Hypocoercivity without confinement: mode-by-mode analysis and decay rates in the Euclidean space

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Abstract.– L^2 *hypocoercivity results for scattering and Fokker-Planck type collision operators are obtained using decoupled Fourier modes. The rates are measured in a space with exponential weights and then extended to larger function spaces by a factorization method. Without confinement, sharp rates of decay are obtained.*

Let us consider the evolution equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F \tag{1}$$

and assume that T and L are respectively anti-Hermitian and Hermitian operators on a complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ with norm $\|\cdot\|$. As in the *hypocoercivity* method of [4] for real valued operators, we consider the Lyapunov functional

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle\mathsf{A}F, F\rangle$$

for some $\delta > 0$, with $\mathsf{A} := \big(1+(\mathsf{T\Pi})^*\mathsf{T\Pi}\big)^{-1}(\mathsf{T\Pi})^*$. Here * denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$ and $\mathsf{\Pi}$ is the orthogonal projection onto the null space of L. We assume that positive constants λ_m, λ_M, and C_M exist, such that, for any $F \in \mathcal{H}$, the following properties hold:
(H1) *microscopic coercivity:* $-\langle\mathsf{L}F, F\rangle \geq \lambda_m\,\|(1-\mathsf{\Pi})F\|^2$,
(H2) *macroscopic coercivity:* $\|\mathsf{T\Pi}F\|^2 \geq \lambda_M\,\|\mathsf{\Pi}F\|^2$,
(H3) *parabolic macroscopic dynamics:* $\mathsf{\Pi T \Pi}\,F = 0$,
(H4) *bounded auxiliary operators:* $\|\mathsf{AT}(1-\mathsf{\Pi})F\| + \|\mathsf{AL}F\| \leq C_M\,\|(1-\mathsf{\Pi})F\|$.
Then for any $t \geq 0$, if F solves (1) with initial datum F_0, we have

$$\mathsf{H}[F(t,\cdot)] \leq \mathsf{H}[F_0]\,e^{-\lambda_\star t}$$

where $\lambda_\star$ is characterized as the smallest $\lambda > 0$ for which there exists some $\delta > 0$ such that $(\delta\,C_M)^2 - 4\,\big(\lambda_m - \delta - \frac{2+\delta}{4}\,\lambda\big)\,\big(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{2+\delta}{4}\,\lambda\big) = 0$ under the additional condition that $\lambda_m - \delta - \frac{1}{4}\,(2+\delta)\,\lambda > 0$.

This abstract hypocoercivity result applies to kinetic equations with various *collision operators* L whose null space is spanned by an *admissible local equilibrium* M, that is, a radially symmetric continuous function such that, additionally, M^{-1} has a growth faster than any polynomial as $|v| \to +\infty$, $\int_{\mathbb{R}^d} M\,\mathrm{d}v = 1$.

Here are two important examples:
▷ *Fokker-Planck operators with general equilibria:* $\mathsf{L}F = \nabla_v \cdot \big[M\,\nabla_v\,\big(M^{-1}\,F\big)\big]$ where M is such that $v \mapsto |\nabla_v\sqrt{M}|^2$ is integrable and a Poincaré inequality holds with respect to the measure $M\,\mathrm{d}v$.

▷ *Scattering collision operators:* $\mathsf{L}F = \int_{\mathbb{R}^d} \sigma(\cdot, v') \big(F(v')\, M(\cdot) - F(\cdot)\, M(v')\big)\, \mathrm{d}v'$. We assume that the *symmetry condition*

$$\int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big)\, M(v')\, \mathrm{d}v' = 0$$

holds and that the *scattering rate* σ is such that $1 \le \sigma(v,v') \le \overline{\sigma}$ for some positive, finite $\overline{\sigma}$. The *microscopic coercivity* property follows from [3].

Next we consider a distribution function $f(t,x,v)$, where x denotes the position variable, $v \in \mathbb{R}^d$ is the velocity variable, and $t \ge 0$ is the time. We shall consider either $x \in \mathbb{T}^d \approx [0, 2\,\pi)^d$ or $x \in \mathbb{R}^d$. In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x

$$f(t,x,v) = \int_{\mathbb{R}^d} \hat{f}(t,\xi,v)\, e^{-i\,x\cdot\xi}\, \mathrm{d}\mu(\xi)$$

where the measure $\mathrm{d}\mu$ is such that $\mathrm{d}\mu(\xi) = (2\pi)^{-d}\, d\xi$ and $d\xi$ is the Lesbesgue measure if $x \in \mathbb{R}^d$, and $\mathrm{d}\mu(\xi) = (2\pi)^{-d} \sum_{z\in\mathbb{Z}^d} \delta(\xi - z)$ is discrete for $x \in \mathbb{T}^d$. Since the collision operator L does not depend on x, the kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \tag{2}$$

is reduced to (1) applied to $F(t,v) = \hat{f}(t,\xi,v)$ for each mode ξ, where ξ is now considered as a parameter, and the transport operator $v\cdot\nabla_x$ is, in Fourier variables, the simple multiplication operator

$$\mathsf{T}F := i\,(v\cdot\xi)\, F\,.$$

With $\Theta = \int_{\mathbb{R}^d} |v\cdot\xi|^2\, M(v)\, \mathrm{d}v$, the operator A is now given by

$$\mathsf{A}F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\, F(v')\, \mathrm{d}v'}{1 + \Theta\,|\xi|^2}\, M\,.$$

Under the above assumptions, for any $t \ge 0$, for any fixed ξ, with we have

$$\|F(t,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}\gamma)} \le 3\, e^{-\,\mu_\xi\, t}\, \|F_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma)}$$

where $\mathrm{d}\gamma = M^{-1}\, \mathrm{d}v$, $\mu_\xi := \frac{\Lambda\,|\xi|^2}{1+|\xi|^2}$, $\Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\, \min\big\{1, \frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\big\}$ with $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ for scattering operators and $\kappa = 2\,\|\nabla_v\sqrt{M}\|_{\mathrm{L}^2(\mathrm{d}v)}/\sqrt{d}$ for Fokker-Planck operators. By the factorization result of [5], the same decay rate is obtained if we replace the measure $\mathrm{d}\gamma$ by

$$\mathrm{d}\gamma_k := \gamma_k(v)\, \mathrm{d}v \quad \text{where} \quad \gamma_k(v) = \pi^{d/2}\, \tfrac{\Gamma((k-d)/2)}{\Gamma(k/2)}\, \big(1 + |v|^2\big)^{k/2}$$

for an arbitrary $k \in (d, +\infty)$. Using Parseval's identity, we obtain that the solution f of (2) on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)$ such that $\iint_{\mathbb{T}^d\times\mathbb{R}^d} f_0\, \mathrm{d}x\, \mathrm{d}v = 1$ satisfies, for any $t \ge 0$,

$$\big\|f(t,\cdot,\cdot) - |\mathbb{T}^d|^{-1} M\big\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le C_k\, \|f_0 - f_\infty\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)}\, e^{-\,\Lambda\, t/4}$$

for some positive constant C_k.

On the whole Euclidean space $\mathbb{R}^d$, we consider the Lyapunov functional

$$f \mapsto \tfrac{1}{2}\, \|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} + \delta\, \langle \mathsf{A}f, f\rangle_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)}$$

where the operator $\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is now defined in the (x,v) variables using $\mathsf{T} := v\cdot\nabla_x$. We can use Plancherel's formula. However, it is clear that without an external potential of confinement, there is no Poincaré inequality to be expected. Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(\mathrm{d}x)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(\mathrm{d}x)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(\mathrm{d}x)}$$

allows us to prove that there exists a constant $C_k > 0$ such that, for any $t \ge 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le C_k\left(\|f_0\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_k;\,\mathrm{L}^1(\mathrm{d}x))}\right)(1+t)^{-\frac{d}{2}}\,.$$

So far we did not assume any sign condition on f. Inspired by the properties of the solutions of the heat equation, a more detailed analysis shows that the zero average solutions of (2) have an improved decay rate. Assume that $f_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d\times\mathbb{R}^d)$ with $\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,v)\,\mathrm{d}x\,\mathrm{d}v = 0$ and let

$$\mathcal{C} := \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_{k+2};\,\mathrm{L}^1(\mathrm{d}x))} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}\gamma_k;\,\mathrm{L}^1(|x|\,\mathrm{d}x))} + \|f_0\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} < \infty\,.$$

Then there exists a constant $c_k > 0$ such that, for any $t \ge 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\gamma_k)} \le c_k\,\mathcal{C}\,(1+t)^{-(1+\frac{d}{2})}\,.$$

For details, see [1]. Further improved estimates will be available in [2].

References

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