A macroscopic model for self-propelled particles with orientation interactions

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Goal : macroscopic description of some animal populations

- Local interactions without leader
- Emergence of macroscopic ordered structures
- ~~Why ?~~ How ?

Follows the work of P. Degond and S. Motsch.

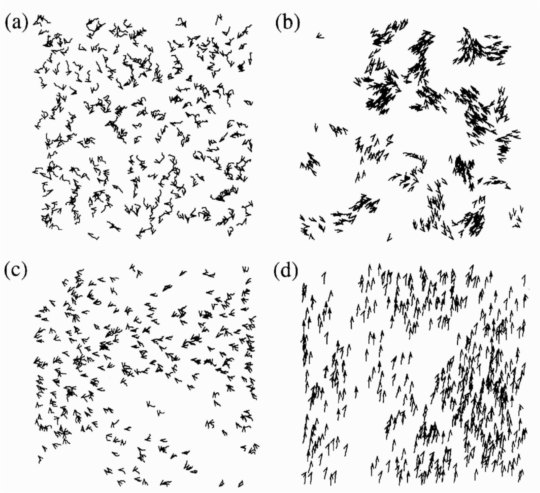
Vicsek's algorithm (discrete in time)

- Particle k at position $X_k \in \mathbb{R}^2$, with orientation $\omega_k \in \mathbb{S}_1$.
- Next time step :
  - New position : $X_k + c\omega_k$ (constant speed c).
  - New orientation : $\overline{\omega_k}$ + « noise ».
- $\overline{\omega_k}$: mean direction of « neighbours » for the k^{th} particle.

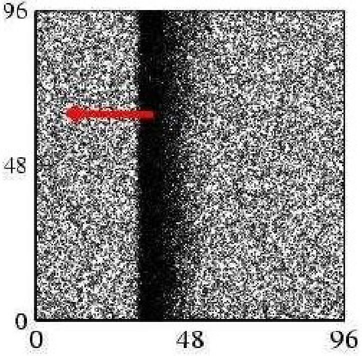
$$\overline{\omega_k} = \frac{J_k}{|J_k|}, \quad J_k = \sum_{\substack{j \text{ such that} \\ |X_j - X_k| \leqslant R}} \omega_j.$$

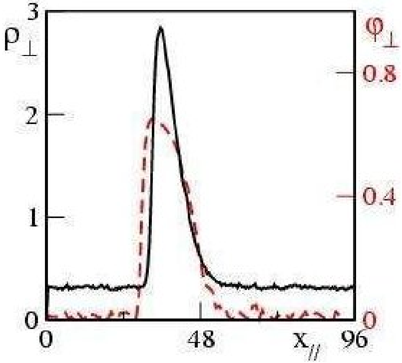
- Noise : angular noise with intensity d.

Phase transition : Vicsek's observations



Travelling in bands : Chaté's observations





A time-continous stochastic system of particles

$$\begin{aligned} \frac{dX_k}{dt} &= \omega_k, \\ d\omega_k &= (\mathrm{Id} - \omega_k \otimes \omega_k)(\nu(\overline{\rho_k})\,\overline{\omega_k}\,dt + \sqrt{2D(\overline{\rho_k})}\,dB_t). \end{aligned}$$

- Observation kernel K, depending on distance between particules and some « angle of vision », is added to the way the « mean direction » $\overline{\omega_k}$ is computed :

$$\overline{\omega_k} = \frac{J_k}{|J_k|}, \quad J_k = \sum_j K\left(|X_j - X_k|, \frac{X_j - X_k}{|X_j - X_k|} \cdot \omega_k\right) \omega_j.$$

- The parameters D and ν now depend on some local density $\overline{\rho_k}$, defined in an analoguous way.

Limit model

We want to replace this dynamical system, at large scale in space and time, by a continuous model taking only in account local variable.
At the end we obtain the following system of partial differential equations, where $\rho(x,t)$ is the « density » of particles, and $\Omega(x,t)$ is their « mean orientation » .

$$\partial_t \rho + \nabla_x \cdot (c_1(\rho)\rho\Omega) = 0,$$
$$\rho\,(\partial_t \Omega + c_2(\rho)(\Omega\cdot\nabla)\Omega) + \lambda(\rho)\,(\mathrm{Id} - \Omega\otimes\Omega)\nabla_x \rho = 0.$$

Mean-field kinetic model

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (Ff) = d(\overline{\rho})\Delta_\omega f,$$

This is a formal derivation from the particle system, with

$$F(x,\omega,t) = \nu(\overline{\rho})(\mathrm{Id} - \omega \otimes \omega)\overline{\omega}, \quad \overline{\omega}(x,\omega,t) = \frac{J(x,\omega,t)}{|J(x,\omega,t)|},$$

$$J(x,\omega,t) = \int_{y\in\mathbb{R}^2,\, \upsilon\in\mathbb{S}_1} K\left(|x-y|, \frac{y-x}{|x-y|}\cdot\omega\right) \upsilon\, f(y,\upsilon,t)\, dy\, d\upsilon\ ,$$

$$\overline{\rho}(x,\omega,t) = \int_{y\in\mathbb{R}^2,\, \upsilon\in\mathbb{S}_1} \widetilde{K}\left(|x-y|, \frac{y-x}{|x-y|}\cdot\omega\right) f(y,\upsilon,t)\, \mathrm{d}y\, \mathrm{d}\upsilon\,.$$

Problems for a more rigorous derivation : deal with nonlinearity and propagation of chaos.

Hydrodynamic scaling

- Divide space and time scales by ε, and expand in ε. The interactions « become local » , we obtain formally :

$$\varepsilon(\partial_t f^\varepsilon + \omega \cdot \nabla_x f^\varepsilon + \kappa\, P(f^\varepsilon) + \widetilde{\kappa}\, R(f^\varepsilon)) = Q(f^\varepsilon) + O(\varepsilon^2)\,.$$

- The differential operators P and R act on variables x and ω, and the operator Q only acts on variable ω.
  The constants κ and $\widetilde{\kappa}$ are related to the observation kernels K and $\widetilde{K}$ (positive if there is a « cone of vision » in front of the particle).

Expressions of the operators

$$
\begin{aligned}
Q(f) &= -\nu(\rho[f])\nabla_\omega \cdot (F_{\Omega[f]} f) + d(\rho[f])\Delta_\omega f, \\
F_\Omega(\omega) &= (\mathrm{Id} - \omega \otimes \omega)\Omega, \\
P(f) &= \nu(\rho[f])\nabla_\omega \cdot (G_{\Omega[f]} f), \\
G_\Omega(\omega) &= (\nabla_x \Omega)^T \omega - (\nabla_x \Omega : \omega \otimes \omega)\, \omega, \\
R(f) &= \nabla_\omega \cdot \Big((\omega \cdot \nabla_x \rho) \left(\dot{\nu}(\rho[f])\, F_{\Omega[f]} f - \dot{d}(\rho[f])\, \nabla_\omega f \right) \Big), \\
\rho[f] &= \int_{\omega \in \mathbb{S}} f(.,\omega)\, \mathrm{d}\omega, \\
\Omega[f] &= \frac{j[f]}{|\, j[f]\, |}, \quad \text{and} \quad j[f] = \int_{\omega \in \mathbb{S}} \omega\, f(.,\omega)\, \mathrm{d}\omega .
\end{aligned}
$$

Equilibria, for the collision operator Q

- Functions f such that $Q(f) = 0$:

$$\mathcal{E} = \{M_{\rho,\Omega}(\omega) \quad | \quad \rho \in \mathbb{R}_+^*, \quad \Omega \in \mathbb{S}_1\},$$

where

$$M_{\rho,\Omega}(\omega) = C(\rho)\exp(\alpha(\rho)\omega \cdot \Omega), \qquad \text{with } \alpha(\rho) = \frac{\nu(\rho)}{\mathrm{d}(\rho)}.$$

- The normalisation constant $C(\rho)$ is chosen such that ρ is the total mass of $M_{\rho,\Omega}$. The flux direction of $M_{\rho,\Omega}$ is Ω.

Collision invariants... must be generalized

- Find all $\psi(\omega)$ such that

$$\int_{\omega\in\mathbb{S}_1} Q(f)\,\psi\,d\omega = 0, \quad \forall f.$$

- Problem : only the constants !
  Not enough equations for ρ et Ω... (we do not have conservation relation in the initial model, except for the mass)

... must be generalized

- Weaker condition, for fixed Ω and ρ. We search functions $\psi_{\rho,\Omega}$ such that

$$\int_{\omega\in\mathbb{S}_1} Q(f)\,\psi_{\rho,\Omega}\,d\omega = 0,\ \forall f \text{ such that } \Omega[f] = \Omega \text{ and } \rho[f] = \rho.$$

- We get a vector space of dimension 2, spanned by the constants and the function $\psi_{\rho,\Omega} = h_\rho(\omega\cdot\Omega)\omega\times\Omega$.

Limit of f^ε

- We assume convergence and regularity. Formal limit $f = M_{\rho,\Omega}$ (since $Q(f) = 0$).
- Integrate the kinetic equation with respect to ω : conservation of mass.

$$\partial_t \rho[f^\varepsilon] + \nabla_x \cdot j[f^\varepsilon] = 0,$$

- Multiply by the generalized collisional invariant associated to $\Omega[f^\varepsilon]$, $\rho[f^\varepsilon]$, and integrate in ω.

$$\Omega \times \int_{\omega \in \mathbb{S}_1} ((\partial_t + \omega \cdot \nabla_x + \kappa\, P + \widetilde{\kappa} R) M_{\rho,\Omega})\, h_\rho(\omega \cdot \Omega)\, \omega \, d\omega = 0.$$

Computations

- Chain rule, we have to compute these kind of integrals (for an arbitrary function γ depending on $\cos\theta = \omega \cdot \Omega$).

$$I_k(\gamma) = \int_{\omega \in \mathbb{S}_1} \omega^{\otimes k}\, \gamma\, M_{\rho,\Omega}\, d\omega ...$$

- Except for the term with $\kappa P + \widetilde{\kappa} R$, where some preliminary tricks using integration by parts must be used.
- We finally get an equation for $\partial_t \Omega$.

Final model

Theorem

The limit of f^ε when $\varepsilon \to 0$ is given (formally) by $f^0 = M_{\rho,\Omega}$, where $\rho = \rho(x,t) \geq 0$ is the total mass of f^0 and $\Omega = \Omega(x,t) \in \mathbb{S}_1$ is the unit vector directing its flux.
Furthermore, $\rho(x,t)$ et $\Omega(x,t)$ satisfy the following first order system of partial differential equations :

$$\partial_t \rho + \nabla_x \cdot (c_1(\rho)\rho\Omega) = 0,$$
$$\rho \left(\partial_t \Omega + c_2(\rho)(\Omega \cdot \nabla)\Omega\right) + \lambda(\rho)\left(\mathrm{Id} - \Omega \otimes \Omega\right)\nabla_x \rho = 0.$$

Expression of convection speeds and parameter λ

$$
\begin{aligned}
c_1(\rho) &= \langle \cos\theta \rangle_{M_\alpha}, \\
c_2(\rho) &= (1 - 2\,\kappa\, d)\,\langle \cos\theta \rangle_{\overline{M}_\alpha} - \kappa\,\nu \langle \cos^2\theta \rangle_{\overline{M}_\alpha}, \\
\lambda(\rho) &= \frac{1}{\alpha} + \\
&\quad \rho\frac{\dot{\alpha}}{\alpha}\left((1 - 2\,\widetilde{\kappa}\, d)\,\langle \cos\theta \rangle_{\overline{M}_\alpha} - \langle \cos\theta \rangle_{M_\alpha} + \widetilde{\kappa}\,\nu \langle \sin^2\theta \rangle_{\overline{M}_\alpha}\right),
\end{aligned}
$$

with the « means » $\langle\cdot\rangle_{M_\alpha}$ and $\langle\cdot\rangle_{\overline{M}_\alpha}$ defined as follows

$$
\begin{aligned}
\langle \gamma(\cos\theta) \rangle_{M_\alpha} &= \frac{\int_0^\pi \gamma(\cos\theta)\exp(\alpha(\rho)\cos\theta)\,\mathrm{d}\theta}{\int_0^\pi \exp(\alpha(\rho)\cos\theta)\,\mathrm{d}\theta}, \\
\langle \gamma(\cos\theta) \rangle_{\overline{M}_\alpha} &= \frac{\int_0^\pi \gamma(\cos\theta) h_\rho(\cos\theta)\exp(\alpha(\rho)\cos\theta)\,\sin^2\theta\,\mathrm{d}\theta}{\int_0^\pi h_\rho(\cos\theta)\exp(\alpha(\rho)\cos\theta)\,\sin^2\theta\,\mathrm{d}\theta}.
\end{aligned}
$$

Non-conservative system, may be non-hyperbolic

- We have to study this system :

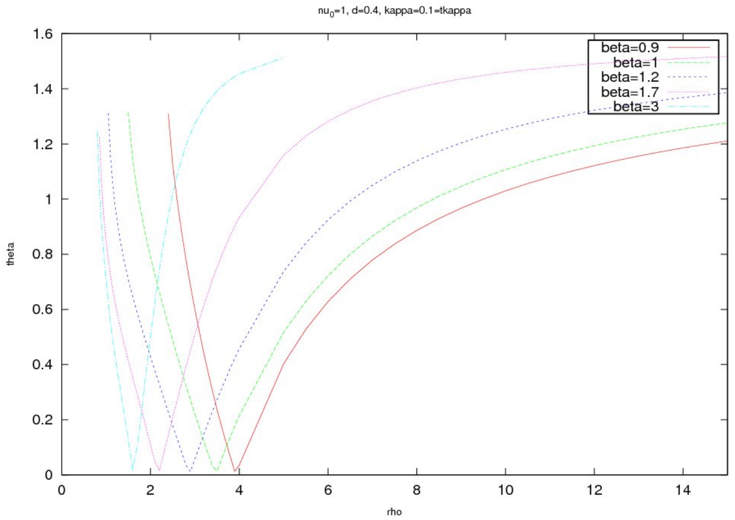
$$\partial_t \rho + \widetilde{c_1(\rho)} \cos\theta \, \partial_x \rho - c_1(\rho)\rho \sin\theta \, \partial_x \theta = 0.$$
$$\partial_t \theta + c_2(\rho) \cos\theta \, \partial_x \theta - \lambda \sin\theta \, \partial_x \ln \rho = 0.$$

- Eigenvalues of $A(\rho, \theta) = \begin{pmatrix} \widetilde{c_1(\rho)} \cos\theta & -c_1(\rho)\rho\sin\theta \\ -\frac{\lambda \sin\theta}{\rho} & c_2(\rho)\cos\theta \end{pmatrix}$:

$$\gamma_\pm = \frac{1}{2}\left[(\widetilde{c_1} + c_2)\cos\theta \pm \left((c_2 - \widetilde{c_1})^2 \cos^2\theta + 4\lambda c_1 \sin^2\theta\right)^{\frac{1}{2}}\right].$$

- In the case $\lambda < 0$ (this can be the case), there is a zone of non-hyperbolicity.

Example : $\nu(\rho) = \rho^\beta$, non-hyperbolicity over the curves



Asymptotic expansion of coefficients

Suppose ν and d are constant.

$$\partial_t \rho + c_1 \nabla_x \cdot (\rho\Omega) = 0,$$
$$\rho\left(\partial_t \Omega + c_2 (\Omega \cdot \nabla)\Omega\right) + \lambda\,(\mathrm{Id} - \Omega \otimes \Omega)\nabla_x \rho = 0.$$

- Speed of the « fluid » : c_1.
- Perturbations in orientation propagate with speed c_2.
- When $d \to 0$ (small noise) :

$$\frac{c_2}{c_1} = 1 - \kappa + \left(\frac{\kappa}{2} - 1\right) d + O(d^2).$$

- The more the observation is directed forward, the more the informations propagate rapidly « backward »...

Summary

The method introduced by P. Degond and S. Motsch (in particular the generalisation of collisonal invariants) works fine, with some little tricks, when we add some features to the initial model.

- No such change in the final model with the introduction of angle of vision.
- But non-hyperbolicity can occur if the parameters depend on a local density.

Future work

- Numerical simulations.
- More rigorous framework for the derivation of the model.

Thanks !