Stability for a nonlinear Fokker-Planck equation with density-dependent diffusion

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Concentration en vitesse et en espace dans les modèles cinétiques et diffusifs (chemotaxis, gravitation, swarming)

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Outline

- **Background and motivations**
- **Main results**
  - Stability of solutions near equilibrium
  - Rate of convergence
- **Key ideas in the proof**
  - Free energy functional
  - Hyperbolic-parabolic argument
  - Energy-spectrum method

A flocking model by Cucker-Smale

Motion of m particles (e.g. birds) with $(x_i, \xi_i) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$\begin{cases} dx_i = \xi_i dt, \\ d\xi_i = \sum_{j=1}^{m} U(|x_j - x_i|)(\xi_j - \xi_i) dt. \end{cases}$$

Proposition (Cucker-Smale 07)

Let

$$U(x) = \frac{C_{n,\gamma}}{(1+|x|^2)^\beta}, \quad x \in \mathbb{R}^n.$$

- $0 \leq \beta < \frac{1}{2}$:

$$\xi_i(t) \to \frac{1}{m} \sum_{j=1}^{m} \xi_j(0), \quad 1 \leq i \leq m. \qquad (*)$$

- $\beta \geq \frac{1}{2}$: $(*)$ *holds conditionally.*

Related work

- **Ha-Tadmor:** From particle to kinetic and hydrodynamic descriptions of flocking.
- **Ha-Liu:** A simple proof of the Cucker-Smale flocking dynamics and mean-field limit.
- **Carrillo-Fornasier-Rosado-Toscani:** Asymptotic flocking dynamics for the kinetic Cucker-Smale model.

Our work will be more related to the study of some nonlinear Fokker-Planck equations.

- **Villani:** Hypocoercivity.
- **Dolbeault-Mouhot-Schmeiser:** Hypocoercivity for kinetic equations with linear relaxation terms.
- **Guo:** The Landau equation in a periodic box.

Cucker-Smale model with non-uniform noise

We consider: $x = (x_1, \cdots, x_m), \xi = (\xi_1, \cdots, \xi_m)$**,**

$$\begin{cases} dx_i = \xi_i dt, \\ d\xi_i = \sum_{j=1}^{m} U(|x_j - x_i|)(\xi_j - \xi_i)dt + \sqrt{2\mu \sum_{j=1}^{m} U(|x_j - x_i|)} dW_i. \end{cases}$$

Here, for the i^{th}**-agent,**

$$\text{strength of noise} \propto d_i(x) =: \sum_{j=1}^{m} U(|x_j - x_i|).$$

So, randomness increases as soon as particles are closer to each other.

Remark

Rewrite

$$d\xi_i = \underbrace{d_i(x)\left[\sum_{j=1}^{m} w_{ij}(x)\xi_j - \xi_i\right] dt}_{\text{alignment}} + \sqrt{2\mu d_i(x)}\,\underbrace{dW_i}_{\text{random}}\,,$$

where

$$\sum_{j=1}^{m} w_{ij}(x) = 1, \quad 1 \le i \le m, \quad \forall x.$$

So, in the model we proposed, the non-uniform noise is such that

$$\text{strength of alignment} \propto \text{strength of noise},$$

and both strengths $\propto d_i(x)$ **for each** i**.**

! Stabilize the existing equilibrium states !

Mean-field limit

Scale

$$U = \frac{\kappa}{m} U_0.$$

Set

$$f^{(m)}(t,x,\xi) = \frac{1}{m}\sum_{i=1}^{m} \delta(x - x_i(t))\delta(\xi - \xi_i(t)),$$

and assume: $\exists\, f(t) \in \mathcal{M}(\mathbb{R}^{2n})$, **s.t.**

$$f^{(m)} \to f(t) \quad \text{in } w^*\text{-}\mathcal{M}(\mathbb{R}^{2n}) \quad \text{as } m \to \infty.$$

Then,

$$\partial_t f + \xi \cdot \nabla_x f + \kappa U_0 * \rho_{\xi f} \cdot \nabla_\xi f = \kappa U_0 * \rho_f \nabla_\xi \cdot (\mu \nabla_\xi f + \xi f),$$

with

$$\rho_f(t,x) = \int_{\mathbb{R}^n} f(t,x,\xi)\, d\xi, \quad \rho_{\xi f}(t,x) = \int_{\mathbb{R}^n} \xi f(t,x,\xi)\, d\xi.$$

Nonlinear Fokker-Planck equation and equilibriums

We consider the Cauchy problem

$$\partial_t f + \xi \cdot \nabla_x f + U * \rho_{\xi f} \cdot \nabla_\xi f = U * \rho_f \nabla_\xi \cdot (\nabla_\xi f + \xi f),$$
$$f(0, x, \xi) = f_0(x, \xi).$$

Assume that U is continuous in x with

$$U(x) = U(|x|) \geq 0, \quad \int_{\mathbb{R}^n} U(x) dx = 1.$$

The existing equilibrium state is a global Maxwellian (after normalization):

$$\mathbf{M} = \mathbf{M}(\xi) = \frac{1}{(2\pi)^{n/2}} \exp\left(-|\xi|^2/2\right).$$

Problem:

! well-posedness and large-time behavior of solutions !

Try to find a Lyapunov functional

One has

$$\frac{d}{dt}E(f) = -D(f),$$

where

$$E(f) = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\left[\frac{|\xi|^2}{2} + \log f\right] f\,dx\,d\xi,$$

$$D(f) = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{U*\rho_f}{f}|\nabla_\xi f + \xi f|^2\,dx\,d\xi - \int_{\mathbb{R}^n} U*\rho_{\xi f}\cdot\rho_{\xi f}\,dx.$$

Difficulty: it is presently unknown if $D(f)$ is non-negative !

Remark: If $U=\delta_0$, then $D(f)=D_0(f)$ with

$$D_0(f) = \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho_f}{f}|\nabla_\xi f + \xi f|^2\,dx\,d\xi - \int_{\mathbb{R}^n}|\rho_{\xi f}|^2\,dx$$

$$= \iint_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho_f}{f}\Big|\nabla_\xi f + \Big(\xi - \frac{\rho_{\xi f}}{\rho_f}\Big)f\Big|^2\,dx\,d\xi \geq 0.$$

Observation at the linearized level

Notice $D(\mathbf{M}) = 0$, and

$$\frac{d}{d\epsilon}D(\mathbf{M}+\epsilon\phi)|_{\epsilon=0} = 0, \quad \frac{d^2}{d\epsilon^2}D(\mathbf{M}+\epsilon\phi)|_{\epsilon=0} = 2\mathbf{L}(\frac{\phi}{\sqrt{\mathbf{M}}}).$$

where $\mathbf{L}$ is a linear self-adjoint operator.

Problem: Does $\mathbf{L}$ have a sign over some Hilbert space ?
(Yes ! Non positive definite !
The density-dependent diffusion is responsible for this.)

In fact, decompose

$$L^2_\xi = \mathcal{N} \oplus \mathcal{N}^\perp, \quad \mathcal{N} = \mathrm{Span}\{\sqrt{\mathbf{M}}, \xi\sqrt{\mathbf{M}}\},$$

and define $\mathbf{P}$ by

$$\mathbf{P} : L^2_\xi \to \mathcal{N}, \quad u \mapsto \mathbf{P}u \equiv \{a^u + b^u \cdot \xi\}\sqrt{\mathbf{M}},$$
$$a^u = \langle\sqrt{\mathbf{M}}, u\rangle, \quad b^u = \langle\xi\sqrt{\mathbf{M}}, u\rangle.$$

Coercivity of the linearized operator

Theorem (D.-Fornasier-Toscani 09)

$-\mathbf{L}$ *is coercive in the sense that* $\exists\, \lambda > 0$, *s.t.*

$$\int_{\mathbb{R}^n} \langle -\mathbf{L}u, u \rangle dx \geq \lambda \|\{\mathbf{I} - \mathbf{P}\}u\|_\nu^2 + \frac{1}{2}\|T_\Delta b^u\|_U^2,$$

for any $u = u(x, \xi)$, *where*

$$\|u\|_\nu^2 = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |\nabla_\xi u(x,\xi)|^2 + \nu(\xi)|u(x,\xi)|^2 d\xi dx d\xi,$$

$$\nu(\xi) = 1 + |\xi|^2,$$

$$\|T_\Delta b^u\|_U^2 = \iint_{\mathbb{R}^n \times \mathbb{R}^n} U(|x-y|)|b^u(x) - b^u(y)|^2 dx dy,$$

$$T_\Delta b^u(x,y) = b^u(x) - b^u(y).$$

So, it is possible to found a perturbation theory for the nonlinear Cauchy problem !

Remarks

- **Is there any $\lambda > 0$ s.t. for any b,**

$$\|T_\Delta b\|_U^2 \geq \lambda \|b\|^2 \quad ?$$

NO! Actually, one can prove

$$\inf_{b \in L_x^2} \|T_\Delta b\|_U^2 = \inf_{b \in L_x^2} \iint_{\mathbb{R}^n \times \mathbb{R}^n} U(|x-y|)|b(x)-b(y)|^2 dx dy = 0.$$

- **Is there any coercivity estimate on the linearized operator L corresponding to the equation with uniform diffusion**

$$\partial_t f + \xi \cdot \nabla_x f + U * \rho_{\xi f} \cdot \nabla_\xi f = U * \rho_f \nabla_\xi \cdot (\xi f) + \Delta_\xi f \quad ?$$

NO! This need a long calculation.

Reformulation

Recall

$$\partial_t f + \xi \cdot \nabla_x f + U * \rho_{\xi f} \cdot \nabla_\xi f = U * \rho_f \nabla_\xi \cdot (\nabla_\xi f + \xi f),$$
$$f(0, x, \xi) = f_0(x, \xi).$$

Set the perturbation $u = u(t, x, \xi)$ **by**

$$f = \mathbf{M} + \sqrt{\mathbf{M}} u.$$

Then,

$$\partial_t u + \xi \cdot \nabla_x u + U * \rho_{\xi\sqrt{\mathrm{M}}u} \cdot \nabla_\xi u = \mathbf{L} u + \Gamma(u, u),$$

where

$$\mathbf{L} u = \underbrace{\Delta_\xi u + \frac{1}{4}(2n - |\xi|^2) u}_{=:\mathrm{L}_{FP} u} + \underbrace{U * \rho_{\xi\sqrt{\mathrm{M}}u} \cdot \xi \sqrt{\mathbf{M}}}_{=:\mathrm{A} u},$$

$$\Gamma(u, u) = U * \rho_{\sqrt{\mathrm{M}}u} \mathbf{L}_{FP} u + \frac{1}{2} U * \rho_{\xi\sqrt{\mathrm{M}}u} \cdot \xi u.$$

Nonlinear asymptotical stability near equilibrium

Our goal is to prove

- **stability:**

  $\exists\, \delta > 0,\ C > 1$ s.t. if $u_0 = \mathbf{M}^{-1/2}(f_0 - \mathbf{M}) \in B_\delta$, where B_δ is a smooth neighborhood of zero, then

  $$\exists\, !u \in C_b([0,\infty); B_{C\delta})$$

  with $u(0) = u_0$.

  local existence + a priori estimates

- **rate of convergence:**

  How fast for $u(t) \to 0$ in some smooth topology ?

  spectral analysis + energy-spectrum method

Main results: nonlinear case

Theorem (D.-Fornasier-Toscani 09)

Let $n \geq 3, N \geq 2[n/2]+2$. U is supposed as before. Assume

$$f_0 \equiv \mathbf{M} + \sqrt{\mathbf{M}} u_0 \geq 0, \quad \|u_0\|_{H^N_{x,\xi}} \leq \delta$$

for some $\delta > 0$.
(i) $\exists ! u(t,x,\xi) \in C([0,\infty); H^N_{x,\xi})$ with

$$f \equiv \mathbf{M} + \sqrt{\mathbf{M}} u \geq 0, \quad \sup_{t\geq 0} \|u(t)\|_{H^N_{x,\xi}} \leq C \|u_0\|_{H^N_{x,\xi}}.$$

for some $C > 1$.
(ii) If $\|u_0\|_{Z_1}$ with $Z_1 = L^2_\xi(L^1_x)$ is bounded and $\|\xi \nabla_x u_0\|$ is small enough, then

$$\sup_{t\geq 0} (1+t)^{\frac{n}{4}} \|u(t)\|_{H^N_{x,\xi}} \leq C \left(\|u_0\|_{H^N_{x,\xi}} + \|u_0\|_{Z_1} \right),$$

for some C.

Iteration

Define an approximate solution sequence $(f^m)_{m=0}^{\infty}$ by solving Cauchy problems iteratively

$$\begin{cases} \partial_t f^{m+1} + \xi \cdot \nabla_x f^{m+1} + U * \rho_{\xi f^m} \cdot \nabla_\xi f^{m+1} \\ \qquad\qquad\qquad = U * \rho_{f^m} \nabla_\xi \cdot (\nabla_\xi f^{m+1} + \xi f^{m+1}), \\ f^{m+1} \equiv \mathbf{M} + \sqrt{\mathbf{M}} u^{m+1}, \\ f^{m+1}|_{t=0} = f_0 \equiv \mathbf{M} + \sqrt{\mathbf{M}} u_0, \end{cases}$$

or equivalently in terms of $u^m(t, x, \xi)$:

$$\begin{cases} \partial_t u^{m+1} + \xi \cdot \nabla_x u^{m+1} + U * b^{u^m} \cdot \nabla_\xi u^{m+1} \\ \qquad\qquad\qquad = \mathbf{L}_{FP} u^{m+1} + \Gamma(u^m, u^{m+1}) + \mathbf{A} u^m, \\ u^{m+1}|_{t=0} = u_0, \end{cases}$$

where $m \geq 0$, and $u^0 \equiv 0$ is set at initial step.

Local existence

Define

$$X(0,T;M)=\left\{\begin{array}{l} v\in C([0,T];H^N(\mathbb{R}^n\times\mathbb{R}^n)): \\ \sup_{0\le t\le T}\|v(t)\|_{H^N_{x,\xi}}\le M,\quad \mathbf{M}+\sqrt{\mathbf{M}}v\ge 0\end{array}\right\}.$$

Theorem

$\exists\, T_*>0,\ \epsilon_0>0,\ M_0>0$ s.t. if $u_0\in H^N_{x,\xi}$ with

$$f_0\equiv \mathbf{M}+\sqrt{\mathbf{M}}u_0\ge 0,\quad \|u_0\|_{H^N_{x,\xi}}\le\epsilon_0,$$

then, for each $m\ge 1$, u^m is well-defined with $u^m\in X(0,T_*;M_0)$. Furthermore, $(u^m)_{m\ge 0}$ is a Cauchy sequence in $C([0,T_*];H^{N-1}_{x,\xi})$, and the corresponding limit function denoted by u belongs to $X(0,T_*;M_0)$, and u is a solution to the Cauchy problem. Meanwhile, $\exists$ at most one solution in $X(0,T_*;M_0)$.

Uniform a priori estimate-1

Define the temporal functionals

$$\mathcal{E}(u(t)) = \|u(t)\|^2_{L^2_\xi(H^N_x)} + \kappa_1 \mathcal{E}_{free}(u(t))$$
$$+\kappa_2 \sum_{1\leq k\leq N} C_k \sum_{\substack{|\beta|=k \\ |\alpha|+|\beta|\leq N}} \|\partial^\alpha_x \partial^\beta_\xi \{\mathbf{I}-\mathbf{P}\}u\|^2,$$

$$\mathcal{D}(u(t)) = \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_x \partial^\beta_\xi \{\mathbf{I}-\mathbf{P}\}u(t)\|^2_\nu + \sum_{|\alpha|\leq N} \|T_\Delta \partial^\alpha_x b^u(t)\|^2_U$$
$$+\|\nabla_x(a^u, b^u)(t)\|^2_{H^{N-1}_x},$$

where $0 < \kappa_2 \ll \kappa_1 \ll 1$**, and**

$$|\mathcal{E}_{free}(u(t))| \leq \|u(t)\|^2_{L^2_\xi(H^N_x)}.$$

Notice

$$\mathcal{E}(u(t)) \sim \|u(t)\|^2_{H^N_{x,\xi}}.$$

Uniform a priori estimate-2

One can show

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda \mathcal{D}(u(t)) \leq C\mathcal{E}(u(t))\mathcal{D}(u(t)).$$

Then, under the a priori assumption on smallness of

$$\sup_{0\leq t\leq T} \mathcal{E}(u(t)),$$

one has the Lyapunov inequality

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda \mathcal{D}(u(t)) \leq 0.$$

for any $0 \leq t \leq T$**.**

Global solutions: Continuity argument

Define

$$T_* = \sup\{t : \sup_{0\le s\le t} \|u(s)\|^2_{H^N_{x,\xi}} \le M\}.$$

Using

local existence + uniform a priori estimates,

then,

$$\mathcal{E}(u_0) \sim \|u_0\|^2_{H^N_{x,\xi}} \ll 1, \; \exists\, M > 0 \Rightarrow T_* = \infty\text{: global solution.}$$

Key point: Construction of free energy functional-1

The free energy functional is responsible to obtain the macroscopic dissipation

$$\|\nabla_x(a^u, b^u)\|^2_{H_x^{N-1}}.$$

Make the macro-micro decomposition:

$$\begin{cases} u(t,x,\xi) = \mathbf{P}u + \{\mathbf{I} - \mathbf{P}\}u, \\ \mathbf{P}u \equiv \{a^u + b^u \cdot \xi\}\sqrt{\mathbf{M}}, \\ a^u = \langle \sqrt{\mathbf{M}}, u \rangle, \quad b^u = \langle \xi\sqrt{\mathbf{M}}, u \rangle, \end{cases}$$

Define $A = (A_{ij}(\cdot))_{n\times n}$ **by**

$$A_{ij}(u) = \int_{\mathbb{R}^n} (\xi_i\xi_j - 1)\sqrt{\mathbf{M}} u \, d\xi.$$

One can obtain a series of equations satisfied by a^u, b^u

Key point: Construction of free energy functional-2

Macro balance laws:

$$\partial_t a^u + \nabla_x \cdot b^u = 0,$$

$$\partial_t b_i^u + \partial_i a^u - (U * b_i^u - b_i^u) + U * a^u b_i^u - U * b_i^u a^u + \sum_{j=1}^{n} \partial_j A_{ij}(\{\mathbf{I} - \mathbf{P}\}u) = 0,$$

Evolution of high-order moments:

$$\partial_t A_{ii}(\{\mathbf{I} - \mathbf{P}\}u) + \partial_i b_i^u - U * b_i^u b_i^u = A_{ii}(l + r),$$

$$\partial_t A_{ij}(\{\mathbf{I} - \mathbf{P}\}u) + \partial_i b_j^u + \partial_j b_i^u - U * b_i^u b_j^u - U * b_j^u b_i^u = A_{ij}(l + r), \quad i \neq j,$$

where

$$l = -\xi \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\}u + \mathbf{L}_{FP}\{\mathbf{I} - \mathbf{P}\}u,$$

$$r = U * a^u \mathbf{L}_{FP}\{\mathbf{I} - \mathbf{P}\}u + \frac{1}{2} U * b^u \cdot \xi \{\mathbf{I} - \mathbf{P}\}u - U * b^u \cdot \nabla_\xi \{\mathbf{I} - \mathbf{P}\}u.$$

Key point: Construction of free energy functional-3

Define $\mathcal{E}_{free}(u(t))$ by

$$\begin{aligned}\mathcal{E}_{free}(u(t)) = 3 &\sum_{|\alpha|\leq N-1}\sum_{j}\sum_{i\neq j}\int_{\mathbb{R}^n} A_{ii}(\partial_x^\alpha \partial_j\{\mathbf{I}-\mathbf{P}\}u)\partial_x^\alpha b_j^u\,dx \\ &-3\sum_{|\alpha|\leq N-1}\sum_{ij}\int_{\mathbb{R}^n} A_{ij}(\partial_x^\alpha \partial_i\{\mathbf{I}-\mathbf{P}\}u)\partial_x^\alpha b_j^u\,dx \\ &+\sum_{|\alpha|\leq N-1}\int_{\mathbb{R}^n}\partial_x^\alpha \nabla_x a^u\cdot \partial_x^\alpha b^u\,dx.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_{free}(u(t)) + \lambda\|\nabla_x(a^u,b^u)\|^2_{H_x^{N-1}} \\ &\leq C\sum_{|\alpha|\leq N}(\|T_\Delta \partial_x^\alpha b^u\|_U^2 + \|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2) \\ &\quad + C\|(a^u,b^u)\|^2_{H_x^N}(\|\nabla_x(a^u,b^u)\|^2_{H_x^{N-1}} + \sum_{|\alpha|\leq N}\|\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}u\|^2).\end{aligned}$$

Representation of solutions

Consider the Cauchy problem of the linearized equation with a nonhomogeneous source:

$$\begin{cases} \partial_t u = \mathbf{B} u + h, \quad t > 0, x \in \mathbb{R}^n, \\ u|_{t=0} = u_0, \quad x \in \mathbb{R}^n, \end{cases}$$

where $n \geq 1$ **is the spatial dimension,** $h = h(t, x, \xi)$ **and** $u_0 = u_0(x, \xi)$ **are given, and the linear operator B is defined by**

$$\mathbf{B} = -\xi \cdot \nabla_x + \mathbf{L}, \quad \mathbf{L} = \mathbf{L}_{FP} + \mathbf{A}.$$

Formally, the solution can be written as the Duhamel formula

$$u(t) = e^{\mathrm{B}t} u_0 + \int_0^t e^{\mathrm{B}(t-s)} h(s) ds.$$

Main result: Linearized case

Set

$$\sigma_{q,m} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

Theorem

Let $1 \le q \le 2$, $n \ge 1$, *and let* $Z_q = L^2_\xi(L^1_x)$.

(i)

$$\|\partial_x^\alpha e^{\mathrm{B}t} u_0\| \le C(1+t)^{-\sigma_{q,m}}(\|\partial_x^{\alpha'} u_0\|_{Z_q} + \|\partial_x^\alpha u_0\|),$$

with $m = |\alpha - \alpha'|$.

(ii) If $\mathbf{P}h = 0$, *i.e. h is microscopic, then*

$$\left\|\partial_x^\alpha \int_0^t e^{\mathrm{B}(t-s)} h(s)ds\right\|^2$$

$$\le C\int_0^t (1+t-s)^{-2\sigma_{q,m}}(\|\nu^{-1/2}\partial_x^{\alpha'} h(s)\|^2_{Z_q} + \|\nu^{-1/2}\partial_x^\alpha h(s)\|^2)ds,$$

with $m = |\alpha - \alpha'|$.

Spectral analysis-1

Define

$$\mathcal{E}^l(\widehat{u}(t,k)) = \|\widehat{u}(t,k)\|^2_{L^2_\xi} + \kappa\,\mathrm{Re}\ \mathcal{E}^l_{free}(\widehat{u}(t,k))$$

for a small constant $\kappa > 0$**, where**

$$\begin{aligned}\mathcal{E}^l_{free}(\widehat{u}(t,k)) = 3\sum_j \sum_{i\neq j} \frac{\mathrm{i}k_j}{1+|k|^2}(A_{ii}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid \widehat{b^u_j}) \\ -3\sum_{ij} \frac{\mathrm{i}k_i}{1+|k|^2}(A_{ij}(\{\mathbf{I}-\mathbf{P}\}\widehat{u}) \mid \widehat{b^u_j}) \\ -\frac{\mathrm{i}k}{1+|k|^2}\cdot(\widehat{b^u} \mid \widehat{a^u}).\end{aligned}$$

Notice $|\mathcal{E}^l_{free}(\widehat{u}(t,k))| \leq C\|\widehat{u}(t,k)\|^2_{L^2_\xi}$**. Then,**

$$\mathcal{E}^l(\widehat{u}(t,k)) \sim \|\widehat{u}(t,k)\|^2_{L^2_\xi}.$$

Spectral analysis-2

One can show

$$\frac{\partial}{\partial t}\mathcal{E}^l(\widehat{u}(t,k)) + \frac{\lambda|k|^2}{1+|k|^2}\mathcal{E}^l(\widehat{u}(t,k)) \le C\|\nu^{-1/2}\widehat{h}(t,k)\|^2_{L^2_\xi},$$

Key idea: Make the Fourier analysis on

$$\begin{aligned}
&\partial_t a^u + \nabla_x \cdot b^u = 0,\\
&\partial_t b^u_i + \partial_i a^u - (U * b^u_i - b^u_i) + \sum_{j=1}^{n}\partial_j A_{ij}(\{\mathbf{I}-\mathbf{P}\}u) = 0,\\
&\partial_t A_{ii}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_i b^u_i = A_{ii}(l+h),\\
&\partial_t A_{ij}(\{\mathbf{I}-\mathbf{P}\}u) + \partial_i b^u_j + \partial_j b^u_i = A_{ij}(l+h),\ \ i \neq j,
\end{aligned}$$

and use Kawashima's hyperbolic-parabolic argument. Then, one has

Spectral analysis-3

$$\frac{\partial}{\partial t}\text{Re }\mathcal{E}^{l}_{free}(\widehat{u}(t,k)) + \frac{|k|^2}{4(1+|k|^2)}(|\widehat{a^u}|^2 + |\widehat{b^u}|^2)$$

$$\leq \frac{1-\text{Re }\widehat{U}}{1+|k|^2}|\widehat{b^u}|^2 + \frac{C}{1+|k|^2}\|\nu^{-1/2}\widehat{h}\|^2_{L^2_\xi} + C\|\{\mathbf{I}-\mathbf{P}\}\widehat{u}\|^2_{L^2_\xi}.$$

Combined with

$$\frac{1}{2}\frac{\partial}{\partial t}\|\widehat{u}(t,k)\|_{L^2_\xi} + \lambda|\{\mathbf{I}-\mathbf{P}\}\widehat{u}|^2_\nu + (1-\text{Re }\widehat{U})|\widehat{b^u}|^2 \leq C\|\nu^{-1/2}\widehat{h}(t,k)\|^2_{L^2_\xi}.$$

The proper linear combination leads to the desired estimate. Then,

$$\|\widehat{u}(t,k)\|^2_{L^2_\xi} \leq Ce^{-\frac{\lambda|k|^2}{1+|k|^2}t}\|\widehat{u_0}(k)\|^2_{L^2_\xi} + C\int_0^t e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}\|\nu^{-1/2}\widehat{h}(s,k)\|^2_{L^2_\xi}ds,$$

for any $t \geq 0$ **and** $k \in \mathbb{R}^n$**.**

Rate of convergence in the nonlinear case

Write the nonlinear equation as

$$u(t) = e^{\mathrm{B}t}u_0 + \int_0^t e^{\mathrm{B}(t-s)}G(s)ds,$$

where the source term G is denoted by

$$G = \Gamma(u,u) - U * b^u \cdot \nabla_\xi u.$$

Energy-spectrum method + Weighted estimates: Use

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda\mathcal{E}(u(t)) \le C\|u(t)\|^2.$$

Lemma

If $\|u_0\|_{Z_1}$ is bounded, then

$$\begin{aligned}\|u(t)\|^2 \le C(\mathcal{E}(u_0) + \|u_0\|_{Z_1}^2)(1+t)^{-\frac{n}{2}} \\ +C\int_0^t (1+t-s)^{-\frac{n}{2}}\mathcal{E}(u(s))[\mathcal{E}(u(s)) + \|\xi\{\mathbf{I}-\mathbf{P}\}u(s)\|^2]ds \\ +C\left[\int_0^t (1+t-s)^{-\frac{n}{4}}\mathcal{E}(u(s))ds\right]^2.\end{aligned}$$

Thanks for your attention!