Stable steady states and self-similar blow up solutions for gravitational Vlasov-Poisson systems

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Scope of the talk

Consider the relativistic gravitational Vlasov-Poisson system (RVP) :

$$\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v),$$

$$\phi_f(t,y) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|x-y|}\rho_f(t,y)dy, \qquad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv.$$

We will present two different types of solutions for this system :

- stable steady states in a subcritical regime (first part of the talk) ;
- self-similar finite time blowing up solutions in a supercritical regime (second part of the talk).

N.B. In all this talk, we shall implicitly consider spherically symmetric solutions.

First consider the classical gravitational Vlasov-Poisson system (VP) :

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0,$$

$$\phi_f(t,y) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \rho_f(t,y)dy, \qquad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)dv.$$

The following quantities are independent of time :

- $\|f(t)\|_{L^q}$ for all $q \in [1,\infty]$, and more generally all $\|j(f)\|_{L^1}$
- $\mathcal{H}(f) = \int_{\mathbb{R}^6} |v|^2 f(t,x,v)dxdv - \int_{\mathbb{R}^3} |\nabla_x \phi_f(t,x)|^2 dx$

A competition between kinetic and potential energies

It has been well known since the early 80’ (Batt, Horst-Hunze, DiPerna-Lions) that (weak) solutions can be constructed in the energy space

$$\mathcal{E}_p = \left\{ f \geq 0 \text{ with } f \in L^1, \quad f \in L^p, \quad |v|^2 f \in L^1 \right\}$$

as long as the kinetic energy remains bounded.

A COMPETITION BETWEEN KINETIC AND POTENTIAL ENERGIES

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as long as the kinetic energy remains bounded.

A key inequality, the interpolation inequality ("Gagliardo-Nirenberg") :

for all $f \in \mathcal{E}_p$ with $p > 9/7$ we have

$$\|\nabla_x \phi_f\|_{L^2}^2 \leq C \| |v|^2 f \|_{L^1}^{1/2} \, \|f\|_{L^1}^{\frac{7p-9}{6(p-1)}} \, \|f\|_{L^p}^{\frac{p}{3(p-1)}}$$

Consequence : the kinetic energy control the potential energy... and weak solutions are global in time.

$$\begin{aligned}\mathcal{H}(f_0) = \mathcal{H}(f(t)) &= \||v|^2 f(t)\|_{L^1} - \|\nabla_x \phi_f(t)\|_{L^2}^2 \\ &\geq \||v|^2 f(t)\|_{L^1} - C\||v|^2 f(t)\|_{L^1}^{1/2}\end{aligned}$$

I will now present three nonlinear Vlasov systems where the kinetic and potential energies have the same strength : the interpolation inequality is said to be critical.

- the classical gravitational Vlasov-Poisson system in dimension 4 :

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v),$$

$$\phi_f(t,y) = -\frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x-y|^2} \rho_f(t,y)\,dy, \qquad \rho_f(t,x) = \int_{\mathbb{R}^4} f(t,x,v)\,dv.$$

Energy : $\mathcal{H}(f) = \int_{\mathbb{R}^8} |v|^2 f(t,x,v)\,dx\,dv - \int_{\mathbb{R}^4} |\nabla_x \phi_f(t,x)|^2\,dx.$

Interpolation : $\|\nabla_x \phi_f\|_{L^2}^2 \leq C \| |v|^2 f \|_{L^1} \, \|f\|_{L^1}^{\alpha_1} \, \|f\|_{L^p}^{\alpha_p}$

$\Longrightarrow$ *The strength of the potential energy has been reinforced by dimension.*

- the relativistic gravitational Vlasov-Poisson system in dimension 3 :

$$\partial_t f + \frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0, \qquad f(t=0,x,v) = f_0(x,v),$$

$$\phi_f(t,y) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \rho_f(t,y) dy, \qquad \rho_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v) dv.$$

Energy :

$$\mathcal{H}(f) = \int_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1 \right) f(t,x,v) dx dv - \int_{\mathbb{R}^3} |\nabla_x \phi_f(t,x)|^2 dx.$$

Interpolation : $\|\nabla_x \phi_f\|_{L^2}^2 \leq C \|\sqrt{1+|v|^2} f\|_{L^1} \, \|f\|_{L^1}^{\beta_1} \, \|f\|_{L^p}^{\beta_p}$

$\Longrightarrow$ *The strength of the kinetic energy has been reduced by special relativity.*

➠ the Manev correction to the Vlasov-Poisson system in dimension 3 :

$$\partial_t f + v \cdot \nabla_x f - \nabla_x(\phi_f + \widetilde{\phi}_f) \cdot \nabla_v f = 0,$$

$$\phi_f = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \rho_f dy, \qquad \widetilde{\phi}_f = -\kappa \int_{\mathbb{R}^3} \frac{1}{|x-y|^2} \rho_f dy.$$

Energy :

$$\mathcal{H}(f) = \int_{\mathbb{R}^6} |v|^2 f(t,x,v) dx dv - \int_{\mathbb{R}^3} |\nabla_x \phi_f(t,x)|^2 dx + \int_{\mathbb{R}^3} \rho_f \, \widetilde{\phi}_f \, dx.$$

Interpolation : $\int_{\mathbb{R}^3} \rho_f \, |\widetilde{\phi}_f| \, dx \leq C \| |v|^2 f \|_{L^1} \, \|f\|_{L^1}^{\gamma_1} \, \|f\|_{L^p}^{\gamma_p}$

$\Longrightarrow$ *The strength of the potential energy has been raised by the Manev correction.*

Invariants of the flow and energy space

The following quantities are independent of time :

- $\|f(t)\|_{L^q}$ for all $q \in [1,\infty]$, and more generally all $\|j(f)\|_{L^1}$

- $$\mathcal{H}(f) = \int_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1\right) f(t,x,v)\,dx\,dv - \int_{\mathbb{R}^3} |\nabla_x \phi_f(t,x)|^2 dx$$

INVARIANTS OF THE FLOW AND ENERGY SPACE

The following quantities are independent of time :

- $\|f(t)\|_{L^q}$ for all $q \in [1, \infty]$, and more generally all $\|j(f)\|_{L^1}$
- $\mathcal{H}(f) = \int_{\mathbb{R}^6} \left(\sqrt{1+|v|^2} - 1\right) f(t,x,v)dxdv - \int_{\mathbb{R}^3} |\nabla_x \phi_f(t,x)|^2 dx$

The interpolation inequality : if $p > 3/2$

$$\begin{aligned} \|\nabla_x \phi_f\|_{L^2}^2 &\leq C_{inter} \||v|f\|_{L^1} \|f\|_{L^1}^{\frac{2p-3}{3(p-1)}} \|f\|_{L^p}^{\frac{p}{3(p-1)}} \\ &\leq C_{inter} \|\sqrt{1+|v|^2} f\|_{L^1} \|f\|_{L^1}^{\frac{2p-3}{3(p-1)}} \|f\|_{L^p}^{\frac{p}{3(p-1)}} \end{aligned}$$

The energy space :

$$\mathcal{E}_p = \left\{ f \geq 0 \text{ with } f \in L^1, \quad f \in L^p, \quad |v|f \in L^1 \right\}$$

CONTROL OF THE KINETIC ENERGY ?

Glassey-Schaeffer (1985) have proved that the Cauchy problem is well-posed in the case of spherically symmetric solutions as long as the kinetic energy remains bounded.

The interpolation inequality can lead to a bound on the kinetic energy :

$$\begin{aligned}\mathcal{H}(f_0) = \mathcal{H}(f(t)) &= \|\sqrt{1+|v|^2} f(t)\|_{L^1} - \|\nabla_x \phi_f(t)\|_{L^2}^2 - \|f(t)\|_{L^1} \\ &\geq \|\sqrt{1+|v|^2} f(t)\|_{L^1} \left(1 - C_{inter} \|f(t)\|_{L^1}^{\frac{2p-3}{3(p-1)}} \|f(t)\|_{L^p}^{\frac{p}{3(p-1)}}\right) - \|f(t)\|_{L^1} \\ &= \|\sqrt{1+|v|^2} f(t)\|_{L^1} \left(1 - C_{inter} \|f_0\|_{L^1}^{\frac{2p-3}{3(p-1)}} \|f_0\|_{L^p}^{\frac{p}{3(p-1)}}\right) - \|f_0\|_{L^1}\end{aligned}$$

This yields a global existence criterion :

$$C_{inter} \|f_0\|_{L^1}^{\frac{2p-3}{3(p-1)}} \|f_0\|_{L^p}^{\frac{p}{3(p-1)}} < 1\,.$$

Blow up

Conversely, Glassey and Schaeffer (1985) have given a well-known argument based on the virial :

$$\int |x|^2 f(t,x,v)dxdv \leq (\mathcal{H}(f_0) + \|f_0\|_{L^1})t^2 + C(f_0)(1+t).$$

If $\mathcal{H}(f_0) + \|f_0\|_{L^1} < 0$ then the solution cannot exist for all time.

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If $\mathcal{H}(f_0) + \|f_0\|_{L^1} < 0$ then the solution cannot exist for all time.

Program : construct in a variational way two types of solutions.

- Stable steady states with satisfy the global criterion (”subcritical”).
- Nearly self-similar blow up solutions (”supercritical’).

Functions under the form $f(x,v) = F\left(\sqrt{1+|v|^2} + \phi_f(x)\right)$ are steady states :

$$\frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0.$$

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A natural construction of such solution via a variational problem : minimize the energy under two constraints

$$\min \left\{ \mathcal{H}(f) \text{ where } f \in \mathcal{E}_p\,, \quad \int f = M_1\,, \quad \int f^p = M_p \right\}.$$

Short bibliography.

- This problem has been well understood for the classical VP system :
  Wolansky, Guo, Rein in 1999–2001
  See also Schaeffer, Dolbeault, Sanchez, Soler, Lemou, FM, Raphael...
- For the RVP system, there are less results. Hadzic and Rein (2007) have constructed stable steady states in a non variational way, by solving nonlinear Poisson equations.

The minimization problem with two constraints

Consider the problem

$$\min \left\{ \mathcal{H}(f) \text{ where } f \in \mathcal{E}_p \,, \quad \int f = M_1 \,, \quad \int f^p = M_p \right\}$$

Two difficulties :

(i) that $\inf \mathcal{H} = -\infty$: ill-posed problem ;

(ii) that $\inf \mathcal{H} = 0$ with no minimizer (minimizing sequences converge to 0).

(i) does not occur.

Let M_1, M_p subcritical in the sense

$$C_{inter} M_1^{\frac{2p-3}{3(p-1)}} \, M_p^{\frac{1}{3(p-1)}} < 1\,.$$

The same calculation as above shows that

$$\begin{aligned} \mathcal{H}(f) \quad &\geq \|\sqrt{1+|v|^2} f\|_{L^1} \left(1 - C_{inter} \|f\|_{L^1}^{\frac{2p-3}{3(p-1)}} \, \|f\|_{L^p}^{\frac{p}{3(p-1)}} \right) - \|f\|_{L^1} \\ &= \|\sqrt{1+|v|^2} f\|_{L^1} \left(1 - C_{inter} M_1^{\frac{2p-3}{3(p-1)}} \, M_p L^{p\,\frac{p}{3(p-1)}} \right) - M_1 \end{aligned}$$

Hence we have $\inf \mathcal{H} \geq -M_1$: the Hamiltonian is bounded from below under this condition.

Note that any minimization problem with only one constraint leads to unbounded Hamiltonian.

(ii) does not occur.

A crucial property of ”breaking homogeneity” prevents (ii) : let

$$f_\lambda(x,v) = f\left(\frac{x}{\lambda}, \lambda v\right).$$

Then

$$\begin{aligned} \lambda \mathcal{H}(f_\lambda) &= \int \frac{|v|^2 f}{\sqrt{\lambda^2 + |v|^2} + \lambda} dx dv - \frac{1}{2}\|\nabla \phi_f\|_{L^2}^2 \\ &\sim -\frac{1}{2}\|\nabla \phi_f\|_{L^2}^2 \quad \text{as } \lambda \to +\infty. \end{aligned}$$

Hence $\inf \mathcal{H} < 0$, which will prevent minimizing sequences from vanishing.

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Hence $\inf \mathcal{H} < 0$, which will prevent minimizing sequences from vanishing.

Note that if we replace $\sqrt{1+|v|^2}-1$ by $|v|$ (ultrarelativistic VP system), we have

$$\mathcal{H}(f_\lambda) = \frac{1}{\lambda}\mathcal{H}(f)$$

and the subcritical problem is not attained by a minimizer : $\inf \mathcal{H} = 0$.

THE EULER-LAGRANGE EQUATION

Theorem. *Under the subcritical assumption on M_1, M_p, every minimizing sequence is relatively compact in the energy space. Moreover, any minimizer Q satisfies the following Euler-Lagrange equation :*

$$\sqrt{1+|v|^2} + \phi_Q = \lambda + \mu Q^{p-1} \quad \text{on } \mathit{Supp}(Q), \quad \lambda, \mu < 0$$

In other words, Q is the polytrope

$$Q = \left(\frac{\sqrt{1+|v|^2} + \phi_Q(x) - \lambda}{\mu} \right)_+^{\frac{1}{p-1}}$$

Proof. Application of concentration-compactness techniques due to P.-L. Lions.

FROM THIS COMPACTNESS THEOREM TO A STABILITY RESULT

If the minimizer is unique (or isolated), one can deduce directly from this theorem the stability of Q by the RVP flow (simple contradiction argument). Crucial : RPV preserves $\mathcal{H}$, $\|f\|_{L^1}$ and $\|j(f)\|_{L^1}$. But the question of uniqueness is open !

From this compactness theorem to a stability result

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➠ New trick based on the rigidity of the flow.

In fact, VPR also preserves any Casimir functional $\int G(f)$, ie $f(t)$ is always equimeasurable with f_0.

Consequence : the flow only selects minimizers which are equimeasurable :

$$\text{meas}\{(x,v),\ Q_1(x,v)>\alpha\} = \text{meas}\{(x,v),\ Q_2(x,v)>\alpha\}, \quad \forall \alpha>0.$$

We conclude the stability proof by showing that equimeasurable minimizers are isolated (in fact, that there are at most two minimizers equimeasurable together).

What happens when the subcritical condition is not satisfied ?

$$C_{inter}\|f\|_{L^1}^{\frac{2p-3}{3(p-1)}}\|f\|_{L^p}^{\frac{p}{3(p-1)}}>1$$

➠ When finite time blow up occurs, velocities are very large and a good model to understand the dynamics is the ultrarelativistic VP system (URVP) : ”all particles have the speed of light”

$$\partial_t f+\frac{v}{|v|}\cdot\nabla_x f-\nabla_x\phi_f\cdot\nabla_v f=0$$

➠ A simple model which displays the same invariance properties is the classical VP system in dimension 4 (VP4D).

$$\partial_t f+v\cdot\nabla_x f-\nabla_x\phi_f\cdot\nabla_v f=0$$

INVARIANTS OF THE SYSTEM, INTERPOLATION INEQUALITIES

For (VP) in dimension 4 ($x \in \mathbb{R}^4, v \in \mathbb{R}^4$), the following quantities still do not depend on t :

- $\|f(t)\|_{L^q}$ for all $q \in [1, \infty]$, or more generally all $\|j(f)\|_{L^1}$
- $\mathcal{H}(f) = \int_{\mathbb{R}^8} |v|^2 f(t,x,v) dx dv - \int_{\mathbb{R}^4} |\nabla_x \phi_f(t,x)|^2 dx$

and the interpolation inequality is also critical :

$$\|\nabla_x \phi_f\|_{L^2}^2 \leq C \||v|^2 f\|_{L^1} \, \|f\|_{L^1}^{\frac{p-2}{2(p-1)}} \, \|f\|_{L^p}^{\frac{p}{2(p-1)}}.$$

We thus have the same phenomenology as for RVP, but with supplementary symmetry properties...

Variational theory

First problem : in dimension 4, the previous minimization problem is ill-posed, the energy is not bounded from below (as for URVP).

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Alternative : we shall consider the question of optimization of the interpolation constant

$$\inf_{f\neq 0} \frac{\| |v|^2 f \|_{L^1} \, \|f\|_{L^1}^{\frac{p-2}{2(p-1)}} \, \|f\|_{L^p}^{\frac{p}{2(p-1)}}}{\|\nabla_x \phi_f\|_{L^2}^2}.$$

This problem is well-posed and admits a 3 parameters family of solutions : $\gamma Q\left(\frac{x}{\lambda}, \mu v\right)$, with Q defined by

$$Q(x,v) = \left(-1 - \frac{|v|^2}{2} - \phi_Q\right)_+^{\frac{1}{p-1}}.$$

Lemma. *If* $\mathcal{H}(f) = \mathcal{H}(Q) = 0$, $\|f\|_{L^1} = \|Q\|_{L^1}$ *and* $\|f\|_{L^p} = \|Q\|_{L^p}$ *then there exists* $\lambda > 0$ *such that* $f(x,v) = Q(\frac{x}{\lambda}, \lambda v)$.

Crucial ! This new invariance parameter λ.

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By concentration-compactness techniques, one can prove the following ”stability result” :

Theorem. *For all* $\varepsilon > 0$, *there exists* $\alpha > 0$ *such that if* f_0 *satisfies*

$$|\mathcal{H}(f_0) - \mathcal{H}(Q)| < \alpha, \quad |\|f_0\|_{L^1} - \|Q\|_{L^1}| < \alpha, \quad |\|f_0\|_{L^p} - \|Q\|_{L^p}| < \alpha$$

then, for some $\lambda(t) > 0$, *we have*

$$\left\| f(t,x,v) - Q\left(\frac{x}{\lambda(t)}, \lambda(t)v\right) \right\|_{\mathcal{E}_p} < \varepsilon.$$

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It remains a free parameter to be controlled : finite-time blow up occurs when $\lambda(t) \to 0$ for $t \to T$.

CONSTRUCTION OF BLOWING UP SELF-SIMILAR SOLUTIONS

Let us search special solutions of (VP4D) under the form

$$f(t,x,v) = g\left(\frac{x}{\lambda(t)}, \lambda(t)v\right).$$

Then g satisfies

$$v \cdot \nabla_x g - \nabla_x \phi_g \cdot \nabla_v g - \lambda\dot{\lambda}\left(x \cdot \nabla_x g - v \cdot \nabla_v g\right) = 0.$$

Construction of blowing up self-similar solutions

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It is natural to set $-\lambda\dot{\lambda} = b$ with $b > 0$, which implies $\lambda = \sqrt{2b(T-t)}$ (blow up as $t \to T$). The self-similar equation reads

$$v\cdot\nabla_x g - \nabla_x\phi_g\cdot\nabla_v g + b\left(x\cdot\nabla_x g - v\cdot\nabla_v g\right) = 0.$$

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$$v \cdot \nabla_x g - \nabla_x \phi_g \cdot \nabla_v g + b\left(x \cdot \nabla_x g - v \cdot \nabla_v g\right) = 0.$$

Generically, we seek a function $g(x,v) = F\left(\frac{|v|^2}{2} + \phi_g(x) + +bx \cdot v\right)$.

4. A model problem : VP in dimension 4

Difficulty : the level sets of $\frac{|v|^2}{2} + \phi_g + bx \cdot v = \frac{|v-bx|^2}{2} - \frac{b^2|x|^2}{2} + \phi_g$ go to the infinity. Because of this tail, such functions g do not have finite energy and mass.

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Trick : in fact, for b small enough, this solution has two connected components. Moreover, one can ”throw out” the tail, which does not create a gravitational field on the other part of the function because of spherical symmetry.

This amounts to seek $g(x,v) = F(\frac{|v|^2}{2} + \phi_g + b\chi(x)x \cdot v)$ where $\chi(x)$ is a truncation.

$\Longrightarrow$ For b small enough, we have $g(x,v) = F\left(\frac{|v|^2}{2} + \phi_g(x) + +bx \cdot v\right)$ on the support of g.

Third variational problem !

We construct such solution by studying the following problem :

$$\min\left\{\int |v|^2 f + b\chi(x)x\cdot vf + f + f^p \text{ where } f\in\mathcal{E}_p \text{ and } \|\nabla_x f\|_{L^2} = C_0\right\}$$

In fine, we construct a self-similar blowing up solution for VP4D :

$$f(t,x,v) = Q_b\left(\frac{x}{\sqrt{2b(T-t)}}, \sqrt{2b(T-t)}v\right).$$

The same can be done for URVP... In order to be able to come back to RVP, we need first a stability result for the blow up profile.

Stability of the self-similar blow up dynamics

Theorem. *For* $b_0 > 0$ *is small enough and* λ_0 *is large enough, there exists* $T > 0$ *and* $\alpha > 0$ *such that if*

$$\left\| f_0 - Q_{b_0}\left(\frac{x}{\lambda_0}, \lambda_0 v\right) \right\|_{\mathcal{E}_p} < \alpha$$

then the solution of VP4D blows up in finite time and

$$f(t,x,v) = \left(Q_{b(t)} + \varepsilon\right)\left(t, \frac{x}{\lambda(t)}, \lambda(t) v\right),$$

where $C_1\sqrt{T-t} \leq \lambda(t) \leq C_2\sqrt{T-t}$,

$$0 < b_0 \leq b(t) \leq 2b_0$$

and the function $\varepsilon(t,x,v)$ *remains small in* L^1.

Sketch of the proof : contrary to the previous proofs, it is not only variational but based on the dynamics of the VP equation. It is inspired by works of Merle and Raphael for NLS in the critical regime.

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- Start with a detailed analysis of the linearized VP flow to detect the algebraic instability directions.
- Apply the modulation theory to write

$$f(t,x,v) = (Q_{b(t)} + \varepsilon)\left(t, \frac{x}{\lambda(t)}, \lambda(t)v\right)$$

so that $\varepsilon(t,x,v)$ is orthogonal to two of these directions.

- The linearized energy enables to control $\varepsilon(t)$.
- The key point is the control of $b(t)$, which relies on the virial identity.
- Control of $\lambda(t)$ by the self-similar equation $\lambda\dot{\lambda} \sim -b$.

5. Coming back to RVP : a stable blow up dynamics

The idea : when the system RVP blows up, we have $\| \sqrt{1+|v|^2} f \|_{L^1} \to +\infty$ whereas $\| f \|_{L^1}$ remains bounded.

$\Longrightarrow$ velocities are large and the behavior of RVP is close to the one of the ultrarelativistic system

(URVP) $$\partial_t f + \frac{v}{|v|} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0$$

Advantage of this system : one can reproduce the previous analysis of VP4D.

The idea : when the system RVP blows up, we have $\| \sqrt{1+|v|^2} f \|_{L^1} \to +\infty$ whereas $\| f \|_{L^1}$ remains bounded.

$\Longrightarrow$ velocities are large and the behavior of RVP is close to the one of the ultrarelativistic system

(URVP) $$\partial_t f + \frac{v}{|v|} \cdot \nabla_x f - \nabla_x \phi_f \cdot \nabla_v f = 0$$

Advantage of this system : one can reproduce the previous analysis of VP4D.

Let

$$f(t,x,v) = g\left(\frac{x}{b(T-t)}, b(T-t)v \right).$$

Then f satisfies URVP iff g satisfies :

$$\frac{v}{|v|} \cdot \nabla_x g - \nabla_x \phi_g \cdot \nabla_v g + b(x \cdot \nabla_x g - v \cdot \nabla_v g) = 0$$

A solution of this equation provides a blowing up solution of URVP.

Theorem.

The system URVP admits a stable family of blow up self-similar solutions under the form

$$f(t,x,v) = Q_b\left(\frac{x}{b(T-t)}, b(T-t)v\right),$$

with

$$Q_b(x,v) = (-|v| - \phi_{Q_b} - b\chi(x)x\cdot v - 1)_+^{1/(p-1)}.$$

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Theorem.

There exists solutions of RVP under the form

$$f(t,x,v) = (Q_{b(t)} + \varepsilon)\left(\frac{x}{\lambda(t)}, \lambda(t)v\right),$$

where $C_1(T-t) \leq \lambda(t) \leq C_2(T-t)$, $0 < b_0 \leq b(t) \leq 2b_0$ *and the function* $\varepsilon(t,x,v)$ *remains small.*

6. Conclusion

M. Lemou, F. M., P. Raphaël, *Stable ground states for the relativistic gravitational Vlasov-Poisson system*, to appear in Comm. Partial Diff. Eq.

- Variational construction of stable steady states, using the argument of ”homogeneity breaking”.
- New argument of stability without uniqueness, using the rigidity of the flow and re-usable to other collisionless kinetic systems.

M. Lemou, F. M., P. Raphaël, *Stable self-similar blow up dynamics for the three dimensional relativistic gravitational Vlasov-Poisson system*, J. Amer. Math. Soc. **21** (2008), no. 4, 1019-1063.

- Construction of self-similar blow up solution for VP4D and URVP in the energy space.
- Characterization of a profile of self-similar blow up solutions for RVP : tells more than the obstructive virial argument (Glassey, Schaeffer 1985).