Trend to equilibrium and particle approximation for a Vlasov-Fokker-Planck equation

— o —

François Bolley (Université Paris-Dauphine)

— o —

Joint work with A. Guillin and F. Malrieu

I - The equation

II - Trend to equilibrium

III - Particle approximation

I - The equation

$$\frac{\partial f_t}{\partial t} + v \cdot \nabla_x f_t - F[f_t] \cdot \nabla_v f_t = \Delta_v f_t + \nabla_v \cdot (f_t (A(v) + B(x))), \qquad t > 0, x, v \in \mathbb{R}^d \quad (VFP)$$

where

$$F[f_t](x) = C *_x f_t(x) = \int_{\mathbb{R}^d} C(x-y)\, f_t(y,w)\, dy\, dw$$

Interpretation 1 : if f_0 is a density on $\mathbb{R}^{2d}$ and if $(x(0), v(0)) \sim f_0$, let

$$\begin{cases} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= -F[f_t](x(t)) + \sqrt{2}\, \frac{dB(t)}{dt} - A(v(t)) - B(x(t)). \end{cases}$$

where $(x(t), v(t)) \sim f_t$. Then f_t evolves according to (VFP).
Interaction force on $x(t)$ is

$$F[f_t](x(t)) = \int_{\mathbb{R}^d} C(x(t) - y)\, f_t(y,w)\, dy\, dw$$

Interpretation 2 :
(VFP) can be seen as the limit behaviour of the N particle system

$$\begin{cases} \dfrac{dx_i(t)}{dt} &= v_i(t) \\ \dfrac{dv_i(t)}{dt} &= -\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} C(x_i(t)-x_j(t)) + \sqrt{2}\,\dfrac{dB_i(t)}{dt} - A(v_i(t)) - B(x_i(t)) \end{cases}$$

for $1 \leq i \leq N$ where $\frac{1}{N}\sum_{j=1}^{N} C(x_i(t)-x_j(t))$ is the drift term generated no more by the distribution f_t in the kinetic level, but by the ”particle level distribution”

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{(x_i(t),v_i(t))}$$

called empirical measure.

II - Trend to equilibrium

1.1. Recall the basics : the Fokker-Planck equation

$$\frac{\partial f_t}{\partial t} = \Delta f_t + \nabla \cdot (f_t(\nabla V(v))), \qquad t > 0, v \in \mathbb{R}^d$$

Interpretation : f_t is the law of $v(t)$ where $dv(t) = \sqrt{2}dB_t - \nabla V(v(t))\, dt$
Long time behaviour for V convex : $f_t \to e^{-V}$
3 ways of quantifying the convergence :

1. L^2 argument / Poincaré inequality : estimate $\int \left|\frac{f_t}{e^{-V}} - 1\right|^2 e^{-V}\, dv$

Poincaré inequality for e^{-V} implies L^2 exp convergence :

$$\int \left|\frac{f_t}{e^{-V}} - 1\right|^2 e^{-V}\, dv \leq e^{-2\lambda t} \int \left|\frac{f_0}{e^{-V}} - 1\right|^2 e^{-V}\, dv$$

2. $L \ln L$ argument / log Sobolev inequality ([Bakry-Emery]) : estimate

$$Ent(f_t|e^{-V}) = \int \frac{f_t}{e^{-V}} \ln \frac{f_t}{e^{-V}}\, e^{-V} = \int f_t \ln f_t + \int fV$$

Log Sobolev inequality for e^{-V} implies entropic convergence whence L^1 convergence :

$$Ent(f_t|e^{-V}) \leq e^{-2\lambda t}\, Ent(f_0|e^{-V})$$

3. in Wasserstein distance : estimate $W_2(f_t, e^{-V})$
* length space / gradient flow argument
[Carrillo-McCann-Villani 04, Ambrosio-Gigli-Savaré 06]
* by hand
Stronger assumption on V : Hess $V(v) \geq \lambda I$, but extends to forces $a(v)$ which are not gradients, for which the steady state is not known and there is no Liapunov functional.

1.2. Extensions

1. L^2 argument extends to inhomogeneous situations by hypocoercivity techniques [Villani 09], [Dolbeault-Mouhot-Schmeiser 09]
2. $L \ln L$ argument extends to
* nonlinear equations such as granular media equations : [Carrillo-McCann-Villani 03]
* linear VFP equation : [Desvillettes-Villani 01], [Villani 09]
* selfconsistent VFP in the torus for small potentials : [Villani 09]
3. Wasserstein distance argument extends to
* granular media equation : [Carrillo-McCann-Villani 04] or by hand
* selfconsistent VFP : now

1.3. Trend to equilibrium for VFP

$$\frac{\partial f_t}{\partial t} + v \cdot \nabla_x f_t - F[f_t] \cdot \nabla_v f_t = \Delta_v f_t + \nabla_v \cdot (f_t(A(v) + B(x))), \qquad t > 0, x, v \in \mathbb{R}^d$$

where

$$F[f_t](x) = C *_x f_t(x) = \int_{\mathbb{R}^d} C(x-y)\, f_t(y,w)\, dy\, dw$$

Theorem For linear like $A(v)$ and $B(x)$, small F, there exists λ such that

$$W_2(f_t, g_t) \leq e^{-\lambda t} W_2(f_0, g_0)$$

for all solutions f_t and g_t with finite second moment in x, v.
In particular there exists a unique steady state f_∞ and all solutions converge exp. fast to it.

Remark Here W_2 is not defined by the usual cost $|x-y|^2 + |v-w|^2$ but by the twisted cost

$$a\,|x-y|^2 + (x-y)\cdot(v-w) + |v-w|^2$$

with a, b depending on A, B and C. For the usual distance we obtain

$$W_2(f_t, g_t) \leq C e^{-\lambda t} W_2(f_0, g_0)$$

Remark

This result gives convergence in a weak sense, not in L^1 norm as in [Villani 09], but

* holds in a noncompact case (not the torus)
* shows existence and uniqueness of stady state, does not use it
* is also a stability result for all solutions
* is really simple

III - The particle system

$$\begin{cases} \dfrac{dx_i(t)}{dt} & = & v_i(t) \\ \dfrac{dv_i(t)}{dt} & = & -\dfrac{1}{N}\displaystyle\sum_{j=1}^{N} C(x_i(t)-x_j(t)) + \sqrt{2}\,\dfrac{dB_i(t)}{dt} - A(v_i(t)) - B(x_i(t)) \end{cases}$$

for $1 \leq i \leq N$, with initially $(x_i(0), v_i(0)) \sim f_0$, for instance independent

For all $t > 0$ the particles are correlated, but for N large :

1. they get independent, in particular $law(2\, particles) \sim law(1\, particle)^{\otimes 2}$
2. by symmetry they all have the same law (true for all N)
3. $law(1\, particle) \sim f_t$
4. the law of one particle is the expectation of the empirical measure
5. the empirical measure gets deterministic
6. the empirical measure is like f_t

Issue : quantify this "propagation of chaos" property

1. Particle point of view

For $1 \leq i \leq N$ the particle $(x_i(t), v_i(t))$ will look like the particle $(\bar{x}_i(t), \bar{v}_i(t))$ evolving according to

$$\begin{cases} \dfrac{d\bar{x}_i(t)}{dt} &= \bar{v}_i(t) \\ \dfrac{d\bar{v}_i(t)}{dt} &= -F[f_t](\bar{x}_i(t)) + \sqrt{2}\,\dfrac{dB_i(t)}{dt} - A(\bar{v}_i(t)) - B(\bar{x}_i(t)) \end{cases}$$

with $\bar{x}_i(0) = x_i(0), \bar{v}_i(0) = v_i(0)$. These are N independent processes, all with law f_t at time t.

Quantified as

$$\sup_{t \geq 0} \mathbb{E}\Big[|x_i(t) - \bar{x}_i(t)|^2 + |x_i(t) - \bar{x}_i(t)|^2\Big] \leq \frac{C}{N}$$

Non uniform in time estimates : classical for diffusive particle systems with Lipschitz coefficients : McKean, Tanaka, Sznitman

Uniform in time estimates : [Malrieu 01] for granular media equations

2. PDE point of view

Goal : simulate the particle system to approximation of the solution f_t of (VFP)

Look for explicit error bounds on the approximation of f_t by the empirical measure : if h is a smooth observable, bound

$$\mathbb{P}\Big[\Big|\frac{1}{N}\sum_{i=1}^{N} h(x_i(t), v_i(t)) - \int_{\mathbb{R}^{2d}} h\, f_t\, dx\, dv\Big| > \varepsilon\Big]$$

in terms of h, N, t, ε and then

$$\mathbb{P}\Big[\Big|\frac{1}{N}\sum_{i=1}^{N} h(x_i(t), v_i(t)) - \int_{\mathbb{R}^{2d}} h\, f_\infty\, dx\, dv\Big| > \varepsilon\Big]$$

in terms of h, N and ε, for the limit profile f_∞.