Well-posedness in a space of measures of kinetic models for collective motion

José Alfredo Cañizo

Universitat Autònoma de Barcelona

October 7, 2009

Main aim

1. Give a well-posedness theory for a variety of models of collective behavior (“swarming”).
2. Give a general theory for the particle approximation / mean-field limit of these models.

Main aim

1. Give a well-posedness theory for a variety of models of collective behavior ("swarming").
2. Give a general theory for the particle approximation / mean-field limit of these models.

What kind of models do we want to include?

1. Our main motivation for this theory comes from models for collective behavior of animals: flocks, swarms... We are interested in behavior that is the collective result of their individual behavior (not, for example, due to them following a “leader”.)
2. Models coming from physics: self-consistent interaction through a potential, friction, transport.

What kind of models do we want to include?

1. Our main motivation for this theory comes from models for collective behavior of animals: flocks, swarms... We are interested in behavior that is the collective result of their individual behavior (not, for example, due to them following a “leader”.)
2. Models coming from physics: self-consistent interaction through a potential, friction, transport.

ODEs & Kinetic models

$$f = f(t,x,v), \qquad \rho = \rho(t,x) = \int f(t,x,v)\,dv.$$

ODEs

$$\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= -\tfrac{1}{N}\textstyle\sum_{j\neq i} \nabla U(x_i - x_j) \\
&\quad + (\alpha - \beta\,|v_i|^2)v_i \\
&\quad + \textstyle\sum_{j\neq i} a_{ij}\,(v_j - v_i)
\end{aligned}$$

Kinetic equation

$$\begin{aligned}
0 = \partial_t f &+ v\nabla_x f \\
&- \mathrm{div}_v\,((\nabla_x U * \rho)f) \\
&+ \mathrm{div}_v((\alpha - \beta\,|v|^2)v\,f) \\
&- \mathrm{div}_v\,((H * f)f)
\end{aligned}$$

$$a_{ij} := \frac{1}{(1 + |x_i - x_j|^2)^\gamma}$$

$$H(x,v) = \frac{v}{(1 + |x|^2)^\gamma}$$

ODEs & Kinetic models

$$f = f(t,x,v), \qquad \rho = \rho(t,x) = \int f(t,x,v)\,dv.$$

ODEs

$$\begin{aligned}
\dot{x}_i &= v_i \\
\dot{v}_i &= -\frac{1}{N}\sum_{j\neq i} \nabla U(x_i - x_j) \\
&\quad + (\alpha - \beta\,|v_i|^2)v_i \\
&\quad + \sum_{j\neq i} a_{ij}\,(v_j - v_i)
\end{aligned}$$

Kinetic equation

$$\begin{aligned}
0 &= \partial_t f + v\nabla_x f \\
&\quad - \mathrm{div}_v\left((\nabla_x U * \rho) f\right) \\
&\quad + \mathrm{div}_v((\alpha - \beta\,|v|^2) v\, f) \\
&\quad - \mathrm{div}_v\left((H * f) f\right)
\end{aligned}$$

$$a_{ij} := \frac{1}{(1 + |x_i - x_j|^2)^\gamma}$$

$$H(x,v) = \frac{v}{(1 + |x|^2)^\gamma}$$

ODEs & Kinetic models

$$f = f(t,x,v), \qquad \rho = \rho(t,x) = \int f(t,x,v)\,dv.$$

ODEs

$$\begin{aligned}\dot{x}_i &= v_i\\ \dot{v}_i &= -\tfrac{1}{N}\textstyle\sum_{j\neq i}\nabla U(x_i - x_j)\\ &\quad + (\alpha - \beta\,|v_i|^2)v_i\\ &\quad + \textstyle\sum_{j\neq i} a_{ij}\,(v_j - v_i)\end{aligned}$$

Kinetic equation

$$\begin{aligned}0 &= \partial_t f + v\nabla_x f\\ &\quad - \mathrm{div}_v\,((\nabla_x U * \rho)f)\\ &\quad + \mathrm{div}_v((\alpha - \beta\,|v|^2)v\,f)\\ &\quad - \mathrm{div}_v\,((H * f)f)\end{aligned}$$

$$a_{ij} := \frac{1}{(1 + |x_i - x_j|^2)^\gamma} \qquad H(x,v) = \frac{v}{(1 + |x|^2)^\gamma}$$

Some references

- Cucker & Smale (IEEE Automatic Control 2007): tendency to the average velocity of neighbours.
- Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado: proof of flocking for $\gamma \leq 1/2$.
- Barbaro, Birnir et. al (2008): model with three regions.
- D'Orsogna, Bertozzi et al. (PRL 2006); Chuang, D'Orsogna et al. (Physica D 2007): attraction-repulsion + friction / self-propulsion. Phenomenology depending on the parameters of the equation.
- Braun & Hepp (CMP 1977), Neunzert (Trans. Fluid Dynamics 1977), Spohn (Rev. Modern Phys. 1980), Dobrushin (Funct. Anal. Appl. 1979)

Some references

- Cucker & Smale (IEEE Automatic Control 2007): tendency to the average velocity of neighbours.
- Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado: proof of flocking for $\gamma \leq 1/2$.
- Barbaro, Birnir et. al (2008): model with three regions.
- D'Orsogna, Bertozzi et al. (PRL 2006); Chuang, D'Orsogna et al. (Physica D 2007): attraction-repulsion + friction / self-propulsion. Phenomenology depending on the parameters of the equation.
- Braun & Hepp (CMP 1977), Neunzert (Trans. Fluid Dynamics 1977), Spohn (Rev. Modern Phys. 1980), Dobrushin (Funct. Anal. Appl. 1979)

Some references

- Cucker & Smale (IEEE Automatic Control 2007): tendency to the average velocity of neighbours.
- Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado: proof of flocking for $\gamma \leq 1/2$.
- Barbaro, Birnir et. al (2008): model with three regions.
- D'Orsogna, Bertozzi et al. (PRL 2006); Chuang, D'Orsogna et al. (Physica D 2007): attraction-repulsion + friction / self-propulsion. Phenomenology depending on the parameters of the equation.
- Braun & Hepp (CMP 1977), Neunzert (Trans. Fluid Dynamics 1977), Spohn (Rev. Modern Phys. 1980), Dobrushin (Funct. Anal. Appl. 1979)

Some references

- Cucker & Smale (IEEE Automatic Control 2007): tendency to the average velocity of neighbours.
- Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado: proof of flocking for $\gamma \leq 1/2$.
- Barbaro, Birnir et. al (2008): model with three regions.
- D'Orsogna, Bertozzi et al. (PRL 2006); Chuang, D'Orsogna et al. (Physica D 2007): attraction-repulsion + friction / self-propulsion. Phenomenology depending on the parameters of the equation.
- Braun & Hepp (CMP 1977), Neunzert (Trans. Fluid Dynamics 1977), Spohn (Rev. Modern Phys. 1980), Dobrushin (Funct. Anal. Appl. 1979)

Some references

- Cucker & Smale (IEEE Automatic Control 2007): tendency to the average velocity of neighbours.
- Ha-Tadmor, Ha-Liu, Carrillo-Fornasier-Toscani-Rosado: proof of flocking for $\gamma \leq 1/2$.
- Barbaro, Birnir et. al (2008): model with three regions.
- D'Orsogna, Bertozzi et al. (PRL 2006); Chuang, D'Orsogna et al. (Physica D 2007): attraction-repulsion + friction / self-propulsion. Phenomenology depending on the parameters of the equation.
- Braun & Hepp (CMP 1977), Neunzert (Trans. Fluid Dynamics 1977), Spohn (Rev. Modern Phys. 1980), Dobrushin (Funct. Anal. Appl. 1979)

The W^1 distance between measures

Look at probability measures μ, ν on $\mathbb{R}^d$ as mass distributions. *A plan* to carry one to the other is a measure π on $\mathbb{R}^d \times \mathbb{R}^d$ such that (abusing notation a bit):

$\pi(x, y) \equiv$ how much mass from point x should go to point y.

Then, the W^1 distance between μ and ν is the *total cost* of carrying one to the other, using the best possible plan:

$$W^1(\mu, \nu) = \inf_{\pi} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} \underbrace{|x - y|}_{\text{how far to go}} \qquad \underbrace{\pi(x, y)}_{\text{how much mass to carry}} \right\}$$

Assumptions

We first look at:

$$0 = \partial_t f + v \nabla_x f + \mathrm{div}_v \left((\nabla_x U * \rho) f \right) + \mathrm{div}_v((\alpha - \beta |v|^2) v f) - \mathrm{div}_v \left((H * f) f \right)$$

Assume ∇U and H...

- ... are locally Lipschitz,
- and have at most linear growth.

Existence & uniqueness

Theorem (Existence)

1. *For any compactly supported measure* $f_0 \in W^1(\mathbb{R}^{2d})$ *there exists a solution*

$$f \in \mathcal{C}([0,+\infty), W^1(\mathbb{R}^{2d}))$$

with initial condition f_0.

2. *For all* $t \geq 0$, f_t *has compact support.*

Continuous dependence & uniqueness

Theorem (Continuous dependence)

For any two solutions f, g, there is $r = r(t)$ such that

$$W^1(f_t, g_t) \leq r(t) W^1(f_0, g_0).$$

$r(t)$ depends on:

- the details of the equation (i.e. α, β, U and H),
- and the size of the support of f and g at $t = 0$.

Proof of continuous dependence

Take a solution f.

Characteristic equations

$$\dot{X} = V,$$
$$\dot{V} = (\nabla U * \rho)(t, X) + V(\alpha - \beta\,|V|^2) + (H * f)f.$$

We define:

$$P_f^t \equiv \text{ flow of characteristics at time } t.$$

Then,

$$f_t = P_f^t \# f_0$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\mathrm{supp}\, f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + \\
&+ C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t\, W_1(f_0, g_0) \\
&\leq C_4(\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)}\, W_1(f_s, g_s)\, ds \\
&+ e^{C_1 t}\, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\mathrm{supp}\, f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)}\, ds + \\
&\quad + C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)}\, ds + L_t\, W_1(f_0, g_0) \\
&\leq C_4(\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)}\, W_1(f_s, g_s)\, ds \\
&\quad + e^{C_1 t}\, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\mathrm{supp}\, f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} ds + \\
&+ C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)} ds + L_t\, W_1(f_0, g_0) \\
&\leq C_4(\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s)\, ds \\
&+ e^{C_1 t}\, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\text{supp } f_0)} + L_t \, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \, \| E[f_s] - E[g_s] \|_{L^\infty(B_R)} \, ds + \\
&\quad + C_3 \int_0^t e^{C_3(t-s)} \, \| H[f_s] - H[g_s] \|_{L^\infty(B_R)} \, ds + L_t \, W_1(f_0, g_0) \\
&\leq C_4(\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s) \, ds \\
&\quad + e^{C_1 t} \, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\mathrm{supp}\, f_0)} + L_t \, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + \\
&+ C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t \, W_1(f_0, g_0) \\
&\leq C_4 (\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s) \, ds \\
&+ e^{C_1 t} \, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\leq W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\leq \left\| P_f^t - P_g^t \right\|_{L^\infty(\operatorname{supp} f_0)} + L_t\, W_1(f_0, g_0) \\
&\leq C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + \\
&+ C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t\, W_1(f_0, g_0) \\
&\leq C_4 (\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s)\, ds \\
&+ e^{C_1 t}\, W_1(f_0, g_0).
\end{aligned}
$$

Proof (II)

$$
\begin{aligned}
W_1(f_t, g_t) &= W_1(P_f^t \# f_0, P_g^t \# g_0) \\
&\le W_1(P_f^t \# f_0, P_g^t \# f_0) + W_1(P_g^t \# f_0, P_g^t \# g_0) \\
&\le \left\| P_f^t - P_g^t \right\|_{L^\infty(\mathrm{supp}\, f_0)} + L_t\, W_1(f_0, g_0) \\
&\le C_2 \int_0^t e^{C_2(t-s)} \left\| E[f_s] - E[g_s] \right\|_{L^\infty(B_R)} \, ds + \\
&+ C_3 \int_0^t e^{C_3(t-s)} \left\| H[f_s] - H[g_s] \right\|_{L^\infty(B_R)} \, ds + L_t\, W_1(f_0, g_0) \\
&\le C_4(\mathrm{Lip}_{2R}(\nabla U) + \mathrm{Lip}_{2R}(H)) \int_0^t e^{C_4(t-s)} W_1(f_s, g_s)\, ds \\
&+ e^{C_1 t}\, W_1(f_0, g_0).
\end{aligned}
$$

Convergence of the particle method

Consider $\{(x_i(t), v_i(t)\}$ solution of the ODEs with N particles, f solution of the PDE. We define:

$$f_t^N := \sum_{i=1}^{N} m_i \, \delta_{(x_i(t), v_i(t))}.$$

Theorem (Mean-field limit)

If $\lim_{N\to\infty} W_1(f_0^N, f_0) = 0,$

then $\lim_{N\to\infty} W_1(f_t^N, f_t) = 0, \qquad t \geq 0.$

Hydrodynamic solutions

Assume *hydrodynamic solutions* of the kinetic eq. with potential + self-propulsion exist:

$$f(t,x,v) = \rho(t,x)\,\delta(v - u(t,x)).$$

Then, ρ and u should satisfy:

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$
$$\partial_t u + (u \cdot \nabla) u = u(\alpha - \beta\,|u|^2) - \nabla U * \rho.$$

Theorem (Hydrodynamic solutions)

1. *Hydrodynamic solutions are unique.*
2. *Solutions with near-monokinetic initial conditions are close to solutions with monokinetic initial conditions.*

Things to be done

- Extend mean-field limit results to models with noise.
- *Qualitative behavior* of these models:
  - Equilibria of the potential interaction?
  - Stability of these equilibria (of “flocks”)?

Things to be done

- Extend mean-field limit results to models with noise.
- *Qualitative behavior* of these models:
  - Equilibria of the potential interaction?
  - Stability of these equilibria (of “flocks”)?

Things to be done

- Extend mean-field limit results to models with noise.
- *Qualitative behavior* of these models:
  - Equilibria of the potential interaction?
  - Stability of these equilibria (of “flocks”)?

A general model

We consider

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot (\mathcal{H}[f] f) = 0.$$

and assume

- $\mathcal{H}$ takes a function $f \in \mathcal{C}([0,T], W^1_c(\mathbb{R}^{2d})$ to a function in $\mathcal{C}([0,T], \mathrm{Lip}_{loc}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d))$.
- It holds:

$$\max_{t\in[0,T]} \|\mathcal{H}[f] - \mathcal{H}[g]\|_{L^\infty(B_R)} \leq C\, \mathcal{W}_1(f,g),$$

$$\max_{t\in[0,T]} \mathrm{Lip}_R(\mathcal{H}[f]) \leq C.$$

The end

Thanks!