**Stability in Gagliardo-Nirenberg-Sobolev inequalities**

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(joint work with Matteo Bonforte, Bruno Nazaret, Nikita Simonov)

Optimal constants and optimal functions are known in some functional inequalities. The next question is the stability issue: is the difference of the two terms controlling a distance to the set of optimal functions ? A famous example is provided by Sobolev's inequalities: in 1991, G. Bianchi and H. Egnell proved that the difference of the two terms is bounded from below by a distance to the manifold of the Aubin-Talenti functions. They argued by contradiction and gave a very elegant although not constructive proof. Since then, estimating the stability constant and giving a constructive proof has always been a challenge.

This contribution focuses on Gagliardo-Nirenberg-Sobolev inequalities. The main tool is based on entropy methods and nonlinear flows. In our method, proving stability amounts to establish, under some constraints, a version of the entropy – entropy production inequality with an improved constant. In simple cases, for instance on the sphere, rather explicit results have been obtained by the *carré du champ* method introduced by D. Bakry and M. Emery. In the Euclidean space, results based on constructive regularity estimates for the solutions of the nonlinear flow have been obtained in a joint research project with Matteo Bonforte, Bruno Nazaret, and Nikita Simonov.

According to [1], on $\mathbb{R}^d$, $d \geq 3$, there is a positive constant $\alpha$ such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathcal{M}} \|\nabla f - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \,, \tag{1}$$

where the left-hand side is the difference of the two terms in Sobolev's inequality, with optimal constant $\mathsf{S}_d$, and $\mathfrak{M}$ denotes the manifold of the optimal Aubin-Talenti functions. Various improvements as, *e.g.*, in [4, 7] have been obtained but the question of *constructive* estimates is still widely open: $\alpha$ is obtained by compactness estimates and contradiction arguments, and no good stability estimate is known so far at least with a strong notion of distance as in the right-hand side of (1). See [6] for a result with a weaker norm.

We consider the family of *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_2^{\theta} \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p) \, \|f\|_{2\,p} \tag{2}$$

with $\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}$, $p \in (1,+\infty)$ if $d = 1$ or 2, $p \in (1, p^*]$ if $d \geq 3$, $p^* = \frac{d}{d-2}$. It is known from [5] that equality is achieved if and only if, up to a multiplication by a constant, a translation and a scaling, $f = \mathsf{g}$ where $\mathsf{g}(x) := \left(1 + |x|^2\right)^{-1/(p-1)}$. We shall denote the corresponding manifold by $\mathfrak{M}$ as when $p = p^*$. Sobolev's inequality when $d \geq 3$, $p = p^*$, the Euclidean Onofri inequality obtained for $d = 2$ by taking the limit as $p \to +\infty$ with $f(x) = \mathsf{g}(x)\left(1 + \frac{1}{2\,p}\,(h(x) - \overline{h})\right)$,

$$\log\left(\int_{\mathbb{R}^2} e^{h-\overline{h}} \, \frac{dx}{\pi\,(1+|x|^2)^2}\right) \leq \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla h|^2 \, dx \quad \text{with} \quad \overline{h} = \int_{\mathbb{R}^2} \frac{h(x)\,dx}{\pi\,(1+|x|^2)^2}$$

and, as $p \to 1_+$, the (scale invariant) *Euclidean logarithmic Sobolev inequality*

$$\frac{d}{2} \log \left( \frac{2}{\pi\, d\, e} \int_{\mathbb{R}^d} |\nabla f|^2\, dx \right) \geq \int_{\mathbb{R}^d} |f|^2 \log |f|^2\, dx$$

are all limit cases of (2). Let us define the *deficit functional*

$$\delta[f] := (p-1)^2\, \|\nabla f\|_2^2 + 4\, \frac{d - p\,(d-2)}{p+1}\, \|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GNS}}\, \|f\|_{2\,p}^{2\,p\,\gamma}$$

with $\mathcal{K}_{\mathrm{GNS}} = C(p,d)\, \mathcal{C}_{\mathrm{GNS}}^{2\,p\,\gamma}$, $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ for some explicit positive constant $C(p,d)$. A scale optimization shows that (2) is equivalent to the inequality $\delta[f] \geq 0$. Stability results for (2) with non-constructive estimates are known from [3, 10].

With $d \geq 1$, $m \in (1-1/d, 1)$, the *fast diffusion* equation in $\mathbb{R}^d$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{3}$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$, $u_0 \in \mathrm{L}^1\big(\mathbb{R}^d, (1+|x|^2)\, dx\big)$, can be interpreted as the gradient flow of the *entropy* $\mathsf{E} := \int_{\mathbb{R}^d} u^m\, dx$ with respect to Wasserstein's distance, as it is known from [8]. By the *carré du champ method* (adapted from the work of D. Bakry and M. Emery), we can relate (2) and (3), and obtain a proof of (2). The key property, inspired by the *Rényi entropy powers* of [9] is based on the one hand on the fact that

$$\mathsf{E}' = (1-m)\, \mathsf{I}$$

where $\mathsf{I} := \int_{\mathbb{R}^d} u\, |\nabla \mathsf{P}|^2\, dx$ is the *Fisher information* and $\mathsf{P} = \frac{m}{m-1}\, u^{m-1}$ is the *pressure variable*, and on the other hand on the identity

$$-\tfrac{d}{dt} \log\Big( \mathsf{I}^{\frac{1}{2}}\, \mathsf{E}^{\frac{1-\theta}{\theta(p+1)}} \Big) = \textstyle\int_{\mathbb{R}^d} u^m\, \big\|\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\, \Delta \mathsf{P}\, \mathrm{Id}\big\|^2\, dx + (m-m_1) \int_{\mathbb{R}^d} u^m \big|\Delta \mathsf{P} + \tfrac{\mathsf{I}}{\mathsf{E}}\big|^2\, dx\,.$$

Hence $\mathsf{I}^{\frac{1}{2}}\, \mathsf{E}^{\frac{1-\theta}{\theta(p+1)}}$ is monotone with a limit given by self-similar *Barenblatt functions* or, equivalently by $\mathsf{g}^{2p}$. With the relations $f = u^{m-1/2}$, $p = 1/(2\,m-1)$, so that $\int_{\mathbb{R}^d} u\, dx = \int_{\mathbb{R}^d} f^{2p}\, dx$, $\mathsf{E} = \int_{\mathbb{R}^d} f^{p+1}\, dx$ and $\mathsf{I} = (p+1)^2 \int_{\mathbb{R}^d} |\nabla f|^2\, dx$, the role of (2) is easily recovered. A rigorous proof goes through the time-dependent rescaling

$$u(t,x) = \frac{1}{\kappa^d\, R^d}\, v\Big(\tau, \frac{x}{\kappa\, R}\Big) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\, \log R(t)$$

so that (3) is changed, with same initial datum $u_0$ and the choice $R(0) = 1$, into the *Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \Big[ v\, \big( \nabla v^{m-1} - 2\, x \big) \Big] = 0\,. \tag{4}$$

With now $f = v^{m-1/2}$ such that $\int_{\mathbb{R}^d} f^{2p}\, dx = \int_{\mathbb{R}^d} \mathsf{g}^{2p}\, dx$ and again $p = \frac{1}{2\,m-1}$, with $\mathcal{Q}[v] := \mathcal{I}[v]/\mathcal{F}[v]$, $\delta[f] \geq 0$ is equivalent to $\mathcal{Q}[v] \geq 4$, where $\mathcal{B} := \mathsf{g}^{2p}$ and

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \big( \mathcal{B}^{m-1}\, (v - \mathcal{B}) - \tfrac{1}{m}\, (v^m - \mathcal{B}^m) \big)\, dx\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v\, \big| \nabla v^{m-1} + 2\, x \big|^2\, dx\,.$$

For any $m \in (1-1/d, 1)$, the main result in [2] is the fact that

$$\mathcal{Q}[v(\tau,\cdot)] > 4 + \eta \quad \forall\, \tau > 0\,, \tag{5}$$

for some $\eta > 0$, under the conditions that $\int_{\mathbb{R}^d} v_0\,(1,x)\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,(1,x)\,dx$ and

$$\mathsf{A}[v_0] := \sup_{r>0} r^{\frac{d-p\,(d-4)}{p-1}} \int_{|x|>r} v_0\,dx$$

is finite, with an explicit dependence of $\eta$ on $\mathsf{A}[v_0]$ and $\mathcal{F}[v_0]$. The method involves:

(i) An *initial time layer* property: if $\mathcal{Q}[v(T,\cdot)] \ge 4+\eta$ for some $\eta > 0$ and $T > 0$, then $\mathcal{Q}[v(\tau,\cdot)] \ge 4 + 4\,\eta\,e^{-4\,T}/(4+\eta-\eta\,e^{-4\,T})$ for any $\tau \in [0,T]$.

(ii) *A threshold time.* Based on a *global Harnack Principle*, there exists some $T > 0$ such that $(1-\varepsilon)\,\mathcal{B} \le v(\tau,\cdot) \le (1+\varepsilon)\,\mathcal{B}$ for any $\tau > T$.

(iii) An *asympotic time layer* property: as a consequence of an improved *Hardy-Poincaré* (spectral gap) inequality, (5) holds for $\eta = \eta(\varepsilon)$, for any $\tau \ge T$.

Rewritten in terms of $f$, the improved entropy – entropy production inequality $\mathcal{Q}[v] > 4+\eta$ is a stability result.

**Theorem 1.** *Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}\big[f^{2p}\big] < \infty$,*

$$\delta[f] \ge \mathcal{C}[f] \inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx\,.$$

The main point is that the dependence of $\mathcal{C}[f]$ on $\mathsf{A}\big[f^{2p}\big]$ and $\mathcal{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f \in \mathfrak{M}$. The critical case $p = p^*$ can also be covered up to an additional scaling. A major open issue is of course to remove the condition $\mathsf{A}\big[f^{2p}\big] < \infty$, which requires a new approach.

## References


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