Measure solutions of the 2D parabolic-elliptic Keller-Segel model

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Joint work with J. Dolbeault (Paris)

Mathematical modelling of chemotaxis ...

The (Patlak- (1953)) Keller-Segel- (1970) model

$$\partial_t \rho + \nabla \cdot (\rho \chi \nabla S - D \nabla \rho) = 0$$
$$\partial_t S = D_S \Delta S + \alpha \rho - \beta S$$

Aggregation phenomena, blow-up in finite time

… stripped to the minimum (in 2D)

$$\partial_t \rho + \nabla \cdot (\rho \nabla S - \nabla \rho) = 0$$

$$S = -\frac{1}{2\pi} \ln(|x|) * \rho$$

Theorem *[Jäger, Luckhaus, Blanchet, Dolbeault, Perthame, Calvez, Carrillo, Masmoudi, Biler, Laurencot, Suzuki, Sugiyama, Herrero, Velazquez, ...]:*
Let $(1 + |x|^2 + \log \rho)\rho(t = 0) \in L^1(\mathbb{R}^2)$, $M = \int \rho(t = 0)dx$. Then
$M < 8\pi \implies$ global existence, self similar dispersion
$M > 8\pi \implies$ concentration in finite time
$M = 8\pi \implies$ global existence, concentration as $t \to \infty$

Preventing blow-up

- Kinetic transport models
  Chalub, Markowich, Perthame, CS (03)
  Results: a) Global existence
  b) Macroscopic limit (local in time): Keller-Segel

- Finite sampling radius. *Hillen, Painter, CS (06)*
  Replace ∇S by

$$\tilde{\nabla} S(x) = \frac{1}{\varepsilon\pi} \int_{|\omega=1|} \omega S(x + \varepsilon\omega) d\omega \longrightarrow \nabla S(x) \text{ as } \varepsilon \to 0\,,$$

  with sampling radius $\varepsilon > 0$.

Preventing blow-up, ctd.

- Finite size effects

$$\partial_t \rho + \nabla \cdot \left(\frac{1}{\varepsilon} G(\varepsilon \rho) \nabla S - \nabla \rho \right) = 0$$

Hillen & Painter (01), Dolak & CS (04): $G(\rho) = \rho(1-\rho)$
Velazquez (02): $G(\rho) \to$ const as $\rho \to \infty$

- Nonlinear diffusion

$$\partial_t \rho + \nabla \cdot (\rho \nabla S - (1 + \varepsilon \rho) \nabla \rho) = 0$$

- Regularization of the Poisson equation

$$S(x,t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y| + \varepsilon) \rho(y,t)\, dy$$

The regularized problem

$$\partial_t \rho^\varepsilon + \nabla \cdot (\rho^\varepsilon \nabla S_\varepsilon[\rho^\varepsilon] - \nabla \rho^\varepsilon) = 0$$

$$S_\varepsilon[\rho](x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y| + \varepsilon)\rho(y)dy$$

$$\rho^\varepsilon(t=0) = \rho_I \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$

Theorem: global existence, $\sup_{x,t} |\rho^\varepsilon(x,t)| \le c/\varepsilon^2$

Proof: $|\nabla S_\varepsilon[\rho^\varepsilon]| \le c/\varepsilon$ by mass conservation

Distributional formulation of the convective flux

$$\int_{\mathbb{R}^2} \phi\rho\nabla S_\varepsilon[\rho]dx = -\frac{1}{4\pi}\int_{\mathbb{R}^4} \frac{(\phi(x)-\phi(y))(x-y)}{|x-y|(|x-y|+\varepsilon)}\rho(x)\rho(y)dy\,dx$$

implies

$$\left|\int_{\mathbb{R}^2} \phi\rho\nabla S_\varepsilon[\rho]dx\right| \le \frac{M^2}{4\pi}|\phi|_{1,\infty} \quad \text{and} \quad \left|\frac{d}{dt}\int_{\mathbb{R}^2} \phi\rho^\varepsilon dx\right| \le c|\phi|_{2,\infty}$$

with $|\phi|_{k,\infty} = \max_{k_1+k_2=k} \sup_{\mathbb{R}^2} |\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}\phi|$

Lemma *(Poupaud 02):* $\{\rho^\varepsilon\}$ is tightly bounded uniform in ε and on bounded time intervals, and it is tightly equicontinuous.

The limit

Also

$$m^{\varepsilon}(x,t) := \int_{\mathbb{R}^2} \frac{(x-y)^{\otimes 2}}{|x-y|(|x-y|+\varepsilon)} \rho^{\varepsilon}(x,t)\rho^{\varepsilon}(y,t)dy$$

is (locally in time) uniformly tightly bounded. Therefore,

$$\rho^{\varepsilon}(t) \rightharpoonup \rho(t) \quad \text{tightly, for every } t\,,$$

$$m^{\varepsilon} \rightharpoonup m \quad \text{tightly, as a measure on } (t_1,t_2)\times\mathbb{R}^2$$

But, in general,

$$m(x,t) \neq \int_{\mathbb{R}^2} \frac{(x-y)^{\otimes 2}}{|x-y|^2} \rho(x,t)\rho(y,t)\,dy = ?$$

The defect measure

$$\nu(x,t) := m(x,t) - \int_{\mathbb{R}^2} \mathcal{K}(x-y)\rho(x,t)\rho(y,t)\,dy$$

with $\mathcal{K}(x) = x^{\otimes 2}/|x|^2$ for $x \neq 0$, and $\mathcal{K}(0) = 0$.

Lemma *(Poupaud 02):* $\nu \geq 0$ and

$$\mathrm{tr}(\nu(x,t)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)$$

The limiting flux

Lemma *(Poupaud 02):* $\rho^\varepsilon \nabla S_\varepsilon[\rho^\varepsilon] \to j[\rho,\nu]$ in the distributional sense with

$$\int_{\mathbb{R}^2} \phi j[\rho,\nu] dx = -\frac{1}{4\pi} \int_{x\neq y} \frac{(\phi(x)-\phi(y))(x-y)}{|x-y|^2} \rho(x)\rho(y) dy\, dx$$

$$-\frac{1}{4\pi} \int_{\mathbb{R}^2} \nu \nabla\phi \, dx$$

The limiting problem

Theorem: The problem

$$\partial_t \rho + \nabla \cdot (j[\rho,\nu] - \nabla\rho) = 0\,, \quad \rho(t=0) = \rho_I$$

has a global weak solution with $\rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2)$ for every $t \geq 0$ and tightly continuous, and with $\nu(t) \in \mathcal{M}(\mathbb{R}^2)^{\otimes 2}$ satisfying $\nu \geq 0$ and

$$\mathrm{tr}(\nu(x,t)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)\,.$$

The strong formulation

Assume

$$\rho(x,t) = \overline{\rho}(x,t) + \sum_n M_n(t)\delta(x - x_n(t))$$

with $\overline{\rho}$, M_n, and x_n smooth, and

$$\nu(x,t) = \sum_n \nu_n(t)\delta(x - x_n(t)) \quad \text{with} \quad \mathrm{tr}(\nu_n) \le M_n^2$$

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Then

$$\nu_n = 4\pi M_n id \quad \Longrightarrow \quad \mathrm{tr}(\nu_n) = 8\pi M_n \le M_n^2$$

The strong formulation, ctd.

$$\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) - \frac{1}{2\pi}\nabla\overline{\rho}\cdot\sum_n M_n(t)\frac{x-x_n}{|x-x_n|^2} = 0$$

$$\dot{M}_n = M_n\overline{\rho}(x=x_n)$$

$$\dot{x}_n = A_n := \nabla S_0[\overline{\rho}](x=x_n) - \frac{1}{2\pi}\sum_{m\neq n} M_m\frac{x_n-x_m}{|x_n-x_m|^2}$$

Long time behaviour for $M > 8\pi$

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\rho\,dx = 4M - \frac{1}{2\pi}\int_{\mathbb{R}^4}(1-\chi_D)\rho\otimes\rho\,dy\,dx - \frac{1}{2\pi}\int_{\mathbb{R}^2} tr(\nu)\,dx$$

Long time behaviour for $M > 8\pi$

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$$= \overline{M}\left(4 - \frac{M}{2\pi} - \frac{1}{2\pi}\sum_n M_n\right) - \frac{1}{\pi}\sum_{m\neq n} M_m M_n \le 0$$

The right hand side vanishes for $\rho = M\delta(x - X_I)$, where X_I is the center of mass (which does not move).

Other regularizations

- Finite sampling radius: same results, same limit

- Finite size effects: formal asymptotics by *Velazquez (02)*:

$$\dot{x}_n = \Gamma(M_n) A_n \,, \qquad 0 < \Gamma(M_n) < 1$$

  with $\Gamma(M_n)$ depending on the regularization.

- Nonlinear diffusion: formal asymptotics can be done

- Kinetic transport models: completely open

Thank you for your attention!