Non-local aggregation models.

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The model.

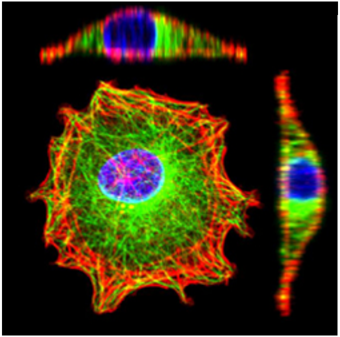
The model.

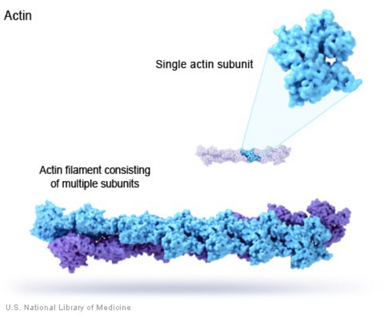
$$\begin{cases} \rho(0,\cdot) = \rho_0, \\ \partial_t \rho = \nabla_x \cdot \left(\rho \nabla_x \left(W *_x \rho + V\right)\right). \end{cases} \quad (1)$$

Civelekoglu G, Edelstein-Keshet L, Modelling the dynamics of F-actin in the cell. *Bull. Math. Biol.* **56**(4), 587–616 (1994).

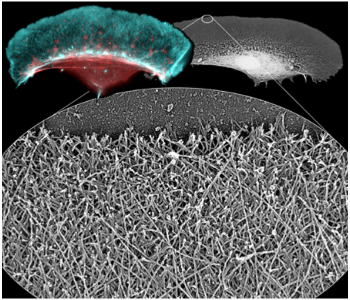
Primi I, Stevens A, Velazquez JJL. Mass-Selection in alignment models with non-deterministic effects. *Comm. Partial Differential Equations* **34**(5), (2009).

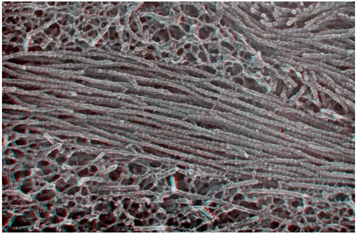
Actin filaments.



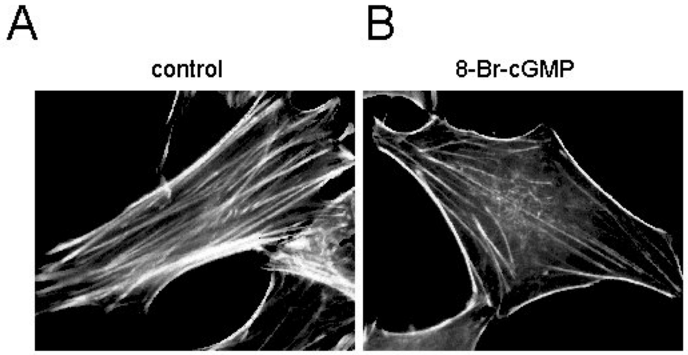


Mesh of actin filaments.

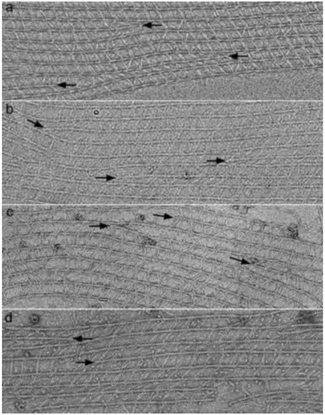


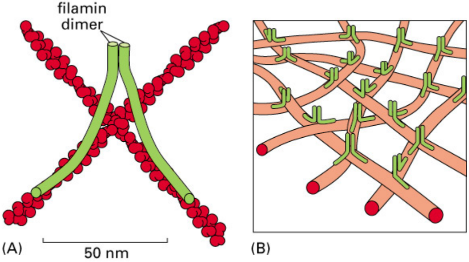


Actin filaments with or without cross-linking proteins.



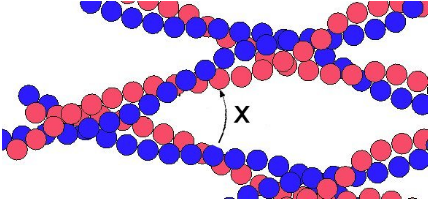
Cross-linking proteins.

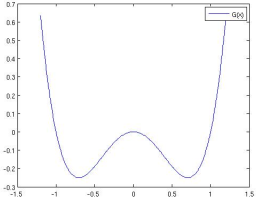




filamin, ABP-50, fibrillin, villin, fascin...

Action of Cross-linking proteins.





$W'(x)$: moment force between two filaments.

Notations.

We the case where the density of proteins is homogeneous in space.

- $\rho(t,\cdot) \in M^1(S^1 \text{ or } \mathbb{R})$. $\rho(t,x)$ is the density of filaments of orientation x. We normalise it by :

$$\int_{\mathbb{R}} d\rho(t,\cdot) = 1,$$

- $W(x)$ is the interaction potential between two filaments. We assume that this potential is symmetric.

The model.

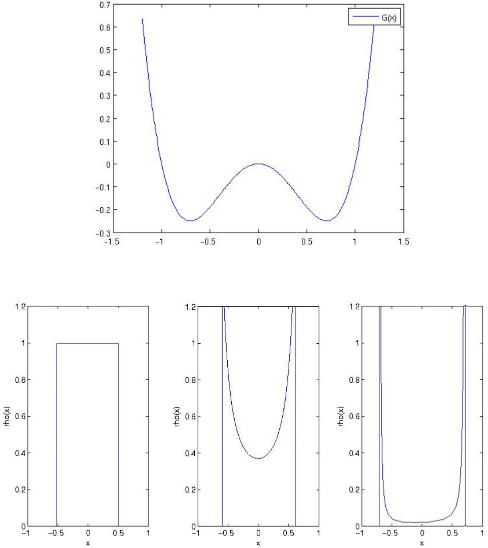
The force applied to one filament is :

$$\partial_x (W * \rho) .$$

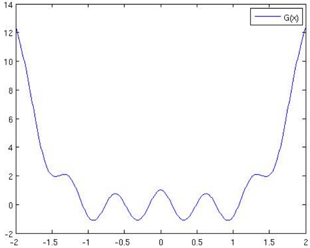
We assume that the rotating speed of a particle is proportional to the moment applied. Then, ρ evolves as :

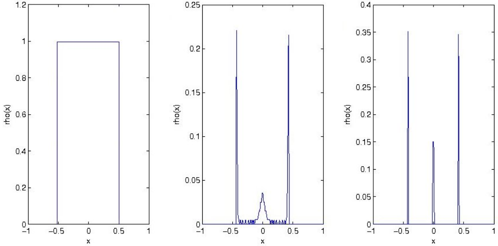
$$\begin{cases} \rho(0,\cdot) = \rho_0, \\ \partial_t \rho = \partial_x (\rho \partial_x (W *_x \rho)) . \end{cases}$$

simulations

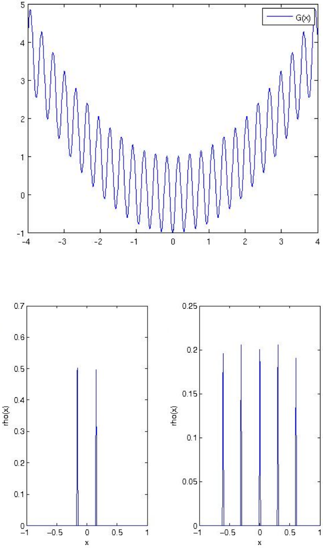


simulations

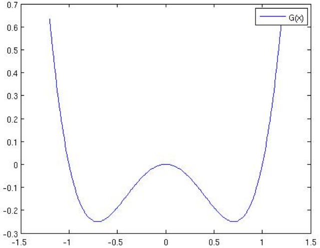


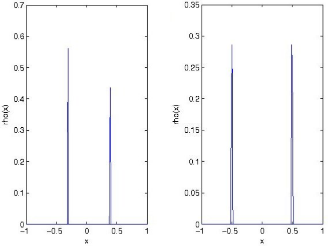


non-uniqueness of steady-states



non-uniqueness of steady-states





Local stability analysis

the solution converges to a set of steady-states

The solution cannot converge to anything but a steady-state:

Proposition

Let $W \in C^2(\mathbb{R})$. Then, For any sequence $t_k \to \infty$, there exists a subsequence, still denoted (t_k), and a steady-state $\bar{\rho} \in M^1(\mathbb{R})$ of (1), such that:

$$W_1\left(\rho(t_k, \cdot), \bar{\rho}\right) \to 0 \quad \text{as } k \to \infty, \tag{2}$$

where W_1 denotes the $1-$Wasserstein distance.

Proof : uses the energy of the system:

$$E(t) := \frac{1}{2} \int \int \rho(t,x)\rho(t,y)W(x-y)\,dx\,dy. \tag{3}$$

steady-states are sums of Dirac masses

Proposition

If W is analytical and W is confining (ie $W(x) \geq Cx^2 + C'$), then every compactly supported steady solution $\bar{\rho}$ of eq. (1) is a finite sum of Dirac masses $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{x_i}$.

Proof :

$$\forall x \in \mathrm{supp}\ \bar{\rho}, \quad 0 = \partial_x (W * \bar{\rho})(x).$$

Then, if $\mathrm{supp}\ \bar{\rho}$ has an accumulation point, $0 = \partial_x (W * \bar{\rho})$. Then $Cte = W * \bar{\rho}$, which is absurd since V and W are confining.

If W is less regular than analytical (C^2), L^1 steady-states may exist, but they cannot be linearly stable (in a sense to be defined).

condition for a sums of Dirac masses to be a steady-state

$$\partial_t \rho = \partial_x \left(\rho \partial_x \left(W *_x \rho \right) \right).$$

Proposition

$\bar{\rho} = \sum_{i=1}^n \bar{\rho}_i \delta_{\bar{x}_i}$, $\bar{\rho}_i \neq 0$ *is steady state of eq. (1) if and only if :*

$$(\bar{\rho}_i)_i \in Ker \left(\left(W'(\bar{x}_i - \bar{x}_j) \right)_{i,j} \right).$$

Proof : for $i = 1, \ldots, n$,

$$\partial_x \left(W *_x \rho \right) (\bar{x}_i) \quad = \quad \sum_j \bar{\rho}_j W' \left(\bar{x}_j - \bar{x}_i \right)$$

Necessary conditions for linear stability 1

Proposition

For a steady solution $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{x}_i}$, $\rho_i \neq 0$ *of eq. (1) to be linearly stable under small dislocations, it is necessary that :*

$$\forall i = 1, \ldots, n, \quad \partial^2_{xx}(W * \rho)(\bar{x}_i) > 0.$$

Necessary conditions for linear stability 2

Proposition

For a steady solution $\bar\rho = \sum_{i=1}^n \bar\rho_i \delta_{\bar x_i}$, $\rho_i \neq 0$ *of eq. (1) to be linearly stable under small perturbations of the* $\bar x_i$, *it is necessary that the linear application* $\mathcal{L}_M$ *defined by the matrix*

$$M = \left(\bar\rho_i W''(\bar x_i - \bar x_j)\right)_{i,j} - diag\left(\left(W''(\bar x_i - \bar x_j)\right)_{i,j} (\bar\rho_j)_j\right),$$

has a spectrum included in $\mathbb{R}^*_- \times i\mathbb{R}$ *when restricted to the hyperspace* $\{(w_i)_{i=1,\dots,n}; \sum_{i=1}^n w_i = 0\}$.

Local stability with support conditions

Proposition

Let $W \in C^2(\mathbb{R})$, and $\bar{\rho} = \sum_{i=1}^n \bar{\rho}_i \delta_{\bar{x}_i}$ a steady-state. If linear stability conditions 1 and 2 are satisfied, then $\bar{\rho}$ is locally stable under a support condition, that is :

$$W_\infty(\rho(t,\cdot),\bar{\rho}) \leq Ce^{-\kappa t},\ \kappa > 0,$$

$$\textit{as soon as } W_\infty(\rho^0,\bar{\rho}) \textit{ is small enough.}$$

Proof: $\rho(t, cdot) = \sum_{i=1}^n \tilde{\rho}_i(t,x)$, with $d(\bar{x}_i, \mathrm{supp}\ \rho_i(t,\cdot))$ small.
We control:

- $(\int x\rho_i(t,x)\,dx)_i$
- $d(\bar{x}_i, \mathrm{supp}\ \rho_i(t,\cdot))$

Local stability for usual M^1 topology

Remark

Steady-states are not locally unique. Thus, we only get the orbital stability of steady-states satisfying linear stability conditions 1 and 2.

Singular potentials

Applications of non-local aggregation equations.

- $W'(0^+) > 0$: Chemotaxis, swarming,
- W regular: granular media, Actin filaments,
- $W'(0^+) < 0$: swarming, crystallisation.

W with an attractive singularity.

After blow-up theory existence theory :

Carrillo JA, Di Francesco M, Figalli A, Laurent T, Slepčev D, Global-in-time weak measure solutions, finite-time aggregation and confinement for nonlocal interaction equations. *preprint UAB* **17**, (2009), submitted.

based on the gradient-flow structure of (1), with the energy:

$$E(t) := \frac{1}{2} \int \int \rho(t,x)\rho(t,y)W(x-y)\,dx\,dy.$$

W with an attractive singularity.

Stable steady-states are finite sums of Dirac masses:

Proposition

Let W be such that $W'(0^+) > 0$, and $\bar{\rho}$ be a compactly supported steady-state of (1). If $supp\ \bar{\rho}$ has an accumulation point x_0 (and a bit more...), then it is locally unstable: For any $\varepsilon > 0$, there exists $\rho^\varepsilon \in M^1(\mathbb{R})$, such that $W_1(\rho^\varepsilon, \bar{\rho}) \leq \varepsilon$ and

$$E(\rho^\varepsilon) < E(\bar{\rho}), \tag{4}$$

where E is the energy defined by (3).

W with an attractive singularity.

The local stability conditions from last section extend to interactions potentials having an attractive singularity:

Proposition

Let W be such that $W'(0^+) > 0$, and $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{x}_i}$, $\rho_i \neq 0$ be a steady-state of (1). $\bar{\rho}$ is locally (for W_∞) stable if the linear stability condition 2 is satisfied.

W with a repulsive singularity.

Existence theory :

Proposition

Let W be such that $W'(0^+) < 0$. Assume that $\rho^0 \in W^{2,\infty}(\mathbb{R})$. Then there exists a unique solution

$$\rho \in L^\infty(\mathbb{R}_+ \times \mathbb{R}) \cap Lip_{loc}(\mathbb{R}_+, W^{2,\infty}(\mathbb{R}))$$

to (1).

Notice that $\rho(t, \cdot)$ is uniformly bounded in $L^\infty(\mathbb{R})$!

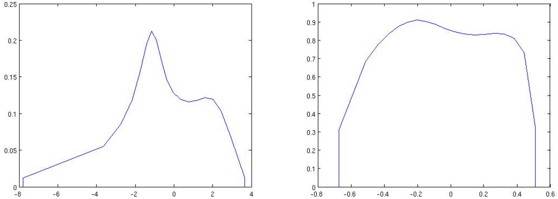
W with a repulsive singularity.

Example

If $W(x) := -|x| + x^2$, *then,*

$$\bar{\rho} := \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$$

is globally stable.

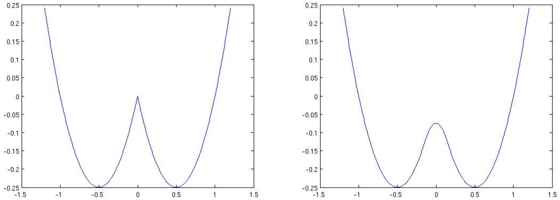


graphs of $\rho(t, \cdot)$ for $t = 0,\ 1.8$.

Link between regular W and W with a repulsive singularity.

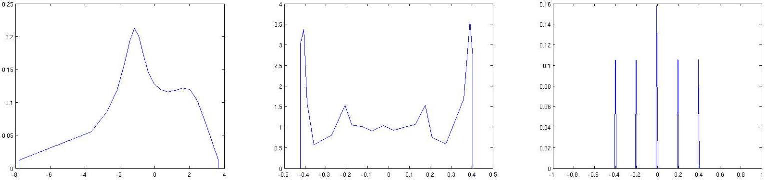
We consider smoothed versions of $W(x) := -|x| + x^2$:

$$W^\varepsilon(x) := \begin{cases} -|x| + x^2 \text{ on } [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^c, \\ -\frac{1}{\varepsilon}x^2 + x^2 - \frac{\varepsilon}{4} \text{ on } [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]. \end{cases}$$

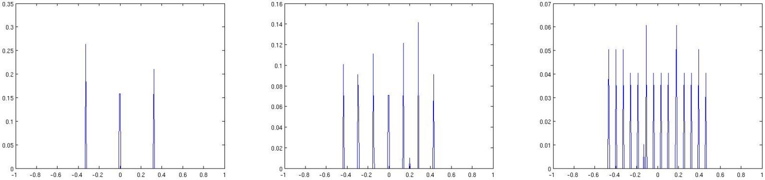


graphs of W and $W^{0.3}$.

Link between regular W and W with a repulsive singularity.



graphs of $\rho(t,\cdot)$ for $W^{0.3}$ and $t = 0,\ 5,\ 20.$



graphs of $\rho(t,\cdot)$ for $W^{0.5}$, $W^{0.2}$, $W^{0.1}$ and t large.

Link between regular W and W with a repulsive singularity.

For $\varepsilon > 0$, let ρ^ε be a steady-state of (1) with W^ε. Then,

$$\rho^\varepsilon \rightharpoonup \bar{\rho} \text{ in } M^1(\mathbb{R}),$$

where $\bar{\rho} = \mathbb{I}_{[-\frac{1}{2},\frac{1}{2}]}$.

Conclusion

- Singular attractive interaction potentials: finite-time convergence to a sum of Dirac masses,
- regular interaction potentials: infinite time convergence to a sum of Dirac masses,
- singular repulsive interaction potentials: convergence to L^∞ steady-states.

Interesting questions.

- Add a diffusion term to (1),
- more singular repulsive potentials,
- More complicated swarming equations,
- ...