma} := \frac{2}{n} e^{1-\frac{2}{n} \mathcal{C}_{\beta, \gamma}}$ , so that  $h' \leq 0$ , and  $p(0) = q$ , (39) is solved by

$$p(s) = 1 + (q-1) e^{4\sigma_{\beta, \gamma} s} \quad \forall s \geq 0. \quad (41)$$

The condition  $p(t_\star) = r$  determines

$$t_\star = \frac{1}{4\sigma_{\beta, \gamma}} \log\left(\frac{r-1}{q-1}\right)$$

such that

$$\|u(t_\star, \cdot)\|_{r, \gamma} = \|u_0\|_{q, \gamma}.$$

If  $t \neq t_\star$ , we use (39) again for  $\sigma > 0$  such that  $r = 1 + (q-1) e^{4\sigma t}$ , i.e.,

$$t = \frac{1}{4\sigma} \log\left(\frac{r-1}{q-1}\right), \quad (42)$$

and obtain

$$\frac{h'}{h} \leq \frac{p'}{p^2} \frac{n}{2} \log\left(\frac{\sigma}{\sigma_{\beta, \gamma}}\right),$$

that is, after integration with respect to  $s \in [0, t]$ ,

$$\|u(t, \cdot)\|_{r, \gamma} = \|u_0\|_{q, \gamma} (t_\star^{-1} t)^{-\frac{n}{2} \frac{r-q}{qr}} \quad \forall t \geq 0. \quad \blacksquare$$

Notice that the choice of  $t_\star$  in (38) is optimal because (38) with  $\sigma = \sigma_{\beta, \gamma}$  means that  $h(s) \leq h(0)$  for any  $s > 0$ , hence  $h'(0) \leq 0$  so that the optimal value of  $\sigma_{\beta, \gamma}$  in (41) determines the optimal constant in (10a). Slightly more subtle is the fact that  $\mathcal{H}_{\beta, \gamma}^{q, r}$  is also the optimal constant. Using (39) with the condition  $p(t) = r$ , we can write that

$$\|u(t, \cdot)\|_{r, \gamma} = h(t) \leq h(0) \exp\left(\int_0^t \frac{p'(s)}{p^2(s)} \left(\mathcal{C}_{\beta, \gamma} - \frac{n}{2} \log\left(\frac{2e}{n\sigma}\right)\right) ds\right)$$

where  $h(0) = \|u_0\|_{q, \gamma}$  and  $\sigma$  can be taken  $s$ -dependent. With the change of variables  $s \mapsto z$ ,  $z = Z(s) = 1/p(s)$ , we can compute

$$\int_0^t \frac{p'(s)}{p^2(s)} \left(\mathcal{C}_{\beta, \gamma} - \frac{n}{2} \log\left(\frac{2e}{n\sigma}\right)\right) ds = \left(\mathcal{C}_{\beta, \gamma} - \frac{n}{2} \log\left(\frac{2e}{n}\right)\right) \left(\frac{1}{q} - \frac{1}{r}\right) - \frac{n}{2} \int_{1/r}^{1/q} \log \sigma(z) dz$$

where, up to a slight abuse of notations, we consider  $\sigma$  as a function of  $z$  and deduce from (40) that

$$\sigma(z) = 4 \frac{p^2}{p'} \frac{p-1}{p^2} = 4 \frac{z(z-1)}{(Z' \circ Z^{-1})(z)},$$

although we do not make use of this identity. Indeed, an infinitesimal variation of  $\int_{1/r}^{1/q} \log \sigma(z) dz$  directly shows that the optimal case is achieved by a constant function  $z \mapsto \sigma(z)$  corresponding to the choice (42). Hence  $\mathcal{H}_{\beta,\gamma}^{q,r}$  as defined in Proposition 4.3 is optimal.

## Appendix A. A purely algebraic computation

For completeness, let us give a proof of Lemma 2.6, (i). We recall that

$$K[D_\alpha p] = \frac{1}{2} L_\alpha |D_\alpha p|^2 - D_\alpha p \cdot D_\alpha L_\alpha p - \frac{1}{n} (L_\alpha p)^2$$

and

$$k[p] = \frac{1}{2} \Delta_\omega |\nabla_\omega p|^2 - \nabla_\omega p \cdot \nabla_\omega \Delta_\omega p - \frac{1}{n-1} (\Delta_\omega p)^2 - (n-2) \alpha^2 |\nabla_\omega p|^2.$$

**Lemma A.1 .** *Let  $d \in \mathbb{N}$ ,  $n \in \mathbb{R}$  such that  $n > d \geq 2$ , and consider a function  $p \in C^3(\mathbb{R}^d \setminus \{0\})$ . Then,*

$$K[D_\alpha p] = \alpha^4 \left(1 - \frac{1}{n}\right) \left| p'' - \frac{p'}{r} - \frac{\Delta_\omega p}{\alpha^2 (n-1) r^2} \right|^2 + \frac{2\alpha^2}{r^2} \left| \nabla_\omega p' - \frac{\nabla_\omega p}{r} \right|^2 + \frac{k[p]}{r^4}.$$

□

This result is a purely algebraic computation which involves no integration by parts and in which, for (CKN), neither  $p$  nor  $m$  plays any role, so that it perfectly makes sense to consider the limit case  $m = 1$  and  $p = 1$ . We recall that the original result of [35, Lemma 5.1] was given in the framework of  $m < 1$ .

**Proof.** By definition of  $K[D_\alpha p]$ , we have

$$\begin{aligned} K[D_\alpha p] &= \frac{\alpha^2}{2} \left[ \alpha^2 p'^2 + \frac{|\nabla_\omega p|^2}{r^2} \right]'' + \frac{\alpha^2 (n-1)}{2r} \left[ \alpha^2 p'^2 + \frac{|\nabla_\omega p|^2}{r^2} \right]' + \frac{1}{2r^2} \Delta_\omega \left[ \alpha^2 p'^2 + \frac{|\nabla_\omega p|^2}{r^2} \right] \\ &\quad - \alpha^2 p' \left( \alpha^2 p'' + \alpha^2 \frac{(n-1)}{r} p' + \frac{\Delta_\omega p}{r^2} \right)' - \frac{1}{r^2} \nabla_\omega p \cdot \nabla_\omega \left( \alpha^2 p'' + \alpha^2 \frac{(n-1)}{r} p' + \frac{\Delta_\omega p}{r^2} \right) \\ &\quad - \frac{1}{n} \left( \alpha^2 p'' + \alpha^2 \frac{(n-1)}{r} p' + \frac{\Delta_\omega p}{r^2} \right)^2, \end{aligned}$$

which can be expanded as

$$\begin{aligned} K[D_\alpha p] &= \frac{\alpha^2}{2} \left[ 2\alpha^2 p''^2 + 2\alpha^2 p' p''' + 2 \frac{|\nabla_\omega p'|^2 + \nabla_\omega p \cdot \nabla_\omega p''}{r^2} - 8 \frac{\nabla_\omega p \cdot \nabla_\omega p'}{r^3} + 6 \frac{|\nabla_\omega p|^2}{r^4} \right] \\ &\quad + \alpha^2 \frac{(n-1)}{r} \left[ \alpha^2 p' p'' + \frac{\nabla_\omega p \cdot \nabla_\omega p'}{r^2} - \frac{|\nabla_\omega p|^2}{r^3} \right] + \frac{1}{r^2} \left[ \alpha^2 p' \Delta_\omega p' + \alpha^2 |\nabla_\omega p'|^2 + \frac{\Delta_\omega |\nabla_\omega p|^2}{2r^2} \right] \\ &\quad - \alpha^2 p' \left( \alpha^2 p''' + \alpha^2 \frac{(n-1)}{r} p'' - \alpha^2 \frac{(n-1)}{r^2} p' - 2 \frac{\Delta_\omega p}{r^3} + \frac{\Delta_\omega p'}{r^2} \right) \\ &\quad - \frac{1}{r^2} \left( \alpha^2 \nabla_\omega p \cdot \nabla_\omega p'' + \alpha^2 \frac{(n-1)}{r} \nabla_\omega p \cdot \nabla_\omega p' + \frac{\nabla_\omega p \cdot \nabla_\omega \Delta_\omega p}{r^2} \right) \\ &\quad - \frac{1}{n} \left[ \alpha^4 p''^2 + \alpha^4 \frac{(n-1)^2}{r^2} p'^2 + \frac{(\Delta_\omega p)^2}{r^4} + 2\alpha^4 \frac{(n-1)}{r} p' p'' + 2\alpha^2 \frac{p'' \Delta_\omega p}{r^2} + 2\alpha^2 \frac{(n-1)}{r^3} p' \Delta_\omega p \right]. \end{aligned}$$

Collecting terms proves the result. ■

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