Congestion in macroscopic models for sheep herds

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- Sheep herds : local interactions $\Rightarrow$ collective movement.

- Congestion : non-overlapping constraint $\Rightarrow$ maximal density ρ^*
  $\Rightarrow$ transition between free and constrained movement
  $\Rightarrow$ incompressibility/compressibility
- Model for the displacement of a sheep herd
  All group memberships have the same speed

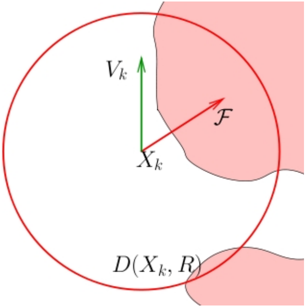
Plan

Plan

Microscopic model

- Attraction-repulsion interactions (no alignement)
- N sheeps : positions $X_k \in \mathbb{R}^2$
  velocities $V_k \in \mathbb{R}^2$, with $|V_k| = 1$

$$\frac{dX_k}{dt} = V_k,$$
$$\frac{dV_k}{dt} = (\mathrm{Id} - V_k \otimes V_k)(\underbrace{\mathcal{F}_k^a}_{\text{attractive term}} - \underbrace{\mathcal{F}_k^r}_{\text{repulsive term}}),$$



- $\mathcal{F}_k^a$ in the direction of the barycenter of the mass distribution in the disc of radius R_a
- $\mathcal{F}_k^r$ in the direction of the barycenter of the mass distribution in the disc of radius R_r

Microscopic model- speed constraint

- Attraction-repulsion interactions (no alignement)
- N sheeps : positions $X_k \in \mathbb{R}^2$
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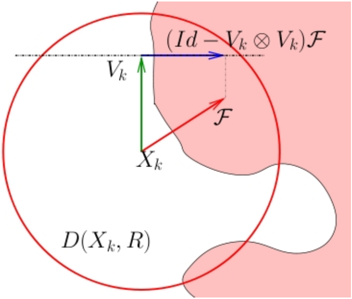


- $|V_k|^2 = 1 \quad \Rightarrow \quad \frac{dV_k}{dt} \perp V_k$

$\Rightarrow$ $(\mathrm{Id} - V_k \otimes V_k)$ = orthogonal projection matrix on the orthogonal plane to V_k.

Microscopic Model - long range attraction, short range repulsion

$$\begin{aligned} \frac{dX_k}{dt} &= V_k, \\ \frac{dV_k}{dt} &= (\mathrm{Id} - V_k \otimes V_k)(\underbrace{\mathcal{F}_k^a}_{\text{attractive term}} - \underbrace{\mathcal{F}_k^r}_{\text{repulsive term}}), \end{aligned}$$

- $\mathcal{F}_k = \nu_k \xi_k$
  $\rightarrow$ ν_k, intensity
  $\rightarrow$ ξ_k = barycenter of mass distrib. in disc $D(X_k, R)$
  $$= \left(\sum_{j, |X_j - X_k| < R} (X_k - X_j) \right) / \left(\sum_{j, |X_j - X_k| < R} 1 \right)$$
- Attraction force : long range and moderate intensity
  Repulsion force : short range and strong intensity

$$R_r \ll R_a \text{ and } \nu_a \ll \nu_r$$

Mean-field Limit

- $f(x, v, t)$ probability distribution function, $x \in \mathbb{R}^2, v \in S^1$
- Great number of interacting particles : $N \to +\infty$ (Mean-field limit)

$$f^N(x, v, t) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - X_k(t))\ \delta(v, V_k(t)) \quad \underset{N\to+\infty}{\longrightarrow} \quad f$$

- f satisfies : $\quad \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot ((\mathrm{Id} - v \otimes v)\,(\mathcal{F}_a - \mathcal{F}_r)\, f) = 0$

$$\mathcal{F}_{a,r}(x, v, t) = \nu_{a,r} \xi_{a,r}, \quad \xi_{a,r}(x, t) = \frac{\int_{D(x, R_{a,r})} (y - x) \rho(y, t) dy}{\int_{D(x, R_{a,r})} \rho(y, t) dy}$$

$$\rho(x, t) = \int_v f(x, v, t) dv = \text{density}$$

Rescaling of the kinetic model

- Large time and space dynamics : hydrodynamic rescaling

$$\tilde{x} = \eta x, \quad \tilde{t} = \eta t, \quad \eta \ll 1$$

Repulsive terms :

1. $R_r = O(\eta)$
2. $\nu_r = O(1)$

$\rightarrow \mathcal{F}_r = \eta \nu_r \frac{\nabla_x \rho}{\rho}$
$\rightarrow$ local repulsive force

Attractive terms :

1. $R_a = O(1)$
2. $\nu_a = O(\eta)$

$\rightarrow \mathcal{F}_a = O(\eta)$
$\rightarrow$ non local attractive force

- Congestion : ρ^* maximal density
  $\nu_r(\rho) \rightarrow +\infty$ as $\rho \rightarrow \rho^*$

Kinetic model

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot \left((\mathrm{Id} - v \otimes v) \left(\mathcal{F}_a - \mathcal{F}_r \right) f \right) = 0$$

$$\mathcal{F}_a = \nu_a \xi_a, \quad \xi_a(x,t) = \frac{\int_{D(x,R_a)} (y-x)\rho(y,t)dy}{\int_{D(x,R_a)} \rho(y,t)dy}$$

$$\mathcal{F}_r = \nu_r(\rho) \frac{\nabla_x \rho}{\rho} =: \nabla_x p(\rho)$$

with p such as $p'(\rho) = \nu_r(\rho)/\rho$

Macroscopic model

- Moments : density $\rho = \int f(x,v,t)dv$
  momentum $\rho u = \int_v v f(x,v,t)dv$
- Monokinetic assumption : $f(x,v,t) = \rho(x,t)\delta(v,u(x,t)),\ |u| = 1.$
  "Locally, only one velocity"

Integration of the kinetic equation leads to

$$|u| = 1$$
$$\partial_t \rho + \nabla_x \cdot \rho u = 0$$
$$\partial_t u + u \cdot \nabla_x u + (Id - u \otimes u)(\nabla_x p(\rho) - \mathcal{F}_a) = 0$$

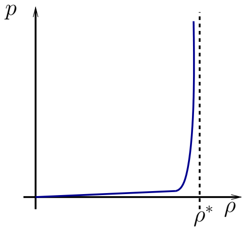
Plan

Pressure localization : asymptotic limit

- Focus on repulsion : $\mathcal{F}_a = 0$

$$\partial_t \rho + \nabla_x \cdot \rho u = 0,$$
$$\partial_t u + u \cdot \nabla_x u + (Id - u \otimes u)\nabla_x p(\rho) = 0$$
$$|u| = 1$$

- $p(\rho) \to +\infty$ as $\rho \to \rho^*$
- For $\rho \ll \rho^*$, no repulsion $\to$ free motion
  For $\rho \sim \rho^*$ $\to$ congestion



- ε : range of p for $\rho \ll \rho^*$
  $\Rightarrow$ We rescale p into εp

Two-phase model

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \rho^\varepsilon u^\varepsilon = 0$$
$$\partial_t u^\varepsilon + u^\varepsilon \cdot \nabla_x u^\varepsilon + (Id - u^\varepsilon \otimes u^\varepsilon)\varepsilon \nabla_x p(\rho^\varepsilon) = 0$$
$$|u^\varepsilon| = 1$$

- $\varepsilon p(\rho^\varepsilon(x,t)) \underset{\varepsilon \to 0}{\longrightarrow} \begin{cases} 0 & \text{if } \rho^\varepsilon(x,t) \to \rho < \rho^* \\ \bar{p}(x,t) & \text{if } \rho^\varepsilon(x,t) \to \rho^* \end{cases}$
- In the limit $\varepsilon \to 0$, two phases :

In the free motion phase $\rho < \rho^*$,

$$|u| = 1$$
$$\partial_t \rho + \nabla_x \cdot \rho u = 0$$
$$\partial_t u + u \cdot \nabla_x u = 0$$
$$\bar{p} = 0$$

Pressureless gaz dynamics

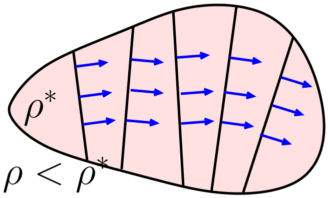
In the congested phase $\rho = \rho^*$,

$$|u| = 1$$
$$\rho = \rho^*, \quad \nabla_x \cdot u = 0$$
$$\partial_t u + u \cdot \nabla_x u + (Id - u \otimes u)\nabla_x \bar{p} = 0$$

Incompressible Euler

The congested phase

- Euler Incompressible equations with speed constraint
- $\nabla_x \cdot u = 0$ and $|u| = 1$
  $\rightarrow$ u constant on lines orthogonal to u



- Elliptic equation satisfied by $\bar{p}$ on each straight lines

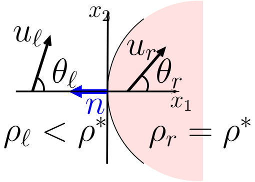
$$-\nabla_x \cdot ((Id - u \otimes u)\nabla_x \bar{p}) = \nabla_x^2 : (u \otimes u)$$

$\rightarrow$ boundary conditions ? Not given by formal asymptotics

Boundary conditions

- 1D Riemann problem accross the interface between the congested region $C_t = \{x, \rho(x) = \rho^*\}$ and non congested regions

$\cos\theta = u \cdot n$



- the 1D system with $\varepsilon > 0$ is not conservative $\rightarrow$ there exist a conservative form

$$\partial_t \rho \; + \; \partial_x(\rho \cos\theta) = 0$$
$$\partial_t \Psi(\cos\theta) \; + \; \partial_x \left(\Phi(\cos\theta) + \varepsilon p(\rho)\right) = 0$$

$$\Psi(u) = \frac{1}{2} \log\left((1+u)/(1-u)\right), \quad \Phi(u) = -\log\left(1/\sqrt{1-u^2}\right)$$

$\rightarrow$ no uniqueness of the conservative form but generic features

Boundary conditions

- Limit $\varepsilon \to 0$ of the Riemann problem solutions
- **Congested** ($\rho = \rho^*$) / **Uncongested** ($0 < \rho < \rho^*$) interface
  $\to$ Rankine Hugoniot conditions gives the pressure jump and interface velocity

$$\bar{p}_{|\partial C_t} = \frac{[\Psi(u \cdot n)][\rho(u \cdot n)]}{[\rho]} - [\Phi(u \cdot n)]$$

$$\sigma = \frac{[\rho(u \cdot n)]}{[\rho]}$$

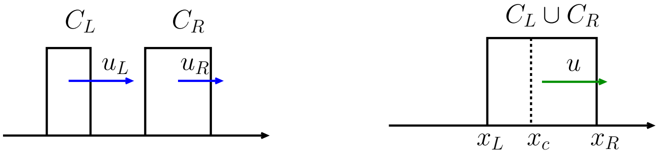
- **Congested** ($\rho = \rho^*$) / **Vacuum** ($\rho = 0$) interface

$$\bar{p}_{|\partial C_t} = 0$$

$$\sigma = u \cdot n$$

Boundary conditions

- **Collision** between two **Congested** regions
  → Riemann Problem does not provide solutions : $\bar{p} = \infty$
  → in 1D, collapsing clusters with a delta pressure in time
    - $\bar{p} = \pi(x)\delta(t - t_c)$, t_c collision time



    - u determined by

$$(\Psi(u) - \Psi(u_L))(x_c - x_L) + (\Psi(u) - \Psi(u_R))(x_R - x_c) = 0$$

    - analogy with two phase-flow models [Bouchut et al.]

  → in 2D, more complicated dynamics...

Plan

- Derivation of a macroscopic model with congestion and speed constraint
- Singular limit in the macroscopic model : free/congested transition
  Study of the compressible-incompressible transition

Outlooks :

- Numerical simulations and comparison with the microscopic model
- grazing time model with moving and motionless sheeps