Asymptotic dynamics of a population density under selection-mutation

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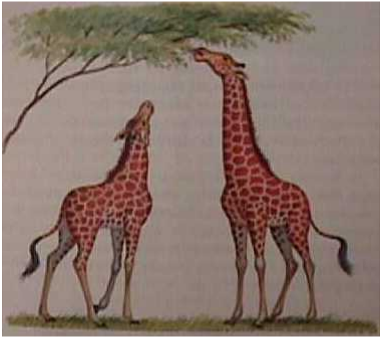
Joint work with B. Perthame and G. Barles

October 2009

Darwinian evolution of a structured population density

- Population models are structured by a parameter representing a phenotipical trait.

- We study the population dynamics under selection and mutations between the traits.



Mathematical modeling (1)

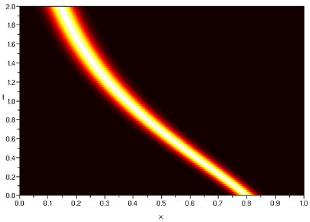
$$\begin{cases} \partial_t n_\epsilon - \epsilon \triangle n_\epsilon = \frac{n_\epsilon}{\epsilon} R(x, I_\epsilon(t)), & x \in \mathbb{R}^d,\ t \geq 0, \\ n_\epsilon(t=0) = n_\epsilon^0 \in \mathcal{L}^1(\mathbb{R}^d), & n_\epsilon^0 \geq 0, \end{cases}$$

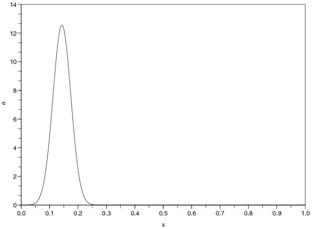
$$I_\epsilon(t) = \int_{\mathbb{R}^d} \psi(x)\, n_\epsilon(t,x) dx.$$

- $x \in \mathbb{R}^d$: A phenotipical trait,
- $n_\epsilon(t,x)$: The density of trait x,
- $I(t)$: The pressure exerted by the population on the ressource,
- $R(x, I)$: The growth and death rates of trait x,
- ϵ : A small parameter that we introduce to consider only rare mutations.

Mathematical modeling (1)

- $R(x, I) = 1 - \frac{x^2}{2} - I.$





- At left : Dynamics of the concentration point. At right : The population density at final time $t = 2$

Mathematical modeling (2)

$$\begin{cases} \partial_t n_\epsilon = \frac{n_\epsilon}{\epsilon} R(x, I_\epsilon(t)) + \frac{1}{\epsilon} \int \frac{1}{\epsilon^d} K(\frac{y-x}{\epsilon})\, b(y, I_\epsilon)\, n_\epsilon(t,y)\, dy, \\ n_\epsilon(t=0) = n_\epsilon^0 \in \mathcal{L}^1(\mathbb{R}^d), \qquad n_\epsilon^0 \geq 0, \end{cases}$$

$$I_\epsilon(t) = \int_{\mathbb{R}^d} \psi(x)\, n_\epsilon(t,x) dx.$$

- $K(z)$: A probability kernel
- ϵ : A small parameter that we introduce to consider only small mutations.
- **Ref :** *G. Barles, S. Mirrahimi, B. Perthame, Concentration in Lotka-Volterra parabolic equations : a general convergence result.Preprint march 2009.*

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Mathematical modeling (1)

$$\begin{cases} \partial_t n_\epsilon - \epsilon \triangle n_\epsilon = \frac{n_\epsilon}{\epsilon} R(x, I_\epsilon(t)), & x \in \mathbb{R}^d,\ t \geq 0, \\ n_\epsilon(t=0) = n_\epsilon^0 \in \mathcal{L}^1(\mathbb{R}^d), & n_\epsilon^0 \geq 0, \end{cases} \tag{1}$$

$$I_\epsilon(t) = \int_{\mathbb{R}^d} \psi(x)\, n_\epsilon(t,x) dx. \tag{2}$$

Some notations

- $n(t,x)$: The weak limit of $n_\epsilon(t,x)$ as ϵ vanishes,
- We expect n to concentrate as Dirac masses,
- A change of variables : $n_\epsilon(t,x) = e^{\frac{u_\epsilon(t,x)}{\epsilon}}$.

Results

Theorem (G. Barles, S. Mirrahimi, B. Perthame)

Assume $(5)-(10)$*. Let* n_ϵ *be the solution to the equations* $(1)-(2)$*, and* $u_\epsilon = \epsilon \ln(n_\epsilon)$*. Then, after extraction of a subsequence,* u_ϵ *converges locally uniformly to a function* $u \in \mathrm{C}(\mathbb{R}^+ \setminus \{0\} \times \mathbb{R}^d)$*, a viscosity solution to the following equation :*

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(x, I(t)), \\ \max_{x \in \mathbb{R}^d} u(t,x) = 0, \quad \forall t > 0, \end{cases} \tag{3}$$

$$I_\epsilon(t) \underset{\epsilon \to 0}{\longrightarrow} I(t) \quad a.e., \qquad \int \psi(x) n(t,x) dx = I(t) \quad a.e.. \tag{4}$$

If additionally $(u_\epsilon^0)_\epsilon$ *is a sequence of uniformly continuous functions which converges locally uniformly to* u^0 *then* $u \in \mathrm{C}(\mathbb{R}^+ \times \mathbb{R}^d)$ *and* $u(0,x) = u^0(x)$*.*

Results

$$\bar{x}(t) \in supp\ n(t, \cdot)$$

$$\implies u(\bar{x}(t), t) = 0$$

$$\implies R(\bar{x}(t), I(t)) = 0$$

Assumptions

$$0 < \psi_m < \psi < \psi_M < \infty, \;\; \psi \in W^{2,\infty}(\mathbb{R}^d), \tag{5}$$

$$\min_{x\in\mathbb{R}^d} R(x, I_m) = 0, \;\; \max_{x\in\mathbb{R}^d} R(x, I_M) = 0, \tag{6}$$

$$-K_1 \leq \frac{\partial R}{\partial I}(x, I) < -K_1^{-1} < 0, \tag{7}$$

$$\sup_{\frac{I_m}{2} \leq I \leq 2I_M} \| R(\cdot, I) \|_{W^{2,\infty}(\mathbb{R}^d)} < K_2, \tag{8}$$

Assumptions

$$I_m \leq \int_{\mathbb{R}^d} \psi(x) n_\epsilon^0(x) \leq I_M, \qquad (9)$$

$$\exists\, A,\ B > 0\,,\ n_\epsilon^0 \leq e^{\frac{-A|x|+B}{\epsilon}}. \qquad (10)$$

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Preliminary results

Theorem

With the assumptions $(5)-(10)$, *and* $I_m - C\epsilon^2 \leq I_\epsilon^0(t) \leq I_M + C\epsilon^2$, *there is a unique solution* $n_\epsilon \in \mathrm{C}(\mathbb{R}^+; L^1(\mathbb{R}^d))$ *to equations* $(1)-(2)$ *and it satisfies*

$$I'_m = I_m - C\epsilon^2 \leq I_\epsilon(t) \leq I_M + C\epsilon^2 = I'_M,$$

where C *is a constant. This solution,* $n_\epsilon(t,x)$, *is nonnegative for all* $t \geq 0$.

Preliminary results

G. Barles, B. Perthame, 2007 :

- With the assumptions (5) – (10), we have a locally uniform BV bound for I_ϵ.
- Particularly, after extraction of a subsequence, $I_\epsilon(t)$ converges a.e. to a function $I(t)$, while ϵ goes to 0.

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By replacing $n_\epsilon = e^{\frac{u_\epsilon}{\epsilon}}$ in equation (1), we deduce that u_ϵ is a smooth solution to the following equation :

$$\begin{cases} \partial_t u_\epsilon - \epsilon \triangle u_\epsilon = |\nabla u_\epsilon|^2 + R(x, I_\epsilon(t)), & x \in \mathbb{R},\ t \geq 0, \\ u_\epsilon(t=0) = \epsilon \ln n_\epsilon^0. \end{cases}$$

Regularity results

Theorem (Regularity of u_ϵ)

Define $v_\epsilon = \sqrt{2D^2 - u_\epsilon}$. *With the assumptions* $(5)-(10)$, *for all* $t_0 > 0$ v_ϵ *are locally uniformly bounded and Lipschitz in* $[t_0, \infty[\times\mathbb{R}^d$,

$$|\nabla v_\epsilon| \leq C(T) + \frac{1}{\sqrt{2t_0}},$$

where $C(T)$ *is a constant depending on* T, K_1, K_2, A *and* B. *Moreover, if we assume that* $(u_\epsilon^0)_\epsilon$ *is a sequence of uniformly continuous functions, then* u_ϵ *are locally uniformly bounded and continuous in* $[0, \infty[\times\mathbb{R}^d$.

Regularity results

- Step 1 An upper bound for u_ϵ,
- Step 2 Regularizing effect in space,
- Step 3 Regularity in space of u_ϵ near $t = 0$,
- Step 4 Local bounds from below for u_ϵ,
- Step 5 Regularizing effect in time.

Regularity results - Regularizing effect in space

Step 2 Let

$$u = f(v).$$

Then we have

$$\partial_t v - \epsilon \triangle v - \left[\epsilon \frac{f''(v)}{f'(v)} + f'(v) \right] |\nabla v|^2 = \frac{R(x, I)}{f'(v)}.$$

We define $p = \nabla v$. By differentiating the previous equation we obtain

$$\begin{aligned}
\partial_t p_i - \epsilon \triangle p_i - 2\left[\epsilon \frac{f''(v)}{f'(v)} + f'(v)\right] \nabla v \cdot \nabla p_i \\
- \left[\epsilon \frac{f'''(v)}{f'(v)} - \epsilon \frac{f''(v)^2}{f'(v)^2} + f''(v)\right] |\nabla v|^2 p_i \\
= -\frac{f''(v)}{f'(v)^2} R(x, I) p_i + \frac{1}{f'(v)} \frac{\partial R}{\partial x_i}.
\end{aligned}$$

Regularity results - Regularizing effect in space

Let $f(v) = -v^2 + 2D^2$, where $D(T) = \sqrt{B + CT}$. Then we have

$$\frac{\partial |p|}{\partial t} - \epsilon \triangle |p| - 2\left[\epsilon \frac{f''(v)}{f'(v)} + f'(v)\right] p \cdot \nabla |p| \\ +2|p|^3 - \frac{K_2}{2D^2}|p| - \frac{K_2}{2D} \leq 0.$$

Thus for $\theta(T)$ large enough we have

$$\frac{\partial (|p| - \theta)}{\partial t} - \epsilon \triangle (|p| - \theta) \\ -2\left[\epsilon \frac{f''(v)}{f'(v)} + f'(v)\right] p \cdot \nabla (|p| - \theta) + 2(|p| - \theta)^3 \leq 0. \qquad (11)$$

Regularity results - Regularizing effect in space

Define the function

$$y(t,x) = y(t) = \frac{1}{2\sqrt{t}} + \theta.$$

Since $y - \theta$ is a solution to (11), and $y(0) = \infty$ and $|p| - \theta$ being a sub-solution we have

$$|\nabla v|(x,t) = |p|(x,t) \leq y(x,t) = \frac{1}{2\sqrt{t}} + \theta(T), \quad 0 < t \leq T.$$

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Asymptotic behavior of u_ϵ

step 1 (Limit) We proved that u_ϵ are locally uniformly bounded and continuous. So by Arzela-Ascoli Theorem after extraction of a subsequence, u_ϵ converges locally uniformly to a continuous function u.

step 2 (Initial condition) We have $u(0,x) = \lim_{\epsilon \longrightarrow 0} u_\epsilon(0,x) = u^0(x)$. So the initial condition is proved.

step 3 $\left(\max_{x\in\mathbb{R}^d} u = 0\right)$

- $u(t,x) \leq 0$: If $0 < a < u(t,x)$, $\Rightarrow$ $n_\epsilon(t,x) \underset{\epsilon\longrightarrow 0}{\longrightarrow} \infty$, $\Rightarrow$ $I_\epsilon(t) \underset{\epsilon\longrightarrow 0}{\longrightarrow} \infty$.
- $\max_{x\in\mathbb{R}^d} u = 0$: If $u(t,x) < -a < 0$, then $I_\epsilon(t) \underset{\epsilon\longrightarrow 0}{\longrightarrow} 0$.

Asymptotic behavior of u_ϵ

step 4 (Limit equation) Properites of Viscosity solutions.

- According to step 1, $u_\epsilon(t,x)$ converge locally uniformly to the continuous function $u(t,x)$ as ϵ vanishes.
- We have $I_\epsilon(s) \longrightarrow I(s)$ a.e. as ϵ goes to 0.
- The function $R(x,I)$ is smooth.

$$\phi_\epsilon(t,x) = u_\epsilon(t,x) - \int_0^t R(x, I_\epsilon(s))ds$$

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A model with local competitions

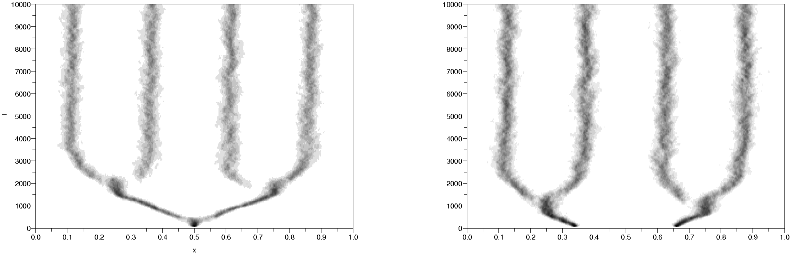
$$\begin{cases} \partial_t n_\epsilon = \epsilon \triangle n_\epsilon + \frac{1}{\epsilon} n_\epsilon (1 - \Phi * n_\epsilon) \\ n_\epsilon(0,x) = n_\epsilon^0(x) \geq 0, \end{cases}$$

where the convolution kernel Φ satisfies

$$\Phi \geq 0, \qquad \int \Phi = 1.$$

A model with local competitions

- In the presence of local competitions we can observe polymorphism and branching.



- A joint work with Emeric Bouin (LJLL) and Pierre Millien (LJLL)

Thank you !