A refined result of flocking for the Cucker-Smale model

J. Rosado

Universitat Autònoma de Barcelona

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Outline

1. Warming Up
2. Collapse of the velocity support
   - Proof for the N-particles system
   - Main Results

ODE and Kinectic Cucker-Smale model

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N-particles ODE system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i & x_i(0) = x_i^0, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} m_j a_{ij} \, (v_j - v_i) & v_i(0) = v_i^0, \end{cases}$$

with the communication rate, $\beta \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Kinetic Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x - y|^2)^{\gamma}} \, f(y, w, t) \, dy \, dw \right)}_{:= \xi(f)(x,v,t)} f(x, v, t) \Big]$$

Some References

- F. Cucker and S. Smale, On the mathematics of emergence, *Japan. J. Math.* **2** (2007) 197–227.
- S.-Y. Ha and J.-G. Liu, A simple proof of the Cucker-Smale flocking dynamics and mean-field limit, to appear in *Comm. Math. Sci.*
- S.-Y. Ha and E. Tadmor, From particle to kinetic and hydrodynamic descriptions of flocking, *Kinetic and Related Models* **1** (2008) 415–435.
- J. A. Cañizo, J. A. Carrillo, J. Rosado, A wel-posedness theory in measures for some kinetic models of collective motion, preprint UAB.
- J.-A. Carrillo, M. Fornasier, J. Rosado and G. Toscani, Asymptotic Flocking Dynamics for the kinetic Cucker-Smale model, preprint UAB.

Remarks and notation

Due to translation invariancy, w.l.o.g. the mean velocity is zero and thus the center of mass is preserved along the evolution, i.e.,

$$\sum_{i=1}^{N_p} m_i v_i(t) = 0 \qquad \text{and} \qquad \sum_{i=1}^{N_p} m_i x_i(t) = x_c$$

for all $t \geq 0$ and $x_c \in \mathbb{R}^d$.

Let us fix any $R_0^x > 0$ and $R_0^v > 0$, such that all the initial velocities lie inside the ball $B(0, R_0^v)$ and all positions inside $B(x_c, R_0^x)$.

Let us define the function $R^v(t)$ to be

$$R^v(t) := \max\left\{|v_i(t)|\,,\, i = 1, \ldots, N_p\right\}.$$

Sketch of the proof

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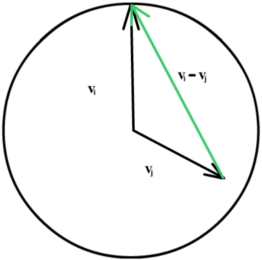
1. $R^v(t) < R_0(t) \quad \forall t > 0.$
2. $|x_i(t) - x_j(t)|$ grows at most linearly for any i, j.
3. $R^v(t) \to 0$ as $t \to \infty$.
4. $\exists R > 0$ st $x_i(t) \in B(x_c, R)$ $\forall t$, $\forall i$.
5. $R^v(t)$ behaves like $\exp\{-Ct\}$ for some constant $C > 0$.

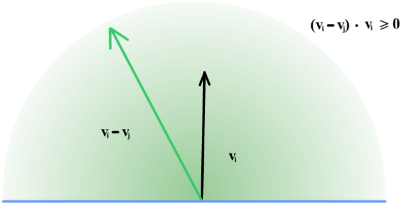
Step 1: $R^v(t) < R_0(t) \quad \forall t > 0$

Choosing the label i to be the one achieving the maximum, we get

$$\frac{d}{dt} R^v(t)^2 = \frac{d}{dt} |v_i|^2 = -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i \right] a(|x_i - x_j|) \,.$$

Because of the choice of the label i, we have that $(v_i - v_j) \cdot v_i \geq 0$ for all $j \neq i$ that together with $a \geq 0$ imply $R^v(t) \leq R_0^v$ for all $t \geq 0$.





Step 2: $|x_i(t) - x_j(t)| \leq C_1 + C_2 t$ for any i, j

Coming back to the equation for the positions,

$$|x_i(t) - x_i^0| \leq R_0^v t \quad \text{for all } t \geq 0 \text{ and all } i = 1, \ldots, N_p.$$

$$\begin{aligned} |x_i(t) - x_j(t)| &= |x_i(t) \pm x_i^0 \pm x_j^0 - x_j(t)| \\ &\leq |x_i(t) - x_i^0| + |x_j(t) - x_i^0| + |x_i^0 - x_j^0| \\ &\leq R_0(t) + R_0(t) + 2R_0^x \end{aligned}$$

Step 3: $R^v(t) \to 0$ as $t \to \infty$

$$a(|x_i - x_j|) \geq \frac{1}{[1 + 4R_0^2(1+t)^2]^\gamma} \text{ for all } t \geq 0 \text{ and all } i, j = 1, \ldots, N_p,$$

with $R_0 = \max(R_0^x, R_0^v)$.

Coming back to the equation for the maximal velocity

$$\begin{aligned} \frac{d}{dt} R^v(t)^2 &= -2 \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i \right] a(|x_i - x_j|) \\ &\leq -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} \sum_{j \neq i} m_j \left[(v_i - v_j) \cdot v_i \right] \\ &= -\frac{2}{[1 + 4R_0^2(1+t)^2]^\gamma} R^v(t)^2 := -f(t)\, R^v(t)^2, \end{aligned}$$

Steps 3: $R^v(t) \to 0$ as $t \to \infty$

Gronwall's lemma:

$$R^v(t) \leq R_0^v \, \exp\left\{ -\frac{1}{2} \int_0^t f(s)\, ds \right\}.$$

For $\gamma \leq 1/2$, the function $f(t)$ is not integrable at ∞ and therefore

$$\lim_{t\to\infty} \int_0^t f(s)\, ds = +\infty$$

and $R^v(t) \to 0$ as $t \to \infty$ giving the convergence to a single point, its mean velocity, of the support for the velocity.

Step 4: $\exists R > 0$ st $x_i(t) \in B(x_c, R)$ $\forall t$, $\forall i$

Again for the position variables, we get

$$\begin{cases} \int_0^t |v_i(s)|\, ds \le C_1 \int_0^t (1+s)^{-1-\epsilon}\, ds & \gamma < 1/2 \\ \int_0^t |v_i(s)|\, ds \le C \int_0^t \frac{1}{1+s}\, ds = C \ln(1+t) & \gamma = 1/2, \end{cases} .$$

There exists $R_1^x > 0$ such that

$$|x_i(t) - x_i^0| \le R_1^x$$

Step 5: $R^v(t) \sim \exp\{-Ct\}$

Now, $a(|x_i(t) - x_j(t)|) \geq a(2\bar{R}^x)$,

$$\begin{aligned}\frac{d}{dt}R^v(t)^2 =& -2\sum_{j\neq i} m_j\left[(v_i - v_j)\cdot v_i\right] a(|x_i - x_j|)\\ \leq& -2a(2\bar{R}^x)\sum_{j\neq i} m_j\left[(v_i - v_j)\cdot v_i\right] = -2a(2\bar{R}^x)R^v(t)^2\end{aligned}$$

from which we finally deduce the exponential decay to zero of $R^v(t)$.

Asymptotic Flocking

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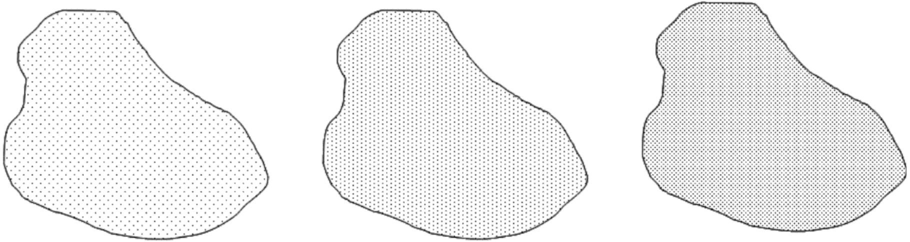
Unconditional Non-universal Flocking Result for general measures

Given $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ compactly supported, then the unique measure-valued solution to the CS kinetic model with $\gamma \leq 1/2$, satisfies the following bounds on their supports:

$$\text{supp}\ \mu(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

for all $t \geq 0$, with $R^x(t) \leq \bar{R}$ and $R^v(t) \leq R_0\, e^{-\lambda t}$ with $\bar{R}^x$ depending only on the initial support radius.

Idea of the proof



We have that

$$\text{supp } \mu_\eta(t) \subset B(x_c(0) + mt, R^x(t)) \times B(m, R^v(t))$$

$$d_{\mathbb{R}^{2d}}(\mu(t), \mu_\eta(t)) \leq \alpha(t)\, d_{\mathbb{R}^{2d}} \left(\mu_0, \sum_{i=1}^{N_p} m_i\, \delta(x - x_i^0)\, \delta(v - v_i^0) \right) \leq \alpha(t)\eta.$$

Then, $\mu_\eta(t) \to \mu(t)$ weakly-$*$ as measures when $\eta \to 0$ for all $t \geq 0$.

Thanks for your attention!