L^2 hypocoercivity, inequalities and applications

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Let us consider the kinetic equation

$$\partial_t f + \mathsf{T} f = \mathsf{L} f \tag{1}$$

where $\mathsf{L}f$ is either the Fokker-Planck operator $\mathsf{L}_1 f = \nabla_v \cdot \big(\mathcal{M}\,\nabla_v(\mathcal{M}^{-1} f)\big)$ or a scattering collision operator $\mathsf{L}f = \int_{\mathbb{R}^d} \sigma(\cdot, v')\,\big(f(v')\,\mathcal{M}(\cdot) - f(\cdot)\,\mathcal{M}(v')\big)\,dv'$, for instance the simplest possible one, the linear BGK operator $\mathsf{L}_2 f = \rho\,\mathcal{M}(v) - f$ where $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$ is the spatial density and $\mathcal{M}(v) = (2\pi)^{-d/2}\,e^{-|v|^2/2}$. The transport operator T on the phase space (with position x and velocity v) can be rewritten as $\mathsf{T} = i\,v\cdot\xi\,\hat f$ in the Fourier variable ξ associated to x. With the operators Π and A defined respectively by $\Pi \hat f := \mathcal{M}\int_{\mathbb{R}^d} \hat f(\xi,w)\,dw$ and $\mathsf{A}\hat f(\xi,v) := -\,i\,\xi\,(1+|\xi|^2)^{-1}\cdot\int_{\mathbb{R}^d} w\,\hat f(\xi,w)\,dw\,\mathcal{M}(v)$, the L^2 entropy, or L^2 Lyapunov functional $\mathsf{H}[\hat f] := \frac12\,\|\hat f\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}\hat f, \hat f\rangle$ where $\|g\|^2 := \iint_{\mathbb{R}^d} |g(w)|^2\,d\gamma$, $d\gamma(w) := \mathcal{M}(w)^{-1}\,dw$, is such that, if f solves (1), then the *entropy – entropy production inequality*

$$\frac{d}{dt}\mathsf{H}[\hat f(t,\xi,\cdot)] \le -\lambda\,\mathsf{H}[\hat f(t,\xi,\cdot)])$$

holds if

$$Q(X,Y) := \Big(1 - \tfrac{\delta\,|\xi|^2}{1+|\xi|^2} - \tfrac{\lambda}{2}\Big)\,X^2 - \tfrac{\delta\,|\xi|}{1+|\xi|^2}\,\big(1+\sqrt3\,|\xi| + \lambda\big)\,X\,Y + \Big(\tfrac{\delta\,|\xi|^2}{1+|\xi|^2} - \tfrac{\lambda}{2}\Big)\,Y^2$$

is a nonnegative quadratic form of X and Y, where $X := \|(1-\Pi)\hat f\|$ and $Y := \|\Pi\hat f\|$. Here it is clear that $\xi\in\mathbb{R}^d$ can be considered as a parameter, that is, we can perform a *mode-by-mode* analysis. Proving the exponential decay of $\mathsf{H}[\hat f]$ for some $\delta = \delta(|\xi|)$ with a rate $\lambda = \lambda(|\xi|)$ is reduced to the discriminant condition which guarantees that $Q \ge 0$. It turns out that $\mathsf{H}[\hat f]$ is equivalent to $\|\hat f\|^2$ if $\delta < 2$ and one can prove by the method of [9, 5] that

$$\|\hat f(t,\xi,\cdot)\|^2 \le C(|\xi|)\,\|\hat f_0(\xi,\cdot)\|^2\,e^{-\lambda(|\xi|)\,t}\quad \forall\,t\ge0\,,\quad \xi\in\mathbb{R}^d\,,$$

where $C(|\xi|) = (2+\delta(|\xi|))/(2-\delta(|\xi|))$. This has been analysed in [5] in terms of asymptotic decay rates of $\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)}$ for a choice of δ which is independent of ξ but can be refined by taking a ξ-dependent value of δ.

Theorem 1. [2] *If f solves* (1) *with $\mathsf{L} = \mathsf{L}_1$ or $\mathsf{L} = \mathsf{L}_2$ for some nonnegative initial datum $f_0 \in \mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d, dx\,d\gamma)\cap \mathrm{L}^2\big(\mathbb{R}^d, d\gamma; \mathrm{L}^1(\mathbb{R}^d,dx)\big)$, then*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)} \le (2\,\pi)^{-d}\,\Psi_{M,Q}(t)\quad \forall\,t\ge0$$

with $M = \|f_0\|_{\mathrm{L}^2(\mathbb{R}^d,d\gamma;\mathrm{L}^1(\mathbb{R}^d,dx))}$, $Q = \|f_0\|_{\mathrm{L}^2(\mathbb{R}^d\times\mathbb{R}^d,dx\,d\gamma)}$, and

$$\Psi_{M,Q}(t) := \inf_{R>0}\Big(\int_0^R C(s)\,e^{-\,\lambda(s)\,t}\,s^{d-1}\,ds\,\omega_d\,d\,M^2 + \sup_{s\ge R} C(s)\,e^{-\,\lambda(R)\,t}\,Q^2\Big)\,.$$

The proof of this result is reminiscent of the proof in [11] of *Nash's inequality*

$$\|u\|^{2+\frac4d}_{\mathrm{L}^2(\mathbb{R}^d)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac4d}_{\mathrm{L}^1(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \tag{2}$$

and the definition of the operator A is inspired by the diffusion limit: see [9, 2]. It is well known that a solution to the heat equation

$$\partial_t u = \Delta u \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

decays according to

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le \mathcal{C}\,\|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)}\,(1+t)^{-\frac{d}{2}}$$

after computing $\frac{d}{dt}\|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)} = -2\,\|\nabla u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)}$ and taking (2) into account. See [7] for a discussion of the optimality of such an estimate. This decay rate can be recovered also at the kinetic level for the solution of (1): see [5].

The next question is of course to understand what happens in presence of an external potential. Let us start at diffusive level with the Fokker-Planck equation

$$\partial_t u = \Delta u + \nabla\cdot(u\,\nabla V) \quad \forall\,(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \tag{3}$$

where V is a given external potential. To fix ideas, we shall assume that $V(x) = |x|^\alpha$ for some $\alpha > 0$ and discuss the cases depending on the value of α.

$\rhd$ For $\alpha \ge 1$, we have the *Poincaré inequality*

$$\int_{\mathbb{R}^d} |v-\bar v|^2\,d\mu_\alpha \le \mathcal{C}\int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

with $\bar v := \int_{\mathbb{R}^d} v\,d\mu_\alpha$ and $d\mu_\alpha := U_\alpha(x)\,dx$, $U_\alpha(x) := Z_\alpha^{-1}\,e^{-|x|^\alpha}$, $Z_\alpha := \int_{\mathbb{R}^d} U_\alpha\,dx$. It is then standard to prove that a solution $u(t,\cdot)$ of (3) is such that $v := u/U_\alpha$ satisfies

$$\int_{\mathbb{R}^d} |v(t,\cdot)-\bar v|^2\,d\mu_\alpha \le \int_{\mathbb{R}^d} |v(0,\cdot)-\bar v|^2\,d\mu_\alpha\,e^{-\frac{2\,t}{\mathcal{C}}} \quad \forall\,t\ge 0\,.$$

$\rhd$ The case $\alpha \in (0,1)$ has been studied in [10] using the weak Poincaré inequality. This approach requires the existence of a uniform bound. Alternatively, we can consider the *weighted Poincaré inequality*

$$\int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha \ge \mathcal{C}\int_{\mathbb{R}^d} |v-\bar v|^2\langle x\rangle^{-\beta}\,d\mu_\alpha \tag{4}$$

with $\beta = 2\,(1-\alpha)$ and the same notations as above for $\bar v$ and $d\mu_\alpha$. Here we use the notation $\langle x\rangle = \sqrt{1+|x|^2}$ and notice that β vanishes as $\alpha \to 1_-$. In order to compensate for the additional weight in the right hand side in (4), it is convenient to introduce a weighted L^2 norm with a weight $\langle x\rangle^k$.

Theorem 2. [4] *Assume that $\alpha \in (0,1)$. If u solves (3) with initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d, d\mu_\alpha) \cap \mathrm{L}^2(\mathbb{R}^d, \langle x\rangle^k\,d\mu_\alpha)$ for some $k>0$, $v = u/U_\alpha$ and $v_0 = u_0/U_\alpha$, then*

$$\int_{\mathbb{R}^d} |v(t,\cdot)-\bar v|^2\;d\mu_\alpha \le \left(\left(\int_{\mathbb{R}^d} |v_0-\bar v|^2\;d\mu_\alpha\right)^{-\beta/k} + \mathcal{K}\,t\right)^{-k/\beta} \quad \forall\,t\ge 0\,,$$

for some constant $\mathcal{K}$ depending on k and u_0.

$\triangleright$ In the limit case as $\alpha \to 0_+$, it makes sense to consider $V(x) = \gamma \log|x|$. In the range $\gamma \in (0,d)$, (3) admits no stationary solution in $\mathrm{L}^1(\mathbb{R}^d)$. In that case, we can again introduce weights and consider the *Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^d} |x|^\gamma\, u^2\, dx \le \mathcal{C} \left(\int_{\mathbb{R}^d} |x|^{-\gamma}\, |\nabla(|x|^\gamma u)|^2\, dx \right)^a \left(\int_{\mathbb{R}^d} |x|^k\, |u|\, dx \right)^{2\,(1-a)}$$

which generalizes Nash's inequality (2). A decay result goes as follows.

Theorem 3. [6] *Let $d \ge 1$ and $\gamma \in (0,d)$, $k \ge \max\{2, \gamma/2\}$. If u solves (3) with initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d, \langle x\rangle^k\, d\mu_\alpha) \cap \mathrm{L}^2(\mathbb{R}^d, d\mu_\alpha)$, then there is a constant $c > 0$ depending on u_0 such that*

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d, |x|^\gamma\, dx)} \le \|u_0\|^2_{\mathrm{L}^2(\mathbb{R}^d, |x|^\gamma\, dx)}\, (1 + c\, t)^{-\frac{d-\gamma}{2}} \quad \forall\, t \ge 0\,.$$

Similar results can be obtained at kinetic level when the transport operator is defined by $\mathsf{T}f = v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f$. With $\mathsf{L} = \mathsf{L}_1$ or $\mathsf{L} = \mathsf{L}_2$, and appropriate estimates involving $\langle x\rangle^k$, results are obtained which are all consistent with a diffusion limit given by (3) and rely on the same functional inequalities. So far we have considered only Maxwellian local equilibria, but a similar discussion can be done when $\mathcal{M}(v) = Z_\beta^{-1} \exp(-|v|^\beta)$ for some $\beta > 0$, depending whether $\beta \ge 1$ or not, and the case $F(v) = \langle v\rangle^{-\gamma}$ has also been studied. Notice that a fractional diffusion limit has to be considered when $\int_{\mathbb{R}^d} |v|^2\, F(v)\, dv$ is infinite. The method also adapts to equations with a Poisson coupling. See [6, 4, 8, 3, 1] for detailed statements.

References

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