Global existence and blowup for the Smoluchowski-Poisson equation with nonlinear diffusion

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The Smoluchowski-Poisson (SP) equation in $\mathbb{R}^d$, $d \geq 2$

$$\begin{aligned} \partial_t u &= \operatorname{div}\ (\nabla u - u\,\nabla\varphi)\ , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\ , \\ \varphi &= E_d * u\ , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\ , \end{aligned}$$

where $d \geq 2$ and E_d is the Poisson kernel

$$E_2(x) := -\frac{1}{2\pi}\ \ln|x| \quad \text{or} \quad E_d := c_d\ |x|^{-(d-2)} \quad \text{if} \quad d \geq 3\ ,$$

(so that $-\Delta\varphi = u$).

- Self-gravitating particles (in astrophysics).
- Parabolic-elliptic Keller-Segel model for chemotaxis.

The Smoluchowski-Poisson equation in $\mathbb{R}^d$, $d \geq 2$

$$\partial_t u = \text{div} \left(\nabla u - u \nabla(E_d * u)\right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d .$$

- Non-negativity: $u_0 \geq 0 \Longrightarrow u \geq 0$,

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- Non-negativity: $u_0 \geq 0 \Longrightarrow u \geq 0$,
- Mass conservation: $\|u(t)\|_1 = M_0 := \|u_0\|_1$,
- Competition between the diffusive term Δu (spreading) and the drift term (concentrating) $\text{ div } (u \, \nabla (E_d * u))$: global existence or finite time blowup.

$$\partial_t u = \Delta u - \nabla u \cdot \nabla (E_d * u) + u^2 \, , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d \, .$$

The generalised Smoluchowski-Poisson (gSP) equation in $\mathbb{R}^d$, $d \geq 2$

The linear diffusion $\text{div}\ (\nabla u) = \Delta u$ is replaced by $\text{div}\ (\nabla u^m)$ with $m > 1$:

$$\partial_t u = \text{div}\ (\nabla u^m - u\, \nabla(E_d * u))\ , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d\ .$$

- Derivation from a Vlasov-Poisson-Fokker-Planck kinetic equation with non-gaussian equilibria (diffusion limit),
- Prevention of crowding (if $a(r)/r \to \infty$ as $r \to \infty$).

A Liapunov functional I

The functional

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \mathcal{A}(u(t,x))\, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x)\, u(t,x)\, dx\,,$$

with

- $\mathcal{A}(r) = r \ln r - r \geq -1$ if $m = 1$,
- $\mathcal{A}(r) = r^m/(m-1) \geq 0$ if $m > 1$.

Two competing terms in $\mathcal{F}$

A Liapunov functional II

Liapunov functional:

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \mathcal{A}(u(t,x))\, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x)\, u(t,x)\, dx\,.$$

At first glance, the "negative" term is quadratic in u and the positive term might dominate it if $\mathcal{A}$ increases faster than quadratically, that is, if $m > 2$.

In fact,

$$m > m_d := \frac{2(d-1)}{d} \quad (m_2 = 1)$$

guarantees global existence (mass conservation and convolution).
[Sugiyama & Kunii (2006), Cieślak & Winkler (2008)]

Is this exponent "optimal"?

Virial identity

Consider the second moment $M_2(t)$ of $u(t)$

$$M_2(t) := \int_{\mathbb{R}^d} |x|^2 \, u(t,x) \, dx \,.$$

Then

- $d = 2$ and $m = m_2 = 1$ ($M_0 = \|u_0\|_1$):

$$\frac{dM_2}{dt}(t) = -\frac{M_0}{4\pi} \, (M_0 - 8\pi) \,,$$

$\longrightarrow$ non-existence of global solutions for $M_0 > 8\pi$.

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- $d \geq 3$ and $m = m_d$:

$$\frac{dM_2}{dt}(t) = 2(d-2) \, \mathcal{F}[u(t)] \leq 2(d-2) \, \mathcal{F}[u_0] \, ,$$

$\longrightarrow$ non-existence of global solutions for u_0 such that $\mathcal{F}[u_0] < 0$.

Outline

1. The gSP equation with $m = m_d$ in $\mathbb{R}^d$, $d \geq 2$
   - The SP equation in $\mathbb{R}^2$
   - The gSP equation with $m = m_d$ in $\mathbb{R}^d$, $d \geq 3$

2. The gSP equation in $\Omega \subset \mathbb{R}^d$, $d \geq 1$
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The SP equation in $\mathbb{R}^2$

$$\partial_t u = \text{ div } \left(\nabla u - u \, \nabla (E_2 * u)\right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^2 .$$

The Liapunov functional:

$$\begin{aligned} \mathcal{F}[u(t)] \quad := \quad & \int_{\mathbb{R}^2} \left(u(t,x) \ln u(t,x) - u(t,x)\right) \, dx \\ & - \frac{1}{2} \int_{\mathbb{R}^2} (E_2 * u)(t,x) \, u(t,x) \, dx . \end{aligned}$$

Global existence

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- Global existence if $\|u_0\|_1 < 8\pi$ by symmetrization techniques. [Diaz, Nagai & Rakotoson (1998)]
- Global existence if $\|u_0\|_1 < 8\pi$ by the logarithmic Hardy-Littlewood-Sobolev inequality. [Dolbeault & Perthame, 2004].

The logarithmic Hardy-Littlewood-Sobolev inequality

$$\int_{\mathbb{R}^2} h \ln h \, dx - \frac{4\pi}{\|h\|_1} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E_2(x-y)\, h(x)\, h(y)\, dydx \geq -\|h\|_1 \left(1 + \ln\pi - \ln\|h\|_1\right)$$

Then

$$\mathcal{F}[u] \geq \left(1 - \frac{\|u_0\|_1}{8\pi}\right) \int_{\mathbb{R}^2} u \ln u \, dx - \frac{\|u_0\|_1^2}{8\pi} \left(1 + \ln\pi - \ln\|u_0\|_1\right),$$

hence a control of $u \ln u$ in L^1 $\longrightarrow$ global existence.

Critical mass $\|u_0\|_1 = 8\pi$

The useful term in the lower bound

$$\mathcal{F}[u] \geq \left(1 - \frac{\|u_0\|_1}{8\pi}\right) \int_{\mathbb{R}^2} u \ \ln u \, dx - C\left(\|u_0\|_1\right)$$

vanishes in the critical case $\|u_0\|_1 = 8\pi$.

What happens if $\|u_0\|_1 = 8\pi$?

$\|u_0\|_1 = 8\pi$

Existence of a one-parameter family of stationary solutions:

$$\frac{8b}{\left(b+|x|^2\right)^2} \in L^1(\mathbb{R}^2) \setminus L^1\left(\mathbb{R}^2; |x|^2\, dx\right), \quad b > 0\,.$$

Theorem

There is a global solution u to the Smoluchowski-Poisson equation and $u(t) \rightharpoonup 8\pi\, \delta_{x_m}$ as $t \to \infty$, x_m being the center of mass of u_0.

[Biler, Karch, L. & Nadzieja (2006), Blanchet, Carrillo & Masmoudi (2008)]

Further properties

- Non-existence of blowing-up self-similar solutions [Naito & Suzuki (2008)]
- Continuation after blowup? [Chavanis & Sire (2004), Velázquez (2004), Dolbeault & Schmeiser (2009)]

The gSP equation with $m = m_d$ in $\mathbb{R}^d$, $d \geq 3$

$$\partial_t u = \text{ div } \left(\nabla u^{m_d} - u \, \nabla (E_d * u)\right) , \quad (t,x) \in (0,\infty) \times \mathbb{R}^d .$$

The Liapunov functional:

$$\mathcal{F}[u(t)] := \int_{\mathbb{R}^d} \frac{u^{m_d}(t,x)}{m_d - 1}) \, dx - \frac{1}{2} \int_{\mathbb{R}^d} (E_d * u)(t,x) \, u(t,x) \, dx .$$

A modified Hardy-Littlewood-Sobolev inequality

Approach: find a functional inequality characterizing the critical mass.

Lemma

There exists $C_* > 0$ such that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{h(x)\, h(y)}{|x-y|^{d-2}}\, dxdy \right| \leq C_* \, \|h\|_m^m \, \|h\|_1^{2/d}$$

for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

[Blanchet, Carrillo & L. (2009)]

A bound from below for $\mathcal{F}$

Introducing the critical mass

$$M_c := \left[\frac{2}{(m-1)\, C_* \, c_d} \right]^{d/2},$$

we have

$$\frac{C_* \, c_d}{2} \, \left(M_c^{2/d} - \|h\|_1^{2/d} \right) \, \|h\|_m^m \leq \mathcal{F}[h]$$

for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

- If $\|u_0\|_1 \leq M_c$ then $\mathcal{F}[u_0] \geq 0$,

A bound from below for $\mathcal{F}$

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for all $h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)$.

- If $\|u_0\|_1 \leq M_c$ then $\mathcal{F}[u_0] \geq 0$,
- If $\|u_0\|_1 < M_c$ then control on the L^m-norm $\longrightarrow$ global existence.

Global existence and blowup

- Global existence if $\|u_0\|_1 < M_1 < M_c$ by Gagliardo-Nirenberg inequalities and finite time blowup if $\|u_0\|_1 > M_2 > M_1$. [Sugiyama (2007)].
- Global existence if $\|u_0\|_1 < M_c$ by the modified Hardy-Littlewood-Sobolev inequality. [Blanchet, Carrillo & L. (2009)]
- If $M > M_c$, then

$$\mu_M := \inf\left\{\mathcal{F}[h] \ : \ h \in L^1(\mathbb{R}^d) \cap L^m(\mathbb{R}^d)\,,\ \|h\|_1 = M\right\} = -\infty\,,$$

and finite time blowup if $\mathcal{F}[u_0] < 0$ by an argument from Weinstein (1986). [Blanchet, Carrillo & L. (2009)]

What happens if $\|u_0\|_1 = M_c$?

Stationary solutions

There is a two-parameter family $\{V_{z,R}\}$ of non-negative and compactly supported stationary solutions such that

$$\|V_{z,R}\|_1 = M_c\,, \quad z \in \mathbb{R}^d\,,\ R > 0\,.$$

[Chavanis & Sire (2008), Blanchet, Carrillo & L. (2009)]

- Unlike for $d = 2$, there are thus global and bounded solutions.
- Minimisers of $\mathcal{F}$ in $\left\{h \in L^1(\mathbb{R}^d) \cap L^{m_d}(\mathbb{R}^d) \ :\ \|h\|_1 = M_c\right\}$.

Global existence: $\|u_0\|_1 = M_c$

Proposition

If $\|u_0\|_1 = M_c$, there is a global solution u.

[Blanchet, Carrillo & L. (2009)]

Open question: If $\|u_0\|_1 = M_c$, what is the large time behaviour of the global solution:

- Convergence to a steady state?
- Blowup in infinite time and concentration to a Dirac mass?

Self-similar blowing-up solutions

Look for solutions of the form

$$u(t,x) = \frac{1}{s(t)^d}\, U\left(\frac{x}{s(t)}\right) \quad \text{and} \quad \varphi(t,x) = \frac{1}{s(t)^{d-2}}\, \Phi\left(\frac{x}{s(t)}\right)$$

for $(t,x) \in [0,T) \times \mathbb{R}^d$ with $s(t) := [d(T-t)]^{1/d}$, $T > 0$.

- There are such solutions with a radially symmetric and non-increasing profile U satisfying $\|U\|_1 = M$ for $M \in (M_c, M_{c,2}]$, $M_{c,2} \in (M_c, \infty)$. In addition, U is compactly supported.
- There are such solutions with a radially symmetric and compactly supported profile U having multiple bumps.

[Blanchet & L.]

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The gSP equation in $\Omega \subset \mathbb{R}^d$, $d \geq 1$

$$
\begin{aligned}
&\partial_t u = \operatorname{div}\left(a(u)\, \nabla u - u\, \nabla \varphi\right), \quad (t,x) \in (0,\infty) \times \Omega\,,\\
&-\Delta \varphi = u - \langle u \rangle\,, \quad \langle \varphi \rangle = 0\,, \quad (t,x) \in (0,\infty) \times \Omega\,,\\
&\partial_\nu u = \partial_\nu \varphi = 0\,, \quad (t,x) \in (0,\infty) \times \partial\Omega\,,\\
&u(0) = u_0\,, \quad x \in \Omega\,,
\end{aligned}
$$

where

- Ω is an open bounded subset of $\mathbb{R}^d$, $d \geq 1$,
- $\langle u \rangle$ denotes the space average of u, and
- $a \geq 0$.

Linear diffusion: $a = 1$

- $d = 1$: global existence.
- $d = 2$ and $\Omega = B(0, 1)$: global existence if $\langle u_0 \rangle < 8\pi$ and finite time blowup if $\langle u_0 \rangle > 8\pi$ and u_0 sufficiently concentrated. [Jäger & Luckhaus (1992), Nagai (1995)]
- $d = 2$: global existence if $\langle u_0 \rangle \leq 4\pi$ and finite time blowup otherwise when u_0 is concentrated either near a point of the boundary or in the interior. [Biler (1998), Gajewski & Zacharias (1998), Nagai (2001), Nagai, Senba & Suzuki (1997), Ohtsuka, Senba & Suzuki (2007)]
- $d \geq 3$: finite time blowup for sufficiently concentrated initial data whatever the value of $\langle u_0 \rangle$ is. [Nagai (1995)]

Blowup profile when $a = 1$ and $d = 2$

Refined description of the dynamics at the blowup time and construction of radially symmetric blowing-up solutions. [Herrero & Velázquez (1996)]

If u blows up at time $T > 0$ then

$$u(t) \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\, \delta_{x_0} + f \quad \text{as} \quad t \to T\,,$$

where

- $\mathcal{S}$ is the set of blowup points which is discrete and finite,
- $m(x_0) = 8\pi$ if $x_0 \in \Omega \cap \mathcal{S}$ and $m(x_0) = 4\pi$ if $x_0 \in \partial\Omega \cap \mathcal{S}$,
- $f \in L^1(\Omega) \cap \mathcal{C}\left(\bar{\Omega} \setminus \mathcal{S}\right)$, $f \geq 0$,

[Senba & Suzuki (2001)]

Nonlinear diffusion

- $d \geq 1$: global existence if

$$a(r) \geq C\,(1+r)^{m-1} \quad \text{and} \quad m > m_d$$

- $d \geq 1$: finite time blowup (for some radially symmetric initial data) by a comparison argument if

$$a(r) \leq C\,(1+r)^{m-1} \quad \text{and} \quad m < m_d\,.$$

[Cieślak & Winkler (2008)]

$$\boxed{m = m_d,\ d \neq 2?}$$

Finite time blowup: $\Omega = B(0,1)$

Assume that there are $m \in [1, m_d]$, $c_1 > 0$, and $c_2 > 0$ such that

$$0 < a(r) \leq c_1\, r^{m-1} + c_2 \quad \text{for} \quad r \geq 0\,.$$

Let $M > 0$. Non-existence of global solutions for some initial data u_0 satisfying $\langle u_0 \rangle = M$ if

- either $1 \leq m < m_d$,
- or $m = m_d$ and $M > M_\star$ for some $M_\star > 0$.

[Nagai (1995)]: $m = 1$, [Cieślak & Winkler (2008)]: $m \in [1, m_d)$, [Cieślak & L. (2009)]

If $a(r) \geq C\,(1+r)^{m_d-1}$ and $\langle u_0 \rangle = M$ is small, then global existence.

An inequality of virial type

A contradiction is obtained by computing the evolution of

$$\int_{B(0,1)} |x|^d \, u(t,x) \, dx$$

for $m = 1$ [Nagai (1995)] and

$$\frac{1}{q} \int_0^1 \left(\frac{M}{d} - U(t,r) \right)^q r^{d-1} \, dr \,, \quad q > 1 \,,$$

$$U(t,r) := \frac{1}{d|B(0,1)|} \int_{B(0,r)} u(t,x) \, dx$$

for $m \in [1, m_d]$ [Cieślak & L. (2009)].

Questions

- Relationship between M_c in the case of $\mathbb{R}^d$ and $M_\star$?
- Threshold for boundary blowup?
- Stability and multiplicity of steady states? (constants are steady states)
- Shape of blowup when $d \geq 3$?

The one dimensional GSP equation

$$
\begin{aligned}
&\partial_t u = \partial_x \left(a(u)\, \partial_x u - u\, \partial_x \varphi\right), \quad (t,x) \in (0,\infty) \times (0,1)\,, \\
&-\partial_x^2 \varphi = u - \langle u \rangle\,, \quad \langle \varphi \rangle = 0\,, \quad (t,x) \in (0,\infty) \times (0,1)\,, \\
&\partial_x u = \partial_x \varphi = 0\,, \quad (t,x) \in (0,\infty) \times \{0,1\}\,, \\
&u(0) = u_0\,, \quad x \in (0,1)\,,
\end{aligned}
$$

with

- $a \in \mathcal{C}^1((0,\infty))$, $a > 0$,
- Initial condition: $u_0 \in \mathcal{C}([0,1])$, $u_0 > 0$, $\langle u_0 \rangle = M > 0$.

Change of unknow function

1. The cumulative distribution function:

$$U(t,x) := \int_0^x u(t,z)\, dz\,, \quad U(t,1) = \langle u(t) \rangle = \langle u_0 \rangle = M\,.$$

is non-decreasing.

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2. The (pseudo-)inverse $F(t,.) : [0,M] \longmapsto [0,1]$ of $U(t,.)$ is given

$$U(t, F(t,y)) = y\,, \quad (t,y) \in [0,\infty) \times [0,M]\,.$$

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$$U(t,F(t,y)) = y\,, \quad (t,y) \in [0,\infty) \times [0,M]\,.$$

3. $f := \partial_y F$ solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,, \quad (t,y) \in (0,\infty) \times (0,M)\,.$$

Alternative formulation

The new unknown f solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,, \quad (t,y) \in (0,\infty) \times (0,M)\,,$$
$$\partial_y f(t,0) = \partial_y f(t,M) = 0\,, \quad t \in (0,\infty)\,,$$
$$f(0,y) = f_0(y) := \frac{1}{u_0(F(0,y))} > 0\,, \quad y \in (0,M)\,,$$

with

$$\Psi'(r) := \frac{1}{r^2}\, a\left(\frac{1}{r}\right)\,, \quad \Psi(1) = 0\,.$$

and

$$\int_0^M f(t,y)\, dy = 1\,.$$

Blowup $\longrightarrow$ “touch-down”

Since

$$f(t,y) = \frac{1}{u(t,F(t,y))}\,, \quad (t,y) \in (0,\infty) \times (0,M)\,,$$

u blows up in finite time $\iff$ f vanishes in finite time.

Remark. $1/M$ is an “unstable” stationary solution to

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,.$$

Global existence

Theorem

Assume that $a \notin L^1(1,\infty)$. Then there is a global solution to the GSP equation. It is bounded in $L^\infty(0,\infty;\mathbb{R}^2)$ if a is not too singular near $r = 0$.

Examples:

$$a(z) = (1+z)^\alpha , \quad \alpha \in [-1,\infty) , \quad \left(a(z) = \frac{1}{z} \right) ,$$

$$a(z) = \frac{1}{(1+z)(\log(1+z))^\beta} , \quad \beta \in (-\infty, 1] .$$

A Liapunov functional

Recall that f solves

$$\partial_t f = \partial_y^2 \Psi(f) - 1 + Mf\,,$$

with $\Psi : (0,1) \to (-\infty, 0)$ since $a \notin L^1(1,\infty)$. Then

$$f > 0 \quad \text{if} \quad \Psi(f) \quad \text{is bounded from below.}$$

Liapunov functional:

$$\frac{1}{2}\int_0^M |\partial_y \Psi(f(t,y))|^2 \ dy + \int_0^M (\Psi(f(t,y)) - M\,\Psi_1(f(t,y)))\ dy$$

with $\Psi_1(1) := 0$ and $\Psi_1'(r) := r\,\Psi'(r) = a(1/r)/r$, $r \in (0,\infty)$.

Proof

- The Liapunov functional controls $\|\partial_y \Psi(f(t))\|_2$ on bounded time intervals.
- The boundedness of $\Psi(f)$ in L^∞ follows by a Poincaré inequality since $\langle f(t) \rangle = 1/M$, and an upper bound on $u(t)$ follows.
- If $a \notin L^1(0,1)$, then there is a positive lower bound for $u(t)$.
- The bounds do not depend on time if, for each $\varepsilon \in (0,1)$, there is $\kappa_\varepsilon > 0$ such that

$$a(r) \leq \varepsilon \, ra(r) + \frac{\kappa_\varepsilon}{r} \quad \text{for} \quad r \in (0,1)\,.$$

Finite time blowup

Theorem

Assume that $a \in L^1(1,\infty)$ and a is not too singular near $r = 0$. For each $M > 0$, there is at least one initial condition u_0 with $\langle u_0 \rangle = M$ for which u blows up in finite time.

Sufficient condition:

$$\sup_{r \in (0,1)} r \int_r^\infty a(s)\, ds < \infty\,.$$

Examples:

$$\begin{aligned} a(r) &\leq C\, r^\alpha \qquad \alpha \in [-2,-1)\,, \\ a(r) &\leq \frac{C}{(1+r)\,(\log(1+r))^\beta}\,, \quad \beta > 1\,, \\ a(r) &= r^{-(2+\alpha)}\,, \quad \alpha > 0\,. \end{aligned}$$

Critical nonlinearity

No critical nonlinearity when $d = 1$?