DUALITY IN SUB-SUPERCRITICAL BUBBLING IN THE BREZIS-NIRENBERG PROBLEM NEAR THE CRITICAL EXPONENT

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1. INTRODUCTION

Let $\Omega \subset I\!\!R^3$ be a bounded domain with smooth boundary and consider the boundary value problem

$$\begin{cases} \Delta u + \lambda u + u^q = 0 & \text{in } \Omega\,, \\ u > 0 & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases} \qquad (1.1)$$

Integrating the equation against a first eigenfunction of the Laplacian yields that a necessary condition for solvability of (1.1) is $\lambda < \lambda_1$. On the other hand, if $1 < q < 5$ and $0 < \lambda < \lambda_1$, a solution may be found as follows. Let us consider the Rayleigh quotient

$$Q_\lambda(u) := \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}} \quad \text{for any } u \in H_0^1(\Omega) \setminus \{0\} \qquad (1.2)$$

and set

$$S_\lambda := \inf_{u \in H_0^1(\Omega)\setminus\{0\}} Q_\lambda(u)\,. \qquad (1.3)$$

The constant S_λ is achieved thanks to the compactness of Sobolev's embedding if $q < 5$, and a suitable scalar multiple of it turns out to be a solution of (1.1). The case $q \geq 5$ is considerably more delicate: for $q = 5$ compactness of the embedding is lost while for $q > 5$ there is no such embedding. This obstruction is not just technical for the solvability question, but essential. Pohozaev [11] showed that if Ω is strictly star-shaped then no solution of (1.1) exists if $\lambda \leq 0$ and $q \geq 5$. Let S_0 be the best constant in the critical Sobolev's embedding,

$$S_0 = \inf_{u \in C_0^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{\left(\int_{\mathbb{R}^3} |u|^6\right)^{1/3}}\,. \qquad (1.4)$$

Let us consider first the case $q = 5$ in (1.2) and define the number

$$\lambda^* = \inf\{\lambda > 0 \ : \ S_\lambda < S_0\}\,. \qquad (1.5)$$

In a well-known paper, [2], Brezis and Nirenberg established that $0 < \lambda^* < \lambda_1$ and, as a consequence, that S_λ is achieved for $\lambda^* < \lambda < \lambda_1$, hence (1.1)

is solvable in this range. When Ω is a ball they find that $\lambda^* = \lambda_1/4$, while no solution exists for $\lambda \leq \lambda^*$.

Let us assume now that $q > 5$. In this case Sobolev's embedding fails and the quantity S_λ may only be interpreted as zero. Thus, no direct variational approach applies to find existence of solutions. Consequences of the analysis of this paper are the following existence and multiplicity results for Problem (1.1) in the super-critical regime when q is sufficiently close to 5.

Theorem 1. (a) *Assume that $\lambda^* < \lambda < \lambda_1$, where λ^* is the number given by (1.5). Then there exists a number $q_1 > 5$ such that Problem (1.1) is solvable for any $q \in (5, q_1)$.*
(b) *Assume that Ω is a ball and that $\lambda_1/4 < \lambda < \lambda_1$. Then, given $k \geq 1$ there exists a number $q_k > 5$ such that Problem (1.1) has at least k radial solutions for any $q \in (5, q_k)$.*

While the result of Part (a) resembles that by Brezis and Nirenberg when $q = 5$, in reality the solution we find has a very different nature: it blows up as $q \downarrow 5$ developing a *single bubble* around certain point inside the domain. The other solutions predicted by Part (b) blow-up only at the origin but exhibit *multiple bubbling*. Let us make this terminology somewhat more precise. By a *blowing-up solution* for (1.1) near the critical exponent we mean an unbounded sequence of solutions u_n of (1.1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to 5$. Setting

$$M_n = \alpha^{-1} \max_\Omega u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

with $\alpha > 0$ to be chosen, we see then that the scaled function

$$v_n(y) = M_n^{-1} u_n(x_n + M_n^{(q_n-1)/2} y),$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - x_n)$. Assuming for instance that x_n stays away from the boundary of Ω, elliptic regularity implies that locally over compacts around the origin, v_n converges up to subsequences to a positive solution of

$$\Delta w + w^5 = 0$$

in entire space, with $w(0) = \max w = \alpha$. It is well-known, see for instance [4] that for the convenient choice $\alpha = 3^{1/4}$, this solution is explicitly given by

$$w(z) = \frac{3^{1/4}}{\sqrt{1 + |z|^2}} . \tag{1.6}$$

which corresponds precisely to an extremal of the Sobolev constant S_0. Coming back to the original variable, we expect that "near x_n" the behavior of $u_n(y)$ can be approximated as

$$u_n(y) = \frac{3^{1/4} M_n}{\sqrt{1 + M_n^4 |y - x_n|^2}} \left(1 + o(1)\right). \tag{1.7}$$

Since the convergence in expanded variables is only local over compacts, it is not clear how far from x_n the approximation (1.7) holds true, even if only one maximum point x_n exists. We say that the solution $u_n(x)$ is a *single bubble* if (1.7) holds with $o(1) \to 0$ uniformly in Ω. The solution predicted by Part (a) has this property around a point of the domain that will be precised below, while those of Part (b) have the form of a "tower" of single bubbles centered at the origin. As we shall see, radial symmetry is not needed for the presence of these solutions: just symmetry with respect to the three coordinate planes around one point of the domain suffices. The

results of [3] concerning asymptotic analysis of radial solutions in a ball when the exponent approaches the critical exponent from below, suggest that the object ruling the location of blowing-up in single-bubble solutions of (1.1) is Robin's function g_λ defined as follows. Let $\lambda < \lambda_1$ and consider Green's function $G_\lambda(x, y)$, solution for given $x \in \Omega$ of

$$-\Delta_y G_\lambda - \lambda\, G_\lambda \ = \delta_x \qquad y \in \Omega \,,$$

$$G_\lambda(x, y) \ = 0 \qquad y \in \partial\Omega \,.$$

Then we define

$$g_\lambda(x) := H_\lambda(x, x)$$

where

$$H_\lambda(x, y) := \frac{1}{4\pi|y - x|} - G_\lambda(x, y) \,.$$

It turns out that $g_\lambda(x)$ is a smooth function which goes to $+\infty$ as x approaches $\partial\Omega$. Its minimum value is not necessarily positive. In fact this number is decreasing in λ. It is strictly positive when λ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. It is suggested in [3] and recently proven by Druet in [9] that the number λ^* given by (1.5) can be characterized as

$$\lambda^* = \sup\{\lambda > 0 \ : \ \min_\Omega g_\lambda > 0\} \,. \tag{1.8}$$

Besides, it is shown in [9] that least energy solutions u_λ for $\lambda \downarrow \lambda^*$ constitute a single-bubble with blowing-up near the set where g_{λ_*} attains its minimum value zero.

We consider here the role of *non-trivial critical values* of g_λ in existence of solutions of (1.1) in dimension $N = 3$. In fact their role is intimate, not only in the critical case $q = 5$ and in the sub-critical $q = 5 - \varepsilon$. More interesting, their connection with solvability of (1.1) for powers above critical is found. In fact phenomena apparently unknown even in the case of the ball is established, which put in evidence an amusing duality between the sub and super-critical cases.

The meaning we give of a non-trivial critical value of g_λ is as follows: Let $\mathcal{D}$ be an open subset of Ω with smooth boundary. We recall that g_λ *links non-trivially in $\mathcal{D}$ at critical level $\mathcal{G}_\lambda$ relative to B and B_0* if B and

B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: if we set $\Gamma = \{\Phi \in C(B, \mathcal{D}) \ / \ \Phi|_{B_0} = \mathrm{Id}\}$, then

$$\sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda := \inf_{\Phi \in \Gamma} \sup_{y \in B} g_\lambda(\Phi(y)),$$

and for all $y \in \partial\mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, there exists a vector τ_y tangent to $\partial\mathcal{D}$ at y such that

$$\nabla g_\lambda(y) \cdot \tau_y \neq 0.$$

Under these conditions a critical point $\bar{y} \in \mathcal{D}$ of g_λ with $g_\lambda(\bar{y}) = \mathcal{G}_\lambda$ in fact exists. It is easy to check that the above conditions hold if

$$\inf_{x \in \mathcal{D}} g_\lambda(x) < \inf_{x \in \partial\mathcal{D}} g_\lambda(x) \quad \text{or} \quad \sup_{x \in \mathcal{D}} g_\lambda(x) > \sup_{x \in \partial\mathcal{D}} g_\lambda(x),$$

namely the case of (possibly degenerate) local minimum or maximum points of g_λ. The critical value $\mathcal{G}_\lambda$ may be taken in these cases respectively as that of the minimum and the maximum of g_λ in $\mathcal{D}$. These also hold if g_λ is C^1-close to a function with a nondegenerate critical point in $\mathcal{D}$. We call $\mathcal{G}_\lambda$ a non-trivial critical level of g_λ in $\mathcal{D}$. This notion of local notrivial critical level was used in analyzing concentration phenomena in a different context in [5].

Theorem 2. *Let us assume that there is a set $\mathcal{D}$ where g_λ has a non-trivial critical level $\mathcal{G}_\lambda$.*
(a) *Assume that $\mathcal{G}_\lambda < 0$, $q = 5 + \varepsilon$. Then Problem (1.1) is solvable for all sufficiently small $\varepsilon > 0$. More precisely, there exists a solution u_ε of (1.1) of the form*

$$u_\varepsilon(y) = \frac{3^{1/4} M_\varepsilon}{\sqrt{1 + M_\varepsilon^4 |y - \zeta_\varepsilon|^2}} (1 + o(1)) \tag{1.9}$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$,

$$M_\varepsilon = 8\sqrt{2(-\mathcal{G}_\lambda)\varepsilon^{-1}}, \tag{1.10}$$

and $\zeta_\varepsilon \in \mathcal{D}$ is such that $g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda$, $\nabla g_\lambda(\zeta_\varepsilon) \to 0$ as $\varepsilon \to 0$.
(b) *Assume that $\mathcal{G}_\lambda > 0$, $q = 5 - \varepsilon$. Then Problem (1.1) has a solution u_ε of (1.1) exactly as in Part (a) but with $M_\varepsilon = 8\sqrt{2\mathcal{G}_\lambda \varepsilon^{-1}}$.*

We observe then that Theorem 1 follows from Part (a) of the above result making use of the characterization (1.8) of the number λ^*. The result of Part (b) recovers the asymptotics found for the radial solution of (1.1) when Ω is a ball and $0 < \lambda < \lambda_1/4$ in Theorem 1 of [3].

Next we state our result concerning the presence of solutions with multiple bubbling from above the critical exponent in a domain exhibiting symmetries, from which Theorem 1 Part (b) is an immediate consequence. We say that $\Omega \subset \mathbb{R}^3$ is *symmetric with respect to the coordinate planes* if for all $(y_1, y_2, y_3) \in \Omega$ we have that

$$(-y_1, y_2, y_3), \ (y_1, -y_2, y_3), \ (y_1, y_2, -y_3) \in \Omega.$$

Theorem 3. *Assume that $0 \in \Omega$, and that Ω is symmetric with respect to the coordinate axes. Assume also that $g_\lambda(0) < 0$ and $q = 5 + \varepsilon$. Then, given $k \geq 1$, there exists for all sufficiently small $\varepsilon > 0$ a solution u_ε of Problem (1.1) of the form*

$$u_\varepsilon(x) = \sum_{j=1}^{k} \frac{3^{1/4} M_{j\varepsilon}}{\sqrt{1 + M_{j\varepsilon}^4 |x|^2}} (1 + o(1))$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ and for $j = 1, \dots, k$,

$$M_{j\varepsilon} \equiv 8 \sqrt{2\,(-g_\lambda(0))\,k^{-1}} \left(\frac{32\sqrt{2}}{\pi} \right)^{j-1} \frac{(k-j)!}{(k-1)!} \, \varepsilon^{\frac{1}{2}-j} \,.$$

The solution predicted by this theorem is a superposition of k bubbles with respective blow-up orders $\varepsilon^{\frac{1}{2}-j}$ for $j = 1, \dots, k$.

Bubbling solutions for semilinear equations near the critical exponent has been the object of various works in the literature. In particular we refer the reader to [1, 7, 6, 10, 12] and references therein for construction of bubbling solutions in relation to Green's function of the domain. The results above do have analogues for dimension $N \geq 4$, which we shall state in the last section. It should be remarked that when $N \geq 4$ we have that $\lambda^* = 0$. The object responsable for concentration and single-bubbling as $\lambda \downarrow 0$ or $\lambda = 0$ and q approaches critical from below, around a non-degenerate critical point of the function g_0, was first established in [12]. The phenomenon of multi-bubbling in the radial case in higher dimensions seems to have been observed first in [7]. A functional analytic setting that allowed to establish the presence of bubbling solutions for slightly supercritical powers and $\lambda = 0$ in the presence of a small hole in the domain was devised in [6].

Next section will be devoted to sketch the proofs of the above results, whose full details will be provided in the forthcoming paper [8]. In the last section we discuss how these results extend to higher dimensions.

2. Sketch of Proofs

Let ζ be a point in Ω. We consider spherical coordinates $y = y(\rho, \theta)$ centered at ζ given by

$$\rho = |y - \zeta| \quad \text{and} \quad \theta = \frac{y - \zeta}{|y - \zeta|} \,. \tag{2.1}$$

We consider now the transformation transformation

$$v(x, \theta) = \mathcal{T}_q(u)(x, \theta) := 2^{\frac{2}{q-1}} e^{-x} u(\zeta + e^{-2x} y) \,.$$

Let us denote by D the ζ-dependent subset of $S = \mathbb{R} \times S^2$ where the variables (x, θ) vary. After these changes of variables, problem (1.1) becomes

$$4\,\Delta_{S^2}\,v + v'' - v + 4\lambda\,e^{-4x}\,v + e^{(q-5)x}\,v^q = 0 \quad \text{in } D \tag{2.2}$$

$$v > 0 \quad \text{in } D\,, \qquad v = 0 \quad \text{on } \partial D\,.$$

Here and in what follows, $' = \frac{\partial}{\partial x}$. Let us write

$$w_{\mu,\zeta}(y) \equiv \frac{3^{1/4}}{\sqrt{1 + \mu^{-2} |y - \zeta|^2}}\, \mu^{-\frac{1}{2}}\,, \tag{2.3}$$

for a point $\zeta \in I\!\!R^3$ and a positive number μ. We observe that

$$\mathcal{T}_5 w_{\mu,\zeta}(x, \theta) = W(x - \xi)\,,$$

where $W(x) = (12)^{1/4} e^{-x} \left(1 + e^{-4x}\right)^{-1/2}$ and $\mu = e^{-2\xi}$. W is the unique solution of the problem

$$\begin{cases} W'' - W + W^5 = 0 & \text{on } (-\infty, \infty)\,, \\[2mm] W'(0) = 0\,, \\[2mm] W > 0\,, \quad W(x) \to 0 \text{ as } x \to \pm\infty\,. \end{cases} \tag{2.4}$$

Let us consider points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$. We look for a solution of (2.2) of the form

$$v(x, \theta) = \sum_{i=1}^{k}(W(x - \xi_i) + \Pi_i(x, \theta)) + \phi(x, \theta)$$

where ϕ is small and $\Pi_i(x, \theta)$ satisfies the boundary value problem

$$-\left(4\,\Delta_{S^2}\,\Pi + \Pi'' - \Pi + 4\,\lambda\,e^{-4x}\,\Pi\right) = 4\,\lambda\,e^{-4x}\,W(x - \xi) \qquad \text{in } D\,, \tag{2.5}$$

$$\Pi = -W(x - \xi) \qquad\qquad\qquad\qquad\qquad \text{on } \partial D\,.$$

A main observation is that $v(x, \theta) \sim \sum_{i=1}^{k} W(x - \xi_i)$ solves (2.2) if and only if, going back in the change of variables, with $\mu_i = e^{-2\xi_i}$,

$$u(y) \sim \mathcal{T}_q^{-1}\left(\sum_{i=1}^{k} W(x - \xi_i)\right) \sim \sum_{i=1}^{k} \frac{3^{1/4}\,\mu_i^{-1/2}}{\sqrt{1 + \mu_i^{-2}|y - \zeta|^2}}\,,$$

solves (1.1). Let us write

$$W_i(x) := W(x - \xi_i)\,, \quad V_i := W_i + \Pi_i\,, \quad V := \sum_{i=1}^{k} V_i\,. \tag{2.6}$$

If we set $v := V + \phi$. then solving (2.2) is equivalent to finding ϕ such that

$$\begin{cases} L_{q,\lambda}\phi = -N_q(\phi) - R_q & \text{in } D \\[2mm] \phi = 0 & \text{on } \partial D \end{cases} \tag{2.7}$$

where, with $c_q \equiv 2^{-(q-5)/2}$,

$$L_{q,\lambda}(\phi) := 4\,\Delta_{S^2}\phi + \phi'' - \phi + 4\,\lambda\,e^{-4x}\,\phi + q\,c_q\,e^{(q-5)x}\,V^{q-1}\,\phi,$$

$$N(\phi) := c_q\,e^{(q-5)x}\left[(V + \phi)_+^q - V^q - q\,V^{q-1}\phi\right]\,,$$

and

$$R_q := c_q \, e^{(q-5)x} \, V^q - \sum_{i=1}^{k} W_i^5 \,. \tag{2.8}$$

Rather than solving (2.2) directly, we consider first the following intermediate problem: Given points $\xi = (\xi_1, \ldots, \xi_k) \in I\!\!R^k$ and a point $\zeta \in \Omega$, find a function ϕ such that for certain constants c_{ij},

$$\begin{cases} L_{q,\lambda}\phi \, = \, -N_q(\phi) \, - \, R_q \, + \, \sum c_{ij} \, Z_{ij} \quad \text{in } D \,, \\[2mm] \phi = 0 \quad \text{on } \partial D \,, \\[2mm] \displaystyle\int_D Z_{ij} \, \phi \, dx \, d\theta = 0 \quad \text{for all } i, j \,, \end{cases} \tag{2.9}$$

where $Z_{ij}(x, \theta) = \mathcal{T}_5(z_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, N+1$, with z_{ij} is the solution of the boundary value problem

$$\begin{cases} -(\Delta z_{ij} + \lambda z_{ij}) = -\Delta \hat{z}_{ij} & \text{in } \Omega \\ z_{ij} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

and the $\hat{z}_{ij}$ are respectively given for $i = 1, \ldots, k$, $j = 1, \ldots, N$, by

$$\hat{z}_{ij}(y) = \frac{\partial}{\partial \zeta_j} w_{\mu_i, \zeta}(y), \quad \hat{z}_{(N+1)j}(y) = \frac{\partial}{\partial \mu_i} w_{\mu_i, \zeta}(y) \,. \tag{2.11}$$

A straightforward computation then yields that Z_{ij} satisfies

$$\begin{cases} -(4 \, \Delta_{S^2} \, Z_{ij} + Z_{ij}'' - Z_{ij} + 4\lambda \, e^{-4x} \, Z_{ij}) = 5 \, W_i^4 \, \hat{Z}_{ij} & \text{in } D \\ Z_{ij} = 0 & \text{on } \partial D \end{cases} \tag{2.12}$$

with $\hat{Z}_{ij} = \mathcal{T}_5(\hat{z}_{ij})$. We will see that problem (2.9) is uniquely solvable if the points ξ_i, ζ satisfy appropriate constraints and q is close enough to 5. To this end we consider first the linear problem in $(\overline{D})$: Find ϕ such that for certain real numbers c_{ij}

$$\begin{cases} L_{q,\lambda}\phi = h + \sum c_{ij} \, Z_{ij} & \text{in } D \,, \\ \phi = 0 & \text{on } \partial D \,, \\ \int_D Z_{ij} \, \phi = 0 & \forall i, j \,. \end{cases} \tag{2.13}$$

We need uniformly bounded solvability in proper functional spaces for Problem (2.13), for a proper range of the ξ_i's and ζ. To this end, it is convenient to introduce the following norm. Given a small but fixed positive number σ let us set us set

$$\|\psi\|_* := \sup_{(x,\theta)\in D} \nu^{-1}(x)|\psi(x, \theta)| \quad \text{with} \quad \nu(x) := e^{-x} + \sum_{i=1}^{k} e^{-\sigma|x-\xi_i|} \tag{2.14}$$

and consider the subspace $\mathcal{C}_*$ of the continuous functions ψ for which $\|\psi\|_*$ is finite. Let $\varepsilon = |q - 5|$. Then we have the following result.

Lemma 1. *Let $\lambda \in (0, \lambda_1 - \delta)$ and $\mathrm{dist}(\zeta, \partial\Omega) > \delta$ for some small positive δ. There exist positive numbers ε_0, δ_0, R_0, and a constant $C > 0$ such that if the points $0 < \xi_1 < \xi_2 < \cdots < \xi_k$ satisfy*

$$R_0 < \xi_1, \quad R_0 < \min_{1 \leq i < k} (\xi_{i+1} - \xi_i), \quad \xi_k < \frac{\delta_0}{\varepsilon_0} \quad \text{if } q \neq 5, \quad \xi_k < \infty, \tag{2.15}$$

then for all $|q - 5| < \varepsilon_0$ and all $h \in C(D)$ with $\|h\|_* < +\infty$, Problem (2.9) admits a unique solution $\phi =: T_{q,\lambda}(h)$. Besides,

$$\|T_{q,\lambda}(h)\|_* \leq C \|h\|_* \quad and \quad |c_{ij}| \leq C \|h\|_*.$$

Consider the map $\xi \mapsto T_\varepsilon$, with values on $\mathcal{L}(\mathcal{C}_*)$. This map is of class C^1. Moreover, there is a constant $C > 0$ such that

$$\|D_\xi T_\varepsilon\|_{\mathcal{L}(\mathcal{C}_*)} \leq C$$

uniformly on ξ and λ satisfying conditions (2.15).

To solve Problem (2.9) we restrict conveniently the range of the parameters ξ_i. Let us consider for a number M large but fixed, the following conditions:

$$\begin{cases} \xi_1 > \frac{1}{2} \log(M\varepsilon)^{-1}, \quad \log(M\varepsilon)^{-1} < \min_{1 \leq i < k}(\xi_{i+1} - \xi_i), \\ \\ \xi_k < k \log(M\varepsilon)^{-1}. \end{cases} \tag{2.16}$$

Useful facts that we easily check is that under relations (2.16), N_q and R_q defined by (2.8), (2.8) satisfy for all small $\varepsilon > 0$ and $\|\phi\|_* \leq 1/4$ the estimates:

$$\|N_q(\phi)\|_* \leq C \|\phi\|_*^2 \quad \text{and} \quad \|R^q\|_* \leq C \left(\varepsilon |\log \varepsilon| + |q - 5| \right), \tag{2.17}$$

provided $\varepsilon = |q - 5|$ is small.

Lemma 2. *Let $\sigma < 1$. Assume that relations (2.16) hold and let $\varepsilon = |q - 5|$. Then there is a constant $C > 0$ such that, for all $\varepsilon \geq 0$ small enough, there exists a unique solution $\phi = \phi(\xi_1, \ldots, \xi_k, \zeta) = \phi(\bar{\xi}, \zeta)$ to problem (2.9) which besides satisfies*

$$\|\phi\|_* \leq C \varepsilon^{1-\sigma} \quad and \quad \|D_{\xi,\zeta}\phi\|_* \leq C \varepsilon^{1-\sigma}.$$

For the proof of the existence part we observe that ϕ solves (2.9) if and only if

$$\phi = T_{q,\lambda}(N_q(\phi) + R_q) =: A_{q,\lambda}(\phi).$$

Thus we need to prove that the operator $A_{q,\lambda}$ defined above is a contraction in a proper region. Let us consider the set

$$\mathcal{F}_r = \{\phi \in C(D) : \|\phi\|_* \leq r \varepsilon^{1-\sigma}\}$$

with r a positive number to be fixed later. From Lemma 1 and (2.17), we get

$$\|A_{q,\lambda}(\phi)\|_* \leq C \|N_q(\phi) + R_q\|_* \leq C \left[(r \varepsilon)^{2(1-\sigma)} + \varepsilon^{1-\sigma} \right] < r \varepsilon^{1-\sigma}$$

for all small ε, provided that r is chosen large enough, but independent of ε. Thus $A_{q,\lambda}$ maps $\mathcal{F}_r$ into itself for this choice of r. Moreover, $A_{q,\lambda}$ turns out to be a contraction mapping in this region. It is indeed easy to check that N_q defines a contraction in the $\|\cdot\|_*$-norm. $\qquad\square$

According to the above result, our problem has been reduced to that of finding points ξ_i and ζ so that the constants c_{ij} which appear in (2.9), for the solution ϕ given by Lemma 2, are all zero. Thus we need to solve the system of equations

$$c_{ij}(\bar{\xi}, \zeta) = 0 \quad \text{for all } i, j. \tag{2.18}$$

If (2.18) holds, then $v = V + \phi$ will be a solution to (2.2). This system turns out to be equivalent to a variational problem, which we introduce next.

Let us consider the functional

$$\mathcal{F}_{q,\lambda}(\bar{\xi}, \zeta) := F_{q,\lambda}(V + \phi), \tag{2.19}$$

where $\phi = \phi(\bar{\xi}, \zeta)$ is given by Lemma 2 and $F_{q,\lambda}$ is defined by

$$F_{q,\lambda}(v) = 2 \int_D |\nabla_\theta v|^2 + \frac{1}{2} \int_D \left[|v'|^2 + |v|^2 \right]$$
$$- 2\lambda \int_D e^{-4x} v^2 - \frac{1}{q+1} \int_D e^{(q-5)x} |v|^{q+1}.$$

Solving system (2.18) turns out to be equivalent to finding a critical point of this functional, namely equivalent to

$$\nabla \mathcal{F}_{q,\lambda}(\bar{\xi}, \zeta) = 0.$$

The key point to find the desired solutions is the following fact, which is a consequence of the estimates given in Lemma 2.

Lemma 3. *The following expansion holds*

$$\mathcal{F}_{q,\lambda}(\bar{\xi}, \zeta) = F_{q,\lambda}(V) + o(\varepsilon),$$

where the term $o(\varepsilon)$ is uniform in the C^1-sense over all points satisfying constraint (2.16), for given $M > 0$.

This result basically says that to obtain a solution of the problem, we need to find a critical point situation for the function $f_{q,\lambda}(\bar{\xi}, \zeta) := F_{q,\lambda}(V)$ which remains under $o(\varepsilon)$ C^1 perturbations. It is convenient to relabel the ξ_i's in this functional, so that we define

$$e_{q,\lambda}(\bar{\mu}, \zeta) := f_{q,\lambda}\left(-\frac{1}{2}\log\mu_1, \dots, -\frac{1}{2}\log\mu_k, \zeta\right). \tag{2.20}$$

A straightforward computation yields that

$$e_{q,\lambda}(\mu, \zeta) = E_{q,\lambda}(U) \tag{2.21}$$

where

$$U(y) = \sum_j [w_{\mu_j,\zeta}(y) + \pi_{\mu_j,\zeta}(y)],$$

with $\pi_{\mu_j,\zeta} = \mathcal{T}_5^{-1}(\Pi_{\xi_j,\zeta})$ and $\xi_j = -\frac{1}{2}\log\mu_j$, and for any $u : \Omega \to \mathbb{R}$,

$$E_{q,\lambda}(u) = \frac{1}{2}\int_\Omega |Du|^2 - \frac{\lambda}{2}\int_\Omega u^2 - \frac{1}{q+1}\int_\Omega u^{q+1}. \qquad (2.22)$$

Next Lemma gives the energy estimate that shows that this critical point situation for the above functional indeed is present under the given assumptions.

Lemma 4. *Assume that $\varepsilon = |q - 5|$ and $\lambda \in (0, \lambda_1)$ such that $g_\lambda(\zeta) \neq 0$. Choose the parameters μ_j as follows*

$$\mu_1 = \varepsilon\,\Lambda_1, \quad \mu_{j+1} = \mu_j\,(\Lambda_{j+1}\,\varepsilon)^2, \quad j = 1, \dots, k-1\,, \tag{2.23}$$

for some positive numbers $\delta < \Lambda_j < \delta^{-1}$.
  Then the following expansion holds,

$$e_{q,\lambda}(\mu,\zeta) = k\,a_0 + \frac{1}{2}\,k(k+1)\,a_4\,\varepsilon\,|\log\varepsilon| + o(\varepsilon) \qquad (2.24)$$

$$+\varepsilon\left\{a_1\,g_\lambda(\zeta)\,\Lambda_1 + k\,a_4\,\log\Lambda_1 + \sum_{j=2}^{k}\Big[(k-j+1)\,a_4\,\log\Lambda_j - a_5\,\Lambda_j\Big]\right\}$$

uniformly with respect to the parameters Λ_j, as $\varepsilon \to 0$. Here a_i's are universal constants and the quantity $o(\varepsilon)$ is uniform in and C^1 senses in (ζ, Λ) for ζ ranging on compact sets.

The proof of this lemma is a rather lengthy computation which uses as a main factor the relationship between the term $\pi_{\mu_j,\zeta}$ and the regular part of Green's function H_λ, or more precisely, the fact that

$$\pi_{\mu_j,\zeta}(y) \sim \mu^{1/2} H_\lambda(\zeta, y)\,.$$

We recall that $g_\lambda(\zeta) = H_\lambda(\zeta, \zeta)$, and this is why this term appears in the expansion of the energy. With the above ingredients we are in a position to provide a proof of our main results.

Proof of Theorem 2. Let $q = 5 + \varepsilon$ and $\lambda \in (0, \lambda_1)$. We need to find a critical point of $\mathcal{E}(\mu, \zeta) = \mathcal{F}_{q,\lambda}(-\frac{1}{2}\log\mu, \zeta)$. We consider the change of variables $\mu = \mu(\Lambda)$ given by $\mu = \Lambda\,\varepsilon$, with $\Lambda > 0$. Thus we are interested in finding critical points of the function $\psi_\varepsilon(\Lambda, \zeta) = \mathcal{E}(\mu(\Lambda), \zeta)$.

With the previous change of variables, the result in Lemma 3 and the expansion given in Lemma 4 read now as

$$\psi_\varepsilon(\Lambda, \zeta) = a_0 + [a_1\,g_\lambda(\zeta)\Lambda + a_4\,\log\Lambda]\,\varepsilon + a_4\,\varepsilon\,|\log\varepsilon| + o(\varepsilon)$$

uniformly with respect to $\Lambda > \delta$ and $\mathrm{dist}(\zeta, \partial\Omega) > \delta$.

Again from Lemma 3, which actually holds with the $o(\varepsilon)$ term in the C^1 sense uniformly on $\Lambda > \delta$ and $\mathrm{dist}(\zeta, \partial\Omega) > \delta$, we obtain

$$\nabla\psi_\varepsilon(\Lambda, \zeta) = \varepsilon\,[\nabla\psi(\Lambda, \zeta) + o(1)]\,, \qquad (2.25)$$

where $\psi(\Lambda, \zeta) = a_1 \, g_\lambda(\zeta)\Lambda + a_4 \, \log \Lambda$. Let D be the region where the assumed nontrivial linking situation for g_λ holds. With no loss of generality we may assume that $g_\lambda(\zeta) \le -\delta$ on D. We observe that for each fixed ζ the above function has a nondegenerate maximum, $\Lambda = \Lambda(\zeta) = -\frac{a_4}{a_1 g_\lambda(\zeta)}$. Thus we get a critical point of ψ if we find one of the function of ζ

$$\psi(\Lambda(\zeta), \zeta) = a_0 + a_4 \left[-1 + \log \left(-\frac{a_4}{a_1 g_\lambda(\zeta)} \right) \right] .$$

The assumed linking structure for g_λ in D gives the presence of such a critical point. Using this and relation (2.25) it is not hard to construct a region $\tilde{D}$ in the (Λ, ζ) variables where a nontrivial linking structure for the functional $\psi_\varepsilon(\Lambda, \zeta)$ is present, and which yields in ζ variable a point for which the value of g_λ in close to $\mathcal{G}_\lambda$ and its gradient is close to zero. This finally yields the result of Part (a) of the Theorem.

As for Part (b), we use the same argument as before, except that now $q = 5 - \varepsilon$, ζ_λ is a critical point for g_λ with positive non-trivial critical value and

$$\psi(\Lambda, \zeta) = a_1 \, g_\lambda(\zeta) \, \Lambda - a_4 \, \log \Lambda \, .$$

Observe that in such a case, a nondegenerate minimum in Λ variable is present, making it possible to argue symmetrically. $\qquad\square$

Proof of Theorem 3. Due to the symmetry of Ω, the constants c_{ij} that appear in (2.9) are automatically zero for any $j = 1, \ldots, N$. We then just need to prove that the $c_{i(N+1)}$'s are zero, for $i = 1, \ldots, k$. Arguing again as in the proof of Theorem 2, we find a solution to problem (1.1) of the form (1.9) after we observe that the function

$$\psi(\Lambda) = a_1 \, g_\lambda(0) \, \Lambda_1 + k \, a_4 \, \log \Lambda_1 + \sum_{j=2}^{k} \left[(k - j + 1) \, a_4 \, \log \Lambda_j - a_5 \, \Lambda_j \right]$$

given by (2.24) in Lemma 4, for the particular change of variable (2.23), has a nondegenerate critical point $(\Lambda_1, \ldots, \Lambda_k) \in I\!\!R_+^k$, which of course presists under small C^1-perturbations, thanks to the assumption $g_\lambda(0) < 0$. $\qquad\square$

3. HIGHER DIMENSIONS

In this section we state extensions of the results obtained in dimension $N = 3$, for the super-critical case. The main difference between $N = 3$ and $N \ge 4$ is that in the latter case the object ruling the single-bubbling phenomenon is the function g_0 (which is always positive), and no longer g_λ.

Let $N \ge 4$ and $\Omega \subset I\!\!R^N$ be a bounded smooth domain. We consider the problem

$$\begin{cases} \Delta u + \lambda \, u + u^{\frac{N+2}{N-2}+\varepsilon} = 0 & \text{in } \Omega \, , \\ u > 0 & \text{in } \Omega \, , \\ u = 0 & \text{on } \partial\Omega \, , \end{cases} \qquad (3.1)$$

for some positive ε. The two results below yield in particular the validity of Theorem 1 for dimensions $N \geq 4$, where the numbers λ^* and $\lambda_1/4$ are now replaced by zero.

Theorem 4. *Assume $N \geq 4$ and let us assume that there is a set $\mathcal{D}$ where g_0 has a non-trivial critical level $\mathcal{G}$. Assume also that $0 < \lambda < \lambda_1$. Then, there exist points ζ_ε in $\mathcal{D}$, with $g_0(\zeta_\varepsilon) \to \mathcal{G}$, $\nabla g_0(\zeta_\varepsilon) \to 0$ and a solution to Problem (1.1) of the form*

$$u_\varepsilon(y) = \alpha_N \left(\frac{1}{1 + (M_\varepsilon)^{\frac{4}{N-2}} |y - \zeta|^2} \right)^{\frac{N-2}{2}} M_\varepsilon \, (1 + o(1)) \,, \tag{3.2}$$

with $o(1) \to 0$ uniformly on Ω as $\varepsilon \to 0$. In (3.2), $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$ and

$$M_\varepsilon = \beta \varepsilon^{-\frac{N-2}{4}} \quad for \, N \geq 5 \,,$$
$$M_\varepsilon = \beta \, \varepsilon^{-\frac{1}{2}} |\log \varepsilon|^{\frac{1}{2}} \quad for \, N = 4 \,,$$

where β is a positive (explicit) number which depends only on λ, N and $\mathcal{G}$.

The corresponding result of multi-bubbling under symmetries extends in exactly the same way, now with the whole range $0 < \lambda < \lambda_1$ allowed.

Theorem 5. *Assume $N \geq 4$, $0 \in \Omega$ and suppose that Ω is symmetric with respect to the coordinate planes. Assume also that $0 < \lambda < \lambda_1$. Then, given an integer $k \geq 1$, there exists a solution to Problem (3.1) of the form*

$$u_\varepsilon(y) = \alpha_N \sum_{j=1}^k \left(\frac{1}{1 + (M_{j\varepsilon})^{\frac{4}{N-2}} |y|^2} \right)^{\frac{N-2}{2}} M_{j\varepsilon} \, (1 + o(1)) \,, \tag{3.3}$$

with $o(1) \to 0$ uniformly on Ω as $\varepsilon \to 0$. Here

$$M_{j\varepsilon} = \beta_j \varepsilon^{-\frac{N-4}{2} - (j-1)} \quad for \, N \geq 5 \,,$$
$$M_{j\varepsilon} = \beta_j \varepsilon^{-\frac{1}{2} - (j-1)} |\log \varepsilon|^{\frac{1}{2}} \quad for \, N = 4 \,,$$

where $\beta_j > 0$ is an explicit number depending on j, k, N, λ and $g_0(0)$.

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