A SHORT REVIEW ON IMPROVEMENTS AND STABILITY FOR SOME INTERPOLATION INEQUALITIES

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Abstract. In this paper, we present recent stability results with explicit and dimensionally sharp constants and optimal norms for the Sobolev inequality and for the Gaussian logarithmic Sobolev inequality obtained by the authors in [22]. The stability for the Gaussian logarithmic Sobolev inequality is obtained as a byproduct of the stability for the Sobolev inequality. Here we also give a new, direct, alternative proof of this result. We also discuss improved versions of interpolation inequalities based on the *carré du champ* method.

1. Introduction and main results

Let us assume that $d \geq 3$ without further notice. The classical *Sobolev inequality* on $\mathbb{R}^d$, can be written as follows:

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\,, \tag{S}$$

where $2^* = \frac{2\,d}{d-2}$ is the Sobolev exponent, $S_d = \frac14\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ is the sharp Sobolev constant, and $|\mathbb{S}^d|$ stands for the volume of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$. Here $\dot{\mathrm{H}}^1(\mathbb{R}^d)$ denotes the closure of $C^\infty_c(\mathbb{R}^d)$ with respect to the seminorm $|f|_{\dot{\mathrm{H}}^1(\mathbb{R}^d)} := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. As proved in [1, 37] (see also [35, 36]), equality in (S) holds if f is one of the *Aubin-Talenti* functions, that is, one of the functions belonging to the $(d+2)$-dimensional *Aubin-Talenti manifold*

$$\mathcal{M} := \Big\{ g_{a,b,c} \,:\, (a,b,c) \in (0,+\infty) \times \mathbb{R}^d \times \mathbb{R} \Big\} \quad \text{with} \quad g_{a,b,c}(x) := c\,\big(a + |x-b|^2\big)^{-\frac{d-2}{2}}\,.$$

In fact, there is equality in (S) if and only if f is in $\mathcal{M}$ according to [32, 29, 14].

Applying the inverse of the stereographic projection and integrating on the sphere $\mathbb{S}^d$ with respect to the uniform probability measure $d\mu_d$, one can rewrite the above inequality as

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \tfrac14\,d\,(d-2)\left(\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,. \tag{1}$$

and state, equivalently, that the only functions in $\mathrm{H}^1(\mathbb{S}^d)$ for which there is equality in (1) are the functions $\omega \mapsto G_{b,c}(\omega) := c\,(1 + b\cdot\omega)^{-(d-2)/2}$ where $b \in B_1 := \{b \in \mathbb{R}^{d+1} \,:\, |b| < 1\}$ and $c \in \mathbb{R}$ are constants. One can embed (1) in the following family of *Gagliardo-Nirenberg-Sobolev inequalities:*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,,\ \forall\, p \in (1,2)\cup(2,2^*]\,. \tag{GNS}$$

As proved by Bidaut-Véron–Véron [7] and Beckner [5] for $p > 2$ (see also [2, 3, 4] if $p \leq 2^\#$), the constant $d/(p-2)$ is the best possible constant in (GNS) (here $2^\# := (2\,d^2+1)/(d-1)^2 < 2^*$ denotes the Bakry-Emery exponent). If $p < 2^*$, the only optimizers in $\mathrm{H}^1(\mathbb{S}^d)$ for (GNS) are the constant functions. The *carré du champ* method used in [2, 3, 4] relies on the linear heat equation, which induces the limitation $p \leq 2^\#$. This limitation is not technical as shown in [25]. The whole range $p \in (1,2)\cup(2,2^*]$ was covered using a *carré du champ* method based on nonlinear diffusion equations

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by Demange and Dolbeault-Esteban-Loss in [20, 24]. The case $p=2$ (that can be obtained from (GNS) by taking the limit $p\to 2$) is the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac d2\int_{\mathbb{S}^d}|u|^2\,\ln\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)\setminus\{0\}\,,\tag{2}$$

where the only optimizers are the constant functions. Inequality (2) can also be proved by the *carré du champ* method based on the heat equation. A remarkable feature of the *carré du champ* method is that additional terms appear in the computations. This has been used in [23] to establish, for all $p\in(2,2^*)$, the *improved interpolation inequalities*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\ge\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\Psi_{d,p}\left(\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)\setminus\{0\},\tag{3}$$

for some convex function $\Psi_{d,p}$ such that $\Psi_{d,p}(0)=\Psi'_{d,p}(0)=0$ and $\Psi_{d,p}(s)>0$ if $s>0$. Inequality (3) is obviously stronger than (GNS) and can be used to prove that the equality case in (GNS) is realized only by constant functions. If we test (3) with $u_\varepsilon(\omega)=1+\varepsilon\,\mathsf{b}\cdot\omega$ for a given $\mathsf{b}\in B_1$, an elementary computation shows that there is a cancellation of the $O(\varepsilon^2)$ terms as $\varepsilon\to 0$ in the left-hand side of the inequality and the first non-zero term is of the order of $O(\varepsilon^4)$. If we denote by Π_1 the $\mathrm{L}^2(\mathbb{S}^d)$ projection on the space generated by the coordinate functions ω_i with $i=1,\,2,\ldots d$, we learn from [10, Theorem 6] that there is an explicit constant $\kappa=\kappa(p,d)\in(0,1)$, depending on d and $p\in[1,2)\cup(2,2^*)$ such that, for all $u\in\mathrm{H}^1(\mathbb{S}^d)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\ge\kappa\left(\frac{\|\nabla\Pi_1 u\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla u\|_{\mathrm{L}^+(\mathbb{S}^d)}\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}+\|\nabla(\mathrm{Id}-\Pi_1)\,u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right).$$

The proof relies on (3) and on a decomposition in spherical harmonics directly inspired by [28] and is not optimal since $\lim_{p\to 2^*}\kappa(p,d)=0$. Let us also notice that Inequality (3) can take various forms. For instance, if $p\in(2,2^\#)$ and $1/\theta=1+(\frac{d-1}{d+2})^2\,(p-1)\,(2^\#-p)/(p-2)$, we read from [21, Ineq. (2.4)] that (3) takes the form

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac{d\,\theta}{p-2}\left(\|u\|^{2/\theta}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{2-2/\theta}_{\mathrm{L}^2(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)\,,$$

which is a strict improvement compared to (GNS) as can be recovered using Hölder inequalities. The case $p=2$ is also covered, except that in the l.h.s. of (3), the deficit of (GNS) has to be replaced by the deficit of (2). Without entering into details, let us quote some related results. By a direct variational approach, an improved inequality like (3) is proved in [28, Theorem 2], in the subcritical range, for some $\Psi_{d,p}(s)\sim s^2$. Using nonlinear flows and appropriate orthogonality constraints, improved inequalities with $\Psi_{d,p}(s)\sim s$ are known from [25]. Both results are unified in [10]. Let us finally mention that improved inequalities are proved in [8] with explicit constants, not on $\mathbb{S}^d$ but on $\mathbb{R}^d$, for the Gagliardo-Nirenberg-Sobolev inequalities using entropy methods and regularization effects for fast diffusion flows, but with some restrictions on the decay of the functions at infinity.

The Gaussian measure can be seen as an infinite dimensional limit $d\to+\infty$ of the uniform probability measure on the sphere of radius $\sqrt d$ tested against functions depending only on a finite number N of coordinates: see for instance [34, 11]. Since $\lim_{d\to+\infty}2^*=2$, it also turns out that (S) has to be replaced by a Gaussian interpolation inequality as follows. On $\mathbb{R}^N$, with $N\ge 1$, let us consider the Gaussian measure $d\gamma(x)=e^{-\pi\,|x|^2}\,dx$. With $\mathrm{L}^2(\gamma):=\mathrm{L}^2(\mathbb{R}^N,d\gamma)$, if $\mathrm{H}^1(\gamma)$ denotes the space of all $u\in\mathrm{L}^2(\gamma)$ with distributional gradient in $\mathrm{L}^2(\gamma)$, the *logarithmic Sobolev inequality* is:

$$\int_{\mathbb{R}^N}|\nabla u|^2\,d\gamma\ge\pi\int_{\mathbb{R}^N}u^2\ln\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma\quad\forall\,u\in\mathrm{H}^1(\gamma)\setminus\{0\}\,.\tag{LSI}$$

According to the result of Carlen [15, Theorem 5], equality holds in (LSI) if and only if for some $a \in \mathbb{R}^N$ and $c \in \mathbb{R}$,

$$u(x) = c\, e^{a\cdot x}\,. \tag{4}$$

Improved forms of the inequality are also known for instance from [27, 26], under some restrictions.

Now let us turn our attention to *stability* issues for (S), (GNS) and (LSI).

• ***Stability for the Sobolev inequality***. In [9] Brezis and Lieb asked the following question:

(Q) Do there exist constants κ, $\alpha > 0$ such that the Sobolev deficit $\boldsymbol{\delta}$ *controls some distance* $\mathbf{d}$ *from the Aubin-Talenti manifold $\mathcal{M}$ according to*

$$\boldsymbol{\delta}(f) := \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \kappa\, \mathbf{d}(f,\mathcal{M})^\alpha \quad ?$$

The 'best possible answer' to this question would involve finding the strongest possible topology to define the distance $\mathbf{d}$ and the best possible constant κ and exponent α. The first answer to Brezis and Lieb's question was given by Bianchi and Egnell in [6]: there is a constant $C^d_{\mathrm{BE}} > 0$ such that

$$\boldsymbol{\delta}(f) \ge C^d_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\,. \tag{5}$$

Similar results for other inequalities have been proved using the strategy of Bianchi and Egnell: see, for example, [18]. The main drawback of this strategy is that no explicit estimate of C^d_{BE} is known nor its dependence on d. Recently, in [22], we proved the following result.

Theorem 1 ([22, Theorem 1.1])**.** *Let $d \ge 3$. There is an explicit constant $\beta > 0$ such that*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \frac{\beta}{d} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\,. \tag{6}$$

This result is dimensionally sharp. Indeed, Theorem 1 can be rewritten as $C^d_{\mathrm{BE}} \ge \beta/d$. On the other hand, it was proved implicitly in [6] (see also [18]) that $C^d_{\mathrm{BE}} \le 4/(4+d)$. This inequality is in fact strict: $C^d_{\mathrm{BE}} < 4/(4+d)$, according to [30], and we learn from [31] that equality in (5), written with the optimal value of C^d_{BE}, is achieved. Hence, Theorem 1 captures the dimensional behavior of C^d_{BE}. For completeness, let us quote the extension of Theorem 1 in [17] to fractional Sobolev inequalities. Using the inverse stereographic projection, the stability result of Theorem 1 can be rewritten on $\mathbb{S}^d$ as

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \tfrac14\, d\,(d-2)\left(\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \ge \frac{\beta}{d} \inf_{(b,c)\in B_1\times\mathbb{R}} \|\nabla u - \nabla G_{b,c}\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)\,.$$

• ***Stability for the Gaussian logarithmic Sobolev inequality***. The interpretation of the Gaussian measure as the limit of uniform probabililty measures on d-dimensional spheres as $d \to +\infty$ and the explicit dimensional dependence of the stability constant of Theorem 1 provides us with a stability result for (LSI).

Theorem 2 ([22, Corollary 1.2])**.** *With $\beta > 0$ as in Theorem 1, for all $N \in \mathbb{N}$, we have*

$$\int_{\mathbb{R}^N} |\nabla u|^2\, d\gamma - \pi \int_{\mathbb{R}^N} u^2 \ln\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma \ge \frac{\beta\,\pi}{2} \inf_{b\in\mathbb{R}^N,\, c\in\mathbb{R}} \int_{\mathbb{R}^N} \left(u - c\, e^{b\cdot x}\right)^2 d\gamma \quad \forall\, u \in \mathrm{H}^1(\gamma)\,. \tag{7}$$

In [22] this result was obtained as a corollary of Theorem 1. In this paper we give a new, direct proof which highlights the strategy of [22] in a slightly simpler setting.

In the next sections, we briefly describe the strategies of [23] and [22] to prove the improvements in the case of subcritical inequalities (Section 2) and the stability results of Theorem 1 (Section 3) and Theorem 2 (Section 4). The new proof of Theorem 2 is given in Section 5.

2. Improved interpolation inequalities

In this section we describe the method used in [23] to prove (3) and similar improvements of the (GNS) inequalities in the subcritical case $p \in (2, 2^*)$. Inequalities (GNS) and their limiting case corresponding to $p = 2$ can be written as:

$$\mathsf{i} \geq d\,\mathsf{e} \quad \text{where} \quad \mathsf{i} := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{and} \quad \mathsf{e} := \begin{cases} \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) & \text{if} \quad p \neq 2\,,\\ \frac12 \int_{\mathbb{S}^d} |u|^2 \ln\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu & \text{if} \quad p = 2\,, \end{cases}$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d)$ and by homogeneity we can assume that $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$. Let $I_p := [0, +\infty)$ if $p \geq 2$ and $I_p := [0, 1/(2-p))$ if $p \in (1, 2)$.

Theorem 3 ([23, Theorem 1.1])**.** *Let $d \geq 3$ and $p \in (2, 2^*)$. With the above notation and conventions, there is an explicit function φ, such that $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi'' > 0$ on I_p, for which*

$$\mathsf{i} \geq d\,\varphi(\mathsf{e})\,. \tag{8}$$

Similar results can be proved in dimension $d = 1$ and $d = 2$: see [20, 23, 21, 10]. Since

$$\mathsf{i} - d\,\mathsf{e} \geq d\left(\varphi(\mathsf{e}) - \mathsf{e}\right) \geq 0\,,$$

it is clear that the equality case in (3) holds if and only if $\mathsf{e} = 0$, that is, if u is a constant. Moreover $\varphi(\mathsf{e}) - \mathsf{e}$ measures a distance to the constants, for instance in $\mathrm{L}^1(\mathbb{S}^d)$ norm using a generalized Ciszár-Kullback-Pinsker inequality, and $\varphi(\mathsf{e}) - \mathsf{e} \sim \varphi''(0)\,\mathsf{e}^2/2$ as $\mathsf{e} \to 0$. To prove (3) in the stronger homogeneous Sobolev norm $\dot{\mathrm{H}}^1(\mathbb{R}^d)$, it is enough to define the convex function

$$\Psi_{d,p}(s) := s - d\,\varphi^{-1}\left(\frac{s}{d}\right)\,.$$

It is elementary to verify that $\Psi_{d,p}(0) = \Psi'_{d,p}(0) = 0$ and that $\Psi_{d,p}(s) > 0$ if $s \neq 0$.

Let us give a sketch of the proof of Theorem 3 based on the method of [23]. As a first step, using Schwarz foliated symmetrization (see for instance [23, Section 2] for references) and cylindrical coordinates on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, we can reduce the problem to functions depending only on one coordinate $z \in [-1, 1]$ corresponding to the South Pole – North Pole axis. In other words, we consider a function $u(\omega) = f(z)$ where $\omega = (\omega_1, \omega_2, \ldots, \omega_{d+1})$ and $z = \omega_{d+1}$, so that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \int_{-1}^1 |f'|^2\left(1 - z^2\right) d\sigma_d \quad \text{and} \quad \|u\|^q_{\mathrm{L}^q(\mathbb{S}^d)} = \int_{-1}^1 |f|^q\,d\sigma_d\,,$$

where $'$ denotes the z- derivative, the Laplace-Beltrami operator is reduced to the *ultraspherical* operator: $\Delta u = \mathcal{L}\,f := \left(1 - z^2\right) f'' - d\,z\,f'$, and the uniform probability measures becomes

$$d\sigma_d(z) := \frac{\Gamma(\frac{d+1}{2})}{\sqrt{\pi}\,\Gamma(\frac{d}{2})}\left(1 - z^2\right)^{\frac d2 - 1} dz\,.$$

The key idea is to prove that $\mathsf{i} - d\,\varphi(\mathsf{e})$ is monotone non-increasing under the action of

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\,\rho^m, \tag{9}$$

where $m = 1$ corresponds to the heat flow, $m > 1$ to the porous medium flow and $m < 1$ to the fast diffusion flow. Here we choose $\rho = |f|^p$ so that $\|u\|_{\mathrm{L}^p(\mathbb{R}^d)}$ is conserved under the action of (9). It is convenient to introduce the exponent β such that

$$m = 1 + \tfrac{2}{p}\left(\tfrac{1}{\beta} - 1\right)$$

and consider the function $w = f^\beta$, such that $w^{\beta\,p} = \rho$, which solves

$$\frac{\partial w}{\partial t} = w^{2-2\beta}\left(\mathcal{L}w + \kappa\,\frac{|w'|^2}{w}\right) \quad \text{with} \quad \kappa := \beta\,(p-2) + 1\,. \tag{10}$$

The *carré du champ* method shows that $\frac{d}{dt}(\mathsf{i} - d\,\mathsf{e}) \le 0$ if the function

$$\gamma(\beta) := -\left(\tfrac{d-1}{d+2}\,(\kappa+\beta-1)\right)^2 + \kappa\,(\beta-1) + \tfrac{d}{d+2}\,(\kappa+\beta-1)$$

takes nonnegative values, which amounts to $m_-(d,p) \le m \le m_+(d,p)$ where

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p + 2 \pm \sqrt{d\,(d-2)\,(p-1)\,(2^*-p)}\right).$$

Using $\lim_{t\to+\infty}(\mathsf{i} - d\,\mathsf{e}) = 0$, we conclude that $\mathsf{i} - d\,\mathsf{e} \ge 0$ for any $t \ge 0$ and, as a special case, for $t = 0$: this proves (GNS) for an arbitrary initial datum. If $m_-(d,p) < m < m_+(d,p)$, we find that $\gamma(\beta) > 0$, which leaves some space for improvement. A more detailed and quite lengthy computation along the flow (10) yields

$$\frac{d}{dt}\big(\mathsf{i} - d\,\varphi(\mathsf{e})\big) \le \frac{\gamma}{\beta^2}\,\frac{\mathsf{i} - d\,\varphi(\mathsf{e})}{\big(1-(p-2)\,\mathsf{e}\big)^\delta}\,\frac{d\mathsf{e}}{dt},$$

where $\delta = 1$ if $1 \le p \le 2$ and $\delta := \frac{2-(4-p)\,\beta}{2\,\beta\,(p-2)}$ if $p > 2$, if φ solves

$$\frac{d\varphi}{ds} = 1 + \frac{\gamma}{\beta^2}\,\frac{\varphi(s)}{\big(1-(p-2)\,s\big)^\delta}, \quad \varphi(0) = 0.$$

This proves (8) for any $t \ge 0$ as a consequence of the monotonicity of $\mathsf{i} - d\,\varphi(\mathsf{e})$ and the fact that $\lim_{t\to+\infty}\big(\mathsf{i} - d\,\varphi(\mathsf{e})\big) = 0$. See [10, Appendix B.4] for detailed justifications. □

3. Stability for the Sobolev inequality. Proof of Theorem 1

In this section we explain the general ideas of the proof of Theorem 1 in [22].

3.1. On the stability proof by Bianchi and Egnell.

The strategy of Bianchi-Egnell to prove (5) is based on two main steps:

(1) A local stability estimate in a neighborhood of $\mathcal{M}$, obtained by a local spectral analysis.
(2) A reduction of the global estimate to the local estimate by the concentration-compactness method based on Lions' analysis (see [33]).

Theorem 1 is a significant improvement of Bianchi-Egnell's result, as it contains a dimensionally sharp lower estimate for the best stability constant. Our strategy is to make the *second step constructive* and the *first one explicit,* with much more detailed estimates.

3.2. Strategy of the proof of Theorem 1.

The proof is divided into several steps:

(1) Local analysis: prove the inequality for *nonnegative* functions close to $\mathcal{M}$ with an explicit remainder term. The analysis is quite involved: it relies on "cuttings" at various heights, the use of uniform bounds on spherical harmonics and some delicate concavity properties.
(2) Local to global extension: prove the inequality for *nonnegative* functions far from $\mathcal{M}$ using the *competing symmetries* method of [16] and a continuous Steiner symmetrization.
(3) Deduce the inequality for *sign-changing* functions from the inequality for *nonnegative* functions by a concavity argument.

• In the first step, in order to obtain uniform estimates as $d \to +\infty$, we need to expand

$$(1+r)^{2^*} - 1 - 2^* r$$

with an accurate remainder term, for all $r \ge -1$. To do that, we "cut r into pieces" by defining

$$r_1 := \min\{r,\gamma\}, \quad r_2 := \min\big\{(r-\gamma)_+, M-\gamma\big\} \quad \text{and} \quad r_3 := (r-M)_+,$$

where γ and M are suitable parameters satisfying $0 < \gamma < M$. Furthermore, define

$$\theta := 2^* - 2 = \tfrac{4}{d-2}.$$

Notice that $\theta \in (0,1]$ if $d \geq 6$ and $\lim_{d\to+\infty} \theta(d) = 0$.

Lemma 4 ([22, Proposition 2.9])**.** *Given $d \geq 6$, $r \in [-1,\infty)$, and $\overline{M} \in [\sqrt{e},+\infty)$, we have*

$$(1+r)^{2^*} - 1 - 2^* r \leq \tfrac12\, 2^*\,(2^*-1)\,(r_1+r_2)^2 + 2\,(r_1+r_2)\,r_3 + \left(1 + C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\right) r_3^{2^*} + \left(\tfrac32\,\gamma\,\theta\,r_1^2 + C_{M,\overline{M}}\,\theta\,r_2^2\right)\mathbb{1}_{\{r\leq M\}} + C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\,,$$

where all the constants in the above inequality are explicit.

One can then prove that there exist computable constants ϵ_1, ϵ_2, k_0, and $\epsilon_0 \in (0,1/\theta)$, such that

$$\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\,\|1+r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \geq \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) + \sum_{k=1}^3 I_k\,,$$

with $\mathrm{A} := \frac14\,d\,(d-2)$ and

$$I_1 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_1|^2 + \mathrm{A}\,r_1^2\right)d\mu_d - \mathrm{A}\,(2^*-1+\epsilon_1\,\theta)\int_{\mathbb{S}^d} r_1^2\,d\mu_d + \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d}(r_2^2+r_3^2)\,d\mu_d\,,$$
$$I_2 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_2|^2 + \mathrm{A}\,r_2^2\right)d\mu_d - \mathrm{A}\left(2^*-1+(k_0+C_{\epsilon_1,\epsilon_2})\,\theta\right)\int_{\mathbb{S}^d} r_2^2\,d\mu_d\,,$$
$$I_3 := (1-\theta\,\epsilon_0)\int_{\mathbb{S}^d}\left(|\nabla r_3|^2 + \mathrm{A}\,r_3^2\right)d\mu_d - \tfrac{2}{2^*}\,\mathrm{A}\,(1+\epsilon_2\,\theta)\int_{\mathbb{S}^d} r_3^{2^*}\,d\mu_d - \mathrm{A}\,k_0\,\theta\int_{\mathbb{S}^d} r_3^2\,d\mu_d\,.$$

Next, one can use spectral gap estimates to prove $I_1 \geq 0$ and the Sobolev inequality to prove $I_3 \geq 0$, noting that the extra coefficient $2/2^* < 1$ gives enough room to accommodate all error terms. Finally, using that $\mu\big(\{r_2>0\}\big)$ is small, an improved spectral gap inequality allows us to show $I_2 \geq 0$.

If $d = 3$, 4, or 5, we can rely on a simpler Taylor expansion that can be found in [22, Proposition 2.7]. As a consequence, the following result has been proved.

Theorem 5 ([22, Theorem 2.1])**.** *There are explicit constants $\epsilon_0 \in (0,1/3)$ and $\delta \in (0,1/2)$ such that for all $d \geq 3$ and for all nonnegative $u = 1+r \in \mathrm{H}^1(\mathbb{S}^d)$ with*

$$\|r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \leq \frac{\delta}{1-\delta}\,,\quad \int_{\mathbb{S}^d} r\,d\mu_d = 0 \quad\textit{and}\quad \int_{\mathbb{S}^d}\omega\,r\,d\mu_d = 0\,,$$

one has

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathrm{A}\,\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \geq \frac{4\,\epsilon_0}{d-2}\left(\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right).$$

Let us define the stability quotient

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}$$

and consider the infimum

$$\mathscr{I}(\delta) := \inf\left\{\mathcal{E}(f)\,:\,0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\setminus\mathcal{M}\,,\quad \inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leq \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\right\}.$$

For a given f satisfying $\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \leq \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, up to a conformal transformation, we can assume that $\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ is realized by the Aubin-Talenti function $g = g_*$ with

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}}\left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}\qquad\forall\,x\in\mathbb{R}^d\,, \tag{11}$$

use the inverse stereographic projection to transform f and g respectively into $u = 1+r$ and 1, and notice that

$$\mathrm{A}\,\|r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \leq \|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,,$$
$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mathrm{A}\,,$$

where the first line follows from (1). We deduce from the condition $\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\le\delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ that

$$\mathrm{A}\;\|r\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}\le\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\mathrm{A}\;\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\le\frac{\delta\,\mathrm{A}}{1-\delta}$$

and apply Theorem 5 to obtain

$$\mathcal{E}(f)=\frac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\mathrm{A}\;\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\mathrm{A}\;\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}}{\|\nabla r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\mathrm{A}\;\|r\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\ge\frac{4\,\epsilon_0}{d-2}\,.$$

This completes the *local* analysis where, by homogeneity, the scale is fixed in terms of $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$. With the notation of Theorem 5,

$$\mathscr{I}(\delta)\ge\frac{4\,\epsilon_0}{d-2}\,.$$

• In Step 2, we deal with nonnegative functions f that are not close to the manifold $\mathcal{M}$, *i.e.*, such that

$$\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}>\delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,. \tag{12}$$

The first ingredient is the method of *competing symmetries* [16] of Carlen and Loss. Consider any nonnegative function $f\in\dot{\mathrm{H}}^1(\mathbb{R}^d)$ and let

$$(Uf)(x):=\left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}}f\left(\frac{x_1}{|x-e_d|^2},\ldots,\frac{x_{d-1}}{|x-e_d|^2},\frac{|x|^2-1}{|x-e_d|^2}\right)\quad\text{where}\quad e_d=(0,\ldots,0,1)\in\mathbb{R}^d\,,$$

and notice that $\mathcal{E}(Uf)=\mathcal{E}(f)$. We also consider the symmetric decreasing rearrangement $\mathcal{R}f=f^*$, with the properties that f and f^* are equimeasurable, and that $\|\nabla f^*\|_{\mathrm{L}^2(\mathbb{R}^d)}\le\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}$.

Theorem 6 ([16, Theorem 3.3])**.** *Let $f\in\mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function with $\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=1$. The sequence $f_n=(\mathcal{R}U)^nf$ is such that $\lim_{n\to+\infty}\|f_n-g_*\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=0$. If $f\in\dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_{\mathrm{L}^2(\mathbb{R}^d)})_{n\in\mathbb{N}}$ is a non-increasing sequence.*

Whether f_n satisfies (12) for all $n\in\mathbb{N}$ or not, we face an alternative.

Lemma 7 ([22, Lemma 3.5])**.** *Let f be as in Theorem 6 and such that (12) holds and let $f_n=(\mathcal{R}U)^nf$. Then either $\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\ge\delta\,\|\nabla f_n\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ for all n, or there exists $n_0\in\mathbb{N}$ such that*

$$\inf_{g\in\mathcal{M}}\|\nabla f_{n_0}-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}>\delta\,\|\nabla f_{n_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\quad\textit{and}\quad\inf_{g\in\mathcal{M}}\|\nabla f_{n_0+1}-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}<\delta\,\|\nabla f_{n_0+1}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\,.$$

In the first case, we have

$$\lim_{n\to+\infty}\|\nabla f_n\|_2^2\le\frac{1}{\delta}\lim_{n\to+\infty}\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_2^2=\frac{1}{\delta}\left(\lim_{n\to+\infty}\|\nabla f_n\|_2^2-S_d\,\|f\|_{2^*}^2\right)$$

where the last equality arises as a consequence of the properties of $(f_n)_{n\in\mathbb{N}}$ (see [22, Lemma 3.4]). Combined with the simple estimate

$$\mathcal{E}(f)=\frac{\|\nabla f\|_2^2-S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2}\ge\frac{\|\nabla f\|_2^2-S_d\,\|f\|_{2^*}^2}{\|\nabla f\|_2^2}\ge\frac{\|\nabla f_n\|_2^2-S_d\,\|f\|_{2^*}^2}{\|\nabla f_n\|_2^2}\,,$$

we can take the limit as $n\to+\infty$ and obtain $\mathcal{E}(f)\ge\delta$. In the second case, we adapt a strategy due to Christ in [19], by building a continuous rearrangement *flow* $(\mathsf{f}_\tau)_{n_0\le\tau<n_0+1}$ with $\mathsf{f}_{n_0}=Uf_n$ such that

$$\|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,,\ \tau\mapsto\|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)}\ \textit{is nonincreasing, and}\ \lim_{\tau\to n_0+1}\mathsf{f}_\tau=f_{n_0+1}\,.$$

Choosing the smallest $\tau_0 \in (n_0, n_0+1)$ such that $\inf_{g\in\mathcal M}\|\nabla(\mathsf f_{\tau_0}-g)\|^2_{\mathrm L^2(\mathbb R^d)}=\delta\,\|\nabla f_{\tau_0}\|^2_{\mathrm L^2(\mathbb R^d)}$ and using $\|f\|_{\mathrm L^{2^*}(\mathbb R^d)}=\|f_{n_0}\|_{\mathrm L^{2^*}(\mathbb R^d)}=\|\mathsf f_{\tau_0}\|_{\mathrm L^{2^*}(\mathbb R^d)}$, this gives:

$$\mathcal E(f)\ge 1-S_d\,\frac{\|f\|^2_{\mathrm L^{2^*}(\mathbb R^d)}}{\|\nabla f\|^2_{\mathrm L^2(\mathbb R^d)}}\ge 1-S_d\,\frac{\|f_{n_0}\|^2_{\mathrm L^{2^*}(\mathbb R^d)}}{\|\nabla f_{n_0}\|^2_{\mathrm L^2(\mathbb R^d)}}\ge 1-S_d\,\frac{\|\mathsf f_{\tau_0}\|^2_{\mathrm L^{2^*}(\mathbb R^d)}}{\|\nabla \mathsf f_{\tau_0}\|^2_{\mathrm L^2(\mathbb R^d)}}=\delta\,\mathcal E(f_{\tau_0})\ge\delta\,\mathscr I(\delta)\,.$$

To build the flow, we refer to [12, 13] and to [22] for further references. The existence of τ_0 requires a discussion that can be found in [22, Section 3.1.2].

Finally, it is simple to prove that for all $\delta\in(0,1)$, $\mathscr I(\delta)\le 1$. Therefore, in all cases, $\mathcal E(f)\ge\delta\,\mathscr I(\delta)$.

• The third step is to remove the positivity assumption of Theorem 1 as in [22, Section 3.2]. Take $f=f_+-f_-$ with $\|f\|_{\mathrm L^{2^*}(\mathbb R^d)}=1$ and define $m:=\|f_-\|^{2^*}_{\mathrm L^{2^*}(\mathbb R^d)}$. Without loss of generality one may assume that $1-m=\|f_+\|^{2^*}_{\mathrm L^{2^*}(\mathbb R^d)}>1/2$. The positive concave function

$$h_d(m):=m^{\frac{d-2}{d}}+(1-m)^{\frac{d-2}{d}}-1$$

satisfies

$$2\,h_d(1/2)\,m\le h_d(m)\,,\quad h_d(1/2)=2^{2/d}-1\,.$$

With $\boldsymbol\delta(f)=\|\nabla f\|^2_{\mathrm L^2(\mathbb R^d)}-S_d\,\|f\|^2_{\mathrm L^{2^*}(\mathbb R^d)}$, one finds $g_+\in\mathcal M$ such that

$$\boldsymbol\delta(f)\ge C^{d,\mathrm{pos}}_{\mathrm{BE}}\,\|\nabla f_+-\nabla g_+\|^2_{\mathrm L^2(\mathbb R^d)}+\frac{2\,h_d(1/2)}{h_d(1/2)+1}\,\|\nabla f_-\|^2_{\mathrm L^2(\mathbb R^d)}\,,$$

and therefore

$$C^d_{\mathrm{BE}}\ge\tfrac12\,\min\left\{\max_{0<\delta<1/2}\,\delta\,\mathscr I(\delta),\,\frac{2\,h_d(1/2)}{h_d(1/2)+1}\right\}\,.$$

• Combining all estimates of the three previous steps completes the proof of Theorem 1. □

4. Stability for the Gaussian logarithmic Sobolev inequality. Proof of Theorem 2

In this section we describe the main steps in the proof of the stability estimate for the Gaussian logarithmic Sobolev inequality (2) by considering the large dimensional limit of (6), as it appears in [22, Section 4]. With $u=f/a$, c, where $g_\star$ is given by (11), Inequality (6) can be rewritten as

$$\int_{\mathbb R^d}|\nabla u|^2\,g_\star^2\,dx+d\,(d-2)\int_{\mathbb R^d}|u|^2\,g_\star^{2^*}\,dx-d\,(d-2)\,\|g_\star\|^{2^*-2}_{\mathrm L^{2^*}(\mathbb R^d)}\left(\int_{\mathbb R^d}|u|^{2^*}\,g_\star^{2^*}\,dx\right)^{2/2^*}$$
$$\ge\frac\beta d\left(\int_{\mathbb R^d}|\nabla u|^2\,g_\star^2\,dx+d\,(d-2)\int_{\mathbb R^d}|u-g_d/g_\star|^2\,g_\star^{2^*}\,dx\right),$$

where $g_d\in\mathcal M$ realizes $\inf_{g\in\mathcal M}\|\nabla f-\nabla g\|^2_{\mathrm L^2(\mathbb R^d)}$. For some parameters a_d, b_d and c_d, we can write that $g_d(x)=c_d\,(a_d+|x-b_d|^2)^{1-d/2}$. We rescale the function u according to

$$u(x)=v\,(r_d\,x)\quad\forall\,x\in\mathbb R^d\,,\quad r_d=\sqrt{\tfrac{d}{2\pi}}$$

and consider the function w_v^d such that $w_v^d\,(r_d\,x)=g_d(x)/g_\star(x)$. Hence

$$\int_{\mathbb R^d}|\nabla v|^2\left(1+\tfrac{1}{r_d^2}\,|x|^2\right)^2d\mu_d\ge 2\,\pi\,(d-2)\left(\left(\int_{\mathbb R^d}|v|^{2^*}\,d\mu_d\right)^{2/2^*}-\int_{\mathbb R^d}|v|^2\,d\mu_d\right)$$
$$+\frac\beta d\left(\int_{\mathbb R^d}|\nabla v|^2\,d\mu_d+2\,\pi(d-2)\int_{\mathbb R^d}|v-w_v^d|^2\,d\mu_d\right)$$

where $d\mu_d = Z_d^{-1}\, g_\star^{2^*}\, dx$ is the probability measure given by

$$d\mu_d(x) := \frac{1}{Z_d}\left(1+\tfrac{1}{r_d^2}\,|x|^2\right)^{-d} dx \quad \text{with} \quad Z_d = \frac{2^{1-d}\sqrt{\pi}}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{d}{2}\right)^{\frac{d}{2}}\,.$$

Our goal is to take the limit $d\to+\infty$ when one considers functions $v(x)$ depending only on $y\in\mathbb{R}^N$, with $x=(y,z)\in\mathbb{R}^N\times\mathbb{R}^{d-N}\approx\mathbb{R}^d$, for some fixed integer N. With $|x|^2=|y|^2+|z|^2$, notice that

$$1+\tfrac{1}{r_d^2}\,|x|^2 = 1+\tfrac{1}{r_d^2}\left(|y|^2+|z|^2\right) = \left(1+\tfrac{1}{r_d^2}\,|y|^2\right)\left(1+\tfrac{|z|^2}{r_d^2+|y|^2}\right)$$

and, as a consequence,

$$\begin{aligned}
&\lim_{d\to+\infty}\left(1+\tfrac{1}{r_d^2}\,|y|^2\right)^{-\frac{N+d}{2}} = e^{-\pi\,|y|^2}\,,\\
&\lim_{d\to+\infty}\textstyle\int_{\mathbb{R}^d}|v(y)|^2\,d\mu_d = \int_{\mathbb{R}^N}|v|^2\,d\gamma\,,\\
&\lim_{d\to+\infty}\textstyle\int_{\mathbb{R}^d}|\nabla v|^2\left(1+\tfrac{1}{r_d^2}\,|x|^2\right)^2 d\mu_d = 4\,\int_{\mathbb{R}^N}|\nabla v|^2\,d\gamma\,,
\end{aligned}$$

where $d\gamma(y):=e^{-\pi\,|y|^2}\,dy$ is the standard *Gaussian probability measure*. However, the function w_v^d depends on d and the main difficulty is to obtain enough estimates on the parameters a_d, b_d and c_d to pass to the limit after integrating in all integrals with respect to z, which completes the proof of (7). See [22, Section 4] for further details. □

5. Stability of the logarithmic Sobolev inequality: a new proof of Theorem 2

Instead of proving Theorem 2 as a consequence of Theorem 1, one can give a direct proof of (7). Just like in Section 3, we prove the quantitative version of the sharp logarithmic Sobolev inequality (Theorem 2) in two steps, one close to and one far from the set of optimizers.

Let us start with a consequence of Theorem 5.

Theorem 8. *There are explicit constants $\eta>0$ and $\delta\in(0,1/2)$ such that for all $N\in\mathbb{N}$ and for all for all nonnegative $u=1+r\in\mathrm{H}^1(\mathbb{S}^d)$ satisfying*

$$\int_{\mathbb{R}^N} r^2\,d\gamma \le \frac{\delta}{1-\delta} \tag{13}$$

and

$$\int_{\mathbb{R}^N} r\,d\gamma = 0 = \int_{\mathbb{R}^N} x_j\,r\,d\gamma\,,\quad j=1,\,2,\ldots,N\,, \tag{14}$$

one has

$$\int_{\mathbb{R}^N}|\nabla u|^2\,d\gamma-\pi\int_{\mathbb{R}^N}|u|^2\ln\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma \ge \eta\int_{\mathbb{R}^N} r^2\,d\gamma\,.$$

The constant δ coincides with the corresponding constant in Theorem 5 and $\eta=2\,\pi\,\epsilon_0$.

Proof. Notice that $x\in\mathrm{L}^2(\gamma)$, so the orthogonality constraints raise no integration issues. We denote $\Sigma_d:=\{x\in\mathbb{R}^{d+1}\,:\,|x|=\rho_d\}$ with $\rho_d:=\sqrt{d/(2\,\pi)}$. The factor of $1/(2\,\pi)$ in the definition of ρ_d is necessary to get the π in the exponent of the Gaussian density. We integrate on Σ_d with respect to the uniform probability measure $d\mu_d$. By rescaling our result in Theorem 5 we find that

$$\begin{aligned}
\int_{\Sigma_d}|\nabla R|^2\,d\mu_d-\pi\,\frac{d-2}{2}\left(\left(\int_{\Sigma_d}(1+R)^{\frac{2\,d}{d-2}}\,d\mu_d\right)^{\frac{d-2}{d}}-\int_{\Sigma_d}(1+R)^2\,d\mu_d\right)&\\
\ge 2\,\pi\,\epsilon_0\int_{\Sigma_d}\left(\frac{1}{\pi}\,\frac{2}{d-2}\,|\nabla R|^2+R^2\right)d\mu_d\,.&
\end{aligned} \tag{15}$$

This inequality is valid for all $R \in \mathrm{H}^1(\Sigma_d)$ such that

$$\left(\int_{\Sigma_d} R^{\frac{2\,d}{d-2}}\,d\mu_d\right)^{\frac{d-2}{d}} \le \frac{\delta}{1-\delta} \tag{16}$$

and

$$\int_{\Sigma_d} R\,d\mu_d = 0 = \int_{\Sigma_d} x_j\,R\,d\mu_d\,,\quad j=1,\ldots,d+1\,. \tag{17}$$

Given a function $r \in \mathrm{H}^1(\gamma)$ and $d > N$, we apply this inequality to the function

$$R_d(x) := r(x_1,\ldots,x_N) - \int_{\Sigma_d} r\,d\mu_d - 2\,\pi\,\frac{d+1}{d}\sum_{n=1}^N x_n \int_{\Sigma_d} y_n\,r(y_1,\ldots,y_N)\,d\mu_d(y)$$

for $x \in \Sigma_d$. This function satisfies the orthogonality conditions (17). Note here that the functions $\sqrt{2\,\pi}\,\sqrt{(d+1)/d}\,x_j$ are L^2-normalized on Σ_d.

We now use the well-known fact that, as $d \to +\infty$, the marginal of $d\mu_d$ corresponding to the first N coordinates converges to $d\gamma$. Thus,

$$\lim_{d\to+\infty}\int_{\Sigma_d}|\nabla r|^2\,d\mu_d = \int_{\mathbb{R}^N}|\nabla r|^2\,d\gamma\,,\qquad \lim_{d\to+\infty}\int_{\Sigma_d} r^2\,d\mu_d = \int_{\mathbb{R}^N} r^2\,d\gamma\,,$$
$$\lim_{d\to+\infty}\int_{\Sigma_d} r\,d\mu_d = \int_{\mathbb{R}^N} r\,d\gamma = 0\,,\qquad \lim_{d\to+\infty}\int_{\Sigma_d} y_n\,r(y_1,\ldots,y_N)\,d\mu_d(y) = \int_{\mathbb{R}^N} y_n\,r\,d\gamma = 0\,.$$

From this we conclude easily that

$$\lim_{d\to+\infty}\int_{\Sigma_d}|\nabla R_d|^2\,d\mu_d = \int_{\mathbb{R}^N}|\nabla r|^2\,d\gamma\,,\quad \lim_{d\to+\infty}\int_{\Sigma_d} R_d^2\,d\mu_d = \int_{\mathbb{R}^N} r^2\,d\gamma\,.$$

With some modest amount of effort one also finds that

$$\lim_{d\to+\infty}\int_{\Sigma_d} R_d^{\frac{2\,d}{d-2}}\,d\mu_d = \int_{\mathbb{R}^N} r^2\,d\gamma\,.$$

Assuming that the inequality in (13) is strict, the same is true for the left side when d is sufficiently large, and consequently the smallness condition (16) holds when d is sufficiently large. Thus, inequality (15) is valid for all sufficiently large d. The equality case in (13) can be obtained at the very end by a simple approximation argument.

Now, we drop the gradient term in the right side and letting $d \to +\infty$ we infer that

$$\int_{\mathbb{R}^N}|\nabla r|^2\,d\gamma - \pi\,\limsup_{d\to+\infty}\frac{d-2}{2}\left(\left(\int_{\Sigma_d}(1+R_d)^{\frac{2\,d}{d-2}}\,d\mu_d\right)^{\frac{d-2}{d}} - \int_{\Sigma_d}(1+R_d)^2\,d\mu_d\right) \ge 2\,\pi\,\epsilon_0\int_{\mathbb{R}^N} r^2\,d\gamma\,.$$

Finally, we verify that

$$\limsup_{d\to+\infty}\frac{d-2}{2}\left(\left(\int_{\Sigma_d}(1+R_d)^{\frac{2\,d}{d-2}}\,d\mu_d\right)^{\frac{d-2}{d}} - \int_{\Sigma_d}(1+R_d)^2\,d\mu_d\right) = \int_{\mathbb{R}^N}(1+r)^2\,\ln\left(\frac{(1+r)^2}{\|1+r\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma\,.$$

In fact, if the orthogonality conditions were not present and the marginals would already be equal to their limit, this would follow from the fact that

$$\lim_{p\to1^+}\frac{1}{p-1}\left(\left(\int_{\mathbb{R}^N} h^p\,d\gamma\right)^{1/p} - \int_{\mathbb{R}^N} h\,d\gamma\right) = \int_{\mathbb{R}^N} h\ln\left(\frac{h}{\int_{\mathbb{R}^N} h\,d\gamma}\right)d\gamma\,,$$

valid on any measure space for any nonnegative function h that satisfies $h \in \mathrm{L}^1 \cap \mathrm{L}^{p_0}(\gamma)$ for some $p_0 > 1$. Proving the latter fact is simple, as well as including the effect of the orthogonality conditions and the convergence of the marginals, so we shall omit it. These remarks complete the proof Theorem 8. □

We emphasize that in the previous proof we did not use Theorem 1, but rather Theorem 5. In this way we avoid having to control the distance to the set of optimizers in the high-dimensional limit, which seems harder than verifying the orthogonality conditions.

Proof of Theorem 2. As in the proof of Theorem 1, we first prove the result for nonnegative functions and then extend it to sign changing solutions. Let us denote by $\kappa^{\rm pos}$ the stability constant in the stability inequality restricted to nonnegative functions.

Step 1. Let η and δ be as in Theorem 8. For $0 \le u \in \mathrm{H}^1(\gamma)$ we distinguish two cases.
• The first case is where

$$\inf_{a\in\mathbb{R}^N,\,c\in\mathbb{R}} \int_{\mathbb{R}^N} (u - c\,e^{a\cdot x})^2\,d\gamma \le \delta \int_{\mathbb{R}^N} u^2\,d\gamma\,.$$

The infimum on the left-hand side is attained at some $a \in \mathbb{R}^N$ and $c \in \mathbb{R}$, as can be checked by optimizing $\int_{\mathbb{R}^N} \big|v - c\,e^{|a|^2/(2\,\pi) - \pi\,|x-a/\pi|^2/2}\big|^2\,dx$ where $v(x) := u(x)\,e^{-\pi\,|x|^2/2}$. Let

$$\tilde u(y) := e^{-\,y\cdot a - \frac{|a|^2}{2\,\pi}}\,u\big(y + \tfrac{a}{\pi}\big)\,.$$

Then, by a simple computation involving an integration by parts and a change of variables,

$$\int_{\mathbb{R}^N} |\nabla \tilde u|^2\,d\gamma - \pi \int_{\mathbb{R}^N} \tilde u^2 \ln\left(\frac{\tilde u^2}{\|\tilde u\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma = \int_{\mathbb{R}^N} |\nabla u|^2\,d\gamma - \pi \int_{\mathbb{R}^N} u^2 \ln\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma\,.$$

Therefore, the deficit of $\tilde u$ coincides with that of u, while the infimum for $\tilde u$ among all functions of the form (4) is attained at the constant $c\,\exp\big(|a|^2/(2\,\pi)\big)$. Finally, by multiplying $\tilde u$ with a constant, we may assume that this constant is equal to one. To summarize, we may assume without loss of generality that the infimum in the theorem is attained at $a = 0$ and $c = 1$.

Let us set $r := u - 1$. Then the minimality implies that r satisfies the orthogonality conditions (14). Moreover, we have

$$\int_{\mathbb{R}^N} r^2\,d\gamma \le \delta \int_{\mathbb{R}^N} u^2\,d\gamma = \delta\left(1 + \int_{\mathbb{R}^N} r^2\,d\gamma\right),$$

so the smallness condition (13) is satisfied and we can apply Theorem 8. This yields the inequality in the theorem with a stability constant η.

• Next, we consider the case where

$$\inf_{a\in\mathbb{R}^N,\,c\in\mathbb{R}} \int_{\mathbb{R}^N} (u - c\,e^{x\cdot a})^2\,d\gamma > \delta \int_{\mathbb{R}^N} u^2\,d\gamma\,.$$

We argue similarly as we did in Section 3 concerning the Sobolev inequality, but there are some differences in this case.

For $f \in \mathrm{L}^2(\gamma)$ we denote by Uf its Gaussian rearrangement, that is, the function on $\mathbb{R}^N$ whose superlevel sets have the form $\{x \in \mathbb{R}^N : x_1 < \mu\}$ for some $\mu \in \mathbb{R}$ and have the same γ-measure as the corresponding superlevel sets of f. Moreover, we denote

$$Vf := e^{\frac{\pi}{2}\,|x|^2}\,\mathcal{R}\left(e^{-\frac{\pi}{2}\,|x|^2}\,f\right),$$

where $\mathcal{R}$ is, as before, the Euclidean rearrangement. Then, as shown in [16, Theorem 4.1], for any $0 \le f \in \mathrm{L}^2(\gamma)$ one has

$$f_n := (VU)^n f \to \|f\|_{\mathrm{L}^2(\gamma)} \quad \text{in} \quad \mathrm{L}^2(\gamma)\,.$$

Moreover, $\|f_n\|_{\mathrm{L}^2(\gamma)} = \|f\|_{\mathrm{L}^2(\gamma)}$ and

$$n \mapsto \int_{\mathbb{R}^N} |\nabla f_n|^2\,d\gamma - \pi \int_{\mathbb{R}^N} f_n^2 \ln\left(\frac{f_n^2}{\|f_n\|^2_{\mathrm{L}^2(\gamma)}}\right) d\gamma$$

is nonincreasing. This is the analogue of Theorem 6 in the present case.

We apply this procedure to our function u, which we assume here to be nonnegative, and obtain a sequence of functions u_n with constant $\mathrm{L}^2(\gamma)$-norm. Moreover, since

$$\inf_{a,\,c} \|u_n - c\,e^{a\cdot x}\|_{\mathrm{L}^2(\gamma)} \le \big\|u_n - \|u\|_{\mathrm{L}^2(\gamma)}\big\|_{\mathrm{L}^2(\gamma)} \to 0 \quad \text{as} \quad n \to +\infty\,,$$

there is an $n_0 \in \mathbb{N}$ such that

$$\inf_{a,\,c} \|u_{n_0} - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)} \ge \delta\,\|u\|^2_{\mathrm{L}^2(\gamma)} > \inf_{a,\,c} \|u_{n_0+1} - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)}\,.$$

This replaces Lemma 7. We have

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |\nabla u|^2\,d\gamma - \pi\int_{\mathbb{R}^N} u^2 \ln\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\inf_{a,\,c}\|u - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)}} &\ge \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,d\gamma - \pi\int_{\mathbb{R}^N} u^2 \ln\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\\ &\ge \frac{\int_{\mathbb{R}^N} |\nabla u_{n_0}|^2\,d\gamma - \pi\int_{\mathbb{R}^N} u_{n_0}^2 \ln\left(\frac{u_{n_0}^2}{\|u_{n_0}\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\,.\end{aligned}$$

We now use a continuous rearrangement flow to connect u_{n_0} to u_{n_0+1}. More precisely, we a consider a family of functions $(\mathsf{u}_\tau)_{n_0\le\tau\le n_0+1}$, where $\mathsf{u}_{n_0} := U u_{n_0}$ and $\mathsf{u}_{n_0+1} := u_{n_0+1}$. We define u_τ as $e^{\frac{\pi}{2}|x|^2}$ times the continuous (Euclidean) rearrangement of $e^{-\frac{\pi}{2}|x|^2}\,U u_{n_0}$ at parameter τ. In the same way as in [22, Lemma 36], one sees that

$$\tau \mapsto \inf_{a,\,c} \|\mathsf{u}_\tau - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)}$$

is continuous, and therefore there is a $\tau_0 \in [n_0, n_0+1)$ such that

$$\inf_{a,\,c} \|\mathsf{u}_{\tau_0} - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)} = \delta\,\|u\|^2_{\mathrm{L}^2(\gamma)}\,.$$

It follows that

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N} |\nabla u_{n_0}|^2\,d\gamma - \pi\int_{\mathbb{R}^N} u_{n_0}^2 \ln\left(\frac{u_{n_0}^2}{\|u_{n_0}\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\\ &\qquad\ge \frac{\int_{\mathbb{R}^N} |\nabla \mathsf{u}_{\tau_0}|^2\,d\gamma - \pi\int_{\mathbb{R}^N} \mathsf{u}_{\tau_0}^2 \ln\left(\frac{\mathsf{u}_{\tau_0}^2}{\|\mathsf{u}_{\tau_0}\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\|u\|^2_{\mathrm{L}^2(\gamma)}}\\ &\qquad\qquad = \delta\,\frac{\int_{\mathbb{R}^N} |\nabla \mathsf{u}_{\tau_0}|^2\,d\gamma - \pi\int_{\mathbb{R}^N} \mathsf{u}_{n_0}^2 \ln\left(\frac{\mathsf{u}_{\tau_0}^2}{\|\mathsf{u}_{\tau_0}\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma}{\inf_{a,\,c}\|\mathsf{u}_{\tau_0} - c\,e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)}}\,.\end{aligned}$$

We can apply the result in the first case to the function u_{τ_0} and infer that the right side is larger or equal than $\kappa^{\rm pos} := \delta\,\eta$. This concludes the proof in the case of nonnegative functions.

Step 2. We now prove the theorem in the general case, that is, for sign-changing functions.

We shall use the notation

$$D(v) := \int_{\mathbb{R}^N} |\nabla v|^2\,d\gamma - \pi\int_{\mathbb{R}^N} v^2 \ln\left(\frac{v^2}{\|v\|^2_{\mathrm{L}^2(\gamma)}}\right)d\gamma \quad \text{for } v \in \mathrm{H}^1(\gamma)\,.$$

Let $u = u_+ - u_- \in \mathrm{H}^1(\gamma)$. By homogeneity we can assume $\|u\|_{\mathrm{L}^2(\gamma)} = 1$. Replacing u by $-u$ if necessary, we can also assume that

$$m := \|u_-\|^2_{\mathrm{L}^2(\gamma)} \in [0, \tfrac12]\,.$$

Then

$$D(u) = D(u_+) + D(u_-) + \pi\, h(m)$$

with

$$h(p) := -\big(p \ln p + (1-p) \ln(1-p)\big)\,.$$

Since the function $p \mapsto h(p)$ is monotone increasing and concave on $[0,1/2]$, it holds that

$$h(p) \geq (2 \ln 2)\, p \quad \forall\, p \in [0, \tfrac{1}{2}]\,.$$

Thus, with $\kappa^{\rm pos}$ denoting the constant from Step 1,

$$\begin{aligned} D(u) &\geq D(u_+) + (2\,\pi\, \ln 2)\, m \geq \kappa^{\rm pos} \inf_{a,\,c} \|u_+ - c\, e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)} + (2\,\pi\, \ln 2)\, \|u_-\|^2_{\mathrm{L}^2(\gamma)} \\ &\geq \frac{1}{2}\, \min\big\{\kappa^{\rm pos},\, 2\,\pi\, \ln 2\big\}\, \inf_{a,\,c} \|u - c\, e^{a\cdot x}\|^2_{\mathrm{L}^2(\gamma)}\,. \end{aligned}$$

This proves the inequality in the general case, with $\kappa = \frac{1}{2}\, \min\big\{\kappa^{\rm pos},\, 2\,\pi\, \ln 2\big\}$. It is straightforward to verify that $\kappa = \beta\,\pi/2$, β being the constant in the statement of Theorem 1. This ends our second proof of Theorem 2 □

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