Hypocoercivity

1 Introduction and state of the art

Our goal is to determine estimates of the rates of convergence to equilibrium for kinetic transport models. The difficulty in this enterprise is the fact that in spatially inhomogeneous kinetic models the decay is the result of dissipation towards a subspace, combined with a conservative mixing effect. If this leads to decay of all solution components, which is not guaranteed in general, the problem is called *hypocoercive* (Villani (2009)). In kinetic theory, the analysis is typically based on Lyapunov functionals arising from the thermodynamical setting of the problem, *i.e.*, entropies or free energies. These may take the mathematical form of relative entropies, measuring the distance to the equilibrium. Actually, for linear problems, which can be interpreted as the evolution of the probability distribution of a Markov process, the decay of relative entropies is a general principle. The dissipation of these *physical* Lyapunov functionals is typically connected to the decay towards the above mentioned subspace, and the dissipation rate is therefore not coercive. Most approaches for proving hypocoercivity are based on the construction of modified Lyapunov functionals with a coercive dissipation rate (see,*e.g.*, [Vil09, MN06, HTT06, DMS15]). A recent strategy is an extension of the classical Bakry-Emery approach (as in [Bau13]), where a modified version has the potential to produce sharp decay rates (see [AE14, AAS15]).

Recently some of these ideas have been extended from the classical examples of kinetic equations to new types of problems such as kinetic models of mixtures interacting by binary collisions as in [DJMZ16] or by chemical reactions in [NS16]. Nonlinear mean-field term have also been considered in [DBGV16, HT16], at least in small masses regimes.

Macroscopic limits with parameters in the diffusive range can be seen as intermediate asymptotics in the analysis of the long-time behavior, and the diffusive character of the macroscopic equations plays an implicit role in the decay to equilibrium. Therefore kinetic models with fractional diffusion macroscopic limits as in [ASS16] are expected to pose new challenges. Hypocoercive methods are also expected to produce intermediate asymptotics results, *i.e.*, to describe solutions which are locally vanishing for large times in the absence of a confining potential strong enough to guarantee the existence of stationary solutions.

2 Hypocoercivity for kinetic models with several conservation laws

This continues an existing cooperation involving J. Dolbeault (Paris), F. Hérau (Nantes), C. Mouhot (Cambridge) and C. Schmeiser (Vienna).

2.1 Hypocoercivity for kinetic models with mass, momentum and energy conservation (Achleitner, Dolbeault, Hérau, Schmeiser)

The abstract approach of [DMS15] has so far only been applied to kinetic models with one conservation law, the conservation of mass. However, the abstract algebraic structure permits more general cases, satisfying the algebraic condition of a *diffusive macroscopic limit.* This is typically satisfied when the null space of the collision operator consists of velocity distributions with vanishing mean velocity, and therefore also for models with mass and energy conservation, such as the BGK model relaxing to a Maxwellian with vanishing mean velocity, or a kinetic Fokker-Planck equation with the temperature self-consistently computed from the distribution function.

The macro–micro decomposition combined with Kawashima's argument on dissipation of the hyperbolic–parabolic system has been considered in [Dua11]. In [DHMS17a], we expect to prove hypocoercivity estimates for linear kinetic models with non-trivial collision operators, that is, dealing with the physical conservation laws of mass, momentum and energy, and a non-trivial transport operator on the whole space which includes a confining potential term. Based on the idea of auxiliary operators of [HTT06], the main idea is to introduce operators corresponds to well-chosen macroscopic moments of the transport operator. This raises a question concerning Korn-Poincaré type estimates in presence of a confining potential which is is invariant, *e.g.*, under rotational symmetry (*cf.* [DHMS17b]). The issue of getting best possible estimates of the constants in the inequality is crucial for obtaining accurate rates of convergence.

Like [DMS15], [AAC16] only considered BGK models with conserved mass, and partly also with conserved energy. But this approach to construct modified entropy functionals, did not apply to BGK equations that also conserve momentum. This important structural restriction has been overcome [AAC17].

2.2 Hypocoercivity for the linearized Boltzmann equation (Achleitner, Dolbeault, Hérau, Schmeiser)

The algebraic structure of the approach of [DMS15] does not apply to the linearized Boltzmann equation, mainly because of momentum conservation. An algebraic approach based on [HTT06], inspired by the commutator calculus of Hörmander's theory for hypoelliptic operators, should provide an alternative method. Compared to [Dua11], the goal is a result without smoothness assumptions on the initial data or, in other words, a result of hypocoercivity without hypoellipticity.

3 Hypocoercivity for nonlinear mean field kinetic models

A long term project is to reveal connections between the various approaches for proving hypocoercivity, to combine their strengths in an *optimal* approach, and to apply these ideas to new model classes. As a preparatory step, some of the researchers involved in the current project have written the review [AS17]. Nonlinear mean field models are one of the desired new model classes.

3.1 Hypocoercivity for Vlasov-Poisson models with collisions (Dolbeault, Hérau, Li, Neumann)

We propose to consider (nondimensionalized) systems of the form

$$\partial_t f + v \cdot \nabla_x f + \nabla_x (V + V_{conf}) \cdot \nabla_v f = \mathsf{L} f \,, \tag{1}$$

$$\Delta_x V = \rho_f \,, \qquad \rho_f = \int_{\mathbb{R}^d} f \, dv \,, \tag{2}$$

where $f(x, v, t) \geq 0$ is the phase-space distribution of a gas of charged particles with position and velocity $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ and time $t \in \mathbb{R}$. The left hand side of (1) describes the Newtonian dynamics of the particles, accelerated by the electric field $\nabla_x (V + V_{conf})(x, t)$, split into a self-consistent part, determined from the Poisson equation (2), and a prescribed confining part, which can be interpreted as created by a background charge density $\Delta_x V_{conf}$. The gas is assumed to interact with an environment at rest and kept at a fixed temperature. The interaction is described by the linear collision operator L, assumed to act only in the velocity direction, to conserve charge, and to promote decay to a Maxwellian velocity distribution with the given environment temperature and vanishing mean velocity. A standard example of (1), (2) is the Vlasov-Poisson-Fokker-Planck (VPFP) system, where L is an elliptic operator in v.

Without the confinement, in dimension $d = 3$, dispersion dominates, such that the long-time asymptotics is described by the kinetic Fokker-Planck equation without force field [CSV96], with algebraic decay of the solution. Confinement [BD95, BAD03] or restriction to a bounded spatial domain [BAD03] may lead to nonvanishing equilibria, and exponential decay can be expected [BAD03].

We propose to adapt the ideas of [DMS15], modifying the relative entropy [BAD03], in order to prove exponential decay to equilibrium without smoothness assumptions on the initial data, improving the results for VPFP of [BAD03]. In particular, we intend to permit not only the Fokker-Planck collision operator, but also models not creating smoothness, such as relaxation. Both the whole space problem with confinement and the bounded domain case without external force will be considered.

If convergence for large times of the solutions of Vlasov-Poisson-Fokker-Planck is well understood in terms of *free energy* functionals for already many years (see for instance [Dol99, BD95, CSV96, BAD03]), the question of the rates is still partially open. Some first results, in case of small masses, have been obtained in [HJ13, HT16], and also in [DBGV16] for a mean field interaction which goes through a wave equation, but degenerates into a (gravitational) Poisson term in a limiting regime.

3.2 A larger class of models with Poisson coupling (Dolbeault, Hérau, Karaki, Li, Neumann)

- *Strong magnetic field and collisional kinetic equations.* When dealing with plasmas, it is crucial to understand the role of magnetic fields. Zeinab Karaki is at the moment working on the topic and will be involved in the proposed collaboration.
- *Models with particles of opposite signs.* The key idea is to use a well adapted norm at diffusive level inspired by the results in [CD12, CD14]. When a system of charged particles satisfies a global electroneutrality condition, faster rates of relaxation are expected.

3.3 Towards improved quantitative results (Achleitner, Dolbeault, Li, Schmeiser)

For applications, a crucial issue is to characterize decay rates as accurately as possible. This is for instance of outstanding importance in semi-conductor physics as delays for entering in stationary regimes are the main limitation for the clock frequency of microprocessors. On a completely different time scale, typical times of relaxation are a fundamental issue in astrophysics and the theory of gravitation.

- *Optimal decay rates for weakly nonlinear models.* In [AE14], hypocoercivity for a class of degenerate Fokker-Planck equations (including the kinetic Fokker-Planck equation) has been proven by a modification of the Bakry-Emery approach [BE85]. Actually the method of [AE14, AAS15] has the potential to provide exponential decay with sharp rates. We intend to adapt this approach to other situations, including the VPFP model, with the hope to improve on the decay rates of [BAD03].
- *Optimal asymptotic rates in models with mean field potentials.* Based on the method used in [CD12, CD14], Xingyu Li has obtain some preliminary results in [Li17] on the characterization of optimal asymptotic rates of convergence of solutions to Nernst-Planck and Debye-Hückel drift-diffusion systems. The method applies to the confined case in any dimension, and also to intermediate asymptotics in dimension two. The next step is again some preliminary work to better understand the various norms that can be build for controlling the rates of convergence, and how to use these norms in presence of a nonlocal drift term. The main application in kinetic theory would be in semi-conductor physics, but applications to gravitational models in astrophysics, which were part of the motivation in [DMOS07, CD14], could be considered in a third step.

4 Hypocoercivity for other models

4.1 Reaction-kinetic models (Bouin, Neumann, Schmeiser)

The classical models for reaction networks coupled with spatial transport are of reaction-diffusion type. The introduction of entropy/entropy-dissipation methods motivated by kinetic theory to the field of reaction-diffusion systems has led to a remarkable development in this field. This includes both the existence theory and the analysis of the long-time behavior. It seems natural to try the extension of some of these results to models with kinetic transport instead of diffusion. This subject offers a large variety of challenges:

- Kinetic models for first order chemical reaction networks: We propose to consider linear systems of kinetic equations for gas mixtures with scattering by an immobile, constant temperature background medium and first order reactions. This is a kinetic version of the reaction-diffusion systems considered in [FPT17], where the reaction network has been assumed to permit so called *complex balance equilibria* based on the assumption of a weakly reversible reaction network, and exponential decay to equilibrium has been shown by entropy/entropy dissipation. We propose to apply the approach of [DMS15] to this situation. Possible extensions are to systems, where some of the chemical species are immobile. This corresponds to the results for degenerate diffusion in [FPT17].
- Nonlinear kinetic-reaction models with large data: We propose to extend the decay results of [NS16] for a system of two kinetic models coupled by a simple quadratic chemical reaction term to far-from-equilibrium initial data. L^2-based theories have to be replaced by the construction of modified Lyapunov functionals starting from the logarithmic entropy. In a first step, initial data bounded in terms of the equilibrium distributions will be assumed, such that at least no new existence theory will be needed.
- Traveling waves in reaction-kinetic models: Recent results on invasion waves for a kinetic version of the KPP-Fisher model (see [BCN15, CHS12]) will be extended to reaction kinetic models with cubic nonlinearities.

4.2 Hypocoercivity for kinetic models involving fractional diffusion (Achleitner, Bouin, Dolbeault, Kanzler, Schmeiser)

Fractional diffusion can arise in various ways in the context of kinetic models. The most obvious are kinetic Fokker-Planck equations with fractional diffusion in the velocity direction. First studies about the return to the equilibrium, concerning also the non-cutoff Boltzmann case, are currently undertaken by Hérau et al. in [HTT17a, HTT17b]. However, fractional diffusion might also arise just in macroscopic limits as a consequence of fat-tailed velocity equilibrium distributions and/or scattering rates decaying for large velocities. Building on our expertise in the study of fractional diffusion (starting with [AHS11]), we propose to consider the following questions:

- Hypocoercivity for fat-tailed equilibrium distributions: The idea is to extend the approach of [DMS15] to relaxation kinetic models, where the equilibrium distribution does not possess finite second order moments. This will require new ideas inspired by the fractional diffusion macroscopic limits of these models [ASS16]. Situations will be studied, where exponential decay can be expected, such as problems on bounded position domains or whole space problems involving a confining potential.
- Decay to equilibrium for degenerate scattering [MMM11]: Whereas in the previous setting the difficulty originated from the macroscopic dynamics, the above mentioned models with decaying scattering rates lack the so-called microscopic coercivity property. As a consequence, exponential decay to equilibrium might not hold, and one has to settle for weaker decay rates.

4.3 The whole space case without confinement: intermediate asymptotics (Bouin, Dolbeault, Schmeiser)

Hypocoercivity methods can be extended to linear kinetic equations with mass conservation and without confinement, such that the long-time behavior has algebraic decay as in the case of the heat equation. In [BDM+17], two alternative approaches will be developed: an analysis based on decoupled Fourier modes and a direct approach where, instead of the Poincaré inequality for the Dirichlet form, the Nash inequality is employed. The former can also be used to provide a proof of exponential decay to equilibrium on the flat torus. Finally, the results can be extended to larger function spaces by a factorization method, using the technique introduced in [GMM]. These results will be extended to kinetic models with fat-tailed equilibrium distributions and/or degenerate scattering rates. In all these cases the decaying solutions are expected to share their asymptotic profile. This will be examined by intermediate asymptotics, which is a well established approach for the corresponding macroscopic models.

Methodology and cooperation perspectives

Hypocoercivity methods applied to kinetic models have started for about 10 years, with several variants. The first year of the project will be devoted to collect and organize existing tools, to share methods, and to develop new tools, like Korn-Poincaré inequalities, weighted Nash and Poincaré inequalities adapted to measures with fat tails, or spectral methods. The second year will be more specifically devoted to the mean field models, reaction-kinetic equations and equations with collisions involving fractional diffusion.

E. Bouin and J. Dolbeault are part of an ANR project on *Entropy, flows, inequalities* which aims at studying large time behaviour and discrete approximations of diffusive and hypocoercive systems in a more probabilistic approach, within a partnership involving french probabilists F. Bolley (LPMA, Paris 6), P. Cattiaux (IMT, Toulouse 3), I. Gentil (ICJ, Université Lyon 1) and A. Guillin (LM, Université de Clermont Auvergne). J. Dolbeault is the french team leader of the *Diffusion* Cost project, also under evaluation.

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