Mathematisches Forschungsinstitut Oberwolfach

Report No. 34/2016

DOI: 10.4171/OWR/2016/34

Calculus of Variations

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10 July – 16 July 2016

Abstracts

Symmetry by flow

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(joint work with M.J. Esteban, M. Loss and M. Muratori)

With the norms $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \big(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\big)^{1/q}$, let us consider the family of *Caffarelli-Kohn-Nirenberg inequalities* introduced in [2] and given by

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \tag{1}$$

in a suitable functional space $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ obtained by completion of smooth functions with support in $\mathbb{R}^d\setminus\{0\}$, w.r.t. the norm given by $\|w\|^2 := (p_\star-p)\,\|w\|^2_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} + \|\nabla w\|^2_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}$. Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters β, γ and p are subject to the restrictions

$$d\ge 2\,,\quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star]\quad\text{with } p_\star:=\frac{d-\gamma}{d-\beta-2} \tag{2}$$

and the exponent $\vartheta=\frac{(d-\gamma)\,(p-1)}{p\,(d+\beta+2-2\,\gamma-p\,(d-\beta-2))}$ is determined by the scaling invariance.

Equality in (1) is achieved by Aubin-Talenti type functions

$$w_\star(x)=\big(1+|x|^{2+\beta-\gamma}\big)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d$$

if we know that *symmetry* holds, that is, if we know that the equality is achieved among radial functions. However, depending on the parameters, to decide whether a minimizer has the full symmetry or not can be difficult. To show that symmetry is broken one can minimize the functional in the *class of symmetric functions* and then check whether the value of the functional can be lowered by perturbing the minimizer away from the symmetric situation. This is the method that has been used to establish that *symmetry breaking* occurs in (1) if

$$\gamma<0\quad\text{and}\quad\beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma \tag{3}$$

where

$$\beta_{\rm FS}(\gamma):=d-2-\sqrt{(\gamma-d)^2-4\,(d-1)}\,.$$

In the critical case $p=p_\star$, the method was implemented by F. Catrina and Z.-Q. Wang in [3], and the sharp result has been obtained by V. Felli and M. Schneider in [6]. The same condition was recently obtained in the subcritical case $p<p_\star$, in [1]. Throughout this report, by *critical* we simply mean that $\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)}$ scales like $\|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}$. One has to observe that proving symmetry breaking by establishing the linear instability is a *local* method, which is based on a painful but rather straightforward linearization around the special function $w_\star$.

A real difficulty occurs when the minimizer in the symmetric class is stable, *i.e.*, all local perturbations that break the symmetry increase the energy: in our case, non-radial perturbations. To establish the optimal symmetry range in (1), and thus determine the sharp constant in the Caffarelli-Kohn-Nirenberg inequalities whenever the optimal functions are radially symmetric, a new method had to be designed. What has been proved in [4] in the critical case $p = p_\star$, and extended in [5] to the sub-critical case $1 < p < p_\star$, is that the symmetry breaking range given in (3) is optimal: symmetry holds in the complementary region of the admissible parameters.

Theorem 1. [4, 5] *Under Condition* (2) *assume that either* $\gamma \geq 0$ *or* $\beta \leq \beta_{\rm FS}(\gamma)$ *if* $\gamma < 0$. *Assume that* $d \geq 2$. *Then all positive solutions in* $\mathrm H^p_{\beta,\gamma}(\mathbb R^d)$ *to*

$$-\,div\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - w^p\right) \quad in \quad \mathbb R^d\setminus\{0\}\,. \tag{4}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star$.

The main ideas of the proof can be summarized into a three steps scheme.

1) The first step is based on a change of variables which amounts to rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise to consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$\alpha = 1 + \tfrac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\tfrac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (1) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm L^{2p,d-n}(\mathbb R^d)} \leq \mathsf K_{\alpha,n,p}\,\|\mathsf D_\alpha v\|^{\vartheta}_{\mathrm L^{2,d-n}(\mathbb R^d)}\,\|v\|^{1-\vartheta}_{\mathrm L^{p+1,d-n}(\mathbb R^d)} \quad \forall\, v \in \mathrm H^p_{d-n,d-n}(\mathbb R^d)\,,$$

with the notations $s = |x|$, $\omega = \frac{x}{s}$ and $\mathsf D_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac1s\,\nabla_\omega v\right)$.

2) Let us consider the derivative of a generalized *Rényi entropy power* functional

$$\mathcal G[u] := \left(\int_{\mathbb R^d} u^m\,d\mu\right)^{\sigma-1} \int_{\mathbb R^d} u\,|\mathsf D_\alpha \mathsf P|^2\,d\mu$$

where $\sigma = \frac2d\,\frac{1}{1-m} - 1$. Here $\mathsf P$ is the *pressure* variable $\mathsf P := \frac{m}{1-m}\,u^{m-1}$ while m and p are related by $p = \frac{1}{2\,m-1}$. Next we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal L_\alpha\,u^m \quad \text{with} \quad \mathcal L_\alpha\,u = -\,\mathsf D_\alpha^*\,\mathsf D_\alpha\,u = \alpha^2\left(u'' + \tfrac{n-1}{s}\,u'\right) + \tfrac{1}{s^2}\,\Delta_\omega\,u \tag{5}$$

in the subcritical range $1 - 1/n < m < 1$ and in the critical case $m = 1 - 1/n$. The key computation is the proof that

$$\begin{aligned}
&-\frac{d}{dt}\,\mathcal G[u(t,\cdot)]\left(\int_{\mathbb R^d} u^m\,d\mu\right)^{1-\sigma}\\
&\geq (1-m)\,(\sigma-1)\textstyle\int_{\mathbb R^d} u^m\left|\mathcal L_\alpha\,\mathsf P - \frac{\int_{\mathbb R^d} u\,|\mathsf D_\alpha \mathsf P|^2\,d\mu}{\int_{\mathbb R^d} u^m\,d\mu}\right|^2 d\mu\\
&\quad + 2\textstyle\int_{\mathbb R^d}\left(\alpha^4\left(1-\frac1n\right)\left|\mathsf P'' - \frac{\mathsf P'}{s} - \frac{\Delta_\omega\,\mathsf P}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega \mathsf P' - \frac{\nabla_\omega \mathsf P}{s}\right|^2\right) u^m\,d\mu\\
&\quad + 2\textstyle\int_{\mathbb R^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega \mathsf P|^2 + c(n,m,d)\,\frac{|\nabla_\omega \mathsf P|^4}{\mathsf P^2}\right) u^m\,d\mu =: \mathcal H[u]
\end{aligned}$$

for some numerical constant $c(n,m,d)>0$. Hence if $\alpha\le\alpha_{\rm FS}:=\sqrt{(d-1)/(n-1)}$, the r.h.s. in $\mathcal H[u]$ vanishes if and only if $\mathsf P$ is an affine function of $|x|^2$.

3) This method has a hidden difficulty. In the above computation, many integrations by parts have to be performed, which require a sufficient decay of the function u and of its derivatives as $|x|\to+\infty$ and also, because of the weight, good properties as $x\to0$. So far, such properties are not known for a general solution of (5). However, we may consider a positive solution to (4) and, up to the above changes of variables, take the corresponding function u as an initial datum for (5). On the one hand, since u is a critical point of $\mathcal G$ under mass constraint, we know that $\frac{d}{dt}\,\mathcal G[u(t,\cdot)]=0$ at $t=0$. On the other hand, because u solves an elliptic PDE, it is possible to establish all regularity and decay estimates that are needed to do the integrations by parts, hence $\mathcal H[u]=0$. In that way we conclude that w is equal to $w_\star$ up to a scaling and a multiplication by a constant, if $\beta\le\beta_{\rm FS}(\gamma)$.

Applying the flow at $t=0$ to a critical point amounts to write the Euler-Lagrange equation and test it with $\mathcal L_\alpha\,u^m$. In other words, what we write is

$$0=\int_{\mathbb R^d}\mathrm d\mathcal G[u]\cdot\mathcal L_\alpha\,u^m\,d\mu\ge\mathcal H[u]\ge0$$

where the last inequality holds because $\mathcal H[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term). If we undo the change of variables, our method amounts to rewrite (4) as

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

for some constants c_1, c_2 and test it against $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$.

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