

On the Long Time Behavior of the Quantum Fokker-Planck equation

C. Sparber¹, J. A. Carrillo², J. Dolbeault³, P. A. Markowich¹

Abstract

We analyze the long time behavior of transport equations for a class of dissipative quantum systems with Fokker-planck type scattering operator, subject to confining potentials of harmonic oscillator type. We establish the conditions under which there exists a thermal equilibrium state and prove exponential decay towards it, using (classical) entropy-methods. Additionally, we give precise dispersion estimates in the cases where no equilibrium state exists.

Key words: open quantum system, Wigner transform, Fokker-Planck operator, long time asymptotics, entropy-dissipation method

1 Introduction

In this paper we analyze a class of *dissipative quantum systems*, modeling the motion of a particle ensemble, say electrons, interacting with a heat bath of oscillators. The resulting irreversible dynamics for the electrons, is a typical example of a so-called *open quantum system* [Da], i.e. a system in which the interaction with the environment is taken into account. The evolution equation, sometimes called *Master equation*, for the density (matrix) operator $R(t)$ of the particles reads

$$\frac{d}{dt}R = -\frac{i}{\hbar}[H, R] + A(R), \quad (1.1)$$

$$R(t=0) = R_0, \quad (1.2)$$

where H is the self-adjoint *Hamiltonian operator* of the free system and A models the effects introduced by the heat bath. The right hand side of (1.1) constitutes the formal generator L of the *quantum dynamical semigroup* acting on R .

¹Inst. f. Mathematik, Univ. Wien, Boltzmanngasse 9, A-1090 Vienna, Austria, e-mails: christof.sparber@univie.ac.at, peter.markowich@univie.ac.at,

²Departamento de Matemática Aplicada, Univ. de Granada, 18071-Granada, Spain, e-mail: carrillo@ugr.es,

³Ceremade, Univ. Paris IX-Dauphine, Place de Lattre de Tassigny 75775, Paris, France, e-mail: dolbeault@ceremade.dauphine.fr

Assuming that the time evolution of the system is *Markovian*, G. Lindblad [Li] gave the most general form of a bounded operator L , such that the semigroup preserves the positivity, hermiticity and the normalization (unit trace) of the density operator R . However, for unbounded operators L , which is the case in our work, the so-called *Lindblad condition* is necessary but *not sufficient* to guarantee the conservation of these properties (see, e.g., [CF] and the references therein).

Using the *Wigner transform* [Wi], dissipative quantum models can be equivalently represented in *phase space*, resulting in a *kinetic transport equation* with interaction terms for the *quasiprobability distribution* of the particles. In this paper we assume that the mechanism coupling particles and environment can be described by a linear scattering operator \mathcal{L}_q of *Fokker-Planck type*. The Lindblad condition therefore reduces to the assumption of a *positive definite diffusion matrix* D .

Up to now, a mathematically rigorous derivation of such a *Quantum Fokker-Planck equation* (QFP) from many-body quantum mechanics is still missing. To the authors knowledge, the only result in this direction is given in [CEFM], which however justifies only a particular case of the class of models considered in this work. Nevertheless there exists a huge amount of a somewhat phenomenological physical literature on this type of equations, which play a relevant role within the areas of quantum optics (laser physics) [Da], [De], [Ri], microelectronics [St], quantum brownian motion [CaLe], [Di], [HuMa], and the description of decoherence and diffusion of quantum states [AnHa], [DGHP].

Rigorous well-posedness and existence of local in time solutions of the frictionless QFP equation, with self-consistent Coulomb interaction, have been studied in a precedent paper of one of the authors [ALMS]. The present work investigates the long time behavior of the linear QFP equation in the presence of friction and an exterior time-independent potential V . The phase space formalism provided by the Wigner transform proves to be particularly helpful for this task, since it allows the use of certain *entropy techniques* established for classical dissipative equations (for an overview on these techniques, see, e.g., [MaVi], [AMTU]). The word "entropy" is used here in a mathematically technical sense and can be seen as a generalization of the classical entropy concept of L. Boltzmann. It should *not* be confused with the physical correct *von Neumann entropy* of quantum states.

By comparison with the classical *Fokker-Planck equation* (FP), we expect the solution of the QFP equation to approach a thermal *equilibrium state* in the long time limit, provided the friction term appearing in \mathcal{L}_q is positive and the exterior potential V is *confining*, i.e. $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$, fast enough. In our work we shall specify the potential V to be *harmonic*. This particular choice allows explicit calculations and is maybe the most fundamental one, from a physical point of view [CEFM]. Using the entropy approach, we will prove the convergence of the solution towards the steady state with a precise exponential rate, under the assumption that the initial data w_0 has bounded "entropy," relatively to the equilibrium state.

This paper is organized as follows. In section 2 we set up the model and collect some preliminaries. In section 3 we specify the potential V to be of harmonic oscillator type and explicitly calculate the corresponding equilibrium state. Exponential decay towards it will be proved in section 4, where we also give precise dispersion estimates in the unconfined cases.

2 The model: preliminaries

We consider a linear dissipative equation modeling the motion of particles, say electrons, under the influence of an electric scalar potential V and a thermal bath of harmonic oscillators in thermal equilibrium. In the following we denote by $\rho(t, \cdot) \in L^2(\mathbb{R}^{2d})$ the N -particle *density matrix function* of the electrons, which is the kernel of the self-adjoint trace-class *density (matrix) operator* $R(t) : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$, i.e.

$$(R(t)f)(x) := \int_{\mathbb{R}^d} \rho(x, y, t) f(y) dy. \quad (2.1)$$

The evolution equation of the electrons is given as a PDE for the density matrix

$$\begin{aligned} \partial_t \rho = & -\frac{i}{\hbar} (H_x - H_y) \rho - \gamma (x - y) \cdot (\nabla_x - \nabla_y) \rho \\ & + \left(D_{qq} |\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2} |x - y|^2 + \frac{2iD_{pq}}{\hbar} (x - y) \cdot (\nabla_x + \nabla_y) \right) \rho, \end{aligned} \quad (2.2)$$

subject to the initial condition

$$\rho(t = 0, x, y) = \rho_0(x, y), \quad x, y \in \mathbb{R}^d. \quad (2.3)$$

Here H_x (resp. H_y) denotes the electron *Hamiltonian*

$$H_x := -\frac{\hbar^2}{2m} \Delta_x + V(x), \quad (2.4)$$

acting on the x (resp. y) variable. The constant m stands for the mass of the individual particles. Equation (2.2) is a generalization of the Caldeira-Leggett master equation for medium temperatures, see [Di], [De].

On a kinetic level this model reads (QFP equation)

$$\partial_t w + \xi \cdot \nabla_x w + \Theta[V]w = \mathcal{L}_q w \quad x, \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \quad (2.5)$$

$$w(t = 0, x, \xi) = w_0(x, \xi), \quad (2.6)$$

where the scattering operator \mathcal{L}_q is defined by

$$\mathcal{L}_q w := \frac{D_{pp}}{m^2} \Delta_\xi w + 2\gamma \operatorname{div}_\xi(\xi w) + D_{qq} \Delta_x w + 2\frac{D_{pq}}{m} \operatorname{div}_x(\nabla_\xi w). \quad (2.7)$$

Here $w(t, \cdot) \in L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ is the *Wigner transform* [Wi] of the corresponding density matrix $\rho(t, \cdot) \in L^2(\mathbb{R}_x^d \times \mathbb{R}_y^d)$, i.e.

$$w(x, \xi, t) := \frac{1}{(2\pi)^d \hbar} \int_{\mathbb{R}^d} \rho \left(x + \frac{\hbar}{2m} y, x - \frac{\hbar}{2m} y, t \right) e^{i\xi \cdot y} dy. \quad (2.8)$$

In the literature $w(t, \cdot)$ is often referred to as a quasiprobability distribution because it generally assumes negative values too [Fo], [Hu].

The (non-local) pseudo-differential operator $\Theta[V]$ is defined by

$$\begin{aligned} \Theta[V]f(x, \xi) := & \frac{i}{(2\pi)^d \hbar} \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \left[V \left(x + \frac{\hbar}{2m} y \right) - V \left(x - \frac{\hbar}{2m} y \right) \right] \\ & f(x, \xi') e^{iy \cdot (\xi - \xi')} d\xi' dy. \end{aligned} \quad (2.9)$$

In order to be consistent with the usual density matrix formulation of *open quantum systems* [Da] in the class of *Lindblad operators* [Li], we assume for the *diffusion constants* $D_{pp} > 0$ and $D_{qq}, D_{pq} \geq 0$. On the other hand, the *friction parameter* γ has to be nonnegative. Additionally we impose the following relation (for more details see [ALMS], [Li])

$$D_{pp}D_{qq} - D_{pq}^2 \geq \frac{\hbar^2\gamma^2}{4} \quad \text{and} \quad D_{pp} > 0 \text{ if } \gamma = 0. \quad (\mathbf{A1})$$

Note that if $\gamma > 0$, condition (A1), the so-called *Lindblad condition*, implies that (2.7) is a uniformly elliptic operator.

Using the Wigner transform the *charge* and *flux densities* associated to the density matrix ρ can be defined (formally, since $w \notin L^1$ in general) in the same way as in classical statistical mechanics. Namely they are given as moments of the Wigner transform, (for details see, e.g., [GaMa], [LiPa]),

$$n(x, t) := \int_{\mathbb{R}^d} w(x, \xi, t) d\xi, \quad (2.10)$$

$$j(x, t) := \int_{\mathbb{R}^d} \xi w(x, \xi, t) d\xi. \quad (2.11)$$

Although not obvious from the above definition, the necessary positivity of n is guaranteed for *physical quasiprobabilities* w , i.e. for w 's, which are indeed the Wigner transformed kernels ρ of density operators R (see [Ar], [LiPa], [Ta] for a more complete account on this).

With the above definitions we obtain, after formally integrating the QFP equation (2.5) w.r.t. ξ , the associated "continuity equation"

$$\partial_t n + \operatorname{div} j = D_{qq} \Delta_x n, \quad (2.12)$$

which obviously implies (again on a formal level) the *conservation of mass*, i.e.

$$M(t) := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} w(x, \xi, t) dx d\xi = \int \int_{\mathbb{R}^d \times \mathbb{R}^d} w_0(x, \xi) dx d\xi. \quad (2.13)$$

In view of this property, we assume for simplicity that

$$M_0 = \int \int_{\mathbb{R}^d \times \mathbb{R}^d} w_0(x, \xi) dx d\xi = 1. \quad (\mathbf{A2})$$

This is of course not a restriction as long as the equation is linear. The continuity equation (2.12) suggests that in this dissipative model the usual definition of the flux density (2.11) needs to be replaced by

$$J(x, t) := j(x, t) - D_{qq} \nabla_x n(x, t) \quad (2.14)$$

$$= \int_{\mathbb{R}^d} (\xi - D_{qq} \nabla_x) w(x, \xi, t) d\xi. \quad (2.15)$$

Thus, instead of (2.12), we obtain the following conservation law

$$\partial_t n + \operatorname{div} J = 0 \quad (2.16)$$

associated to the QFP equation.

Remark. In physical units, the friction and diffusion constants are usually given by (see, e.g., [Di], [De])

$$\gamma = \frac{\lambda}{2m}, \quad D_{pp} = \lambda k_B T, \quad D_{qq} = \frac{\lambda \hbar^2}{12m^2 k_B T}, \quad D_{pq} = \frac{\lambda \Omega \hbar^2}{12\pi m k_B T}. \quad (2.17)$$

Here $\lambda > 0$ is the coupling constant of the heat bath, k_B the Boltzmann constant, T the temperature of the bath and Ω the cut-off frequency of the reservoir oscillators. In terms of these constants the Lindblad condition **(A1)** reads

$$\Omega \leq \frac{k_B T}{\hbar} \quad (2.18)$$

which implies the validity of our model at medium/high temperatures.

In the classical limit $\hbar \rightarrow 0$: $D_{qq} = D_{pq} = 0$ and (at least formally) the pseudo-differential operator simplifies to

$$\Theta[V]f \rightarrow -\frac{1}{m} \nabla_x V \cdot \nabla_\xi f. \quad (2.19)$$

Thus we recover the well known kinetic Fokker-Planck equation for the limiting classical phase space probability distribution $w^{cl}(t, \cdot) \in \mathcal{M}^+(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ (the cone of positive bounded Borel measures)

$$\partial_t w^{cl} + \xi \cdot \nabla_x w^{cl} - \frac{1}{m} \nabla_x V \cdot \nabla_\xi w^{cl} = \frac{D_{pp}}{m^2} \Delta_\xi w^{cl} + 2\gamma \operatorname{div}_\xi(\xi w^{cl}). \quad (2.20)$$

Also, the flux density J formally simplifies to the classical one, i.e. $J \rightarrow j$, as $\hbar \rightarrow 0$. For a rigorous theory of such *homogenization limits* see, e.g., [LiPa], [GMMP] and for details on the extensively studied FP equation, see for example [Ri].

Furthermore note that the Lindblad condition **(A1)** disqualifies the classical FP scattering operator [CEFM] as a relevant quantum mechanical model for the environment interaction.

In this work we are concerned with the following solution concept for our IVP.

Definition 2.1. A function $w \in C(\mathbb{R}_0^+; L^p(\mathbb{R}_x^d \times \mathbb{R}_\xi^d))$, with $1 \leq p < \infty$, is a mild solution of the IVP (2.5), (2.6) with $w_0 \in L^p(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, if and only if

$$w(t, x, \xi) = \int \int_{\mathbb{R}^d \times \mathbb{R}^d} w_0(x_0, \xi_0) G(t, x, \xi, x_0, \xi_0) dx_0 d\xi_0, \quad (2.21)$$

where the Green's function G satisfies equation (2.5) for all fixed $(x_0, \xi_0) \in \mathbb{R}^{2d}$, all $t > 0$, with an initial condition

$$\lim_{t \searrow 0} G(t, x, \xi, x_0, \xi_0) = \delta(x - x_0, \xi - \xi_0). \quad (2.22)$$

that has to be understood in a weak sense. If additionally $w \in C^1(\mathbb{R}^+; C^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d))$, then w is called a classical solution.

Remark. In classical physics the theory of kinetic equations focuses on L^1 -solutions. This implies mass conservation, since classical phase-space distributions are pointwise positive functions. Having in mind the definition of the Wigner transform, the L^2 -norm is more convenient in our quantum mechanical context.

3 Harmonic oscillator potentials

In the next two sections we choose a normalization such that $\hbar = m = 1$, for simplicity. We moreover assume the confining potential to be of the following class

$$V(x) = \frac{\omega_0^2}{2}|x|^2 + ax + b, \quad a, b \in \mathbb{R}, \quad \omega_0 \geq 0. \quad (\text{A3})$$

An easy calculation shows that, maybe after an appropriate shift in the x -variable, the pseudo-differential operator $\Theta[V]$ is given by

$$\Theta[V]w = -\omega_0^2 x \cdot \nabla_x w. \quad (3.1)$$

The QFP equation thus simplifies to

$$\partial_t w + \xi \cdot \nabla_x w - \omega_0^2 x \cdot \nabla_x w = \mathcal{L}_q w \quad x, \xi \in \mathbb{R}^d, t \in \mathbb{R}^+ \quad (3.2)$$

$$w(t=0, x, \xi) = w_0(x, \xi), \quad (3.3)$$

which can be equivalently written in the more compact form

$$\partial_t w = \operatorname{div}_{(x,\xi)} (D \nabla_{(x,\xi)} w + P(x, \xi) w), \quad (3.4)$$

$$w(0, x, \xi) = w_0(x, \xi), \quad (3.5)$$

where the *diffusion matrix* D and the vector-valued *drift* P are given by

$$D := \begin{pmatrix} D_{qq} \mathbb{I}_d & D_{pq} \mathbb{I}_d \\ D_{pq} \mathbb{I}_d & D_{pp} \mathbb{I}_d \end{pmatrix}, \quad P(x, \xi) := \begin{pmatrix} -\xi \\ \omega_0^2 x + 2\gamma \xi \end{pmatrix}. \quad (3.6)$$

Here \mathbb{I}_d denotes the identity matrix in \mathbb{R}^d . Note that the Lindblad condition (A1) guarantees that the diffusion matrix is positive definite if $\gamma > 0$ and then (3.2) is parabolic in the phase-space coordinates $(x, \xi) \in \mathbb{R}^{2d}$.

3.1 Fundamental solution

Now consider only the first order part of the operator (3.2), resp. (3.4). The associated characteristic ODE's are given by

$$\dot{X} = \xi, \quad X(t=0) = x_0, \quad (3.7)$$

$$\dot{\xi} = -(\omega_0^2 X + 2\gamma \xi), \quad \xi(t=0) = \xi_0. \quad (3.8)$$

This system defines the characteristic flow $\Phi_t(x_0, \xi_0) = [X_t(x_0, \xi_0), \dot{X}_t(x_0, \xi_0)]$ in phase space $\mathbb{R}_x^d \times \mathbb{R}_\xi^d$. This flow can be explicitly calculated, depending on the size of the friction constant γ .

Lemma 3.1. *Consider the system (3.7), (3.8) with $\gamma \geq 0$, then the dissipative flow $\Phi_t : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is given by:*

1. If $0 \leq \gamma < \omega_0$, then, with $\omega := \sqrt{\omega_0^2 - \gamma^2}$,

$$\Phi_t(x_0, \xi_0) = \frac{e^{-\gamma t}}{\omega} \begin{bmatrix} x_0(\omega \cos(\omega t) + \gamma \sin(\omega t)) + \xi_0 \sin(\omega t), \\ \xi_0(\omega \cos(\omega t) - \gamma \sin(\omega t)) - x_0 \omega_0^2 \sin(\omega t) \end{bmatrix}. \quad (3.9)$$

2. If $\gamma > \omega_0$, then, with $\omega := \sqrt{\gamma^2 - \omega_0^2}$,

$$\begin{aligned} \Phi_t(x_0, \xi_0) &= \frac{e^{-\gamma t}}{\omega} [x_0(\omega \cosh(\omega t) + \gamma \sinh(\omega t)) + \xi_0 \sinh(\omega t), \\ &\quad \xi_0(\omega \cosh(\omega t) - \gamma \sinh(\omega t)) - x_0 \omega_0^2 \sinh(\omega t)]. \end{aligned} \quad (3.10)$$

3. If $\gamma = \omega_0$ then

$$\Phi_t(x_0, \xi_0) = e^{-\gamma t} [(\gamma t + 1)x_0 + t\xi_0, (1 - \gamma t)\xi_0 - \gamma^2 t x_0]. \quad (3.11)$$

Proof. The proof follows from straightforward calculations. \square

Now, using lemma 2.1, we obtain an explicit representation of Green's function G of the QFP equation with harmonic oscillator potential, depending on the size of γ .

Proposition 3.1. *Let $\gamma \geq 0$ and let conditions (A1), (A3) hold, then the Green's function G associated to (3.2) is, for every fixed $t > 0$, a pointwise positive function, given by*

$$G(t, x, \xi, x_0, \xi_0) = e^{2d\gamma t} F(t, X_{-t}(x, \xi) - x_0, \dot{X}_{-t}(x, \xi) - \xi_0), \quad (3.12)$$

with

$$F(t, x, \xi) = \frac{\exp\left(-\frac{\nu(t)|x|^2 + \lambda(t)|\xi|^2 + \mu(t)(x \cdot \xi)}{4\lambda(t)\nu(t) - \mu^2(t)}\right)}{(2\pi)^d (4\lambda(t)\nu(t) - \mu^2(t))^{d/2}} \in C^1(\mathbb{R}^+; \mathcal{S}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)). \quad (3.13)$$

In (3.12) we denote by X_{-t}, \dot{X}_{-t} the components of the inverse characteristic flow Φ_{-t} , which satisfies $\Phi_{-t} \circ \Phi_t = \text{id}$:

$$\Phi_{-t}(x, \xi) = [X_{-t}(x, \xi), \dot{X}_{-t}(x, \xi)].$$

The associated functions $\lambda, \nu, \mu : \mathbb{R}^+ \rightarrow \mathbb{R}$ are defined by the following expressions:

$$\lambda(t) := \int_0^t \left(D_{qq} \alpha^2(s) + D_{pp} \beta^2(s) + 2D_{pq} \alpha(s) \beta(s) \right) ds, \quad (3.14)$$

$$\nu(t) := \int_0^t \left(D_{qq} \dot{\alpha}^2(s) + D_{pp} \dot{\beta}^2(s) + 2D_{pq} \dot{\alpha}(s) \dot{\beta}(s) \right) ds, \quad (3.15)$$

$$\mu(t) := 2 \int_0^t \left(D_{qq} \alpha(s) \dot{\alpha}(s) + D_{pp} \beta(s) \dot{\beta}(s) + D_{pq} \frac{d}{ds} (\alpha(s) \beta(s)) \right) ds, \quad (3.16)$$

where the functions $\alpha, \beta : \mathbb{R}^+ \rightarrow \mathbb{R}$ are given by:

$$\alpha(t) := d^{-1} \text{div}_x (X_{-t}(x, \xi)), \quad \beta(t) := d^{-1} \text{div}_\xi (X_{-t}(x, \xi)). \quad (3.17)$$

Proof. The proof is similar to the calculations given in [Bo], [Ho]. First note that by definition of α and β , we can write $X_{-t}(x, \xi) = \alpha(t)x + \beta(t)\xi$. Thus if G is the fundamental solution of (3.2), the linear transformation (3.12) guarantees that

F has to be a fundamental solution of the following PDE with time dependent coefficients

$$\partial_t F(t, x, \xi) = \left(\frac{d\lambda}{dt}(t)\Delta_x + \frac{d\nu}{dt}(t)\Delta_\xi - \frac{d\mu}{dt}(t)\nabla_x \cdot \nabla_\xi \right) F(t, x, \xi).$$

where the functions λ, ν, μ are calculated depending on the choice of γ . A Fourier transform now shows that

$$(\mathcal{F}F)(t, k, \eta) \equiv \hat{F}(t, k, \eta) := \int_{\mathbb{R}^d \times \mathbb{R}^d} F(t, x, \xi) e^{-i(x \cdot k + \xi \cdot \eta)} dx d\xi$$

is a solution of

$$\partial_t \hat{F}(t, k, \eta) = - \left(\frac{d\lambda}{dt}(t)|k|^2 + \frac{d\nu}{dt}(t)|\eta|^2 - \frac{d\mu}{dt}(t)(k \cdot \eta) \right) \hat{F}(t, k, \eta).$$

This equation can easily be integrated and thus, using that $\hat{F}(t, 0, 0) = 1$ for all $t \geq 0$, we obtain

$$\ln \left(\hat{F}(t, k, \eta) \right) = - (\lambda(t)|k|^2 + \nu(t)|\eta|^2 - \mu(t)(k \cdot \eta)).$$

After some lengthy and tedious calculations (where one checks that $4\lambda\nu \geq \mu^2$), an inverse Fourier transform gives

$$\begin{aligned} F(t, x, \xi) &= (2\pi)^{-2d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i(x \cdot k + \xi \cdot \eta)} \hat{F}(t, k, \eta) dk d\eta \\ &= (2\pi)^{-d} \frac{e^{-\frac{|x|^2}{4\lambda}}}{(4\pi\lambda)^{d/2}} \int_{\mathbb{R}^d} e^{-i\frac{\mu}{2\lambda}(x \cdot \eta) - \frac{1}{4\lambda}(4\lambda\nu - \mu^2)|\eta|^2 + i\xi \cdot \eta} d\eta \\ &= \frac{\exp \left(-\frac{\nu(t)|x|^2 + \lambda(t)|\xi|^2 + \mu(t)(x \cdot \xi)}{4\lambda(t)\nu(t) - \mu^2(t)} \right)}{(2\pi)^d (4\lambda(t)\nu(t) - \mu^2(t))^{d/2}}, \end{aligned}$$

which is the desired result. \square

Note that the quantum mechanical effects in F and consequently in G only enter in form of the constants $D_{qq}, D_{pq} \sim \hbar^2$, which appear in the auxiliary functions λ, ν, μ . In other words we obtain Green's function for the classical Fokker-Planck equation in a square-well potential by setting $D_{qq} = D_{pq} = 0$ in the above expressions. The pointwise positivity of G is a consequence of the minimum principle for parabolic equations of Fokker-Planck type [Ev].

From the above proposition we draw the following consequences (among which we obtain the conservativity of the quantum dynamical semigroup, associated to the harmonically confined QFP equation).

Corollary 3.1. *Let $\gamma \geq 0$ and assume (A1)-(A3). Then for every initial condition $w_0 \in L^p(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ with $1 \leq p < \infty$, there exists a unique classical solution $w \in C(\mathbb{R}_0^+; L^p(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)) \cap C^1(\mathbb{R}^+; C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d))$ with $M(t) = M_0 \equiv 1$. Moreover if w_0, n_0 are non-negative a.e., so are $w(t, \cdot), n(t, \cdot)$, for all $t \in \mathbb{R}^+$.*

Proof. For convenience we use the notation: $y := (x, \xi)$ as well as $y_0 := (x_0, \xi_0)$. With the following linear change of variables

$$G(t, \Phi_t(y), y_0) = e^{2d\gamma t} F(t, y - y_0) \in C^1(\mathbb{R}^+, \mathcal{S}(\mathbb{R}^{2d})),$$

we can express our solution in the form

$$w(t, \Phi_t(y)) = e^{2d\gamma t} (w_0 * F(t, y)).$$

A straightforward computation now shows that the Jacobian determinant of the mapping $\Phi_t(\cdot)$ is given by

$$\det \left(\frac{\partial \Phi_t(y)}{\partial y} \right) = \exp(-2d\gamma t).$$

With these preparations and using Young's inequality [LiLo], we obtain

$$\| w(t) \|_p = \| w_0 * F(t) \|_p \leq \| w_0 \|_p \| F(t) \|_1 < \infty,$$

since for each fixed $t \in \mathbb{R}^+$: $F(t, \cdot) \in \mathcal{S}(\mathbb{R}^{2d}) \subset L^1(\mathbb{R}^{2d})$. More precisely we have

$$\| F(t) \|_p^p = \int_{\mathbb{R}^{2d}} |F(t, y)|^p dy = p^{-1},$$

for all $1 \leq p < \infty$. For $p = 1$ this implies, after a simple calculation, that $M(t) = M_0 \equiv 1$, by (A2). Since $G(t, \cdot)$ is pointwise positive, we clearly obtain that if w_0 , resp. n_0 are a.e. non-negative, so are $w(t, \cdot)$, $n(t, \cdot)$. \square

Remark. If the initial condition w_0 is the Wigner transform of a pure quantum state ψ_0 , it is well known [Hu], [LiPa] that $w_0 \geq 0$ pointwise, if and only if ψ_0 is a Gaussian. A similar characterization for mixed states has not been found yet. From the above result it is easy to deduce the *conservativity* (i.e. conservation of hermiticity, positivity and normalization of the density matrix ρ) of the quantum dynamical semigroup corresponding to (3.2), using the inverse Wigner transform

$$\rho(x, y, t) = (2\pi)^{-d} \int_{\mathbb{R}^d} w \left(\frac{x+y}{2}, \xi, t \right) e^{i\xi \cdot (x-y)} d\xi, \quad (3.18)$$

which is defined in the sense of the usual L^2 -Fourier transform [Fo], [LiLo].

3.2 Stationary states.

As in the classical case, we expect that the competing effects of the confining potential plus the positive friction and the dissipating behavior of the operator result in a *thermal equilibrium state* as $t \rightarrow \infty$.

Definition 3.1. A (thermal) equilibrium state is a steady state, i.e. stationary solution $w \in L^2(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ of the QFP equation (2.5), such that

$$J(x) = \int_{\mathbb{R}^d} (\xi - D_{qq} \nabla_x) w(x, \xi) d\xi = 0. \quad (3.19)$$

In principle it could be possible that there exist stationary solutions of the QFP equation, which are not equilibrium states, however the following proposition and corollary show that this is not the case.

Furthermore, we shall see that the physical intuitive assumption, that the mass of the initial state is equal to the mass of the steady state, guarantees its uniqueness.

Proposition 3.2. *Let $\gamma > 0$, $\omega_0 > 0$ and assume (A1), (A3), then the unique solution w_∞ of the stationary QFP equation, satisfying*

$$M_\infty := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} w_\infty(x, \xi) dx d\xi = 1, \quad (3.20)$$

is given by the following non-isotropic Gaussian function

$$w_\infty(x, \xi) = \frac{\gamma\omega_0}{(2\pi)^d \sqrt{Q}} \exp\left(-\frac{\gamma}{Q} \left[Q_{11}\omega_0^2|x|^2 + 2Q_{12}\omega_0 x \cdot \xi + Q_{22}|\xi|^2 \right]\right), \quad (3.21)$$

such that $w_\infty \in \mathcal{S}(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, with $Q := Q_{11}Q_{22} - Q_{12}^2$ and

$$Q_{11} := D_{pp} + \omega_0^2 D_{qq}, \quad (3.22)$$

$$Q_{12} := 2\omega_0\gamma D_{pq}, \quad (3.23)$$

$$Q_{22} := D_{pp} + \omega_0^2 D_{qq} + 4\gamma(D_{pq} + \gamma D_{qq}). \quad (3.24)$$

Proof. The proof is much simpler in Fourier-space where the steady state \hat{w}_∞ is explicitly given by

$$\hat{w}_\infty(k, \eta) = \exp\left(-\frac{D_{pp}}{4\gamma\omega_0^2} (|k|^2 + \omega_0^2|\eta|^2) - \frac{D_{pq}}{\omega_0^2} |k|^2 - D_{qq} \left[\left(\frac{\gamma}{\omega_0^2} + \frac{1}{4\gamma}\right) |k|^2 + \frac{\omega_0^2}{4\gamma} |\eta|^2 + k \cdot \eta \right]\right).$$

First it is straightforward to check that this is indeed a solution of the Fourier transformed stationary QFP equation

$$\omega_0^2 \eta \cdot \nabla_k \hat{w}_\infty + (2\gamma\eta - k) \cdot \nabla_\eta \hat{w}_\infty = - (D_{pp}|\eta|^2 + D_{qq}|k|^2 + 2D_{pq}k \cdot \eta) \hat{w}_\infty$$

and then that it satisfies the Fourier transformed mass normalization condition $\iint w_\infty dx d\xi = 1$, i.e.

$$\hat{w}_\infty(0, 0) = 1. \quad (3.25)$$

Thus, it remains to prove the uniqueness of the steady state. To do so we consider the corresponding characteristic system

$$\dot{k} = \omega_0^2 \eta, \quad k(s=0) = k_0 \quad (3.26)$$

$$\dot{\eta} = 2\gamma\eta - k, \quad \eta(s=0) = \eta_0. \quad (3.27)$$

Having in mind that, by assumption, $\gamma > 0$ we check that the real parts of the eigenvalues $\lambda_{1,2} := \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$ of this system are positive, which, by the standard theory of ODE's, implies that $(k, \eta) = (0, 0)$ is a source of the characteristic flow (a similar argument holds if $\gamma = \omega_0$). Thus the condition (3.25) is necessary and sufficient to guarantee the uniqueness of the solution \hat{w}_∞ . \square

From the above proposition we draw the following consequences:

Corollary 3.2. *Under the same assumptions as above we have:*

1. *The state w_∞ is also an equilibrium state, i.e. $J_\infty(x) = 0$ for all $x \in \mathbb{R}^d$.*

2. If either $\omega_0 = 0$ (free motion case) or $\gamma = 0$ (frictionless case) or both are equal to zero, no nontrivial L^1 -steady state exists.

Proof. It is a lengthy but straightforward calculation to show that the current-density J_∞ associated to w_∞ vanishes identically.

In the limiting cases $\gamma = 0$ or $\omega_0 = 0$, we consider the characteristic curves given by the ODE system (3.26), (3.27). Along these curves the steady state varies according to

$$\frac{d}{ds} \hat{w}_\infty(k_s, \eta_s) = -\vartheta_s(k_0, \eta_0) \hat{w}_\infty(k_s, \eta_s), \quad (3.28)$$

where $\eta_s = \eta_s(k_0, \eta_0)$, $k_s = k_s(k_0, \eta_0)$ denote the (vector valued) characteristic curves starting for $s = 0$ at the point (k_0, η_0) . By $\vartheta_s(k_0, \eta_0)$, we mean

$$\vartheta_s(k_0, \eta_0) := D_{pp} |\eta_s(k_0, \eta_0)|^2 + D_{qq} |k_s(k_0, \eta_0)|^2 + 2D_{pq}(k_s \cdot \eta_s)(k_0, \eta_0).$$

We integrate equation (3.28) and obtain the Fourier transformed steady state parametrized by $s \in \mathbb{R}$

$$\hat{w}_\infty(k_s(k_0, \eta_0), \eta_s(k_0, \eta_0)) = \hat{w}_\infty(k_0, \eta_0) \exp\left(-\int_0^s \vartheta_\tau(k_0, \eta_0) d\tau\right). \quad (3.29)$$

In case the potential vanishes, i.e. $\omega_0 = 0$ (and $\gamma \geq 0$), we have for all $s \in \mathbb{R}$ that $k_s(k_0, \eta_0) = k_0$ (free motion) and

$$\eta_\tau(k_0, \eta_0) = \begin{cases} \eta_0 - k_0 \tau & \text{if } \gamma = 0, \\ \eta_0 \exp(2\gamma\tau) - \frac{k_0}{2\gamma} & \text{if } \gamma > 0. \end{cases}$$

Now let $\omega_0 = \gamma = 0$ and set $k_0 = 0$ in equation (3.29). We obtain

$$\hat{w}_\infty(0, \eta_0) = \hat{w}_\infty(0, \eta_0) \exp(-D_{pp} |\eta_0|^2 s), \quad \forall \eta_0 \in \mathbb{R}^d, s \in \mathbb{R},$$

which implies

$$\hat{w}_\infty(0, \eta_0) = 0, \quad \forall \eta_0 \in \mathbb{R}^d,$$

in contradiction to (3.25). The same type of argument holds in the case $\omega_0 = 0$, $\gamma > 0$, if we set $\eta_0 = k_0/2\gamma$ in equation (3.29).

In the frictionless case, i.e. $\gamma = 0$, $\omega_0 > 0$, the characteristics are circles in the (k, η) plane. Thus for $s = 2\pi/\omega_0$ we have $(k_s(k_0, \eta_0), \eta_s(k_0, \eta_0)) = (k_0, \eta_0)$. However if we integrate over one period, i.e. setting $s = 2\pi/\omega_0$ in (3.29), we obtain for all $k_0, \eta_0 \in \mathbb{R}^d$

$$\hat{w}_\infty(k_0, \eta_0) = \hat{w}_\infty(k_0, \eta_0) \exp\left(-\frac{\pi}{\omega_0^3} (D_{pp} + \omega_0^2 D_{qq}) (|k_0|^2 + \omega_0^2 |\eta_0|^2)\right)$$

which clearly implies, since $D_{pp}, D_{qq} > 0$, that \hat{w}_∞ identically vanishes. \square

The above corollary in particular shows that, as expected, the presence of a positive friction together with a confining potential are crucial to guarantee the existence of an admissible, i.e. with finite mass, equilibrium state.

Note that although the solution w of (2.5) in general will not be nonnegative, the steady state w_∞ is nevertheless a pointwise *positive* function, because of Assumption **(A2)**. The associated density matrix ρ_∞ is a particular example of a *mixed* quantum state with positive Wigner transform. It is explicitly given by

$$\rho_\infty(x, y) = \frac{\gamma\omega_0}{(16\pi^3)^{d/2}\sqrt{\gamma Q_{22}}} e^{-\frac{1}{4\gamma Q_{22}}[\gamma^2\omega_0^2(x+y)^2 + Q(x-y)^2] + i\omega_0\frac{Q_{12}}{Q_{22}}\left(\frac{x^2-y^2}{2}\right)}. \quad (3.30)$$

Observe that the equilibrium density n_∞ , which is obtained from the diagonal (i.e. $x = y$) of the steady state density matrix ρ_∞ , is a real valued Gaussian function.

Remark. As a special limit case, we observe that in the classical limit, i.e. $D_{qq} = D_{pq} = 0$, the equilibrium state simplifies to the well known stationary solution w_∞^{cl} of the classical kinetic Fokker-Planck equation (2.20) with a harmonic oscillator potential (c.f. [Ri]), i.e.

$$w_\infty^{cl}(x, \xi) = \frac{\omega_0\gamma}{(2\pi)^d D_{pp}} e^{-\frac{\gamma}{D_{pp}}(|\xi|^2 + \omega_0^2|x|^2)}. \quad (3.31)$$

Note that, in contrast to the classical case, the quantum steady state is *not* a function of the classical energy

$$H(x, \xi) := \frac{1}{2}|\xi|^2 + \frac{1}{2}\omega_0^2|x|^2. \quad (3.32)$$

For the classical kinetic FP equation, the fact that $w_\infty^{cl} = w_\infty^{cl}(H)$, implies that the transport and the classical FP scattering operator vanish independently when applied to w_∞^{cl} . In our quantum mechanical framework this is no longer true, since there the steady state w_∞ results from a cancellation of the transport operator and the scattering term \mathcal{L}_q .

4 Long time behavior

We are now in the position to describe the long time behavior of the linear QFP equation in the cases where both friction and an external potential of harmonic oscillator type are present, and in cases where one is missing. We will start with the latter situation.

4.1 Dispersion estimates in the unconfined case

We want to address the unconfined or dispersive cases, i.e. either $\gamma = 0$ or $\omega_0 = 0$ (or both). By comparison with the classical FP equation, we expect that the particles escape to infinity and thus that the macroscopic density n decays to 0 as $t \rightarrow \infty$. More precisely we have the following theorem.

Theorem 4.1. *Let be either $\gamma = 0$ or $\omega_0 = 0$, or both, and assume **(A1)**, **(A3)**:*

1. *If $w_0 \in L^1 \cap L^2$, then the solution w of the QFP equation satisfies*

$$\|w(t)\|_p \leq C_p R_w(t)^{-\frac{d}{2q}} \|w_0\|_1, \quad 1 \leq p \leq \infty, \quad (4.1)$$

where C_p is a constant independent of w_0 , $p^{-1} + q^{-1} = 1$ and

$$R_w(t) := e^{-4\gamma t} (4\lambda(t)\nu(t) - \mu^2(t)), \quad (4.2)$$

which is a pointwise positive function with $R_w(t) \rightarrow \infty$ as $t \rightarrow \infty$.

2. Consequently we have for the corresponding density, that

$$\|n(t)\|_p \leq C_p R_n(t)^{-\frac{d}{2q}} \|w_0\|_1, \quad 1 \leq p \leq \infty, \quad (4.3)$$

where again the rate $R_n(t) \rightarrow \infty$ as $t \rightarrow \infty$. Explicitly

$$R_n(t) := 2 \left(\lambda(t)\tilde{\alpha}^2(t) + \mu(t)\tilde{\alpha}(t)\tilde{\beta}(t) + \nu(t)\tilde{\beta}^2(t) \right), \quad (4.4)$$

using

$$\tilde{\alpha}(t) := d^{-1} \operatorname{div}_{x_0} (X_t(x_0, \xi_0)), \quad \tilde{\beta}(t) := d^{-1} \operatorname{div}_{\xi_0} (X_t(x_0, \xi_0)).$$

Proof. The estimate in claim no. 1 is trivial in the case $p = \infty$ from the formula of the fundamental solution in (3.12) and (2.21). The estimate for $p = 1$ is due to the following property of the Green's function

$$\int \int_{\mathbb{R}^d \times \mathbb{R}^d} G(t, x, \xi, x_0, \xi_0) dx_0 d\xi_0 = 1$$

and a trivial estimate over (2.21). The estimates for $1 < p < \infty$ are then obtained by interpolation. The function $R_w(t)$ can be computed explicitly in each of the three cases and is given by:

1. If $\omega_0 > \gamma = 0$, then with $\varphi = \varphi(t) = 2t\omega_0$,

$$\begin{aligned} R_w(t) &= \frac{1}{4} \left(D_{qq}^2 + \frac{1}{\omega_0^4} D_{pp}^2 \right) (\varphi^2 + 2 \cos \varphi - 2) \\ &\quad + \frac{2}{\omega_0^2} D_{pq}^2 (\cos \varphi - 1) + \frac{1}{2\omega_0^2} D_{pp} D_{qq} (\varphi^2 - 2 \cos \varphi + 2). \end{aligned}$$

2. If $\gamma > \omega_0 = 0$, then with $\chi = \chi(t) = e^{-2t\gamma}$,

$$\begin{aligned} R_w(t) &= \frac{1}{4\gamma^4} (D_{pp}^2 + 4\gamma D_{pp} D_{pq}) (t\gamma(1 - \chi^2) - (1 - \chi)^2) \\ &\quad - \frac{1}{\gamma^2} D_{pq}^2 (1 - \chi)^2 + \frac{t}{\gamma} D_{qq} (1 - \chi^2) \end{aligned}$$

3. If $\gamma = \omega_0 = 0$, then

$$R_w(t) = -4 D_{pq}^2 t^2 + 4 D_{pp} D_{qq} t^2 + \frac{1}{3} D_{pp}^2 t^4.$$

In all cases, R_w diverges as $t \rightarrow \infty$ and thus claim no. 1 is proved.

To prove claim no. 2 we first compute the integral of the fundamental solution w.r.t. ξ , which gives

$$\int_{\mathbb{R}^d} G(t, x, \xi, x_0, \xi_0) d\xi = R_n^{-d/2}(t) \mathcal{N} \left(\frac{x - X_t(x_0, \xi_0)}{\sqrt{R_n(t)}} \right), \quad (4.5)$$

where X_t is defined as in lemma 3.1 and

$$\mathcal{N}(\sigma) := (2\pi)^{-d/2} \exp\left(-\frac{\sigma^2}{2}\right).$$

We can then deduce a formula for the evolution of the macroscopic density, using (3.12), (2.21) and the above computation (4.5), to obtain

$$n(t, x) = R_n^{-d/2}(t) \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{N}\left(\frac{x - X_t(x_0, \xi_0)}{\sqrt{R_n(t)}}\right) w_0(x_0, \xi_0) dx_0 d\xi_0.$$

The dispersion estimates are then straightforward from this expression. On the other hand, the function $R_n(t)$ again can be computed explicitly:

1. If $\omega_0 > \gamma = 0$, then with $\varphi = \varphi(t) = 2t\omega_0$,

$$2\omega_0^3 R_n(t) = 2\omega_0 D_{pq} (1 - \cos \varphi) + D_{pp} (\varphi - \sin \varphi) + \omega_0^2 D_{qq} (\varphi + \sin \varphi).$$

2. If $\gamma > \omega_0 = 0$, then with $\chi = \chi(t) = e^{-2t\gamma}$,

$$8\gamma^3 R_n(t) = D_{pp} (4\chi + 4\gamma t - 3 - \chi^2) + 16\gamma^3 t D_{qq} + 8\gamma D_{pq} (\chi + 2\gamma t - 1).$$

3. If $\gamma = \omega_0 = 0$, then

$$R_n(t) = 2 D_{qq} t + 2 D_{pq} t^2 + \frac{2}{3} D_{pp} t^3.$$

In all three cases R_n diverges as $t \rightarrow \infty$ and thus claim no. 2 is also proved. \square

The behavior at $t = +\infty$ of the rate functions can be obtained directly from the explicit computations in the previous result, depending on the diffusion constants. In the following corollary we will consider only the physical most important case, namely $D_{pp} > 0$.

Corollary 4.1. *Assume $D_{pp} > 0$ and (A1), (A3), then we, have as $t \rightarrow +\infty$:*

1. *If $\omega_0 > \gamma = 0$, then $R_w(t) = O(t^2)$ and $R_n(t) = O(t)$.*
2. *If $\gamma > \omega_0 = 0$, then $R_w(t) = O(t)$ and $R_n(t) = O(t)$.*
3. *If $\gamma = \omega_0 = 0$, then $R_w(t) = O(t^4)$ and $R_n(t) = O(t^3)$.*

Remark. The computation of n can also be used to measure the convergence rates towards the steady state macroscopic density n_∞ in the confined case. This computation however is quite involved and we leave the details to the reader, since we will take a more elegant approach in the next subsection.

Nevertheless the advantage of such an explicit calculation would be that one can prove exponential decay of the solution, even for the classical FP equation (this can also be seen by comparison with spectral theoretical approaches [Ri]), whereas the entropy method used in section 4.2 only gives a suboptimal rate in this case [DeVi].

4.2 Exponential decay towards equilibrium

We now assume the presence of friction and of the confining potential. Like in the classical case, we expect exponential decay of the solution towards the equilibrium state, which is usually proved using spectral theory.

However in this work we shall follow a different approach, the so-called *entropy-entropy-dissipation method* for classical FP type equations (see [AMTU] and references therein). As we shall see it can be successfully applied in our quantum mechanical context too (for an overview of the classical applications, see [MaVi]). First let us rewrite the drift-vector P given by (3.6) in the following way

$$P(x, \xi) = D(\nabla A(x, \xi) + F(x, \xi)), \quad (4.6)$$

where the gradient is taken w.r.t. $(x, \xi) \in \mathbb{R}^{2d}$ and A is defined as the normalized potential appearing in the expression of the equilibrium state, more precisely

$$w_\infty(x, \xi) \equiv \exp(-A(x, \xi)). \quad (4.7)$$

Consequently the QFP equation (3.2) takes the standard form of a *non-symmetric drift-diffusion equation*

$$\partial_t w = \operatorname{div} [D(\nabla w + w(\nabla A + F))], \quad t \in \mathbb{R}^+ \quad (4.8)$$

$$w(t = 0, x, \xi) = w_0(x, \xi). \quad (4.9)$$

The vector field F is explicitly given by

$$F(x, \xi) := D^{-1}P(x, \xi) - \nabla A(x, \xi), \quad (4.10)$$

where D^{-1} is the inverse of the diffusion matrix, A is defined by (4.7) and P is the drift given in (3.6).

The rewritten QFP equation (for harmonic oscillator potentials) with equilibrium state given by (4.7) can now be identified as a special case of FP type equations, see [AMTU]. Therefore we introduce the concept of *relative entropies* in the same sense as in the quoted work:

Definition 4.1. *Let \mathcal{A} be either \mathbb{R} or \mathbb{R}^+ . Let $\varphi \in C(\bar{\mathcal{A}}) \cap C^4(\mathcal{A})$ satisfy*

$$\varphi(1) = 0, \quad \varphi'' \geq 0 \text{ with } \varphi'' \neq 0 \text{ and } (\varphi''')^2 \leq \frac{1}{2}\varphi''\varphi^{IV} \text{ on } \mathcal{A}. \quad (4.11)$$

Assume moreover that $f \in L^1(\mathbb{R}^n)$, $g \in L^1_+(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} f \, dx = \int_{\mathbb{R}^n} g \, dx = 1$ and $f/g \in \bar{\mathcal{A}}$ $g(dx)$ a.e. Then

$$e_\varphi(f | g) := \int_{\mathbb{R}^n} \varphi\left(\frac{f}{g}\right) g(dx) \quad (4.12)$$

is an admissible relative entropy of f w.r.t. g and generating function φ .

Two relative entropies which are frequently used are the *logarithmic relative entropy*, associated to the generating function

$$\varphi_1(\alpha) := \alpha \ln \alpha - \alpha + 1, \quad \alpha \in \mathbb{R}^+. \quad (4.13)$$

and the *quadratic relative entropy* with generator

$$\varphi_2(\alpha) := k(\alpha - 1)^2, \quad \alpha \in \mathbb{R}, \quad k > 0. \quad (4.14)$$

Indeed it is known, see [AMTU], that (up to a positive multiplicative constant) for every admissible entropy generator φ , there exist constants $K_1, K_2 > 0$ such that for all $\alpha \in \mathbb{R}^+$, it holds that

$$K_1\varphi_1(\alpha) \leq \varphi(\alpha) \leq K_2\varphi_2(\alpha). \quad (4.15)$$

Remark. The above defined entropies should not be confused with the quantum mechanical *von Neumann entropy* $S := -\text{Tr}(R \ln R)$, where R is the density operator of the particle ensemble [Th] (also note that in contrast to the physical convention the minus sign in front of (4.12) is dropped).

There is however a notion of entropy for quantum states, called *Wehrl entropy* [We], which is closely related to the logarithmic entropy defined above (using the pointwise nonnegative *Husimi transform* of w , see, e.g., [LiPa]) and which can be interpreted as a measure of *coherence and localization of quantum states*. For details, see [AnHa], [GnZy], [SlZy] and the references given therein.

Since the solution of the QFP equation in general is not pointwise positive, it seems that we need to look for an admissible entropy on all of \mathbb{R} . This would imply, see [AMTU], that the only admissible entropy useful for our purpose is the quadratic one (4.14). We circumvent this shortcoming by decomposing the initial condition w_0 into its negative and positive parts. More precisely we write

$$w_0(x, \xi) = w_0^+(x, \xi) - w_0^-(x, \xi) \text{ a.e.} \quad (4.16)$$

where now w_0^+, w_0^- are both non-negative functions with mass M_0^+, M_0^- respectively. Clearly condition **(A2)** implies

$$1 = M_0^+ - M_0^-. \quad (4.17)$$

Having in mind that the QFP equation is linear, we denote by

$$w^\pm(t, x, \xi) := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} w_0^\pm(x_0, \xi_0) G(t, x, \xi, x_0, \xi_0) dx_0 d\xi_0 \quad (4.18)$$

the mild solution of (4.8) corresponding to w_0^\pm . Corollary 2.1 implies that $w^\pm(t, \cdot) \geq 0$ a.e. and we shall now apply the entropy-entropy-dissipation method to each of the two functions w^+, w^- . Note that the steady state associated to $w^\pm(t, \cdot)$ is given by $w_\infty^\pm = M_0^\pm w_\infty$.

In order to use the results established in [AMTU], we first need to check that the following property is fulfilled.

Lemma 4.1. *Assume $\gamma > 0, \omega_0 > 0$ as well as **(A1)**, **(A3)**, $y := (x, \xi) \in \mathbb{R}^{2d}$. Then it holds that*

$$\text{div}_y(DFw_\infty) = 0 \quad \text{on } \mathbb{R}^{2d} \quad (4.19)$$

where F is defined by (4.10).

Proof. To prove the claim, we note, using equation (4.7), that condition (4.19) is equivalent to

$$\text{div}_y(D\nabla w_\infty + Pw_\infty) = 0,$$

which is the stationary QFP equation for potentials of the form **(A3)**. Thus the claim is true by definition of A, F and w_∞ . \square

With these preparations we can now state the main result of this section.

Theorem 4.2. *Assume $\gamma > 0$, $\omega_0 > 0$ and assume (A1)-(A3). Let the initial data w_0 be such that $w_0(x, \xi) = w_0^+(x, \xi) - w_0^-(x, \xi)$ a.e., $w_0^\pm \in L^1 \cap L^2$ and assume*

$$e_\varphi(w_0^\pm | M^\pm w_\infty) < \infty. \quad (\text{A4})$$

Then there exists a $\kappa > 0$ such that

$$e_\varphi(w^\pm(t, \cdot) | M_0^\pm w_\infty) \leq e^{-2\kappa t} e_\varphi(w_0^\pm | M_0^\pm w_\infty), \quad t > 0. \quad (4.20)$$

As a consequence, the solution of (3.2) converges exponentially towards the equilibrium state. More precisely it holds that:

1. If $e_{\varphi_1}(w_0^\pm | w_\infty) < \infty$, with φ_1 defined in (4.13), then

$$\|w(t, \cdot) - w_\infty\|_1 \leq C e^{-\kappa t}, \quad C \in \mathbb{R}^+, \quad t > 0. \quad (4.21)$$

2. More generally, if $e_{\varphi_2}(w_0^\pm | w_\infty) < \infty$, with φ_2 defined in (4.14), then

$$\|w(t, \cdot) - w_\infty\|_p \leq C e^{-\kappa t} \quad C \in \mathbb{R}^+, \quad t > 0, \quad (4.22)$$

with $1 \leq p \leq 2$.

Proof. The proof of the first claim (4.20) is a consequence of the following *convex Sobolev inequality* [AMTU] (in which we use the notation $d\rho_\infty = \rho_\infty dx$)

$$\int_{\mathbb{R}^{2d}} \varphi\left(\frac{\rho}{\rho_\infty}\right) d\rho_\infty \leq \frac{1}{2\kappa} \int_{\mathbb{R}^{2d}} \varphi''\left(\frac{\rho}{\rho_\infty}\right) \left| \nabla\left(\frac{\rho}{\rho_\infty}\right) \right|^2 d\rho_\infty,$$

which holds for every admissible entropy generator φ (c.f. definition 4.1), every function $\rho \in L^1(\mathbb{R}^{2d})$, with $M = M_\infty$ and $\rho_\infty := \exp(-A(x, \xi)) \in L^1_+(\mathbb{R}^{2d})$, with uniformly convex potential A , i.e.

$$\frac{\partial^2 A}{\partial(x, \xi)^2} \geq \kappa_1 \mathbb{I}_{2d}, \quad (4.23)$$

where $\kappa_1 > 0$ and \mathbb{I}_{2d} denotes the identity matrix in \mathbb{R}^{2d} . Using this inequality, we can now estimate (recall that $w_\infty^\pm = M^\pm w_\infty$)

$$\int_{\mathbb{R}^{2d}} \varphi\left(\frac{w^\pm}{w_\infty^\pm}\right) dw_\infty^\pm \leq \frac{1}{2\delta\kappa_1} \int_{\mathbb{R}^{2d}} \varphi''\left(\frac{w^\pm}{w_\infty^\pm}\right) \nabla^\top\left(\frac{w^\pm}{w_\infty^\pm}\right) D \nabla\left(\frac{w^\pm}{w_\infty^\pm}\right) dw_\infty^\pm,$$

where $\delta > 0$ is the smallest eigenvalue of D given by

$$\delta := \frac{1}{2} \left(D_{pp} + D_{qq} - \sqrt{(D_{pp} - D_{qq})^2 + 4D_{pq}^2} \right), \quad (4.24)$$

since by definition of δ , it holds that $\delta I \leq D$. Provided lemma 4.1, the results of [AMTU] imply the exponential decay of the relative entropy with a rate $\kappa \equiv \delta\kappa_1$ and thus (4.20) is proved.

Using the well known *Csiszár-Kullback inequality* [AMTU], [Cs], [Ku], we further obtain

$$\|w^\pm(t, \cdot) - M_0^\pm w_\infty\|_1 \leq K e^{-\kappa t}, \quad K \in \mathbb{R}^+, t > 0.$$

Since by **(A2)** we have $1 = M_0^+ - M_0^-$, this implies

$$\begin{aligned} \|w(t, \cdot) - w_\infty\|_1 &= \|w^+(t, \cdot) - w^-(t, \cdot) - (M_0^+ - M_0^-)w_\infty\|_1 \\ &\leq \|w^+(t, \cdot) - M_0^+ w_\infty\|_1 + \|w^-(t, \cdot) - M_0^- w_\infty\|_1 \\ &\leq C e^{-\kappa t}, \quad C \in \mathbb{R}^+. \end{aligned}$$

For entropy generators of the form (4.14), we get from (4.20)

$$\|w^\pm(t, \cdot) - M_0^\pm w_\infty\|_{2,\delta} \leq e^{-\kappa t} \|w_0^\pm - M_0^\pm w_\infty\|_{2,\delta} \quad \forall t > 0,$$

where $\|\cdot\|_{2,\delta}$ is the L^2 -norm with weight $\delta = 1/w_\infty$, i.e.

$$\|f\|_{2,\delta}^2 := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x, \xi)|^2 w_\infty^{-1}(dx, d\xi).$$

As above, the use of the triangle inequality allows us to conclude that exponential decay (with rate κ) of w holds in this weighted L^2 -norm. Since $1/w_\infty \geq 1$ pointwise, we have $\|f\|_2 \leq \|f\|_{2,\delta}$ (here we have used the Lindblad condition **(A1)** to show that the constant in front of $1/w_\infty$ is indeed greater than 1).

This implies exponential decay in the usual L^2 -norm and thus, by interpolation, we obtain the exponential convergence of w towards the steady state in all L^p -norms with $1 \leq p \leq 2$. \square

It should be noted, that the main drawback of the above theorem is the fact that it only holds for initial data with positive and negative part bounded in relative entropy **(A4)**. Furthermore one should have in mind that the above theorem fails, if $\gamma = 0$ or $\omega_0 = 0$, since then A is no longer uniformly convex. However we have already seen that in these cases no finite-mass steady state, different from zero, exists.

The precise value of the rate κ can be obtained by the following transformation. Define a new potential \tilde{A} by

$$\tilde{A}(\tilde{x}, \tilde{\xi}) := A\left(\sqrt{D}(x, \xi)^\top\right), \quad (4.25)$$

where \sqrt{D} is the square root of D in the sense of positive definite matrices. Then, it holds that

$$\frac{\partial^2 \tilde{A}}{\partial(\tilde{x}, \tilde{\xi})^2} \geq \kappa \mathbf{I}, \quad (4.26)$$

which gives the optimal rate κ as the smallest eigenvalue of the Hessian of \tilde{A} . For details, see [AMTU] or [Ri].

Remark. In contrast to our result, the convergence to the equilibrium state for classical kinetic FP equations (2.20) is, by similar entropy methods, obtained with a suboptimal rate of order $O(t^{-\infty})$, see [DeVi]. This is due to the fact, that for the classical FP equation, in addition to the thermal equilibrium state

(3.31), there exist other states with zero entropy dissipation. Indeed explicit calculations show that κ depends on \hbar and converges to 0 as $\hbar \rightarrow 0$.

We hope to extend the above results to more general, maybe nonlinear, potentials V in forthcoming works. Although quite simple, the presented harmonic oscillator case is expected to be an essential prerequisite in this analysis.

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