Lieb-Thirring inequalities with improved constants

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Abstract

Following Eden and Foias we obtain a matrix version of a generalised Sobolev inequality in one-dimension. This allows us to improve on the known estimates of best constants in Lieb-Thirring inequalities for the sum of the negative eigenvalues for multi-dimensional Schrödinger operators.

Key-words: Sobolev inequalities; Schrödinger operator; Lieb-Thirring inequalities.

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1 Introduction

Let $H$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - V$$

For a real-valued potential $V$ we consider Lieb-Thirring inequalities for the negative eigenvalues $\{\lambda_n\}$ of the operator $H$

$$\sum |\lambda_n|^{\gamma} \leq L_{d,\gamma} \int_{\mathbb{R}^d} V_+(x)^{d/2+\gamma} \, dx ,$$

where $V_+ = (|V| + V)/2$ is the positive part of $V$.

Eden and Foias have obtained in [3] a version of a one-dimensional generalised Sobolev inequality which gives best known estimates for the constants in the inequality (2) for $1 \leq \gamma < 3/2$. The aim of this short article is to extend the method from [3] to a class of matrix-valued potentials. By using ideas from [6] this automatically improves on the known estimates of best constants in (2) for multidimensional Schrödinger operators.

Lieb-Thirring inequalities for matrix-valued potentials for the value $\gamma = 3/2$ were obtained in [6] and also in [2]. Here we state a result corresponding to $\gamma = 1$.

**Theorem 1.** Let $V \geq 0$ be a Hermitian $M \times M$ matrix-function defined on $\mathbb{R}$ and let $\lambda_n$ be all negative eigenvalues of the operator (1). Then

$$\sum |\lambda_n| \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \text{Tr} \left[ V(x)^{3/2} \right] \, dx.$$
Remark 1. The constant $\frac{2}{3\sqrt{3}}$ should be compared with the Lieb-Thirring constant found in [7] for a class of single eigenvalue potentials and with the constant obtained in [5] which is twice as large as the semi-classical one

$$\frac{4}{3\sqrt{3}\pi} < \frac{2}{3\sqrt{3}} < 2 \times \frac{2}{3\pi} = 2 \times \frac{1}{2\pi} \int_{\mathbb{R}} (1 - \xi^2)_+ d\xi.$$  
This is about $0.2450 \cdots < 0.3849 \cdots < 0.4244 \ldots$.

Remark 2. Note that the values of the best constants for the range $1/2 < \gamma < 3/2$ remain unknown.

Let $\mathcal{A}(x) = (a_1(x), \ldots, a_d(x))$ be a magnetic vector potential with real valued entries $a_k \in L^2_{\text{loc}}(\mathbb{R}^d)$ and let

$$H(\mathcal{A}) = (i \nabla + \mathcal{A})^2 - V,$$
where $V \geq 0$ is a real-valued function.

Denote the ratio of $2/3\sqrt{3}$ and the semi-classical constant by

$$R := \frac{2}{3\sqrt{3}} \times \left( \frac{2}{3\pi} \right)^{-1} = 1.8138 \ldots$$

By using the Aizenmann-Lieb argument [1], a “lifting” with respect to dimension [6], [5], and Theorem 1 we obtain the following result:

**Theorem 2.** For any $\gamma \geq 1$ and any dimension $d \geq 1$, the negative eigenvalues of the operator $H(\mathcal{A})$ satisfy the inequalities

$$\sum |\lambda_n|^{\gamma} \leq L^{d,\gamma}_{d,c} \int_{\mathbb{R}^d} V(x)^{d/2 + \gamma} dx,$$
where

$$L^{d,\gamma}_{d,c} \leq R \times L^{1}_{d,c} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - |\xi|^2)^{\gamma} d\xi.$$  

**Remark 3.** Theorem 2 allows us to improve on the estimates of best constants in Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials recently obtained in [4].

2 One-dimensional generalised Sobolev inequality for matrices

Let $\{\phi_n\}_{n=0}^N$ be an orthonormal system of vector-functions in $L^2(\mathbb{R}; \mathbb{C}^M)$, $M \in \mathbb{N}$,

$$(\phi_n, \phi_m)_{L^2(\mathbb{R}; \mathbb{C}^M)} = \sum_{j=1}^M \int_{\mathbb{R}} \phi_n(x, j) \overline{\phi_m(x, j)} dx = \delta_{nm},$$
where $\delta_{nm}$ is the Kronecker symbol. Let us introduce an $M \times M$ matrix $U$ with entries

$$u_{j,k}(x, y) = \sum_{n=0}^{N} \phi_n(x, j) \phi_n(y, k).$$

Clearly

$$U(x, y)^* = U(y, x). \quad (4)$$

The fact that the functions $\phi_n$ are orthonormal can be written in a compact form

$$\int_{\mathbb{R}} U(x, y) U(y, z) dy = U(x, z). \quad (5)$$

The two properties (4) and (5) prove that $U(x, y)$ is the Schwartz kernel of an orthogonal projection $P$ in $L^2(\mathbb{R}; \mathbb{C}^M)$ whose image is the subspace of vector-functions spanned by $\{\phi_n\}_{n=1}^{N}$.

**Theorem 3.** Let us assume that the vector-function $\phi_n$, $n = 1, 2, \ldots N$, are from the Sobolev class $H^1(\mathbb{R}; \mathbb{C}^M)$. Then

$$\int_{\mathbb{R}} \text{Tr} \left[ U(x, x)^3 \right] dx \leq \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{\mathbb{R}} |\phi'_n(x, j)|^2 dx .$$

**Proof.**

$$\frac{d}{dy} \text{Tr} \left[ U(x, y) U(y, x) U(x, x) \right]$$

$$= \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] + \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \quad (6)$$

By integrating (6) and taking absolute values one obtains

$$\frac{1}{2} \text{Tr} \left[ U(x, z) U(z, x) U(x, x) \right]$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} \left| \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] + \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \right| dy$$

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and
\[
\frac{1}{2} \text{Tr} \left[ U(x, z) U(z, x) U(x, x) \right]
\leq \frac{1}{2} \int_{\mathbb{R}} \left| \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right| dy
+ \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] dy.
\]

Taking absolute values and adding the two inequalities yields for any \( z \in \mathbb{R} \)
\[
\left| \text{Tr} \left[ U(x, z) U(z, x) U(x, x) \right] \right|
\leq \frac{1}{2} \int_{\mathbb{R}} \left| \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \right| dy
+ \frac{1}{2} \int_{\mathbb{R}} \left| \text{Tr} \left[ U(x, y) \left( \frac{d}{dy} U(y, x) \right) U(x, x) \right] \right| dy. \quad (7)
\]

Note that we have reproven the inequality
\[
|f(x)|^2 \leq \int_{\mathbb{R}} |f(y) f'(y)| dy
\]
for traces of matrices. By using properties of traces, the Cauchy-Schwarz inequality for matrix-functions and also properties (4) and (5), we find that for all \( x \in \mathbb{R} \)
\[
\left( \int_{\mathbb{R}} \left| \text{Tr} \left[ \left( \frac{d}{dy} U(x, y) \right) U(y, x) U(x, x) \right] \frac{d}{dy} \right| dy \right)^2
\leq \int_{\mathbb{R}} \left[ \frac{d}{dy} U(x, y)^* \frac{d}{dy} U(x, y) \right] dy \int_{\mathbb{R}} \left[ U(x, y)^* U(x, x)^2 U(x, y) \right] dy, \]
\[
= \int_{\mathbb{R}} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(x, y) \right] dy \int_{\mathbb{R}} \left[ U(x, y)^2 U(x, y) U(x, x) \right] dy
\]
\[
= \int_{\mathbb{R}} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(x, y) \right] dy \text{ Tr} \left[ U(x, x)^3 \right],
\]
and similarly
\[
\left( \int_{\mathbb{R}} \left| \text{Tr} \left[ U(x, y) \frac{d}{dy} U(y, x) U(x, x) \right] \frac{d}{dy} \right| dy \right)^2
\leq \int_{\mathbb{R}} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(y, x) \right] dy \text{ Tr} \left[ U(x, x)^3 \right].
\]
Thus, using this, and setting $x = z$ in (7), we arrive at
\[
\left| \text{Tr} \left[ U(x, x)^3 \right] \right| \leq \int_{\mathbb{R}} \text{Tr} \left[ \frac{d}{dy} U(x, y) \frac{d}{dy} U(y, x) \right] dy .
\]
Integrating with respect to $x$ we finally obtain
\[
\int_{\mathbb{R}} \left| \text{Tr} \left[ U(x, x)^3 \right] \right| dx 
\leq \sum_{n,k=1}^{N} \sum_{i,j=1}^{M} \int_{\mathbb{R}} \phi_n(x, i) \overline{\phi_n'(y, j)} \phi_k(y, j) \overline{\phi_k(x, i)} dx dy
\]
\[= \sum_{n=1}^{N} \sum_{j=1}^{M} \int_{\mathbb{R}} |\phi_n'(x, j)|^2 dx,
\]
which completes the proof. \hfill \square

3 Lieb-Thirring inequalities for Schrödinger operators with matrix-valued potentials

Let us assume that $V \in C_0^\infty(\mathbb{R}; \mathbb{C}^{M \times M})$, $V \geq 0$, be a $M \times M$ Hermitian matrix-valued potential with entries $\{v_{ij}\}_{i,j=1}^{M}$. Then the negative spectrum of the Schrödinger operator $H = -\frac{d^2}{dx^2} - V$ in $L^2(\mathbb{R}; \mathbb{C}^M)$ is finite. For general potentials the result is obtained by an approximation argument.

Denote by $\{\phi_n\}$ the orthonormal system of eigen-vector functions corresponding to the eigenvalues $\{\lambda_n\}_{n=1}^{N}$
\[
-\frac{d^2}{dx^2} \phi_n - V \phi_n = \lambda_n \phi_n .
\]
Clearly,
\[
\sum_n \lambda_n = \sum_{n,j} \int_{\mathbb{R}} |\phi_n'(x, j)|^2 dx - \text{Tr} \left[ \int_{\mathbb{R}} V(x) U(x, x) dx \right]
\]
and by Hölder’s inequality for traces,
\[
\int_{\mathbb{R}} \text{Tr} \left[ V(x) U(x, x) \right] dx \leq \left( \int_{\mathbb{R}} \text{Tr} \left[ V(x)^{3/2} \right] dx \right)^{2/3} \left( \int_{\mathbb{R}} \text{Tr} \left[ U(x, x)^3 \right] dx \right)^{1/3},
\]
so that using Theorem 3
\[
\sum_n \lambda_n \geq X - \left( \int_{\mathbb{R}} \text{Tr} \left[ V(x)^{3/2} \right] dx \right)^{2/3} X^{1/3}
\]

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with \( X := \int_{\mathbb{R}} \text{Tr} \left[ U(x, x)^3 \right] \, dx \). Minimising the right hand side with respect to \( X \) we finally complete the proof of Theorem 1
\[
\sum_n \lambda_n \geq -\frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \text{Tr} \left[ V(x)^{3/2} \right] \, dx.
\]

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